EISENSTEIN SERIES FOR PRINCIPAL CONGRUENCE SUBGROUPS OF $\mathrm{GL}(2, \mathbb{F}_q[T])$

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Abstract. We determine the zeroes of Drinfeld-Goss Eisenstein series for the principal congruence subgroups $\Gamma(N)$ of $\Gamma = \mathrm{GL}(2, \mathbb{F}_q[T])$ on the Drinfeld modular curve $X(N)$.

0. Introduction. In recent years, the study of Eisenstein series, both for the classical modular group $\mathrm{SL}(2, \mathbb{Z})$ and the Drinfeld modular group $\mathrm{GL}(2, \mathbb{F}_q[T])$ and the arithmetic of their zeroes led to remarkable and surprising results, see [1, 2, 4, 9, 10, 11, 21, 22, 24].

In the present paper we deal with the case of Eisenstein series for the principal congruence subgroup

$$\Gamma(N) = \{\gamma \in \Gamma \mid \gamma \equiv 1 \,(\mathrm{mod}\, N)\}$$

of $\Gamma = \mathrm{GL}(2, A)$ for some $N \in A := \mathbb{F}_q[T]$.

While the classical Eisenstein series

$$E^{(k)}(z) = \sum_{a,b \in \mathbb{Z}}{}' = \frac{1}{(az+b)^k}$$

(the $\sum'$ denotes the sum over all $(a, b) \neq (0, 0)$) have all their zeroes in the standard fundamental domain on the unit circle (equivalently: their j-invariants belong to the interval $[0, 1728]$), and the Drinfeld-Goss Eisenstein series [18]

$$E^{(k)}(z) = \sum_{a,b \in A}{}' \frac{1}{(az+b)^k}$$

have a similar property $[2, 9]$, the situation drastically changes once we replace $\Gamma = \mathrm{GL}(2, A)$ by $\Gamma(N)$ as above. Here the basic functions are partial sums of $E^{(k)}$ subject to congruence conditions. For technical reasons, we work with the equivalent functions

$$E_u^{(k)}(z) := \sum_{\substack{a,b \in \mathbb{F}_q(T) \\ (a,b) \equiv u \,(\bmod\, A \times A)}}{}' \frac{1}{(az+b)^k},$$

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where $u = N^{-1}(u_1, u_2)$ with $u_i \in A$, $\deg u_i < \deg N$ $(i = 1, 2)$. It turns out that these *Eisenstein series with level* N have their zeroes in the standard fundamental domain $\mathcal{F}$ in specified subdomains $\mathcal{F}_s$ "far away from the unit circle".

The description is given in Theorem 3.1, our main result. The distribution pattern of the zeroes is governed by the Goss polynomial $G_k(X)$ (see section 2) of the lattice A. Our results depend on the determination of the Newton polygon of $G_k(X)$ over the valued field $K_\infty = \mathbb{F}_q((T^{-1}))$, which has been carried out in [13] for the case of a prime field $\mathbb{F}_q$. The general case will be given in [14].

The paper is organized as follows.

In section 1 we collect the necessary definitions, notations and background on Drinfeld modular forms and curves.

In section 2 we review facts about Goss polynomials and determine the vanishing order of $E_u^{(k)}$ at the cusp ∞.

Section 3 is devoted to the statement and proof of the main result Theorem 3.1, which describes the location of the zeroes of $E_u^{(k)}$ in the fundamental domain $\mathcal{F}$ in terms of Goss polynomials. We also calculate the spectral norm of $E_u^{(k)}$ along $\mathcal{F}$ (Corollary 3.9).

Section 4 gives the overall picture of the zeroes of $E_u^{(k)}$ on the modular curve $X(N)$.

We conclude in section 5 with a more detailed study of the two extremal cases where the weight k equals $q + 1$ (the first non-trivial case; if $1 \leq k \leq q$ then $E_u^{(k)} = (E_u^{(1)})^k$ has no non-cuspidal zeroes) or where the conductor N has degree one.

The present study suggests an abundance of natural questions, for example about the arithmetic nature of the zeroes, about similar results for other congruence subgroups of Γ, e.g., the Hecke congruence subgroups $\Gamma_0(N)$, or about the analogous number-theoretical case.

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Notations.

$\mathbb{F} = \mathbb{F}_q$ = finite field with q elements, q = power of the prime p
$A = \mathbb{F}[T]$ = polynomial ring in an indeterminate T,
  $A_s = \{a \in A \mid \deg a \leq s\}$
$K = \mathbb{F}(T)$ = quotient field of A
$K_\infty = \mathbb{F}((T^{-1}))$ = completion of K at the place at infinity, with ring of integers $O_\infty = \mathbb{F}[[T^{-1}]]$ and its absolute value $| \, . \, |$ normalized such that $|T| = q$

C_∞ = completed algebraic closure of K_∞ w.r.t. $|\ .\ |$
$\Omega = C_\infty \backslash K_\infty$ the Drinfeld upper half-plane
$|\ .\ |_i \colon C_\infty \longrightarrow \mathbb{R}_{\geq 0}$ the "imaginary part" function, $|z|_i = \inf_{x \in K_\infty} |z - x|$
N a fixed non-constant element of A, of degree δ
$\Gamma = \mathrm{GL}(2, A)$ the Drinfeld modular group, which acts on the projective line $\mathbb{P}^1(C_\infty)$ through fractional linear transformations
$\Gamma(N) = \{\gamma \in \Gamma \mid \gamma \equiv 1 \,(\mathrm{mod}\, N)\}$ the principal congruence subgroup with conductor N
$\Gamma_\infty = \{\gamma \in \Gamma \mid \gamma = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\}$, the stabilizer group of ∞ in Γ
$Z = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \mid a \in \mathbb{F}^*\} \hookrightarrow \Gamma$, the kernel of the action on $\mathbb{P}^1(\mathbb{C}_\infty)$
$G(N) = \Gamma/\Gamma(N) \cdot Z$
$\mathbb{Q}_{\geq 0} = \{a \in \mathbb{Q} \mid a \geq 0\}$

