ON FINITE UNIONS AND FINITE PRODUCTS WITH THE D-PROPERTY

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Abstract. We show that the product of a subparacompact $\mathcal{C}$-scattered space and a Lindelöf D-space is D. In addition, we show that every regular locally D-space which is the union of a finite collection of subparacompact spaces and metacompact spaces has the D-property. Also, we extend this result from the class of locally D-spaces to the wider class of $\mathcal{D}$-scattered spaces. All the results are shown in a direct way.

1. Introduction

All spaces under consideration are Hausdorff and regular. Our terminology is standard. Terms not defined here can be found in [3].

An *open neighbourhood assignment* (ONA) for a space X is a function η from X to the topology of X such that $x \in \eta(x)$ for every $x \in X$. If Y is a subset of X, we write $\eta[Y] = \bigcup\{\eta(y) : y \in Y\}$. Then, we say that X is a *D-space*, if for every open neighbourhood assignment η for X there is a closed discrete subset D of X such that $\eta[D] = X$.

It is obvious that every compact space is a D-space. However, it is not known whether every Lindelöf space is D, and it is also unknown whether the D-property is implied by paracompactness, subparacompactness or metacompactness.

Recall that a space X is *scattered*, if every nonempty (closed) subspace of X has an isolated point. More generally, suppose that K is a class of spaces such that for every X in K, each closed subspace of X is also in K. Then we say that a space X is K-*scattered*, if for every nonempty closed subspace Y of X there are a point $y \in Y$ and a neighbourhood U of y in Y such that U with the relative topology of Y is in K. We denote by $\mathcal{C}$ the class of compact spaces and by $\mathcal{D}$ the class of D-spaces. Clearly, the class of $\mathcal{C}$-scattered spaces contains every locally compact space and every scattered space, and the class of $\mathcal{D}$-scattered spaces contains every $\mathcal{C}$-scattered space and every D-space.

It is not known whether the union of two D-spaces is a D-space. It is known that the finite unions of (some generalised) metric spaces are D (see [1],[2] and [9]). Also, a study of the D-property in several types of unions of $\mathcal{C}$-scattered spaces was carried out in [7], [8] and [9].

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For a study of the D-property in products of spaces, we refer the reader to [4], [6] and [10]. In particular, due to results shown in [6] and [10], it can be proved that for every natural number n, any box product of scattered spaces of height $\leq n$ is a D-space. In this paper, we shall prove that the product of a subparacompact $\mathcal{C}$-scattered space and a Lindelöf D-space has the D-property. So, in particular, the product of the Sorgenfrey line with any subparacompact $\mathcal{C}$-scattered space is D.

In addition, we shall prove here in a direct way, i.e. without using topological games, that every $\mathcal{D}$-scattered space which is the union of a finite collection of subparacompact spaces and metacompact spaces has property D. Then, we obtain as a corollary that a space is D if it is a finite union of $\mathcal{C}$-scattered spaces each of which being either subparacompact or metacompact. Previously, by means of stationary strategies in topological games, it was proved by Peng that every space which is a finite union of subparacompact $\mathcal{C}$-scattered spaces is D (see [9]) and that every $\mathcal{D}$-scattered space which is a finite union of metacompact spaces has also property D (see [8, Section 2]).

The above results on finite unions can not be extended to infinite unions of spaces, since the space constructed in [5] provides us an example of a locally compact scattered space which is a countable union of paracompact spaces but does not have property D.

The organisation of this paper is as follows. In Section 2, we show our result on finite products. In Section 3, we prove that every $\mathcal{D}$-scattered space which is the union of a finite collection of subparacompact spaces has property D. And in Section 4, we extend this last result to finite unions of subparacompact spaces and metacompact spaces.

We shall use without explicit mention the well-known facts that "D-space", "subparacompact" and "metacompact" are closed hereditary.

2. A result for finite products

In this section, our aim is to prove the following result.

Theorem 2.1. *If X is a subparacompact $\mathcal{C}$-scattered space and Y is a Lindelöf D-space, then $X \times Y$ is a D-space .*

In order to prove Theorem 2.1, we need some preparation. First we say that a space X is *countably* D, if for every ONA η for X there is a countable closed discrete subset D of X such that $\eta[D] = X$. It is easy to check that a space X is Lindelöf D iff X is countably D.

