# ON STABILITY OF SOLUTIONS TO SYSTEMS OF CONVEX INEQUALITIES

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## 1. Introduction

Let $T$ be an arbitrary set. We associate with it the space $\ell_\infty(T)$ of all uniformly bounded real-valued functions $a = (a_t)$ on $T$ with the sup-norm:

$$\|a\| = \sup_{t\in T} |a_t|.$$

Let further $X$ be a Banach space and for any $t \in T$ a function $\varphi_t$ on $X$ will be given. We assume that *all these functions are convex, proper and lower semi-continuous.* Let furthermore $Y$ be another Banach space, and let $A\colon X \to Y$ be a linear bounded operator. The object of our interest is the following system of inequalities and equalities on $X$:

$$\varphi_t(x) \le p_t, \quad \forall t \in T, \quad Ax = y \tag{1}$$

where $p = (p_t) \in \ell_\infty$. We shall consider $\varphi = (\varphi_t)_{t\in T}$ and $A$ as data and $p$ and $y$ as parameters whose nominal value is $(\mathbf{0}, \overline{y})$ for some $\overline{y} \in Y$. By $\mathbf{0}$ we mean the element of $\ell_\infty$ whose components are zeros for all $t \in T$: $\mathbf{0}_t = 0$, $\forall t$.

By $\mathcal{S}(p, y)$ we denote the set of solutions of (1). This is obviously a convex closed set that in principle can be empty. We shall assume however that $\mathcal{S}(\mathbf{0}, \overline{y}) \ne \emptyset$. The following is the list of problems to be addressed in the paper:

(**P1**) Lipschitz stability of solutions to (1) at $(\mathbf{0}, \overline{y})$. Specifically: *given an $\overline{x} \in \mathcal{S}(\mathbf{0}, \overline{y})$, find the Lipschitz modulus of $\mathcal{S}$ at $((\mathbf{0}, \overline{y}), \overline{x})$.* We recall that the Lipshitz modulus of $\mathcal{S}$ at $((\mathbf{0}, \overline{y}), \overline{x})$ (with respect to a given norm $\|(p, y)\|$ in $\ell_\infty \times Y$) is the lower bound of $K > 0$ having the property that for some $\varepsilon > 0$ (depending on $K$)

$$\mathcal{S}(p', y') \bigcap B(\overline{x}, \varepsilon) \subset \mathcal{S}(p, y) + K\|(p, y) - (p', y')\|B.$$

Here $B(x, \varepsilon)$ is the (closed) ball of radius $\varepsilon$ around $x$ and $B$ stands for the unit ball ( here obviously in $X$).

(**P2**) Effect of perturbations of the system. Specifically, *what will happen with the Lipschitz modulus of $\mathcal{S}(\mathbf{0}, \overline{y})$ if we replace $\varphi_t(x)$ by $\varphi_t(x) + \langle x_t^*, x - \overline{x}\rangle$ and $Ax$ by $Ax + \Lambda(x - \overline{x})$* ($\Lambda\colon X \to Y$ also being a bounded linrear operator)?

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The research was completed during the author's stay in the Centre de Recerca Matematica at Univ. Autnoma de Barcelona in October 2010.

(**P3**) Distance to infeasibility. Specifically, *what is the minimal norm of the perturbations* $(x_t^*, \Lambda)$ *that make infeasible the system*

$$\varphi_t(x) + \langle x_t^*, x - \overline{x}\rangle \le p_t, \quad (A + \Lambda)x = y.$$

These (and related problems) have been actively discussed in the regularity theory of variational analysis both for systems involving only convex objects similar to (1) (e.g. [2, 3, 5, 6, 8, 11, 12] and for more general objects (see e.g. [15, 17] for monographic accounts and [1, 10] for surveys). The paper offers solutions to the problems for systems (1) under no additional (to the mentioned above) assumptions on the data. In particular, the most recent results containing formulas for regularity moduli obtained in [3, 6] (for linear systems) and [5] (for systems of convex inequalities) follow from our results without much effort.

However the generality of the setting is not the only, and actually not the main contribution of the paper. More important is the understanding that all information about the stability related behaviour of (1) comes from analysis of a much simpler system containing just one numerical inequality

$$\Phi(x) \le \xi, \quad Ax = b,$$

where $\Phi(x) = \sup_t \varphi_t(x)$. In terms of the system (1) this means that there is no need to look at the effect of all possible perturbations of the inequalities: constant perturbations with $p_t \equiv const$ are enough. A similar observation has been made in a very recent paper by Cánovas, Gómez-Senent and Parra [5] for the case of a pure inequality system on $I\!R^n$ with $T$ being a compact Hausdorff space and $(t, x) \to \varphi_t(x)$ continuous on $T \times I\!R^n$ (and of course convex as functions of $x$) under the assumption that the Slater constrained qualification is satisfied.

The other main point to be emphasized is that, compared to earlier publications, even relating to the systems of linear inequalities, our proofs are much shorter and less technical. The core of our approach is a combination of some elementary but convenient properties of $\ell_\infty(T)$ and the techniques provided by the quantitative regularity theory for set-valued mappings with convex graphs. This theory is now well developed. Its three main ingredients are the exact formula for the surjection modulus of such mappings (containing earlier Robinson's estimates for constrained systems), the concept of perfect regularity and its dual characterization, both obtained in Ioffe-Sekiguchi [12] and the formula for the distance to infeasibility in Dontchev-Lewis-Rockafellar [7]. As we shall see, all three work very efficiently for (1).

## 2. Preliminaries.

1. **Regularity of convex multifunction.** The key (and the only!) fact of the general regularity theory needed for our future discussions is that that the Lipschitz modulus of a set-valued mapping at a certain point of its graph is reciprocal of the modulus of surjection of the inverse mapping at the same point. To be precise, let $X$ and $Y$ be metric spaces, and let $F\colon X \rightrightarrows Y$ be a set-valued mapping. Let $\Psi\colon Y \rightrightarrows X$ stand for the inverse mapping, that is $\Psi(y) = \{x \in X : \ y \in F(x)\}$.