1. Modular forms and curves [6, 7, 9, 18, 19].

Recall that the Drinfeld half-plane Ω carries a natural structure of C_∞-analytic space, so the notion of an analytic (holomorphic, meromorphic) function on Ω is meaningful. We define the following analytic subspaces of Ω:

$$\mathcal{F} := \{z \in \Omega \mid |z| = |z|_i \geq 1\} \tag{1.1}$$

and for $s \in \mathbb{Q}_{\geq 0}$

$$\mathcal{F}_s := \{z \in \Omega \mid |z| = |z|_i = q^s\}.$$

Then $\mathcal{F}$ is the disjoint union of the $\mathcal{F}_s$, and is a fundamental domain for the action of Γ on Ω, that is, each $z \in \Omega$ is Γ-equivalent with at least one and at most finitely many $z' \in \mathcal{F}$. The $\mathcal{F}_s$ are rational subdomains, isomorphic with a "Riemann sphere" $\mathbb{P}^1(C_\infty)$ minus $q+1$ disjoint open balls if $s \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ (resp. minus 2 disjoint open balls if $s \notin \mathbb{N}_0$), see [3, 16, 17]. Note that for $z \in \mathcal{F}$ and $a, b \in K_\infty$ the following useful formula holds:

$$|az + b| = \max\{|az|, |b|\}. \tag{1.2}$$

We also need

$$\begin{aligned} \Gamma_s &= \{\gamma \in \Gamma \mid \gamma(\mathcal{F}_s) \cap \mathcal{F} \neq \emptyset\} = \{\gamma \in \Gamma \mid \gamma(\mathcal{F}_s) = \mathcal{F}_s\} \\ &= \mathrm{GL}(2, \mathbb{F}),\ s = 0 \\ &= \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Gamma \mid a, d \in \mathbb{F}^*,\ b \in A_s\},\ s > 0. \end{aligned} \tag{1.3}$$

Further, $\Gamma_s(N) := \Gamma_s \cap \Gamma(N)$ has size

$$\#\Gamma_s(N) = q^{\max([s] - \delta + 1, 0)}, \tag{1.4}$$

where $\delta = \deg N \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $[s]$ = largest integer $n \leq s$.

Given a discrete A-submodule X of C_∞ (i.e., the intersection of Λ with each ball

$B(0,s)$ with finite radius s is finite; such Λ are called *A-lattices*), let

$$e_\Lambda(z) := z \prod_{0\neq\lambda\in\Lambda} (1 - z/\lambda) \tag{1.5}$$

be its *lattice function.* The product converges, locally uniformly, and defines an entire, surjective, $\mathbb{F}$-linear function $e_\Lambda\colon C_\infty \longrightarrow C_\infty$, which apparently is Λ-periodic and may be written as

$$e_\Lambda(z) = \sum_{i\geq 0} \alpha_i(\Lambda) z^{q^i},\ \alpha_0(\Lambda) = 1. \tag{1.6}$$

Taking logarithmic derivatives, we get the identity of meromorphic functions

$$\frac{e'_\Lambda(z)}{e_\Lambda(z)} = \frac{1}{e_\Lambda(z)} = \sum_{\lambda\in\Lambda} \frac{1}{z-\lambda} =: t_\Lambda(z). \tag{1.7}$$

We define the *uniformizer at* ∞

$$t(z) := t_A(z) = \sum_{a\in A} \frac{1}{z-a}; \tag{1.8}$$

it yields an isomorphism of analytic spaces

$$A\backslash\mathcal{F} \xrightarrow{\cong} B(0,1) \setminus \{0\}, \tag{1.9}$$

where the left hand side is the set of equivalence classes modulo the action of A on $\mathcal{F}$ by shifts $z \longmapsto z+a$ and the right hand side the pointed ball with radius 1 around zero. Regarded as a function on $\mathcal{F}$, $|t(z)|$ depends only on $|z| = |z|_i$, and is a strictly decreasing function of $|z|$. Similarly, we let

$$t_N(z) := \frac{1}{e_{NA}(z)} = \sum_{a\in NA} \frac{1}{z-a}, \tag{1.10}$$

which yields $NA \setminus \mathcal{F} \xrightarrow{\cong} B(0,r) \setminus \{0\}$ with some r.

1.11 Remark. For arithmetical purposes it is useful to choose other normalizations of t resp. t_N, which involve transcendental constants and correspond to the classical $e^z \rightsquigarrow e^{2\pi i z}$. That renormalization is however irrelevant for our purpose, as is the precise value of the radius r above.

For a function f of Ω and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \gamma \in \Gamma$, we put as usual

$$f_{[\gamma]_k}(z) := (cz+d)^{-k} f(\frac{az+b}{cz+d}), \tag{1.12}$$

which defines a right action of Γ on functions.

A *modular form of weight k* for $\Gamma(N)$ is a holomorphic function $f\colon \Omega \longrightarrow C_\infty$ that satisfies

(1.13) (i) for each $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \gamma \in \Gamma(N))$, $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$;

(ii) for each $\gamma \in \Gamma$, the function $f_{[\gamma]_k}$ has a power series expansion, convergent for $|z|_i \gg 0$:

$$f_{[\gamma]_k}(z) = \sum_{i \geq 0} a_i t_N^i(z).$$

Note that $|z|_i$ large is equivalent with $|t_N(z)|$ small, so the above expansion is nothing else than the Laurent expansion of $f_{[\gamma]_k}$ on the pointed ball $NA \setminus \mathcal{F} \xrightarrow{\cong} B(0,r) \setminus \{0\}$. It suffices to check property (ii) for γ running through a set of representatives in the finite set

$$\text{cusps}(N) := \Gamma/\Gamma(N)\Gamma_\infty. \tag{1.14}$$

We further let $M_k(N)$ be the C_∞-vector space of modular forms of weight k for $\Gamma(N)$ and $M(N) = \bigoplus_{k \geq 0} M_k(N)$ the algebra of all modular forms.

1.15 Example. Let k be a natural number and $0 \neq u \in (K/A)^2$ a class with $Nu = 0$. The *Eisenstein series*

$$E_u^{(k)}(z) := \sum_{\substack{(a,b) \in K^2 \\ (a,b) \equiv u \,(\bmod\, A^2)}} \frac{1}{(az+b)^k}$$

converges locally uniformly on Ω and defines an element $0 \neq E_u^{(k)}$ of $M_k(N)$. Its study and notably the determination of its zeroes is our main objective. We represent the row vector u by $\frac{1}{N}(u_1, u_2)$ with $u_i \in A$ not both zero, $d_i := \deg u_i < \delta = \deg N$ $(i = 1, 2)$. Further, we will restrict to considering $E_u^{(k)}$ with u *primitive of level* N, i.e., $N'u \neq 0$ for proper divisors N' of N; otherwise, we replace N by $N/\gcd(u_1, u_2, N)$.

An easy calculation yields the fundamental property for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \gamma \in \Gamma$:

$$E_u^{(k)}(\gamma z) = (cz+d)^k (E_{u\gamma}(z), \text{ that is, } E_u^{(k)})_{[\gamma]_k} = E_{u\gamma}^{(k)}, \tag{1.16}$$

where $u\gamma$ is the effect of right matrix multiplication of u with γ. We abbreviate

$$E_u(z) := E_u^{(1)}(z) = \sum_{(a,b) \equiv u \,(\bmod\, A^2)} \frac{1}{az+b},$$

which by (1.7) equals $e_u^{-1}(z)$, with

$$e_u(z) := e_{Az+A}\left(\frac{u_1 z + u_2}{N}\right). \tag{1.17}$$

This shows in particular that E_u has no zeroes as a function on Ω.