Now, we consider the extension of the Cantor-Bendixson process for topological spaces defined in [11]. For any space X and any ordinal α, we define the *α-derivative* X^α as follows: $X^0 = X$; if $\alpha = \beta + 1$, $X^\alpha = \{x \in X^\beta : x$ does not have a compact neighbourhood in $X^\beta\}$; and if α is a limit, $X^\alpha = \bigcap\{X^\beta : \beta < \alpha\}$.

The following lemma is straightforward from the definition.

Lemma 2.1. *Assume that X is a $\mathcal{C}$-scattered space and x is a point of X such that $x \in X^\alpha \setminus X^{\alpha+1}$ for some ordinal α. Then, there is a neighbourhood U of x with $U \cap X^{\alpha+1} = \emptyset$.*

By using Lemma 2.1, it is easy to check that a space X is $\mathcal{C}$-scattered iff there is an ordinal α such that $X^\alpha = \emptyset$. Then, we define the *height of* a $\mathcal{C}$-scattered space X by $\mathrm{ht}(X) =$ the least ordinal α such that $X^\alpha = \emptyset$.

We shall prove Theorem 2.1 proceeding by transfinite induction on the height of the $\mathcal{C}$-scattered space X. The following lemma will be needed.

Lemma 2.2. *Assume that Y is a Lindelöf D-space. Assume that X is a space such that there is a compact set $A \subseteq X$ in such a way that for every open set U in X with $A \subseteq U$, $(X \setminus U) \times Y$ is D. Then, $X \times Y$ is D.*

Proof. In order to show that $X \times Y$ is D, assume that η is an ONA for $X \times Y$ such that for every $(x, y) \in X \times Y$, $\eta(x, y) = U \times V$ where U is a basic neighbourhood of x in X and V is a basic neighbourhood of y in Y. We will write $P_0(U \times V) = U$ and $P_1(U \times V) = V$. We define the ONA η' for Y as follows. First, for every $y \in Y$ we consider a finite subset $H(y)$ of A such that

$$A \subseteq \bigcup\{P_0(\eta(x, y)) : x \in H(y)\}.$$

Then, for every $y \in Y$, we define $\eta'(y) = \bigcap \big\{P_1(\eta(x, y)) : x \in H(y)\big\}$.

Since Y is Lindelöf D, there is a countable closed discrete subset E of Y such that $\eta'[E] = Y$. Put $E = \{y_n : n \geq 0\}$. For every $n \in \omega$, we define

$$U_n = \bigcup\{P_0(\eta(x, y_n)) : x \in H(y_n)\},$$
$$V_n = \eta'(y_n).$$

Now, proceeding by induction on $n \in \omega$, we define a closed discrete subset D_n of $X \times Y$ as follows. We put

$$D_0 = \{(x, y) : y \in E, x \in H(y)\}.$$

Note that D_0 is a closed discrete subset of $X \times Y$, since E is closed discrete in Y. Then, we put $W_0 = \eta[D_0]$. Now assume that $n = m + 1$ where $m \geq 0$. Let $W_m = \bigcup\{\eta[D_k] : k \leq m\}$. By the assumption of the lemma, $(X \setminus U_m) \times Y$ is a closed D-subset of $X \times Y$. Hence as W_m is open in $X \times Y$, $((X \setminus U_m) \times Y) \setminus W_m$ is also D in $X \times Y$. So, there is a closed discrete subset D_n in $((X \setminus U_m) \times Y) \setminus W_m$ such that $\eta[D_n] \supseteq ((X \setminus U_m) \times Y) \setminus W_m$.

We put $D = \bigcup\{D_n : n \geq 0\}$.

Claim 1. $\eta[D] = X \times Y$.