We fix a certain point $(\overline{x},\overline{y}) \in \text{Graph } F$ and recall that the *rate* or *modulus of surjection* of $\Phi$ at $(\overline{x},\overline{y})$, denoted usually by $\text{sur}F(\overline{x}|\overline{y})$, is the upper bound of $r>0$ such that for some $\varepsilon>0$ the inclusion $B(y,rt)\subset F(B(x,t))$ holds for all $(x,y)\in \text{Graph } F\bigcap\big(B(\overline{x},\varepsilon)\times B(\overline{y},\varepsilon)\big)$ and all $t\in(0,\varepsilon)$. If no such $r$ exists we set $\text{sur}F(\overline{x}|\overline{y})=0$. We say that $F$ is *regular at* $(\overline{x},\overline{y})$ if $\text{sur}F(\overline{x}|\overline{y})>0$.

The *Lipschitz modulus* of the inverse mapping at $(\overline{y},\overline{x})$, denoted by $\text{lip}\Psi(\overline{y}|\overline{x})$, is the lower bound of $K>0$ such that for some $\varepsilon>0$ we have

$$\Psi(x)\bigcap B(\overline{y},\varepsilon)\subset \Psi(x')+K\|x-x'\|B$$

if $x,x'$ belong to $B(\overline{x},\varepsilon)$. If no such $K$ exists, we set $\text{lip}\Psi(\overline{y}|\overline{x})=\infty$. Then under the convention that $0\cdot\infty=1$ the equality $\text{sur}\Phi(\overline{x}|\overline{y})\cdot\text{lip}\Psi(\overline{y}|\overline{x})=1$ is valid unconditionally (see e.g [10] for a short proof).

Calculation of the exact value of the rate of surjection is possible only in certain specific cases. The case of set-valued mappings with convex graphs is one of them. Namely, the following theorem holds true.

**Theorem 1** (Ioffe-Sekiguchi [12], Th. 1). *Let $F\colon X\rightrightarrows Y$ be a set-valued mapping with convex graph. If* $\text{int } F(X)\neq\emptyset$, *then for any* $(x,y)\in\text{Graph } F$

$$\text{sur}F(x|y)=\lim_{\lambda\to 0}\inf_{\|y^*\|=1}\inf_{x^*}\left(\|x^*\|+\frac{1}{\lambda}S_{\text{Graph } F-(x,y)}(x^*,y^*)\right). \qquad (2)$$

Here $S_Q(x^*)$ stands for the *support function* of $Q$: $S_Q(x^*)=\sup_{x\in Q}\langle x^*,x\rangle$.

In certain cases, calculation of the surjection rate of a convex set-valued mapping can be further simplified. The key for this simplification is the notion of perfect regularity introduced in the same paper [12]. Recall that, given a convex set $Q\subset X$ and an $x\in Q$, the *tangent cone* $T(Q,x)$ *to* $Q$ *at* $x$ is the closed cone generated by the set $Q-x$. In other words, $T(Q,x)$ is the closure of the set of all vectors $\lambda(u-x)$, where $\lambda\geq 0$ and $u\in Q$. Clearly, convexity of $Q$ implies convexity of the tangent cone to $Q$ at every point. The polar of the tangent cone, $N(Q,x)=T^{\circ}(Q,x)=\{x^*\in X^*:\ \langle x^*,h\rangle\leq 0,\ \forall\, h\in T(Q,x)\}$ is called the *normal cone to* $Q$ *at* $x$.

If $Q\subset X\times Y$, that is to say, $Q$ is the graph of a certain set-valued mapping $F\colon X\rightrightarrows Y$, and $(x,y)\in Q$, then we can view the tangent cone $T(Q,(x,y))$ as the graph of some other set-valued mapping from $X$ into $Y$, which associates with every $h\in X$ the collection of $v\in Y$ such that $(h,v)\in T(Q,(x,y))$. This mapping is often called the *(contingent) derivative of* $F$ *at* $(x,y)$ and is denoted $DF(x,y)$. We shall denote by $\text{sur}DF(x,y)$ the rate of surjction of $DF(x,y)$ at $(0,0)$.

Likewise, the normal cone $N(Q,(x,y))$ can be viewed as the graph of a set-valued mapping from $Y^*$ into $X^*$, namely associating with every $y^*\in Y^*$ the set $\{x^*:(x^*,-y^*)\in N(Q,(x,y))\}$. This mapping is called the *coderivative of* $F$ *at* $(x,y)$[1].

[1]For set-valued mappings with convex graphs the concept of a coderivative under a different name was introduced by Pschenichnyi in late 60s []. Mordukhovich in 1980 in [14] extended the definition to arbitrary set-valued mappings. The very term "coderivative" was introduced four years later in [9]

As follows from the definition, $Q \subset x + T(Q,x)$ if $Q$ is a convex set and $x \in Q$. Therefore for any set-valued mapping with convex graph and any $(x,y) \in \text{Graph } F$ we have $\text{sur}F(x|y) \le \text{sur}DF(x,y)$. A set-valued mapping $F$ with convex graph is called *perfectly regular* (see [12]) at $(x,y) \in \text{Graph } F$ if

$$\text{sur}F(x|y) = \text{sur}DF(x,y).$$

It was shown in [12] (Theorem 3) that *a mapping with convex locally closed graph is perfectly regular at $(x,y)$ if and only if*

$$\text{sur}F(x|y) = \inf\big\{\|x^*\| : \ x^* \in D^*F(x,y)(y^*), \ \|y^*\| = 1\big\} \tag{3}$$

(with the standard convention that $\inf \emptyset = \infty$).

The following proposition offers a sufficient condition for perfect regularity.

**Proposition 2** ([12], Prop. 5). *Let $F\colon X \rightrightarrows Y$ be a set-valued mapping with convex and locally closed graph. Suppose there is a closed convex subset $Q^*$ of the unit sphere in $Y^*$ such that for some $(\overline{x},\overline{y}) \in \text{Graph } F$*

$$S_{\text{Graph } F-(\overline{x},\overline{y})}(x^*,y^*) < \infty \ \& \ \|y^*\| = 1 \quad \Rightarrow \quad y^* \in Q^*.$$

*Then $F$ is perfectly regular at $(\overline{x},\overline{y})$.*

2. **Radius of regularity and distance to infeasibility**. Following [7] we define the *radius of regularity* of a set-valued mapping $F\colon X \rightrightarrows Y$ at $(\overline{x},\overline{y}) \in \text{Graph } F$ as the lower bound of norms of linear bounded operators $A\colon X \to Y$ such that $F + A$ is not regular at $(\overline{x},\overline{y})$:

$$\text{rad}F(\overline{x}|\overline{y}) = \inf\big\{\|A\| : \ \text{sur}(F+A)(\overline{x}|\overline{y} + A\overline{x}) = 0\big\}.$$