Next, we discuss modular curves. We let $X(N)$ be the smooth connected algebraic curve over C_∞ (the *principal modular curve of level* N, see [6, 18]) whose C_∞-points are given by

$$X(N)(C_\infty) = \Gamma(N) \setminus \Omega \,\dot{\cup}\, \Gamma(N) \setminus \mathbb{P}^1(K).$$

As Γ acts transitively on $\mathbb{P}^1(K)$, we may identify

$$\mathrm{cusps}(N) = \Gamma/\Gamma(N)\Gamma_\infty \xrightarrow{\cong} \Gamma(N) \setminus \mathbb{P}^1(K), \tag{1.18}$$

which we call the set of *cusps* of $X(N)$. Its cardinality is

$$\#\mathrm{cusps}(N) = (q-1)^{-1}|N|^2 \prod_{\substack{P|N \\ P\text{monic, prime}}} (1-|P|^{-2}). \tag{1.19}$$

The function t_N of (1.10) serves as a uniformizer at the cusp ∞, and the behavior of e.g. modular forms $f \in M_k(N)$ at the cusp $\gamma\infty$ ($\gamma \in \Gamma$) is described through the behavior of $f_{[\gamma]_k}$ at ∞.

Similarly, the principal modular curve $X(1)$ of level 1 has points

$$X(1)(C_\infty) = \Gamma \setminus \Omega \,\dot\cup\, \{\infty\} \xrightarrow{\cong} \mathbb{P}^1(C_\infty),$$

where the identification is given by the Drinfeld j-invariant $j\colon \Gamma \setminus \Omega \xrightarrow{\cong} C_\infty$ defined and discussed e.g. in [5, 8, 19]. The curve $X(N)$ is a ramified Galois cover of $X(1)$ with group

$$G(N) := \Gamma/\Gamma(N)Z \xrightarrow{\cong} \{\gamma \in \mathrm{GL}(2, A/N) \mid \det\gamma \in \mathbb{F}^*\}/Z, \tag{1.20}$$

where Z ist the group of $\mathbb{F}^*$-valued scalar matrices, regarded simultaneously as a subgroup of Γ and of $\mathrm{GL}(2, A/N)$. Studying the ramification of $X(N)$ over $X(1)$, one finds [5, 18]:

$$g(N) = 1 + \frac{|N|-q-1}{q+1}\#\mathrm{cusps}(N) \tag{1.21}$$

for the genus $g(N)$ of $X(N)$. There is a line bundle $\mathcal{M}$ over $X(N)$, of degree

$$\deg(\mathcal{M}) = (q^2-1)^{-1}\#G(N) = (q+1)^{-1}|N|\,\#\mathrm{cusps}(N), \tag{1.22}$$

such that $M_k(N)$ equals the space $H^0(X(N), \mathcal{M}^{\otimes k})$ of sections of the k-fold tensor product $\mathcal{M}^{\otimes k}$ [7], VII 6.1, [18]. The order of vanishing of E_u (= pole order of e_u) at the cusps of $X(N)$ is described in [6] Korollar 2.2, see (2.12). It is the aim of the present work to give an overall picture of the zeroes (both cuspidal and non-cuspidal) of all the $E_u^{(k)}$.

Let now $z \in \Omega$ be Γ-equivalent with $z' \in \mathcal{F}$. Then $z' \in \mathcal{F}_s$ with a well-defined $s \in \mathbb{Q}_{\geq 0}$ (i.e., $|z'| = |z'|_i = q^s$ is independent of the choice of $z' \in \mathcal{F}$). We define the *type*

$$\mathrm{type}(z) := s, \tag{1.23}$$

which yields a function type: $X(N) \longrightarrow \mathbb{Q}_{\geq 0} \cup \{\infty\}$ (with the obvious declaration type $(z) = \infty$ for cusps z). We may now state a weak form of our main result.

1.24 Theorem. *All the zeroes of $E_u^{(k)}$ on $X(N)$ are at points with type $i \in \mathbb{N} = \{1, 2, 3 \ldots\}$ or $i = \infty$ (i.e., at cusps).*

In Theorem 3.1 and Proposition 2.12 we will describe in detail which and how often types $i \in \mathbb{N} \cup \{\infty\}$ occur as zeroes of $E_u^{(k)}$. In view of (1.16), possibly replacing u by γu, we may restrict to studying the behavior of $E_u^{(k)}$ on the fundamental domain $\mathcal{F}$.

2. Goss polynomials.

Let $\Lambda \subset C_\infty$ be an A-lattice with lattice function $e_\Lambda(z) = \sum_{i \geq 0} \alpha_i(\Lambda) z^{q^i}$ and $t_\Lambda(z) = \frac{1}{e_\Lambda(z)} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}$ as in (1.5) to (1.8). The following result has been proven in [19], see also [8].

2.1 Proposition. *There exists a series of polynomials $G_{k,\Lambda}(X) \in C_\infty[X]$ ($k = 1, 2, 3, \ldots$) such that we have an identity of meromorphic functions*

$$\sum_{\lambda \in \Lambda} \frac{1}{(z - \lambda)^k} = G_k(t_\Lambda(z)).$$

These Goss polynomials $G_k = G_{k,\Lambda}$ *satisfy*

(2.2) G_k *is monic of degree k with $G_k(0) = 0$;*

(2.3) *putting $G_k(X) = 0$ for $k \leq 0$, the recurrence*

$$G_k(X) = X(G_{k-1}(X) + \alpha_1 G_{k-q}(X) + \alpha_2 G_{k-q^2}(X) + \cdots)$$

with the coefficients $\alpha_i = \alpha_i(\Lambda)$ of $e_\Lambda(z)$ holds;

(2.4) $G_{pk}(X) = (G_k(X))^p$ *($p = \operatorname{char} \mathbb{F} = \operatorname{char} K$);*

(2.5) $X^2(G_k'(X) = kG_{k+1}(X)$*;*

(2.6) $G_k(X) = X^k$ *if $k \leq q$.*

2.7 Remark. For some questions it is useful to know how the quantities $e_\Lambda, t_\Lambda, G_{k,\Lambda}$ change if the lattice Λ is replaced by $\Lambda' = c \cdot \Lambda$ with $0 \neq c \in C_\infty$. The relevant (and easily proved) formulas can be found in [13] 2.20.

Recalling the notation of section 1, the identity

$$E_u(z) = t_\Lambda\left(\frac{u_1 z + u_2}{N}\right)$$

holds with the lattice $\Lambda = Az + A$. Therefore (2.4)+(2.6) yield the following immediate consequence:

2.8 Corollary. *Suppose that $k = k_1 \cdot p^n$ with $1 \leq k_1 \leq q$. Then*

$$E_u^{(k)} = E_u^k$$

holds. In particular, $E_u^{(k)}$ has no non-cuspidal zeroes. □

From now on, we focus on the Goss polynomials of the A-lattice A, which are crucial for our purposes. Therefore, $G_k(X) = G_{k,A}(X)$ will always refer to the lattice A; it is obvious from definitions that it has coefficients in K_∞. The next result has been shown in [13] in the special case where $q = p$ is prime; the proof of the general case will be given in [14].