Let $(x, y) \in X \times Y$. First, assume that $x \in \bigcap\{U_n : n \geq 0\}$. Let $k \in \omega$ such that $y \in V_k$. Since $x \in U_k$, we deduce that $x \in P_0(\eta(u, y_k))$ for some $u \in H(y_k)$. Hence,

$$(x, y) \in P_0(\eta(u, y_k)) \times V_k \subseteq P_0(\eta(u, y_k)) \times P_1(\eta(u, y_k)) = \eta(u, y_k) \subseteq \eta[D_0].$$

Now, assume that $x \notin \bigcap\{U_n : n \geq 0\}$. Let n be the least m such that $x \notin U_m$. Then as

$$\eta[D_{n+1}] \supseteq ((X \setminus U_n) \times Y) \setminus W_n,$$

we infer that $(x, y) \in \bigcup\{\eta[D_k] : k \leq n+1\}$.

Claim 2. D is closed discrete.

Assume that $z \in X \times Y$. By Claim 1, there is an $m \in \omega$ such that $z \in W_m = \bigcup\{\eta[D_k] : k \leq m\}$. Let n be the least m with this property. By the way in which D_k is defined, $D_k \cap W_n = \emptyset$ if $k > n$. Then as each D_k is closed discrete in $X \times Y$, there is a neighbourhood U of z such that $(U \setminus \{z\}) \cap D = \emptyset$. □

Proof of Theorem 2.1. We proceed by induction on $\alpha = \mathrm{ht}(X)$. If $\alpha = 0$, then $X = \emptyset$, and so we are done. Assume that $\alpha > 0$. For every $x \in X$, consider a closed neighbourhood V_x of x such that if $x \in X^\gamma \setminus X^{\gamma+1}$ then $V_x \cap X^{\gamma+1} = \emptyset$ and $V_x \cap X^\gamma$ is a compact set. Then, let U_x be an open neighbourhood of x with $U_x \subseteq V_x$. Since X is subparacompact, there is a covering $\mathcal{P} = \bigcup\{\eta_n : n \in \omega\}$ of X satisfying the following:

(1) each element of $\mathcal{P}$ is a closed subset of X,
(2) $\mathcal{P}$ is a refinement of $\{U_x : x \in X\}$,
(3) η_n is discrete in X for every $n \in \omega$.

First, assume that α is a limit ordinal. By conditions (1) and (2), every element V of $\mathcal{P}$ is a closed subspace of X of height $< \alpha$, and so $V \times Y$ is D by the induction hypotheses. For every $n \in \omega$, let $E_n = \bigcup\{V \times Y : V \in \eta_n\}$. Clearly, E_n is closed in $X \times Y$. Also as E_n is a discrete union of closed D-subspaces of $X \times Y$, we infer that E_n is D. Hence as $X \times Y = \bigcup\{E_n : n \geq 0\}$, the space $X \times Y$ is a countable union of closed D-subspaces, and thus $X \times Y$ is D.

Now, assume that α is a successor ordinal $\beta + 1$. Put $Z = X^\beta$. Assume that $V \in \mathcal{P}$. If $V \cap Z = \emptyset$, we deduce from the induction hypotheses that $V \times Y$ is D. And if $V \cap Z \neq \emptyset$, then $V \cap Z$ is a compact set, and hence we deduce from Lemma 2.2 and the induction hypotheses that $V \times Y$ is also D. Now, proceeding as above, we infer that $X \times Y$ is D. □

On the other hand, by using an argument similar to the one given in the proofs of Lemma 2.2 and Theorem 2.1, and by using the fact that a space X is Lindelöf D iff X is countably D, we can prove the following result.

Theorem 2.2. *If X is a Lindelöf $\mathcal{C}$-scattered space and Y is a Lindelöf D-space, then $X \times Y$ is Lindelöf and D.*

As an immediate consequence of Theorem 2.2, we obtain that every Lindelöf $\mathcal{C}$-scattered space is D. Also, it was shown in [11, Theorem 1.4] that the product of two $\mathcal{C}$-scattered spaces is $\mathcal{C}$-scattered. So, we obtain in a direct way that any finite product of Lindelöf $\mathcal{C}$-scattered spaces is Lindelöf $\mathcal{C}$-scattered.

3. Unions of subparacompact spaces

We say that a space X is *locally* D, if every point $x \in X$ has a neighbourhood U such that U with the relative topology of X is a D-space.

If x is a point of a space X and U is a neighbourhood of x such that U with the relative topology of X is a D-space, we will say that U is a *D-neighbourhood of* x.