The main fact we need is

**Proposition 3**. *If $F\colon X \rightrightarrows Y$ is a set-valued mapping with convex locally closed graph which is perfectly regular at $(\overline{x},\overline{y}) \in \text{Graph } F$, then*

$$\text{rad}F(\overline{x}|\overline{y}) = \text{sur}F(\overline{x}|\overline{y}).$$

*Proof.* The proposition is an easy consequence of the definition and Theorem 2.9 of [7][2]. □

The concept of distance of feasibility refers to the minimal perturbation that makes the inclusion $y \in F(x)$ infeasible. It was first introduced by Renegar [16] for linear systems and then extended by Dontchev-Lewis-Rockafellar in [7] to arbitrary set-valued mappings with convex graphs. Namely, given a $\overline{y} \in F(X)$, the *distance of infeasibility* for the system $\overline{y} \in F(x)$ is the minimal norm of pairs $(A,y)$ with $A\colon X \to Y$ being a bounded linear operator and $y \in Y$ such that $\overline{y} + y$ no longer belongs to the range of $F+A$: $\overline{y}+y \notin (F+A)(X)$. The specific value of the distance to infeasibility depends, of course on the choice of the norm in $\mathcal{L}(X,Y) \times Y$. We shall use, as in [7] the max norm $\|(A,y)\| = \max\{\|A\|, \|y\|\}$. It was shown in [7]

[2]This fact extends to arbitrary mappings if we define perfect regularity not throught the contingent derivative but by (3).

that the distance to infeasibility is also connected with the rate of surjection of some convex process associated with $F$. We need however another result: a direct formula for the distance of infeasibility in terms of $F$ itself.

**Proposition 4** (Dontchev-Lewis-Rockafellar [7], Th. 4.8). *Let $F\colon X \rightrightarrows Y$ be a set-valued mapping with convex and closed graph. Set $\|(A,y)\| = \max\{\|A\|, \|y\|\}$. Then the distance to infeasibility for the system $\overline{y} \in F(x)$ is equal to*

$$\inf_{\|y*\|=1}\ \inf_{x^*} \max\big\{\|x^*\|, S_{\text{Graph } F}(x^*,y^*) + \langle y^*, \overline{y}\rangle\big\}.$$

3. **Elementary facts about** $\ell_\infty(T)$. This is obviously a Banach space. Denote by $\mathcal{K}$ the cone of nonnegative elements of $\ell_\infty$, that is $p = (p_t) \in \mathcal{K}$ if $p_t \geq 0$ for all $t$, and by $\mathcal{K}^*$ the dual cone of $\mathcal{K}$ (the opposite of the polar of $K$): $p^* \in (\ell_\infty)^*$ belongs to $\mathcal{K}^*$ if and only if $\langle p^*, p\rangle \geq 0$ for all $p \in \mathcal{K}$. We usually write $p^* \geq 0$ for $p^* \in \mathcal{K}^*$. As well known, the dual space $(\ell_\infty)^*$ can be represented as the collection of all bounded finitely additive measures on $T$ .

The space $\ell_\infty$ ordered by $\mathcal{K}$ is a conditionally complete Banach lattice: any bounded set has an exact upper bound and the norm is a monotone function on $\mathcal{K}$. The function $\mathbf{1}$ identically equal to 1 on $T$ is the unique *unit* on $\ell_\infty$: $\langle p^*, \mathbf{1}\rangle = \|p^*\|$ for $p^* \in \mathcal{K}$ and $Q \subset \ell_\infty$ is bounded if and only if $|a| = (|a_t|) \leq \lambda\mathbf{1}$ for some $\lambda > 0$. Thus the set

$$P^* = \big\{p^* \in (\ell_\infty)^* :\ p^* \geq 0,\ \|p^*\| = 1\big\}$$

is convex and weak$^*$ closed, hence weak$^*$ compact. The same is clearly valid for the set $\{p^* \in (\ell_\infty)^* :\ p^* \geq 0,\ \|p^*\| \leq 1\}$.

## 3. Systems of convex inequalities.

In this section we consider systems containing only inequalities with convex lower semi-continuous functions, that is

$$\varphi_t(x) \leq p_t, \quad \forall\, t \in T \tag{4}$$

and denote as before by $\mathcal{S}(p)$ the set of solutions of (4). We shall assume that $\varphi_t$ satisfy the following *uniform boundedness condition*

(**UB**) $$\inf_{t\in T}\inf_{x\in X} \varphi_t(x) > -\infty.$$

It will be clear that in the context of the this paper the assumption does not really impose any restrictions on the choice of possible $\varphi_t$. For instance, if we are interested in Lipschitz stability of $\mathcal{S}$ at $(\overline{x}, \mathbf{0})$, then replacing $\varphi_t$ by $\max\{-1, \varphi_t(x)\}$ does not make any change.

We associate with $(\varphi_t)$ the set valued mapping

$$F(x) = \big\{a \in \ell_\infty :\ a_t \geq \varphi_t(x),\ \forall\, t\big\}$$

from $X$ into $\ell_\infty(T)$ and the function

$$\Phi(x) = \sup_{t\in T} \varphi_t(x).$$

It is clear that the graph of $F$ is closed and convex,

$$F(x) + \mathcal{K} \subset F(x), \quad \forall\, x \tag{5}$$

and dom $F = \{x : F(x) \neq \emptyset\} =$ dom $\Phi$. As the interior of $\mathcal{K}$ is nonempty, it follows from (5) that the range of $F$ is either empty or its interior is nonemty. Equally obvious is that $\Phi$ is a lower semicontinuous convex function bounded from below (by (**UB**)). If $F(x) \neq \emptyset$ for a certain $x$, then the vector $\varphi(x) = (\varphi_t(x))$ belongs to $\ell_\infty$ (also by (**UB**)).