2.9 Theorem. *Let $0 \neq x \in C_\infty$ be a zero of $G_k(X) = G_{k,A}(X)$. Then there exists some $n \in \mathbb{N}_0$ such that $\log_q |x| = -q(\frac{q^n-1}{q-1})$.*

In terms of the Newton polygon of the polynomial $G_k(X)$ over the valued field K_∞ ([23], Ch. II), the theorem may be phrased as follows: All the slopes of the Newton polygon of $G_k(X)$ have the form $-q(\frac{q^n-1}{q-1})$ for some $n \in \mathbb{N}_0$. (In fact, the possible n are less or equal to $\log_q(k-1) - 1$, see [13].)

Given k, we define

$$\begin{array}{lll} & \gamma(k) & := \text{multiplicity of } 0 \text{ as a zero of } G_k(X), \\ & & \text{and for } n \geq 0 \\ (2.10) & \gamma_n(k) & := \text{number of zeroes } x \text{ of } G_k(X) \text{ (counted} \\ & & \text{with multiplicity) with } \log_q |x| = -q(\frac{q^n-1}{q-1}) \\ & & = \text{width of the segment with slope } -q(\frac{q^n-1}{q-1}) \\ & & \text{of the Newton polygon of } G_k(X). \end{array}$$

By the theorem, $k = \gamma(k) + \sum_{n \geq 0} \gamma_n(k)$. Explicit formulas for these numbers in terms of the q-adic expansion of $k-1$ can be found in [13] and [14].

(2.11) As in (1.15), we let $0 \neq u \in (K/A)^2$ with $Nu = 0$ be represented by $\frac{1}{N}(u_1, u_2)$ with $u_i \in A$ of degree $d_i < \delta = \deg A$ $(i = 1, 2)$. We put $\deg 0 = -\infty$ and evaluate formulas containing $-\infty$ in the usual fashion. In particular, $q^{d_1} = |u_1| = 0$ if $u_1 = 0$.

2.12 Proposition. *The vanishing order of $E_u^{(k)}$ at the cusp ∞ equals $|u_1|\gamma(k)$.*

Proof. In what follows, we calculate formally and interchange limits and summation orders. The estimates justifying these operations are almost trivial, due to our non-archimedean situation, and are left to the reader. We have

$$\begin{aligned} E_u^{(k)}(z) &= \sum_{a,b \in A \times A} \frac{1}{\left((\frac{u_1}{N} + a)z + \frac{u_2}{N} + b\right)^k} = \sum_a \sum_b \cdots \\ &= \sum_a G_{k,A}\left(t_A\left(\left(\frac{u_1}{N} + a\right)z + \frac{u_2}{N}\right)\right). \end{aligned}$$

Suppose that $u_1 = 0$. The terms corresponding to $a \neq 0$ in the double sum vanish upon $|z|_i \longrightarrow \infty$, which implies

$$E_u^{(k)}(\infty) = \sum_{b \in A} \frac{1}{\left(\frac{u_2}{N} + b\right)^k} = \left(\frac{u_2}{N}\right)^{-k} + \text{ smaller terms,}$$

which thus doesn't vanish. Let now $u_1 \neq 0$. As $\deg u_1 < \deg N$, Lemma 2.13 and the definition of $\gamma(k)$ show that $G_{k,A}(t_A(\frac{u_1}{N}z + \frac{u_2}{N}))$ is a power series in the uniformizer t_N at infinity with precise vanishing order $|u_1|\gamma(k)$, while the terms $G_{k,A}(t_A(\frac{u_1}{N} + a)z + \frac{u_2}{N}$ with $a \neq 0$ have strictly larger vanishing orders when regarded as power series in t_N. □

2.13 Lemma. *Let c, d be elements of A, $c \neq 0$. The function $t_A(\frac{c}{N}z + \frac{d}{N})$ may be expanded as a power series in $t_N(z)$ of shape $C \cdot t_N^{|c|}+$ terms of higher order in t_N with some constant $C \neq 0$.*

Proof. The assertion is a well-known fact, and we give the proof for the reader's convenience only, who is assumed to be familiar with the basic theory of Drinfeld modules as e.g. presented in [20] Ch. IV or [7] Ch. IV.

Let ρ be the rank-one Drinfeld module that corresponds to the lattice NA. It yields for each $c \in A$ an operator polynomial $\rho_c(X)$ of degree $|c|$ such that

$$e_{NA}(cz) = \rho_c(e_{NA}(z)) \tag{1}$$

holds. Further, the lattice functions of A and NA are related by

$$e_{NA}(Nz) = Ne_A(z). \tag{2}$$

Now

$$t_A\left(\frac{c}{N}z + d\right) = \frac{1}{e_A\left(\frac{cz+d}{N}\right)} \overset{(2)}{=} \frac{N}{e_{NA}(cz+d)} \overset{(1)}{=} \frac{N}{\rho_c(e_{NA}(z)) + e_{NA}(d)}.$$

Taking into account that $e_{NA}(z) = t_N(z)^{-1}$ and expanding by $t_N^{|c|}$, we get $Nt_N^{|c|}$ divided by a polynomial in t_N with non-vanishing absolute term. □

3. The zeroes of $E_u^{(k)}$ on $\mathcal{F}$.

We keep the notation of (1.15) and (2.11): $0 \neq u \in (K/A)^2$ with $Nu = 0$, represented by $\frac{1}{N}(u_1, u_2)$ with $d_i = \deg u_i < \delta = \deg N$ and $N'u \neq 0$ for all proper divisors N' of N. Our goal is to prove the following result.

3.1 Theorem. *Suppose that $u_1 \neq 0$. For $i = 0, 1, 2, \ldots$, the Eisenstein series $E_u^{(k)}$ has $\gamma_i(k)q^{i+1}$ zeroes (counted with multiplicity) on $\mathcal{F}_{\delta-d_1+i}$ and no other zeroes on $\mathcal{F}$. If $u_1 = 0$, $E_u^{(k)}$ has no zeroes on $\mathcal{F}$.*

In view of (1.16) and (2.12) we then know the location of all the zeroes of all $E_u^{(k)}$ on $X(N) = \Gamma(N) \setminus (\Omega \cup \mathbb{P}^1(K))$. In particular, Theorem 1.24 is a consequence of Theorem 3.1. As the group $\Gamma_{\delta-d_1+i}(N)$ (which by (1.4) has order $q^{\max(i+1-d_1,0)}$) acts without fixed points on $\mathcal{F}_{\delta-d_1+i}$, $E_u^{(k)}$ has $\gamma_i(k)q^{\min(d_1,i+1)}$ zeroes on

$$\Gamma_{\delta-d_1+i}(N) \setminus \mathcal{F}_{\delta-d_1+i} \hookrightarrow \Gamma(N) \setminus \Omega \hookrightarrow X(N).$$

Before proving the theorem, we collect some more information.

3.2 Lemma. *Consider the functions* $z \longmapsto t(\frac{u_1z+u_2}{N})$ *and* $E_u = E_u^{(1)}$ *on* $\mathcal{F}$ *and their absolute values. Then*

(i) $|E_u(z)| = |t(\frac{u_1z+u_2}{N})|$.