First we show the following result, whose proof is a modification of the argument given in [7, Theorem 2.1].

Theorem 3.1. *If a locally D-space X is the union of a finite collection of subparacompact spaces, then X is D.*

Proof. Assume that $X = X_1 \cup \cdots \cup X_k$ is a locally D-space where $X_1, \ldots, X_k$ are subparacompact. We proceed by induction on k. If $k = 0$, then $X = \emptyset$, and so we are done. Now suppose that the statement holds for $k = l$ for some $l \geq 0$, and let us show that it also holds for $k = l + 1$. Assume that η is an ONA for X. As X is regular and locally D, we may assume that $\mathrm{Cl}_X(\eta(x))$ is D for every $x \in X$.

Now since each X_i is subparacompact, for $1 \leq i \leq k$ there is a covering $\mathcal{P}_i = \bigcup\{\eta_{ij} : j \in \omega\}$ of X_i satisfying the following:

(1) each element of $\mathcal{P}_i$ is a closed subset of X_i,
(2) $\mathcal{P}_i$ is a refinement of $\{\eta(x) \cap X_i : x \in X_i\}$,
(3) η_{ij} is discrete in X_i for every $j \in \omega$.

For $1 \leq i \leq k$ and $n \geq 0$, we put $\gamma_{in} = \{\mathrm{Cl}_X(U) : U \in \eta_{in}\}$. Since $\mathrm{Cl}_X(\eta(x))$ is D for every $x \in X$, by using (2), we deduce that every element of γ_{in} is D for $1 \leq i \leq k$ and $n \in \omega$. We put $F_{in} = \{x \in X : \gamma_{in}$ is not locally finite at $x\}$ for $1 \leq i \leq k$ and $n \in \omega$. Clearly each F_{in} is closed in X, and hence F_{in} is locally D. Also, we deduce from (3) that each $F_{in} \subseteq X \setminus X_i$, and so F_{in} is D by the induction hypotheses. Note also that if U is an open set with $F_{in} \subseteq U$, then $\{F \setminus U : F \in \gamma_{in}\}$ is a locally finite collection of D-subspaces, and hence $\bigcup\{F \setminus U : F \in \gamma_{in}\}$ is D.

Now, for every $n \in \omega$, let $\gamma_n = \bigcup\{\gamma_{in} : 1 \leq i \leq k\}$. Since $\bigcup\{F_{in} : 1 \leq i \leq k\} = \{x \in X : \gamma_n$ is not locally finite at $x\}$, proceeding as in the proof of [7, Theorem 2.1], we can construct for every $n \in \omega$ a closed discrete subset D_n of X such that $\bigcup\{\eta[D_m] : m \leq n\} \supseteq \bigcup(\gamma_0 \cup \cdots \cup \gamma_n)$ and in such a way that $D_n \cap \bigcup\{\eta[D_m] : m < n\} = \emptyset$. We put $D = \bigcup\{D_n : n \in \omega\}$. Then, it is easy to check that D is as required. □

Now, our aim is to extend Theorem 3.1 from locally D-spaces to $\mathcal{D}$-scattered spaces.

We define the *D-derivative* X^* of a space X as the set of all $x \in X$ such that x does not have a D-neighbourhood in X. Clearly,

$$X \setminus X^* = \bigcup\{U \in \tau_X : \mathrm{Cl}_X(U) \text{ has property D}\}.$$

Then, we extend the Cantor-Bendixson process for topological spaces by using the notion of D-derivative. If X is a space and α is an ordinal, we define $X^{(\alpha)}$ as follows. $X^{(0)} = X$; if $\alpha = \beta + 1$, $X^{(\alpha)} = (X^{(\beta)})^*$; and if α is a limit, $X^{(\alpha)} = \bigcap\{X^{(\beta)} : \beta < \alpha\}$.

The following lemma is straightforward from the definition.