**Proposition 5**. *The mapping $F$ is perfectly regular at any point of its graph.*

*Proof.* Clearly, Graph $F$ is a closed set as all $\varphi_t$ are lower semicontinuous. Take an $(\overline{x}, \overline{a}) \in$ Graph $F$ and suppose that $S_{\text{Graph } F-(\overline{x},\overline{y})}(x^*, -p^*) = K < \infty$ for some $x^*,\ p^*$ with $\|p^*\| = 1$. The inequality means that $\langle x^*, x - \overline{x}\rangle - \langle p^*, a - \overline{a}\rangle \leq K$ for all $(x, a) \in$ Graph $F$. By (5), $\langle p^*, p\rangle \geq -K$ for all $p \in \mathcal{K}$ which may happen only if $p^* \geq 0$. Thus $p^* \in P^*$. Apply Proposition 2 with $Q = P^*$. □

An immediate consequence of the proposition is that sur$F(\overline{x}|\mathbf{0}) = \infty$ if $\Phi(\overline{x}) < 0$. Indeed in this case the tangent cone to $F(\overline{x})$ at $\mathbf{0}$ is the whole of $\ell_\infty$ which means that sur$DF(\overline{x}, \mathbf{0}) = \infty$. Therefore our main attention in the future discussions will be given to the nontrivial case $\Phi(\overline{x}) = 0$.

The following function $p^* \circ F$ defined for a given a $p^* \geq 0$ plays central role in our discussions:

$$(p^* \circ F)(x) = \inf_{a \in F(x)} \langle p^*, a\rangle = \begin{cases} \langle p^*, \varphi(x)\rangle, & \text{if} \quad F(x) \neq \emptyset; \\ \infty, & \text{otherwise.} \end{cases}$$

In particular, if $p^* = 0$, then $(p^* \circ F)(x)$ is the *indicator* of dom $F$ which is the function equal to zero on dom $F$ and $+\infty$ outside of dom $F$. It is clear that $p^* \circ F$ is a convex function (as so are all $\varphi_t$ and $p^* \geq 0$).

In what follows we fix a certain $\overline{x} \in X$ and assume that $\Phi(\overline{x}) \leq 0$, that is $\varphi_t(\overline{x}) \leq 0$ for all $t$.

**Proposition 6**. *If $\Phi(\overline{x}) \leq 0$, then $(x^*, -p^*) \in N(\text{Graph } F, (\overline{x}, \mathbf{0}))$ if and only if*

$$p^* \geq 0, \quad x^* \in \partial(p^* \circ F)(\overline{x}), \quad (p^* \circ F)(\overline{x}) = 0. \tag{6}$$

*In particular, if $\Phi(\overline{x}) < 0$, then $(x^*, -p^*) \in N(\text{Graph } F, (\overline{x}, \mathbf{0}))$ if and only if $p^* = 0$ and $x^* \in N(\text{dom } F, \overline{x})$.*

*Proof.* Suppose $(x^*, -p^*) \in N(\text{Graph } F, (\overline{x}, \mathbf{0}))$, that is

$$\langle x^*, x\rangle - \langle p^*, a\rangle \leq \langle x^*, \overline{x}\rangle, \quad \forall\, (x, a) \in \text{Graph } F. \tag{7}$$

By (5) $p^* \geq 0$. As $\varphi(x)$ is the minimal element of $F(x)$ whenever $F(x) \neq \emptyset$, it follows that

$$\langle x^*, x\rangle - (p^* \circ F)(x) \leq \langle x^*, \overline{x}\rangle, \quad \forall\, x \tag{8}$$

which means that

$$(p^* \circ F)^*(x^*) \leq \langle x^*, \overline{x}\rangle. \tag{9}$$

On the other hand, the inequality $(p^*\circ F)(x)+(p^*\circ F)^*(x^*)\ge\langle x^*,x\rangle$ holds unconditionally. Therefore, as $(p^*\circ F)(\overline{x})\le 0$ by the assumption, (9) implies that

$$(p^*\circ F)(\overline{x})=0 \quad \text{and} \quad x^*\in\partial(p^*\circ F)(\overline{x}).$$

This completes the proof of (6).

Conversely, let (6) hold. Then for any $x$

$$\langle x^*,x\rangle-(p^*\circ F)(x)\le(p^*\circ F)^*(x^*)=\langle x^*,\overline{x}\rangle$$

which is (8). But (8) implies (7) because $\langle p^*,a\rangle\ge(p^*\circ F)(x)$ if $a\in F(x)$. □

Denote by $\Delta$ the set of "Dirac measures" on $T$, which are functionals $\delta_t\in(\ell_\infty)^*$ defined by

$$\langle\delta_t,a\rangle=a_t, \qquad \forall a\in\ell_\infty.$$

Recall:

$$P^*:=\big\{p^*\in(\ell_\infty)^*: \quad p^*\ge 0,\ \|p^*\|=1\big\}.$$

We claim that $\langle p^*,a\rangle\le\sup\{a_t:\ t\in T\}$ for any $p^*\in P^*$ and any $a\in\ell_\infty$. In particular

$$(p^*\circ F)(x)\le\Phi(x), \quad \forall x \tag{10}$$

if $p^*\in P^*$. Indeed, if $a'\ge a$, then $\langle p^*,a'\rangle\ge\langle p^*,a\rangle$. It follows that for an $a=(a_t)$ with $a_t\equiv\alpha$ we have $\langle p^*,a\rangle=\alpha\|p^*\|$ and the claim immediately follows.

**Proposition 7**. $P^*=\mathrm{cl}^*(\mathrm{conv}\ \Delta)$.

*Proof.* It is obvious that $P^*$ is convex and weak$^*$-closed (even weak$^*$-compact) and that $\Delta\subset P^*$, so we only have to prove that $P^*\in\mathrm{cl}^*(\mathrm{conv}\ \Delta)$. Assuming the contrary we find an $x^*\in P^*\backslash\mathrm{cl}^*(\mathrm{conv}\ \Delta)$.Then there is an $a\in\ell_\infty$ such that

$$\langle p^*,a\rangle>\sup\{a_t:\ t\in T\}.$$

But as we have just seen this cannot be true. □

**Proposition 8**. *For any $x^*$ the function $p^*\mapsto(p^*\circ F)^*(x^*)$ on $(\ell_\infty)^*$ is convex and weak$^*$-lower semicontinuous on its domain.*

*Proof.* Convexity follows from the obvious inequality

$$\begin{aligned}&\sup_x\big(\langle x^*,x\rangle-((\alpha p_1^*+(1-\alpha)p_2^*)\circ F)(x)\big)\\ &\qquad=\sup_x\big(\alpha(\langle x^*,x\rangle-(p_1^*\circ F)(x))+(1-\alpha)(\langle x^*,x\rangle-(p_2^*\circ F)(x))\big)\\ &\qquad\le\alpha\sup_x(\langle x^*,x\rangle-(p_1^*\circ F)(x))+(1-\alpha)\sup_x(\langle x^*,x\rangle-(p_2^*\circ F)(x)).\end{aligned}$$