(ii) $\log_q |t(\frac{u_1z+u_2}{N})| = -q^{\frac{q^{s+d_1-\delta}-1}{q-1}}$, $s \geq \delta - d_1$
$= \min(\delta - d_1 - s, \delta - d_2)$, $s < \delta - d_1$
for $z \in \mathcal{F}_s$, $s \in \mathbb{N}_0$.

(iii) $\log_q |t(\frac{u_1z+u_2}{N})|$ *depends only on* $s = \log_q |z|$. *Regarded as a function of* $s \in \mathbb{Q}_{\geq 0}$, *it is linear on intervals* $[i, i+1] \cap \mathbb{Q}_{\geq 0}$, $i \in \mathbb{N}_0$.

Proof. (i) As $E_u(z) = e_{Az+A}(\frac{u_1z+u_2}{N})^{-1}$ and $t(\frac{u_1z+u_2}{N}) = e_A(\frac{u_1z+u_2}{N})^{-1}$, both numbers differ by the factor $\prod_{\substack{a,b\in A\\ a\neq 0}}(1 - \frac{u_1z+u_2}{N(az+b)})$. Since $|u_1z + u_2| < |Naz| \leq |N(az+b)|$, that factor has absolute value 1.
(ii) This follows from an elementary (but tedious) calculation, using (1.2), and is left to the reader.
(iii) This is a general property of invertible holomorphic functions, see e.g. [25], but results in our case from the calculation that shows (ii). □

We thus have control on $|t(\frac{u_1z+u_2}{N})|$. In particular, for $z \in \mathcal{F}$:

$$\left| t\left(\frac{u_1z+u_2}{N}\right)\right| > 1 \Leftrightarrow |z| < q^{\delta - d_1}. \tag{3.3}$$

Next, let $\mathcal{F}_s$ be one of the subspaces described in (1.1). For a holomorphic function f on $\mathcal{F}_s$, let

$$\|f\|_s = \sup\{f(z) \mid z \in \mathcal{F}_s\} = \max\{f(z) \mid z \in \mathcal{F}_s\} \tag{3.4}$$

denote the spectral norm on $\mathcal{F}_s$.

3.5 Lemma. *Suppose that* f *may be written as* $f = f_p + f_c$ *with a holomorphic principal part* f_p *and a complementary part* f_c *that satisfy* $\|f\|_s = \|f_p\|_s > \|f_c\|_s$. *Then the number of zeroes of* f *on* $\mathcal{F}_s$ *(counted with multiplicity) agrees with the number of zeroes of* f_p *on* $\mathcal{F}_s$.

Proof. Without restriction, $\|f\|_s = 1$. Let $x_1, \ldots, x_n$ (resp. $y_1, \ldots, y_m$) be the zeroes of f (resp. f_p) on $\mathcal{F}_s$, each counted with multiplicities. Then ([3], Théorème I.2.2) we can write

$$f(z) = \prod_{1\leq i\leq n}(z - x_i)g(z),$$

where $|g| = \|f\|_s = 1$ is constant on $\mathcal{F}_s$. Similarly, $f_p(z) = \prod_{1\leq j\leq m}(z - y_j)g_p(z)$ with $|g_p| = 1$ on $\mathcal{F}_s$. Since the canonical reductions $\overline{f}$ and $\overline{f}_p$ of f and f_p agree, we find n = number of zeroes of $\overline{f}$ = number of zeroes of $\overline{f}_p = m$. □

As in the proof of (2.12), we write

$$E_u^{(k)}(z) = \sum_{a\in A} G_k\left(t\left(\left(\frac{u_1}{N}+a\right)z+\frac{u_2}{N}\right)\right). \tag{3.6}$$

3.7 Lemma. *Put $f_p(z) := G_k(t(\frac{u_1z+u_2}{N}))$ and $f_c := E_u^{(k)} - f_p$. For each $s \in \mathbb{Q}_{\geq 0}$, the (in-)equalities*

$$\|f_p\|_s = \|E_u^{(k)}\|_s > \|f_c\|_s$$

hold.

Proof. For s given, let $\sigma := \log_q |t(\frac{u_1z+u_2}{N})|$ be the constant absolute value of the function $z \longmapsto t(\frac{u_1z+u_2}{N})$ on $\mathcal{F}_s$. Write the Goss polynomial

$$G_k(X) = \prod_{1\leq i\leq n}(X-x_i)\prod_{1\leq j\leq m}(X-y_j) \quad (n+m=k) \tag{1}$$

with zeroes x_i and y_j that satisfy $|x_i| < q^\sigma$, $|y_j| \geq q^\sigma$. Note that $n > 0$ since $G_k(X)$ is divisible by X (and even by X^2 if $k > 1$). Replacing the term f_p, which corresponds to $a = 0$ in (3.6), by $G_k(t((\frac{u_1}{N}+a)z+\frac{u_2}{N}))$ with $a \neq 0$, the quantity

$$\left|e_A\left(\frac{u_1}{N}+a\right)z+\frac{u_2}{N}\right| = \left|e_A\left(\frac{u_1z+u_2}{N}\right)+e_A(az)\right| = |e_A(az)|$$

becomes strictly larger, as follows from (3.2). Hence for the reciprocals:

$$q^{\sigma_a} := \left|t\left(\left(\frac{u_1}{N}+a\right)z+\frac{u_2}{N}\right)\right| < \left|t\left(\frac{u_1z+u_2}{N}\right)\right| = q^\sigma.$$

Since these functions are invertible on $\mathcal{F}_s$, the absolute values are constant on $\mathcal{F}_s$ and agree with the spectral norms. We read off from (1) that $\|G_k(t((\frac{u_1}{N}+a)z+\frac{u_2}{N}))\|_s$ decreases compared to $\|f_p\|_s$ by a factor smaller or equal to $\prod_{1\leq i\leq n}(\sup(q^{\sigma_a},|x_i|)q^{-\sigma}) < 1$. As $\sigma_a \longrightarrow -\infty$ with increasing deg a, we are done. □

Proof of Theorem 3.1. From the preceding lemmas, the number of zeroes of $E_u^{(k)}$ and of $f_p(z) = G_k(t(\frac{u_1z+u_2}{N}))$ on $\mathcal{F}_s$ agree. We abbreviate $\tau(z)$ for $t(\frac{u_1z+u_2}{N})$.

If $u_1 = 0$, then $\tau(z)$ and $f_p(z) = G_k(\tau(z))$ are constant, and there are no zeroes of f_p on $\mathcal{F}$. Thus suppose $u_1 \neq 0$. By (2.9) there are precisely $\gamma_i(k)$ many values τ of $\tau(z)$ with $\log_q|\tau| = -q(\frac{q^i-1}{q-1})$ $(i = 0,1,2,\ldots)$ which are zeroes of $G_k(X)$, and no other zeroes. By (3.2), these arise on $\mathcal{F}_{\delta-d_1+i}$.