Lemma 3.1. *Assume that X is a $\mathcal{D}$-scattered space and x is a point of X such that $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$ for some ordinal α. Then, there is a neighbourhood U of x with $U \cap X^{(\alpha+1)} = \emptyset$.*

By using Lemma 3.1, we obtain that a space X is $\mathcal{D}$-scattered iff there is an ordinal α such that $X^{(\alpha)} = \emptyset$. Then, we define the *rank of* a $\mathcal{D}$-scattered space X by $\mathrm{rank}(X) =$ the least ordinal α such that $X^{(\alpha)} = \emptyset$.

Theorem 3.2. *If a $\mathcal{D}$-scattered space X is the union of a finite collection of subparacompact spaces, then X is D.*

Proof. Suppose that $X = X_1 \cup \cdots \cup X_k$ is $\mathcal{D}$-scattered and $X_1, \ldots, X_k$ are subparacompact. We proceed by induction on k. If $k = 0$, $X = \emptyset$, and so we are done. Now assume that the statement holds for $k = l$ for some $l \geq 0$. In order to show that the statement holds for $k = l + 1$, we proceed by transfinite induction on the rank α of X. The case $\alpha = 0$ is trivial. Suppose that $\alpha > 0$ and that the statement holds for $\mathcal{D}$-scattered spaces of rank $< \alpha$ which are unions of at most k subparacompact spaces. First, assume that $\alpha = \beta + 1$ is a successor ordinal. Let η be an ONA for X. Put $Z = X^{(\beta)}$. Then Z is a closed locally D-subspace of X, and hence Z is D by Theorem 3.1. Let D be a closed discrete subset of Z such that $\bigcup\{\eta(x) \cap Z : x \in D\} = Z$. Let $Y = X \setminus \eta[D]$. Since Y is closed in X and $\mathrm{rank}(Y) < \alpha$, we infer that Y is D by the induction hypotheses. Let E be a closed discrete subset of Y such that $\bigcup\{\eta(x) \cap Y : x \in E\} = Y$. Then, we see that $D \cup E$ is a closed discrete subset of X and $\eta[D \cup E] = X$.

Now, assume that α is a limit ordinal. By using Lemma 3.1 and the induction hypotheses we deduce that X is locally D, and so X is D again by Theorem 3.1. □

It is known that any space which is the union of finitely many $\mathcal{C}$-scattered spaces is also $\mathcal{C}$-scattered (see [11, Lemma 1.1]). So we obtain as a direct consequence of Theorem 3.2 the following result, which was proved by Peng in [9] by means of topological games.

Corollary 3.1. *If a space X is the union of a finite collection of subparacompact $\mathcal{C}$-scattered spaces, then X is D.*

We want to remark that the proof for Theorem 3.2 improves the direct proof given in [7, Theorem 2.1] for scattered spaces.

4. Unions of subparacompact and metacompact spaces

In this section, we shall prove the following result.

Theorem 4.1. *If a $\mathcal{D}$-scattered space X is the union of a finite collection of subparacompact spaces and metacompact spaces, then X is D.*

The following lemma and Theorem 4.2 will be useful to prove the above theorem. Recall that if $\mathcal{P}$ is an open covering of a space X then, for every $x \in X$, $\text{ord}(x, \mathcal{P}) = |\{P \in \mathcal{P} : x \in P\}|$.

Lemma 4.1. *Assume that $\mathcal{P}$ is an open covering of a space X such that the closure of every element of $\mathcal{P}$ is D. Then, for every $n \geq 1$, the set $\{x \in X : ord(x, \mathcal{P}) \leq n\}$ is D.*

Proof. Fix $n \geq 1$. Let $F = \{x \in X : \text{ ord}(x, \mathcal{P}) \leq n\}$. Clearly, F is closed in X. Let η be an ONA for F. For $1 \leq i \leq n$ we put $F_i = \{x \in X : \text{ ord}(x, \mathcal{P}) = i\}$.

We construct sets $D_1, \ldots, D_n$ such that for $1 \leq i \leq n$, D_i is a closed discrete subset of $F_i \setminus \eta[D_1 \cup \cdots \cup D_{i-1}]$ and $F_i \subseteq \eta[D_1 \cup \cdots \cup D_i]$. First, we define D_1. Note that for every $U \in \mathcal{P}$ with $U \cap F_1 \neq \emptyset$, $\text{Cl}_X(U \cap F_1) = U \cap F_1$. Then as $\text{Cl}_X(U \cap F_1)$ is a closed subset of $\text{Cl}_X(U)$ and $\text{Cl}_X(U)$ is D, we infer that $U \cap F_1$ is D. Hence F_1 is a discrete union of D-subspaces, and so F_1 is D. Thus, there is a closed discrete subset D_1 of F_1 with $F_1 \subseteq \eta[D_1]$.