On the other hand, if $\varphi(x)\in\ell_\infty$, then $p^*\mapsto\langle p^*,\varphi(x)\rangle$ is linear and weak$^*$-continuous. It follows that for any $x^*$ the function $p^*\mapsto(p^*\circ F)^*(x^*)$ is an upper bound of affine and weak$^*$-continuous functions corresponding to $p^*\in\mathcal{K}^*$. □

**Proposition 9**. *Suppose that $\Phi(\overline{x})=0$. Then $x^*\in\partial\Phi(\overline{x})$ if and only if there is a $p^*\in P^*$ such that $(x^*,-p^*)\in N(\mathrm{Graph}\ F,(\overline{x},0))$.*

*Proof.* Let $p^* \in P^*$ and $(x^*, -p^*) \in N(\text{Graph } F, (\overline{x}, 0))$. Then (10) holds, so that

$$(p^* \circ F)^*(u^*) \geq \Phi^*(u^*), \quad \forall u^* \in X^*.$$

On the other hand by Proposition 6 $x^* \in \partial(p^* \circ F)(\overline{x})$ and $(p^* \circ F)(\overline{x}) = \Phi(\overline{x}) = 0$. Thus

$$\Phi^*(x^*) \leq (p^* \circ F)^*(x^*) = \langle x^*, \overline{x}\rangle - (p^* \circ F)(\overline{x}) = \langle x^*, \overline{x}\rangle - \Phi(\overline{x})$$

which shows that $x^* \in \partial\Phi(\overline{x})$.

To prove the opposite implication, we first observe that,

$$\Phi(x) = \sup_{p^* \in P^*} (p^* \circ F)(x).$$

Indeed, on the one hand, we have (10). On the other hand, $(\delta_t \circ F)(x) = \varphi_t(x)$ if $F(x) \neq \emptyset$ which together with the fact that otherwise $(p^* \circ F)(x) = \infty$ for any $p^* \geq 0$ gives the opposite inequality.

As $P^*$ is a convex and weak*-compact set, it follows, in view of the minimax theorem of Sion [18], that

$$\begin{aligned} \Phi^*(x^*) &= \sup_x (\langle x^*, x\rangle) - \sup_{p^* \in P^*} (p^* \circ F)(x)) \\ &= \sup_x \inf_{p^* \in P^*} (\langle x^*, x\rangle - \sup_{p^* \in P^*} (p^* \circ F)(x)) \\ &= \inf_{p^* \in P^*} \sup_x (\langle x^*, x\rangle - \sup_{p^* \in P^*} (p^* \circ F)(x)) \\ &= \inf_{p^* \in P^*} (p^* \circ F)^*(x^*). \end{aligned} \tag{11}$$

By Proposition 7 the infimum in the last expression is attained, so that $\Phi^*(x^*) = (p^* \circ F)^*(x^*)$ for some $p^* \in P^*$. If now $x^* \in \partial\Phi(\overline{x})$, we therefore get

$$(p^* \circ F)^*(x^*) = \Phi^*(x^*) = \langle x^*, \overline{x}\rangle - \Phi(\overline{x}) \leq \langle x^*, \overline{x}\rangle - (p^* \circ F)(\overline{x})$$

which may be valid only if $x^* \in \partial(p^* \circ F)(\overline{x})$ and $(p^* \circ F)(\overline{x}) = \Phi(\overline{x}) = 0$. □

Summarizing the results of Propositions 6 and 9, we get the following theorem.

**Theorem 10**. *Suppose that the functions $\varphi_t$ are convex and lower semicontinuous for all $t \in T$ and $\Phi(\overline{x}) = 0$ for some $\overline{x} \in X$. Then the following conditions are equivalent:*

*(a)* $x^* \in \partial\Phi(x)$;

*(b)* $x^* \in \partial(p^* \circ F)(\overline{x})$ *for some* $p^* \in P^*$ *such that* $(p^* \circ F)(\overline{x}) = 0$;

*(c)* $(x^*, -p^*) \in N(\text{Graph } F, (\overline{x}, 0))$ *for some* $p^* \in P^*$;

*(d)* $(x^*, \langle x^*, \overline{x}\rangle) \in \text{epi } \Phi^*$.

*Proof.* Propositions 1 and 6 show that (a), (b) and (c) are equivalent, and (d) is an obvious reformulation of (a) if $\Phi(\overline{x}) = 0$. □

**Remark 11**. Condition (d) can be equivalently written as

$$(x^*, \langle x^*, \overline{x}\rangle) \in \mathrm{cl}^*\mathrm{conv}\,\Big(\bigcup_{t\in T} \mathrm{Graph}\ \varphi_t^*\Big). \tag{12}$$

Indeed, as $\Phi(\overline{x}) = 0$, (d) is obviously the same as $(x^*, \langle x^*, \overline{x}\rangle) \in \mathrm{Graph}\ \Phi^*$. On the other hand, the epigraph of $\Phi^*$ is the weak$^*$ closure of the convex hull of the union of epi $\varphi_t^*$. (12) is an immediate consequence of these two facts.

**Theorem 12**. *Assume that the functions $\varphi_t$ are convex and lower semi-continuous and $\mathbf{0} \in F(\overline{x})$, that is $\Phi(\overline{x}) \le 0$. Then*

$$\mathrm{sur}F(\overline{x}|\mathbf{0}) = \begin{cases} d(0, \partial\Phi(\overline{x})) = \min\{\|x^*\| :\ x^* \in \partial\Phi(\overline{\alpha})\} & \text{if } \Phi(\overline{x}) = 0;\\ \infty, & \text{if } \Phi(\overline{x}) < 0\end{cases}$$

*and therefore*

$$\mathrm{lip}\mathcal{S}(\mathbf{0}, \overline{x}) = \sup\big\{\|x^*\|^{-1} :\ x^* \in \partial\Phi(\overline{\alpha})\big\}.$$

*if* $\Phi(\overline{x}) = 0$ *and* $\mathrm{lip}\mathcal{S}(\mathbf{0}, \overline{x}) = 0$ *if* $\Phi(\overline{x}) < 0$.