How many $z \in \mathcal{F}_{\delta-d_1+i}$ are there that give rise to the same value of $\tau(z)$?

We have for $z, z' \in \mathcal{F}_{\delta-d_1+i}$:

$$\begin{aligned}\tau(z)=\tau(z') \Leftrightarrow e_A\left(\frac{u_1z+u_2}{N}\right) &= e_A\left(\frac{u_1z'+u_2}{N}\right)\\ \Leftrightarrow e_A\left(\frac{u_1(z-z')}{N}\right) &\Leftrightarrow z-z' \in \frac{N}{u_1}A\end{aligned}$$

Hence the map τ is m-to-one on $\mathcal{F}_{\delta-d_1+i}$, with

$$\begin{aligned} m &= \#\left\{\frac{w}{u_1} \in \frac{N}{u_1}A \;\middle|\; \left|\frac{w}{u_1}\right| \leq q^{\delta-d_1+i}\right\}\\ &= \#\{w \in NA \mid \deg w \leq \delta+i\} = q^{i+1}.\end{aligned}$$

Therefore there are precisely $\gamma_i(k)q^{i+1}$ zeroes of f_p, thus of $E_u^{(k)}$, on $\mathcal{F}_{\delta-d_1+i}$ $(i=0,1,2,\ldots)$, and no other zeroes on $\mathcal{F}$. $\square$

The spectral norm defines a function

$$\begin{aligned}\nu_u^{(k)}: \mathbb{Q}_{\geq 0} &\longrightarrow \mathbb{R}\\ s &\longmapsto \|E_u^{(k)}\|_s.\end{aligned} \tag{3.8}$$

Recall that $G_k(X)$ is exactly divisible by $X^{\gamma(k)}$. Let $g_{\gamma(k)}$ be the coefficient of $X^{\gamma(k)}$ and $\psi(k) := -\log_q |g_{\gamma(k)}| \in \mathbb{N}_0$ its ∞-adic valuation. We further need $\omega(k)$, the largest i such that $\gamma_i(k) > 0$ (which is less than $\log_q(k-1)$ [13]).

3.9 Corollary. *The function $\nu_u^{(k)}$ enjoys the following properties:*

(i) $\log_q|\nu_u^{(k)}|$ *is linear on intervals $[i,i+1]\cap\mathbb{Q}_{\geq 0}$, $i \in \mathbb{N}_0$;*

(ii) $\nu_u^{(k)}$ *is non-increasing;*

(iii) *if $u_1 = 0$ then $\nu(s) = |E_u^{(k)}(z)| = 1$ for each $s \in \mathbb{Q}_{\geq 0}$ and $z \in \mathcal{F}$.*

From now on, suppose $u_1 \neq 0$. Then

(iv) $\nu_u^{(k)}(s) = |E_u^{(k)}(z)|$ *for each $z \in \mathcal{F}_s$ if $s \notin \{\delta-d_1+i \mid \gamma_i(k)\neq 0\}$.*

Let $s \in \mathbb{N}_0$.

(v) *If $s \leq \delta - d_1$ then $\log_q \nu_u^{(k)}(s) = k\cdot\min(\delta-d_1-s, \delta-d_2)$;*

(vi) *if $s \geq \delta-d_1+\omega(k)$ then $\log_q \nu_u^{(k)}(s) = -\gamma(k)\frac{q^{s-\delta+d_1}-1}{q-1} - \psi(k)$.*

Proof. (i) follows from (3.1), i.e., the fact that $E_u^{(k)}$ has its zeroes in $\mathcal{F}$ only in $\bigcup_{s\in\mathbb{N}}\mathcal{F}_s$.

(ii) results from (3.2) (the non-increasingness of $\log_q|t(\frac{u_1z+u_2}{N})|$ as a function of $\log_q|z|$) and (3.7).

(iii) has already been shown in the proof of (3.1).

(iv) comes from the description of zeroes of $E_u^{(k)}$.

(v) If $z \in \mathcal{F}_s$ with $s \in \mathbb{N}_0$, $s < \delta-d_1$ then $\log_q|t(\frac{u_1z+u_2}{N})| = \min(\delta-d_1-s, \delta-d_2) > 0$ (cf. (3.2)), so $t(\frac{u_1z+u_2}{N})$ is larger in absolute value than the zeroes of $G_k(X)$, and

$|E_u^{(k)}(z)| = |G_k(t(\frac{u_1z+u_2}{N}))|$ is determined through the leading term of G_k.
(vi) For $z \in \mathcal{F}_s$ with $s > \delta - d_1 + \omega(k)$, $|t(\frac{u_1z+u_2}{N})| < |x|$ for each zero $x \neq 0$ of $G_k(X)$, hence $|E_u^{(k)}(z)|$ is given by the lowest order term of G_k. □

3.10 Remark. Combining the explicit description of the Newton polygon of $G_k(X)$ given in [13] and [14] with (3.2), it is possible to work out the precise value of $\nu_u^{(k)}(s) = \|E_n^{(k)}\|_s$ also for s on the critical strip $\{\delta - d_1, \delta - d_1 + 1, \delta - d_1 + \omega(k)\}$.

4. Distribution of the zeroes of $E_u^{(k)}$ on $X(N)$.

Recall that $G(N) = \Gamma/\Gamma(N) \cdot Z$ is the group of the ramified Galois covering of $X(N)$ over $X(1) \xrightarrow{\cong} \mathbb{P}^1(C_\infty)$. It acts transitively on the set of cusps

$$\text{cusps}(N) = \Gamma/\Gamma(N) \cdot \Gamma_\infty$$

and on

$$\text{(4.1)} \qquad \text{Eis}(N) := \{u \in (K/A)^2 \mid Nu = 0\}_{\text{prim}}/Z,$$

where $\{\ldots\}_{\text{prim}}$ refers to those u for which $N'u \neq 0$ for all proper divisors N' of N. As non-primitive u's give rise to Eisenstein series $E_u^{(k)}$ of strictly smaller level than N and Z-equivalent u, u' yield essentially the same Eisenstein series (i.e., $u' = cu$ with $c \in \mathbb{F}^* \xrightarrow{\cong} Z$ implies $E_{u'}^{(k)} = c^{-k}E_u^{(k)}$), we use Eis$(N)$ as an index set for them. Both cusps(N) and Eis(N) have the same cardinality

$$\text{(4.2)} \qquad \#\text{Eis}(N) = \#\text{cusps}(N) = (q-1)|N|^2 \prod_{\substack{P|N \\ P \text{ monic, prime}}} (1 - |P|^{-2}).$$

(In [6] sect. 3 it is shown how one can find a common set of representatives in $G(N)$ for both sets which actually is a subgroup of $G(N)$.)