Now, assume that $1 < i \leq n$ and $D_1, \ldots, D_{i-1}$ have been constructed. Note that by the definition of F_i, since $F_1 \cup \cdots \cup F_{i-1} \subseteq \eta[D_1 \cup \cdots \cup D_{i-1}]$, we infer that $F_i \setminus \eta[D_1 \cup \cdots \cup D_{i-1}]$ is closed in X. Put $F_i' = F_i \setminus \eta[D_1 \cup \cdots \cup D_{i-1}]$. It is easy to check that if $U_1, \ldots, U_i$ are distinct elements of $\mathcal{P}$ with $U_1 \cap \cdots \cap U_i \cap F_i' \neq \emptyset$, then $\text{Cl}_X(U_1 \cap \cdots \cap U_i \cap F_i') = U_1 \cap \cdots \cap U_i \cap F_i'$. Now as the closure in X of every element of $\mathcal{P}$ is D, we deduce that $U_1 \cap \cdots \cap U_i \cap F_i'$ is D. Therefore, F_i' is a discrete union of D-subspaces, and so F_i' is D. Thus, there is a closed discrete subset D_i of $F_i \setminus \eta[D_1 \cup \cdots \cup D_{i-1}]$ such that $F_i \setminus \eta[D_1 \cup \cdots \cup D_{i-1}] \subseteq \eta[D_i]$, and hence $F_i \subseteq \eta[D_1 \cup \cdots \cup D_i]$.

Put $D = D_1 \cup \cdots \cup D_n$. Clearly, D is a closed discrete subset of F and $\eta[D] = F$. □

Theorem 4.2. *If a locally D-space X is the union of a finite collection of subparacompact spaces and metacompact spaces, then X is D.*

Proof. Assume that $X = Y_1 \cup \cdots \cup Y_m \cup Z_1 \cup \cdots \cup Z_n$ is a locally D-space with $0 \leq m, n < \omega$ where $Y_1, \ldots, Y_m$ are subparacompact and $Z_1, \ldots, Z_n$ are metacompact. We proceed by induction on $r = m+n$. If $r = 0$, then $X = \emptyset$, and so we are done. So, suppose that the statement holds for $r \geq 0$ and let us show that it also holds for $r+1$. Assume that η is an ONA for X. As X is regular and locally D, we may assume that $\text{Cl}_X(\eta(x))$ is D for every $x \in X$. Since each Z_i is metacompact, for $1 \leq i \leq n$ there is a collection $\mathcal{P}_i$ of open sets in X such that $\{U \cap Z_i : U \in \mathcal{P}_i\}$ is a refinement of $\{\eta(x) \cap Z_i : x \in Z_i\}$ and $\{U \cap Z_i : U \in \mathcal{P}_i\}$ is a point-finite open cover of Z_i. Without loss of generality, we may assume that for $1 \leq i \leq n$ and $U \in \mathcal{P}_i$ there is an $x \in Z_i$ such that $U \subseteq \eta(x)$, and hence $\text{Cl}_X(U)$ is D.

Let $T = X \setminus (\bigcup \mathcal{P}_1 \cup \cdots \cup \bigcup \mathcal{P}_n)$. Clearly, $T \subseteq Y_1 \cup \cdots \cup Y_m$. Also, as T is closed in X, T is locally D. So, by Theorem 3.1, we infer that T is D. Therefore, there is a closed discrete subset A in T such that $T \subseteq \eta[A]$.