*Proof.* The case $\Phi(\overline{x}) < 0$ is trivial as in this case the tangent cone to $F(\overline{x})$ at $\mathbf{0}$ is the whole of $Y$, and the result is immediate from Proposition 5. In the nontrivial case $\Phi(\overline{x}) = 0$ the result follows from Theorem 10 (and Proposition 5). □

Consider a special case of (4) with

$$\varphi_t(x) = \langle a_t, x\rangle - \beta_t; \quad a_t \in X^*,\ b_t \in R. \tag{13}$$

The Fenchel conjugate of $\varphi_t$ is equal to $\beta_t$ plus the indicator of $\{a_t\}$. Applying this along with Remark 11, we get

**Corollary 13**. *Let $\overline{x} \in X$ solves (4) with $\varphi_t$ given by (13) and $\Phi(\overline{x}) = 0$. Set*

$$C = \mathrm{cl}^*\mathrm{conv}\,\Big(\bigcup_{t\in T}(a_t, \beta_t)\Big).$$

*Then* $\mathrm{sur}F(\overline{x}|\mathbf{0}) = \inf\{\|x^*\| :\ (x^*, \langle x^*, \overline{x}\rangle) \in C\}$.

The corollary strengthens the recent result of [6] in two important respects. First it shows that no qualification condition is needed for the result. In [6] it is assumed that the strong Slater condition is satisfied, that is there are $x$ and $\varepsilon > 0$ such that $\varphi_t(x) < -\varepsilon$ for all $t$. (Of course the absence of this condition trivially implies that $\mathrm{sur}F(\overline{x}|\mathbf{0}) = 0$ but the point is that there is no need in a priori verification of the Slater condition which may be as painstaking as the calculation of the rate of surjection). Secondly we do not assume that either $a_t$ or $b_t$ are uniformly bounded.

The questions about the radius of regularity of $F$ at $(\overline{x}, 0)$ and the distance to feasibility for the system $0 \in F(x)$ can also be easily answered. In the proposition below we consider the max-norm $\|(x^*, \alpha)\| = \max\{\|x^*\|, |\alpha|\}$ in $X^* \times I\!R$.

**Proposition 14**. *The radius of regularity of $F$ at $(\overline{x},\mathbf{0})$ is equal to $d(0,\partial\Phi(\overline{x}))$, and the distance to feasibility for the system $\mathbf{0}\in F(x)$ is*

$$d((0,0),\text{epi }\Phi^*)=\inf\big\{\|(x^*,\alpha)\|:\ (x^*,\alpha)\in\text{epi }\Phi^*\big\}=\inf_{x^*}\max\{\|x^*\|,\Phi^*(x^*)\big\}.$$

*In particular for systems of linear inequalities with $\varphi_t$ defined by (13), the distance to infeasibility of the system $\mathbf{0}\in F(x)$ is*

$$d((0,0),C)=\inf\big\{(\|(x^*,\alpha)\|:\ (x^*,\alpha)\in C\big\}.$$

*Proof.* The first statement is an immediate consequence of Propositions 3 and 5 and Theorem 12. To prove the second we apply Proposition 4. We have

$$S_{\text{Graph }F}(x^*,-p^*)=\begin{cases}(p^*\circ F)(x^*), & \text{if } p^*\geq 0;\\ \infty, & \text{otherwise.}\end{cases}$$

The arguments needed for the proof of the equality basically repeat what was said in the beginning of the proof of Proposition 6. We have therefore

$$\begin{aligned}&\inf_{\|p^*\|=1}\inf_{x^*}\max\big\{\|x^*\|,S_{\text{Graph }F}(x^*,-p^*)\big\}\\ &=\inf_{x^*}\inf_{p^*\in P^*}\max\big\{\|x^*\|,(p^*\circ F)^*(x^*)\big\}\\ &=\inf_{x^*}\max\big\{\|x^*\|,\inf_{p^*\in P^*}(p^*\circ F)^*(x^*)\big\}=\max\big\{\|x^*\|,\Phi^*(x^*)\big\}\end{aligned}$$

(the second equality due to (11)). A reference to Proposition 4 completes the proof. □

## 4. Equality and inequality constraints.

We are now turn to the general system (1) containing a linear operator equality along with a system of convex inequalities. As above we start by introducing the set-valued mapping $G\colon X\rightrightarrows\ell_\infty(T)\times Y$:

$$G(x)=\big\{(a,y)\in\ell_\infty(T)\times Y:\ a_t\geq\varphi_t(x),\ \forall\, t\in T,\quad y=Ax\big\}=F(x)\times\big\{A(x)\big\}.$$

As in the case of $F$, our purpose will be to find estimates for the rate of surjection of $G$ at a certain given point of the graph. To this end, we fix the max norm in $\ell_\infty\times Y$: $\|(a,y)\|=\max\{\|a\|,\|y\|\}$, so that the dual norm in $(\ell_\infty)^*\times Y^*$ is $\|(p^*,y^*)\|=\|p^*\|+\|y^*\|$.

We shall need "$\varepsilon$-versions" of some of the implications of Theorem 8. Recall that given a function $f$ on $X$, the $\varepsilon$-subdifferential of $f$ at $x$ is $\partial_\varepsilon f(x)=\{x^*:\ f(x)+f^*(x^*)\leq\langle x^*,x\rangle+\varepsilon\}$.

**Proposition 15**. *We posit the assumptions of Theorem 10. Let $\varepsilon\geq 0$, $p^*\in P^*$ and $(p^*\circ F)(\overline{x})\geq-\varepsilon$. If under these conditions $x^*\in\partial_\varepsilon(p^*\circ F)(\overline{x})$, then $x^*\in\partial_{2\varepsilon}\Phi(\overline{x})$.*

*Proof.* Indeed, as we have seen $(p^*\circ F)^*(x^*)\geq\Phi^*(x^*)$ and, on the other hand, $(p^*\circ F)(\overline{x})\geq\Phi(\overline{x})-\varepsilon$ by the assumption. Thus

$$\Phi(x^*)\leq\langle x^*,\overline{x}\rangle-(p^*\circ F)(\overline{x})+\varepsilon\leq\langle x^*,\overline{x}\rangle-\Phi(\overline{x})+2\varepsilon$$

as claimed. □

We are ready to prove the second main result of the paper. Set

$$\Omega(x) = \big\{(\alpha, y^*) \in I\!R \times Y^* : \ \alpha \geq 0,\ \alpha\Phi(x) = 0,\ \alpha + \|y^*\| = 1\big\}.$$