Now let us count the total number of zeroes of $E_u^{(k)}$. Choose a set R of representatives for $G(N)$ in Γ. In view of (1.16) and $X(N) = \bigcup_{\gamma \in R} \gamma(\mathcal{F} \cup \{\infty\})$, non-cuspidal zeroes of $E_u^{(k)}$ on $X(N)$ are described by pairs (γ, z), where $\gamma \in R$ and z is a zero of $E_{u\gamma}^{(k)}$ on $\mathcal{F}$. Two such pairs, (γ_1, z_1) and (γ_2, z_2), yield the same zero if and only if $\gamma\gamma_1 z_1 = \gamma_2 z_2$ with some $\gamma \in \Gamma(N)$. If so, type(z_1) = type(z_2), i.e., z_1 and z_2 belong to the same $\mathcal{F}_s$ ($s \in \mathbb{N}$) and are equivalent under Γ_s. On the other hand, if $z_1 \in \mathcal{F}_s$ is a zero of $E_{u\gamma_1}^{(k)}$ and $z_2 = \beta z_1$ with $\beta \in \Gamma_s$, then there exists a unique $\gamma_2 \in R$ such that z_2 is a zero of $E_{u\gamma_2}^{(k)}$. Hence the equivalence class of (γ, z) has length $\#\Gamma_s/Z = (q-1)q^{s+1}$. We thus find (where we abuse language and write $\#\{\dots\}$ for the number of zeroes counted with multiplicity):

$$\begin{aligned}
&\#\{\text{non-cuspidal zeroes of } E_u^{(k)} \text{on } X(N)\} \\
&= \sum_{\gamma\in R}\sum_{s\in\mathbb{N}} \frac{\#\{\text{zeroes of } E_{u\gamma}^{(k)} \text{ on } \mathcal{F}_s\}}{(q-1)q^{s+1}} \\
&= (q-1)q^{\delta} \sum_{v\in\mathrm{Eis}(N)}\sum_{s\in\mathbb{N}} \frac{\#\{\text{zeroes of } E_{v}^{(k)} \text{ on } \mathcal{F}_s\}}{(q-1)q^{s+1}},
\end{aligned}$$

as each $E_v^{(k)}$ occurs $(q-1)q^{\delta}$ times as $E_{u\gamma}^{(k)}$ when γ runs through R.

For $v =$ class of $N^{-1}(v_1, v_2)$ with $\deg v_i < \delta$ $(i = 1, 2)$, we let $d_1 = d_1(v) = \deg v_1$. With (3.1) the expression becomes

$$\begin{aligned}
&q^{\delta} \sum_{v\in\mathrm{Eis}(N)}\sum_{i\geq 0} \frac{\gamma_i(k)q^{i+1}}{q^{\delta-d_1(v)+i+1}} = \sum_{v\in\mathrm{Eis}(N)} q^{d_1(v)} \sum_{i\geq 0}\gamma_i(k) \\
&= \sum_{v\in\mathrm{Eis}(N)} |v_1| \sum_{i\geq 0}\gamma_i(k).
\end{aligned}$$

A similar calculation, based on (2.12), yields $\sum_{v\in\mathrm{Eis}(N)} |v_1|\gamma(k)$ for the number of cuspidal zeroes of $E_u^{(k)}$ on $X(N)$. Together

$$\begin{aligned}
&\#\{\text{zeroes of } E_u^{(k)} \text{ on } X(N)\} \\
&= \sum_{v\in\mathrm{Eis}(N)} |v_1| \Big(\sum_{i\geq 0}\gamma_i(k) + \gamma(k)\Big) = k \sum_{v\in\mathrm{Eis}(N)} |v_1|.
\end{aligned}$$

As may be verified by elementary means (although this is rather delicate), but also follows from Korollar 2.2 in [6], the identity

$$\sum_{v\in\mathrm{Eis}(N)} |v_1| = |N| \frac{\#\mathrm{cusps}(N)}{q+1} = \deg(\mathcal{M}) \tag{4.4}$$

holds. Hence the above calculation is (of course ...) compatible with (1.22). Beyond the sheer number, it exhibits a more precise picture of the location of the zeroes, which will be exemplified in the next section.

5. Examples.

We consider in more detail the two extremal cases where either the weight k or the conductor N is as small as possible without leading to a trivial situation. We keep the notation of the preceding sections.

5.1 Example. Let $k = q + 1$ and $u \in (K/A)^2$ be primitive of level N. Here $G_k(X) = G_{q+1}(X) = X^{q+1} + \alpha_1 X^2$ with a constant $\alpha_1 \in O_\infty$ of absolute value 1, so $\gamma(q+1) = 2$, $\psi(q+1) = 0 = \omega(q+1)$ (cf. (3.8)). If $u_1 = 0$ then $E_u^{(q+1)}$ has no zeroes on $\mathcal{F} \cup \{\infty\}$ and $|E_u^{(q+1)}| = 1$ on $\mathcal{F}$. Thus suppose $u_1 \neq 0$. Then $E_u^{(q+1)}$ has $(q-1)q$ zeroes on $\mathcal{F}_{\delta-d_1}$ and no other zeroes on $\mathcal{F}$. The formulas of (3.9) yield for $z \in \mathcal{F}_s$, $s \in \mathbb{N}_0$:

$$\begin{aligned}
\log_q |E_u^{(q+1)}(z)| &= (q+1)\min(\delta - d_1 - s, \delta - d_2), && s < \delta - d_1 \\
&= -2q\tfrac{q^{s-\delta+d_1}-1}{q-1}, && s > \delta - d_1 \\
\|E_u^{(q+1)}\|_{\delta-d_1} &= 1
\end{aligned}$$

The zeroes of $E_u^{(q+1)}$ are of type s with $s = \delta - d_1$ between 1 and δ. The considerations of Section 4 show that

$$\begin{aligned}
&\#\{x \in X(N) \mid x \text{ is a non-cuspidal zero of type } s \text{ of } E_u^{(q+1)}\} \\
&= (q-1)\#\{v \in \mathrm{Eis}(N) \mid d_1(v) = \delta - s\},
\end{aligned}$$

which apart from s and $\delta = \deg N$ depends in general on the splitting type of N. However for $s = \delta$ that number is $(q-1)\#\{v \in \mathrm{Eis}(N) \mid d_1(v) = 0\} = (q-1)|N|$. Hence the number (counted with multiplicity) of *all* zeroes of type δ of all $E_u^{(q-1)}$ ($u \in \mathrm{Eis}(N)$) is $(q-1)|N|\#\mathrm{Eis}(N) = \#G(N)$.

As $G(N)$ acts on the corresponding set $\mathcal{Z}$, and acts fixed-point free (the only fixed points of $G(N)$ are at cusps and at elliptic points, of type 0), $\mathcal{Z}$ forms one orbit of $G(N)$, of size $\#G(N)$. We have thus shown the following result.