Let $Y = X \setminus \eta[A]$. We see that Y is a closed subset of X with $Y \subseteq \bigcup \mathcal{P}_1 \cup \cdots \cup \bigcup \mathcal{P}_n$. We construct sets $D_1, \dots, D_n$ in such a way that for $1 \leq i \leq n$, D_i is a closed discrete subset of $Y \setminus (\eta[D_1] \cup \cdots \cup \eta[D_{i-1}] \cup \bigcup \mathcal{P}_i)$ such that $\eta[D_i]$ covers $Y \setminus (\eta[D_1] \cup \cdots \cup \eta[D_{i-1}] \cup \bigcup \mathcal{P}_i)$. First, we construct D_1. Let $F = Y \setminus \bigcup \mathcal{P}_1$. Since F is a closed subset of X and $F \cap Z_1 = \emptyset$, by the induction hypotheses, we deduce that F is D. So, there is a closed discrete subset D_1 of F with $F \subseteq \eta[D_1]$. Now, suppose that $1 < i \leq n$ and $D_1, \dots, D_{i-1}$ have been constructed. Let $F = Y \setminus (\eta[D_1] \cup \cdots \cup \eta[D_{i-1}] \cup \bigcup \mathcal{P}_i)$. Since F is a closed subset of X and $F \cap Z_i = \emptyset$, again by the induction hypotheses, we infer that the required closed discrete subset D_i exists.

Note that, proceeding by induction on i, we can easily check that for $1 \leq i \leq n$, $Y \setminus (\eta[D_1] \cup \cdots \cup \eta[D_i]) \subseteq \bigcup \mathcal{P}_1 \cap \cdots \cap \bigcup \mathcal{P}_i$.

Let $B = D_1 \cup \cdots \cup D_n$. Clearly, B is a closed discrete subset of X. Let $Z = Y \setminus \eta[B]$. It follows that Z is a closed subset of X such that $Z \subseteq \bigcup \mathcal{P}_1 \cap \cdots \cap \bigcup \mathcal{P}_n$. Our purpose is to construct a closed discrete subset C of Z such that $Z \subseteq \eta[C]$. For $1 \leq i \leq n$, we put $\mathcal{P}'_i = \{U \cap Z : U \in \mathcal{P}_i\}$. And for $1 \leq i \leq n$ and $j \in \omega$, we define

$$H_{ij} = \{x \in Z : \operatorname{ord}(x, \mathcal{P}'_i) \leq j\}.$$

Since $Z \subseteq \bigcup \mathcal{P}'_1 \cap \cdots \cap \bigcup \mathcal{P}'_n$, each H_{ij} is closed in Z. Also, note that $\mathrm{Cl}_Z(U \cap Z)$ is D for $1 \leq i \leq n$ and $U \in \mathcal{P}_i$, because $\mathrm{Cl}_Z(U \cap Z)$ is a closed subset of $\mathrm{Cl}_X(U)$ and $\mathrm{Cl}_X(U)$ is D. Then, by Lemma 4.1, H_{ij} is D for $1 \leq i \leq n$ and $j \in \omega$.

Now, put $Y'_i = Y_i \cap Z$ for $1 \leq i \leq m$. Since each Y'_i is subparacompact, for $1 \leq i \leq m$ there is a covering $\mathcal{V}_i = \bigcup\{\eta_{ij} : j \in \omega\}$ of Y'_i satisfying the following:

(1) each element of $\mathcal{V}_i$ is a closed subset of Y'_i,
(2) $\mathcal{V}_i$ is a refinement of $\{\eta(x) \cap Y'_i : x \in Y'_i\}$,
(3) η_{ij} is discrete in Y'_i for every $j \in \omega$.

For $1 \leq i \leq m$ and $j \in \omega$, we define $\gamma_{ij} = \{\mathrm{Cl}_Z(U) : U \in \eta_{ij}\}$ and $F_{ij} = \{x \in Z : \gamma_{ij}$ is not locally finite at $x\}$. By the argument given in the proof of Theorem 3.1, for $1 \leq i \leq m$ and $j \in \omega$, F_{ij} is a closed D-subset of Z and each element of γ_{ij} is also D.

Let $\mathcal{H} = \{H_{ij} : 1 \leq i \leq n, j \in \omega\}$ and $\mathcal{K} = \{\bigcup \gamma_{ij} : 1 \leq i \leq m, j \in \omega\}$. Note that $(Z_1 \cup \cdots \cup Z_n) \cap Z \subseteq \bigcup \mathcal{H}$ and $(Y_1 \cup \cdots \cup Y_m) \cap Z \subseteq \bigcup \mathcal{K}$.