**Theorem 16**. *Assume as before that $\varphi_t$ for any $t \in T$ is a convex lower semi-continuous function, the uniform boundedness assumption is satisfied and $\Phi(x) = \sup_t \varphi_t(x)$. Let $A\colon X \to Y$ be a linear bounded operator, and let $\Phi(\overline{x}) \leq 0,\ A\overline{x} = \overline{y}$ for some $\overline{x} \in X,\ \overline{y} \in Y$, so that $(\overline{x}, \mathbf{0}, \overline{y}) \in \text{Graph } G$. Then*

$$\text{sur}G(\overline{x}|(\mathbf{0}, \overline{y})) \leq \inf\big\{\|x^* + A^*y^*\| : \ x^* \in \partial(\alpha\Phi)(\overline{x})),\ (\alpha, y^*) \in \Omega(\overline{x})\big\};$$

$$\text{sur}G(\overline{x}|(\mathbf{0}, \overline{y})) \geq \lim_{\varepsilon\to 0}\inf\big\{\|x^* + A^*y^*\| : \ x^* \in \partial_\varepsilon(\alpha\Phi)(\overline{x})),\ (\alpha, y^*) \in \Omega(\overline{x})\big\};$$

*Moreover, if* $\dim Y < \infty$*, equality holds in both relations. In other words, in this case*

$$\text{sur}G(\overline{x}|(\mathbf{0}, \overline{y})) = \inf\big\{\|x^* + A^*y^*\| : \ x^* \in \partial(\alpha\Phi)(\overline{x})),\ (\alpha, y^*) \in \Omega(\overline{x})\big\};$$

**Remark 17**. Here we identify $(0 \cdot \Phi)(\cdot)$ with the indicator of dom $\Phi$ (obviously equal to dom $F$). The theorem also applies to the degenerate cases when $T = \emptyset$ or $Y = \{0\}$. In the first case $\Phi(x) \equiv -\infty$ and the condition $\alpha\Phi(\overline{x}) = 0$ may only be satisfied when $\alpha = 0$, and we may adopt the convention that $0 \times (-\infty) = 0$ to make the conclusion formally valid. If $Y = \{0\}$, then also $Y^*$ may contain only the zero vector and we get the already established result with $\alpha = 1$.

*Proof.* Denote by $\mathcal{A}$ the contingent derivative of $G$ at $(\overline{x}, (0, \overline{y}))$. Then

$$\text{sur}G(\overline{x}|(\mathbf{0}, \overline{y})) \leq \text{sur}\mathcal{A}(0|(0, 0)).$$

(since Graph $G \in (xb|(\mathbf{0}, \overline{y})) +$ Graph $\mathcal{A}$). On the other hand, as follows from (3) and Theorem 3 of [12], every closed convex process is perfectly regular at the origin. But the normal cone to the graph of $\mathcal{A}$ at the origin coincides with the normal cone to the graph of $G$ at $(\overline{x}, (0, \overline{y}))$. This means that $x^* \in D^*\mathcal{A}(0|(0, 0))(p^*, y^*)$ if and only if

$$\langle x^*, x - \overline{x}\rangle - \langle p^*, a\rangle - \langle y^*, y - \overline{y}\rangle \leq 0, \quad \forall (x, a, y) \in \text{Graph } G,$$

which is the same as

$$\langle x^* - A^*y^*, x - \overline{x}\rangle - \langle p^*, a\rangle \leq 0, \quad \forall (x, a) \in \text{Graph } F.$$

By Proposition 4 the latter is equivalent to

$$p^* \geq 0, \quad x^* - A^*y^* \in \partial(p^* \circ F)(\overline{x}), \quad (p^* \circ F)(\overline{x}) = 0.$$

As follows from Proposition 7 the relations $u^* \in \partial(p^* \circ F)(\overline{x}),\ (p^* \circ F)(\overline{x}) = \Phi(\overline{x}) = 0$ and $\|p^*\| = \alpha$ are equivalent to $u^* \in \partial(\alpha\Phi(\overline{x}))$ if $\alpha > 0$. For $\alpha = 0$ the equivalence follows from the fact that both $p^* \circ F$ and $\alpha\Phi$ reduce in this case to the indicator of dom $\Phi$. Thus

$$\text{sur}\mathcal{A}(0|(\mathbf{0}, 0)) = \inf\{\|x^*\| : \ x^* - A^*y^* \in \partial(\alpha\Phi(\overline{x})),\ \alpha + \|y^*\| = 1\}.$$

Replacing $x^*$ by $x^* + A^*y^*$, we get the first of the two declared inequalities.

The proof of the second inequality needs Theorem 1. According to the formlula for the surjection modulus of a convex set-valued mapping provided by the theorem

$$\mathrm{sur}G(\overline{x}|(\mathbf{0},\overline{y})) = \lim_{\lambda\to 0}\inf\left\{\|u^*\| + \frac{1}{\lambda}S_Q(u^*,p^*,y^*):\ \|p^*\|+\|y^*\|=1\right\}, \tag{14}$$

where $Q = \text{Graph}\ G - (\overline{x},0,\overline{y})$ and $S_Q$ is the *support function* of $Q$, that is

$$S_Q(u^*,p^*,y^*) = \sup\left\{\langle u^*, x-\overline{x}\rangle + \langle p^*, a\rangle + \langle y^*, y-\overline{y}\rangle:\ (x,a,y)\in \text{Graph}\ G\right\}$$

$$= \sup\left\{\langle u^* - A^*y^*, x-\overline{x}\rangle + \langle p^*, a\rangle:\quad (x,a)\in\text{Graph}\ F\right\}.$$

As above we see that if $S_Q(u^*,-p^*,-y^*)<\infty$, then $p^*\geq 0$ and

$$\begin{aligned} S_Q(u^*,-p^*,-y^*) &= \sup_x(\langle u^*-A^*y^*, x-\overline{x}\rangle - (p^*\circ F)(x)) \\ &= (p^*\circ F)^*(u^*-A^*y^*) - \langle u^*-A^*y^*,\overline{x}\rangle. \end{aligned} \tag{15}$$