5.2 Proposition. *Let $u, v \in (K/A)^2$ be primitive of level N and inequivalent modulo Z (i.e., $v \neq cu$, $c \in \mathbb{F}^*$). The sets of zeroes of type $\delta = \deg N$ of $E_u^{(q+1)}$ and $E_v^{(q+1)}$ are disjoint, and all these zeroes are simple.*

5.3 Remarks. (i) It would be interesting to know whether such properties (simplicity of non-cuspidal zeroes of $E_u^{(k)}$, disjointness of the corresponding divisors) hold in greater generality. Of course, (2.4) and (2.8) yield some restrictions.
(ii) Since $\mathcal{Z} = \{x \in X(N) \text{ of type } \delta | \exists u \in \mathrm{Eis}(N) \text{ s.t. } E_u^{(q+1)}(x) = 0\}$ forms an orbit under $G(N)$, it corresponds to one point $j(\mathcal{Z})$ on the modular curve $X(1) \xrightarrow{\cong} \mathbb{P}^1(C_\infty)$ without level. From [12] 2.3 we see that $\log_q |j(\mathcal{Z})| = q^{\delta+1}$. It is worthwhile to determine that number and, more generally, the j-invariants of other zeroes of $E_u^{(k)}$ and to study their arithmetic. See (5.8) for a special case.

Finally, we treat the case where the weight k is unrestricted but $\delta = \deg N = 1$, without restriction, $N = T$.

5.4 Example. The case $N = T$.
The modular curve $X(T)$, of genus 0, is a well-studied object, see e.g. [1, 2, 15].

There are natural identifications

$$(5.5)\qquad \begin{array}{lcl} G(T) & = & \mathrm{PGL}(2,\mathbb{F}) \\ \mathrm{cusps}(T) & = & G(T)/B \xrightarrow{\cong} \mathbb{P}^1(\mathbb{F}) \\ & & \text{class of } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longmapsto (a:c) \\ \mathrm{Eis}(T) & = & B \setminus G(T) \xrightarrow{\cong} \mathbb{P}^1(\mathbb{F}), \\ \text{class of } T^{-1}(u_1,u_2) & \longmapsto & \qquad (u_1 : u_2) \end{array}$$

where $B = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \subset G(T)$ is the standard Borel subgroup and $u_1, u_2 \in \mathbb{F}$. (The description in [6] sect. 3 might be helpful. It applies to general conductors.) Going through the identifications we find:

(5.6) $E_u^{(k)}$ with $u = T^{-1}(u_1, u_2)$ vanishes at the cusp $(a : c)$ of $X(T)$ if and only if $u_1 a + u_2 c \neq 0$. In this case, the vanishing order is $\gamma(k)$.

For each cusp $(a : c)$ let $\alpha_{a:c} = \left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{F}) \hookrightarrow \Gamma$ be a representative, and let $R = \{\alpha_{a:c} \mid (a : c) \in \mathrm{cusps}(T)\}$. Then

$$X(T) = \bigcup_{\alpha \in R} \alpha(\mathcal{F} \cup \{\infty\}),$$

where the intersection of $\alpha(\mathcal{F} \cup \{\infty\})$ and $\beta(\mathcal{F} \cup \{\infty\})$ for $\alpha, \beta \in R$, $\alpha \neq \beta$, is in $\alpha(\mathcal{F}_0) = \beta(\mathcal{F}_0) = \mathcal{F}_0$. This corresponds to the fact that the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}(2, K_\infty)$ divided out by $\Gamma(T)$ is a star composed of $q+1$ half lines $\bullet - - - \bullet - - - \bullet - - \cdots$ glued together in their origins, see [15] and [1]. For $u = T^{-1}(u_1, u_2)$ as above, the zeroes of $E_u^{(k)}$ on $X(T)$ are

- $\gamma(k)$ zeroes at each of the q cusps $(a : c)$ with $u_1 a + u_2 c \neq 0$;
- $\gamma_i(k)$ zeroes (counted with multiplicity) on $\alpha_{a:c}(\mathcal{F}_{1+i})$, for each of the q representatives $\alpha_{a:c}$ with $u_1 a + u_2 c \neq 0$.

The conjunction of the two examples is the case where

$$(5.7)\qquad N = T \text{ and } k = q+1\ .$$

There are precisely $\#\mathrm{Eis}(T) \times \gamma_0(q+1) \times \deg \mathcal{M} = (q+1)(q-1)q = \#G(T)$ non-cuspidal zeroes of $E_u^{(q+1)}$, $u \in \mathrm{Eis}(T)$, all different, which form a complete orbit under $G(T)$. Here we can directly calculate the j-invariant.

5.8 Proposition. *Let $z \in X(T)$ be a non-cuspidal zero of $E_u^{(q+1)}$ for some $u \in \mathrm{Eis}(T)$. Then $j(z) = \frac{(T^q - T)^{q+1}}{T^q - 2T}$.*

Proof. As in the proof of (2.13), we use rudiments of the theory of Drinfeld modular forms and the corresponding notation, see [8] or [9].

Fix $0 \neq u \in (K/A)^2$ with $Tu = 0$, let $E := E_u$ and $e := E^{-1}$. Then

$$(1)\qquad E_u^{(q+1)}(z) = G_{q+1,\Lambda}(E(z))$$

with the lattice $\Lambda := Az + A$ in C_∞ (cf. (2.8)).

$$(2)\qquad G_{q+1,\Lambda}(X) = X^{q+1} + \alpha_1 X^2,$$

where $\alpha_1 = \alpha_1(Az + A)$ is the first coefficient of $e_\Lambda(\omega) = \sum_{i\geq 0} \alpha_i(Az + A)\omega^{q^i}$. Regarded as a function of z, α_1 is a modular form of weight $q - 1$ for Γ.

Let ϕ be the Drinfeld module corresponding to Λ, given by the operator polynomial

$$\phi_T(X) = TX + gX^q + \Delta X^{q^2}$$

with $g, \Delta \in C_\infty$, $\Delta \neq 0$. Again, g and Δ depend in such a way on z that they are modular forms of weights $q - 1$ and $q^2 - 1$, respectively. In fact, from the functional equation of e_Λ,

$$e_\Lambda(Tz) = \phi_T(e_\Lambda(z), \tag{3}$$

we find

$$\alpha_1(z) = \frac{1}{T^q - T} g(z). \tag{4}$$

Also from (3) and (1.7), $e = E^{-1}$ is a T-division point of ϕ, i.e., $\phi_T(e) = 0$, and since e has no zeroes,

$$T + ge^{q-1} + \Delta e^{q^2-1} = 0 \tag{5}$$

identically on Ω. From (1), (2), (4) we see

$$E_u^{(q+1)}(z) = 0 \Leftrightarrow E_u^{q-1} + \frac{g(z)}{T^q - T} = 0 \Leftrightarrow g(z) = \frac{T - T^q}{e^{q-1}(z)}. \tag{6}$$

Thus, if z is a zero then (5) and (6) imply

$$\Delta(z) = \frac{T^q - 2T}{e^{q^2-1}(z)},$$

and so

$$j(z) = \frac{g^{q-1}(z)}{\Delta(z)} = \frac{(T^q - T)^{q+1}}{T^q - 2T}$$

as stated. □

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