Then, proceeding by induction on $k \in \omega$, we construct a closed discrete subset E_k of Z such that $E_k \cap \bigcup\{\eta[E_l] : l < k\} = \emptyset$ and $Z \subseteq \bigcup\{\eta[E_k] : k \in \omega\}$. We shall carry out the construction in such a way that if k is odd then $\eta[E_0 \cup \cdots \cup E_k]$ will cover some element of $\mathcal{H}$, and if $k > 0$ and k is even then $\eta[E_0 \cup \cdots \cup E_k]$ will cover some element of $\mathcal{K}$. First, let us consider two bijections $h_1 \colon \omega \longrightarrow \{1, \dots, n\} \times \omega$ and $h_2 \colon \omega \longrightarrow \{1, \dots, m\} \times \omega$. We put $E_0 = \emptyset$. Assume that $k > 0$ and $E_0, \dots, E_{k-1}$ have been constructed. Put $V = \eta[E_0 \cup \cdots \cup E_{k-1}]$. Assume that

$k = 2l+1$ for some $l \geq 0$. Let $h_1(l) = (i, j)$. Since H_{ij} is D, $H_{ij} \setminus V$ is also D, and so there is a closed discrete subset E_k of $H_{ij} \setminus V$ such that $H_{ij} \setminus V \subseteq \eta[E_k]$, and hence $H_{ij} \subseteq \eta[E_0 \cup \cdots \cup E_{k-1} \cup E_k]$. Note that as H_{ij} is closed in Z, E_k is also closed in Z. Now, suppose that $k = 2(l+1)$ for some $l \geq 0$. Let $h_2(l) = (i, j)$. Since F_{ij} is D, $F_{ij} \setminus V$ is also D, and hence there is a closed discrete subset E_{ij} of $F_{ij} \setminus V$ such that $F_{ij} \setminus V \subseteq \eta[E_{ij}]$. Let $W = V \cup \eta[E_{ij}]$. Then as $\{F \setminus W : F \in \gamma_{ij}\}$ is a locally finite collection of closed D-subsets of Z, there is a closed discrete subset D_{ij} of $\bigcup\{F \setminus W : F \in \gamma_{ij}\}$ such that $\bigcup\{F \setminus W : F \in \gamma_{ij}\} \subseteq \eta[D_{ij}]$. We put $E_k = E_{ij} \cup D_{ij}$. It is easily seen that E_k is a closed discrete subset of Z.

We put $C = \bigcup\{E_k : k \geq 0\}$. It follows from the construction that $Z \subseteq \eta[C]$. Also as for $k \geq 0$, $E_k \cap \bigcup\{\eta[E_l] : l < k\} = \emptyset$ and E_k is a closed discrete subset of Z, it follows that C is a closed discrete subset of Z.

Finally, we put $D = A \cup B \cup C$. It is easy to see that D is a closed discrete subset of X and $\eta[D] = X$. □

Proof of Theorem 4.1. By using Theorem 4.2, we can proceed by means of an argument parallel to the one given in the proof of Theorem 3.2. □

The following result is an immediate consequence of Theorem 4.1.

Corollary 4.1. *Suppose that X is the union of a finite collection of subparacompact spaces and metacompact spaces. Then:*

(a) X is D if and only if X is $\mathcal{D}$-scattered.

(b) If X is not D, then there is an uncountable closed subspace Y of X such that no point of Y has a D-neighbourhood.

Also, by using again [11, Lemma 1.1], we obtain the following corollary from Theorem 4.1.

Corollary 4.2. *If a space X is the union of a finite collection $\{X_i : i = 1, \ldots, k\}$ of $\mathcal{C}$-scattered spaces where X_i is either subparacompact or metacompat for $1 \leq i \leq k$, then X is a D-space.*

We want to remark that, by using Lemma 4.1, we can prove that every submetacompact $\mathcal{D}$-scattered space is D. However, we do not know whether every space which is a finite union of submetacompact $\mathcal{C}$-scattered spaces has the D-property.

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