Observe further that $\mathrm{sur}G(\overline{x}|(\mathbf{0},\overline{y}))\leq \mathrm{sur}A$ if $\dim Y>0$, in other words, if the equality part of (1) is present, and the surjection rate of a linear operator is necessarily finite. Therefore $\mathrm{sur}G(\overline{x}|(\mathbf{0},\overline{y}))<\infty$ and for some $K>0$ the inequality $S_Q(u^*,p^*,y^*)\leq K\lambda$ holds whenever $(u^*,p^*,y^*)$ realize, say $\lambda$-infimum in (14). On the other hand $S_Q(u^*,p^*,y^*)\geq 0$ for all $u^*,p^*,y^*$ as $Q$ contains the origin of $X\times \ell_\infty(T)\times Y$. Therefore

$$\mathrm{sur}G(\overline{x},(\mathbf{0},\overline{y}))\geq \lim_{\varepsilon\to 0}\inf\left\{\|u^*\|:\ S_Q(u^*,p^*,y^*)\leq\varepsilon,\ \|p^*\|+\|y^*\|=1\right\} \tag{16}$$

Combining (15) with (16) and setting $x^*=u^*-A^*y^*$, we get

$$\begin{aligned}\mathrm{sur}G(\overline{x},(\mathbf{0},\overline{y}))\geq \lim_{\varepsilon\to 0}\big\{\|x^*+A^*y^*\|:\ p^*\geq 0,\ \|p^*\|+\|y^*\|=1,\\ (p^*\circ F)^*(x^*)\leq\langle x^*,\overline{x}\rangle+\varepsilon\big\}.\end{aligned} \tag{17}$$

We note next that $(p^*\circ F)(\overline{x})\geq -\varepsilon$ if $(p^*\circ F)^*(x^*)\leq\langle x^*,\overline{x}\rangle+\varepsilon$ and on the other hand, the latter inequality together with the fact that $(p^*\circ F)(\overline{x})\leq\Phi(\overline{x})\leq 0$ implies that $x^*\in\partial_\varepsilon(p^*\circ F)(\overline{x})$ and therefore by Proposition 8 that $x^*\in\partial_{2\varepsilon}\|p^*\|\Phi(\overline{x})$ if $p^*\neq 0$. If $p^*=0$, the equality, as in the first part of the proof follows from the fact that both $p^*\circ F$ and $\|p^*\|\Phi$ reduce to the indicator of dom $\Phi$. Setting $\alpha=\|p^*\|$, we get from (17) the second of the declared inequalities.

If finally $\dim Y<\infty$, take $x_n^*,p_n^*,y_n^*$ realizing infimum in (16) for $\varepsilon=n^{-1}$. Let $x^*,p^*,y^*$ be a weak$^*$ cluster point of the sequence. Then $p^*\geq 0$ $\|p^*\|+\|y^*\|=1$, $x^*+A^*y^*\in\partial\Phi(\overline{x})$ and $\|x^*+A^*y^*\|\leq\liminf\|x_n^*+A^*y_n^*\|$. Thus, setting $\alpha=\|p^*\|$, we see from the second inequality in the statement that $\alpha+\|A^*y^*\|=1$ and $\mathrm{sur}G(\overline{x}|(0,\overline{y})\geq\|x^*+A^*y^*\|$. The reference to the first inequality completes the proof of the theorem. □

We observe that here again *no qualification condition is required by the theorem.* Its specialization for the case of linear inequalities is technically not much more complicated that in the pure inequality case of the previous section. In particular, for the case $\dim Y<\infty$ we get

**Corollary 18**. *Suppose* $\dim Y < \infty$ *and* $\varphi_t(x) = \langle a_t, x\rangle - \beta_t$. *Let* $\overline{x} \in X$ *be such that* $\varphi_t(\overline{x}) \le 0$ *for all* $t \in T$. *Set* $A\overline{x} = \overline{y}$. *then*

$$\text{sur}G(\overline{x}|(\mathbf{0},\overline{y})) = \inf\big\{\|u^*\| : \ (u^*, \langle u^*, \overline{x}\rangle) \in E\big\},$$

*where*

$$E = \text{cl}^*\Big(\text{conv }\{\alpha(a_t.\beta_t) + (A^*y^*, \langle y^*\overline{y}\rangle) : \ \alpha \ge 0,\ \alpha + \|y^*\| = 1,\ t \in T\}\Big).$$

An alternative way to prove the part of the theorem relating to $A$ with finite dimensional range is offered by the following proposition.

**Proposition 19**. *If* $\dim Y < \infty$, *then* $G$ *is perfectly regular at any point of its graph. Hence in this case* $\text{rad}G(\overline{x}, (\mathbf{0}, \overline{y})) = \text{sur}G(\overline{x}|(\mathbf{0}, \overline{y}))$.

*Proof.* This is an immediate consequence of Proposition 2. Indeed, on the one hand, the set $Q = \{(-p^*, y^*) \in (\ell_\infty)^* \times y^* : \ p^* \ge 0,\ \|p^*\| + \|y^*\| = 1\}$ is compact in the product of the weak*-topology in $(\ell_\infty)^*$ and standard topology of $Y^*$ and, on the other hand, as we have seen, $(p^*, y^*) \in Q$ if $\|p^*\| + \|y^*\| = 1$ and $S_{\text{Graph } G-(\overline{x},0,\overline{y})}(x^*, -p^*, y^*) < \infty$. □

Thus the problem of radius of regularity in this case is trivially solved by this proposition. The question about the radius of regularity of $G$ in case of infinitely many equalities remains however open. Finally, as far as the distance to infeasibility is concerned, the same simple arguments as in the proof of Proposition 14 lead to

**Proposition 20**. *Under the assumptions of Theorem 16 the distance to infeasibility of the system* $(\mathbf{0}, \overline{y}) \in G(x)$ *is*

$$\inf_{x^*} \max\big\{\|x^*\|, \inf\{\alpha\Phi^*(x^* - A^*y^*) + \langle y^*, \overline{y}\rangle : \ \alpha + \|y^*\| = 1\}\big\}.$$

*In particular, if* $\overline{y} = 0$ *the distance to infeasibility is*

$$\inf\big\{\|(x^* + A^*y^*, \beta)\| : \ (x^*, \beta) \in \text{epi }(\alpha\Phi^*);\ \alpha \ge 0,\ \alpha + \|y^*\| = 1\big\}.$$

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