RELAXATION METHODS FOR SOLVING LINEAR INEQUALITY SYSTEMS: CONVERGING RESULTS

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Abstract. The problem of finding a feasible solution to a linear inequality system arises in numerous contexts. In [12] an algorithm, called extended relaxation method, that solves the feasibility problem, has been proposed by the authors. Convergence of the algorithm has been proven. In this paper, we consider a class of extended relaxation methods depending on a parameter and prove their convergence. Numerical experiments have been provided, as well.

1. Introduction

In various numerical problems one is confronted with the task to find at least of one solution of *linear semi-infinite systems* (LSIS's for short) of the form:

$$\sigma = \{a_t' x \geq b_t, t \in T\} \tag{1}$$

where, T is an arbitrary nonempty index set, $x \in \mathbb{R}^n$, $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$ are arbitrary mappings, and its solution set is represented by F. If not stated otherwise, given the LSIS's, σ, we will suppose that $a_t \neq 0_n$ for all $t \in T$, such that each inequality represents a closed half-space, and F is not empty.

The LSIS's arise in many different frames, both in mathematical theory (see [10], [7], [18], [2]), and in the practice (see [13], [4]). The study of the solutions of linear semi-infinite programming problems, started in 1920's (see [11], [5], [6], [17], [20], [16]), requires as a first step to solve the feasibility problem, i.e., to find an initial point, which is a solution of (1). Numerical methods looking for a feasible solutions of LSIS's, with a finite index set T, has been studied in [1] and [19]. Agmon (see [1]) proposes different numerical procedures to solve the feasibility problem and give several converging results. One of the proposed scheme is the so called *over and under projection with a fixed ratio.* However, in case that T is a infinite set, no numerical method for solving the feasibility problem is reported until Jeroslow proposed a *projection method* in 1979, only for $T = \mathbb{N}$ [15]. Later Hu, in [14], has considered the same projection method. Both authors, under different conditions on the nominal data, have proven convergence of the algorithms. In a resent paper [12], we have extended the idea of the

Research partially supported by CONACyT of MX, Grant 55681 and Ministry of Science and Innovation of SP, Grant MTM2008-06695-C03-01.

relaxation method when $|T| < \infty$. So, an algorithm, called *extended relaxation method* (ERM) has been proposed. The main differences with the two previous papers, which represents the ERM, are that instead of the projection at each step of the algorithm of the current iteration, we consider, as a next point, its reflection with respect to a certain hyperplane. And second, we have substantial difference in the relaxation rule, i.e., how to decide when we have to make the next step. A convergence theorem has also been obtained. In this present work, inspired of the over and under projection with a fixed ratio methods, proposed by Agmon, we consider a class of *extended relaxation methods* depending on a parameter λ. In Section 2, we give some preliminaries and in Section 3 we present the description of the methods and prove the convergence of the algorithms for $\lambda \in (0, 2]$. Finally, in Section 4, we give three examples and numerical results illustrating the behavior of the proposed algorithms.

2. Preliminaries

The Euclidean distance from $\bar{x}$ to the hyperplane, $H = \{x \in \mathbb{R}^n | a'x = b\}$, such that $a'\bar{x} < b$ will be denoted by $d(\bar{x}, H) = \frac{b - a'\bar{x}}{\|a\|}$. The geometric idea for ordinary systems is as follows: suppose an arbitrary initial point, $\bar{x} \in \mathbb{R}^n$, such that it is not a solution of the system (i.e., $\bar{x} \notin F$); and we try to find the farthest halfspace (among these defined by the inequalities in the system σ), with boundary H, of $\bar{x}$; the next point arises from $\bar{x}$ by going along the projection vector of $\bar{x}$ onto the hyperplane H, with a distance $\lambda d(\bar{x}, H)$, where $\lambda > 0$ is a prefixed parameter.

If $\mu := d(\bar{x}, H) > 0$ is the Euclidean distance from $\bar{x}$ to the hyperplane H, the next point will be $\bar{x} + \mu\lambda \frac{a}{\|a\|}$, where a is the vector from the definition of H.

It is clear that, for infinite systems and a fixed x^r, $g(t, x^r) := a_t' x^r - b_t$ is a non linear function, called slack or marginal function, and the techniques involving calculations of global maxima, for example, Matlab routines, only provide approximations. So, finding the farthest hyperplane is not an easy work, and then to avoid this problem, we propose a relaxation method, in which it is not necessary to calculate the farthest hyperplane. In fact, if a sequence $\{x^k\}$ is generated, such that, from a current point, x^r, the next point is of the form $x^{r+1} = x^r + \varepsilon\lambda \frac{a}{\|a\|}$ we need only that at each step ε (together with its corresponding vector a) has to be sufficiently close, in some sense, to μ.

In particular, if the parameter $\lambda = 2$ ($\lambda = 1$), the next point is the reflection of $\bar{x}$ from the hyperplane (the projection of $\bar{x}$ into H, respectively). The case $\lambda = 1$, has been widely studied in [14] and [15]. The authors of this paper, deal with systems where $T \subset \mathbb{R}^m$, and assume the restrictive conditions $\sup\{\| a_t \| \, |t \in T\} < \infty$ and $\inf\{b_t \, |t \in T\} > -\infty$. The second paper considers only LSIS's with $T = \mathbb{N}$. In [12], we propose method for $\lambda = 2$, where we consider an arbitrary index set T and only suppose that the feasible set F is of a full dimensionality. We can even drop this restriction, if the *Strong Slater* condition, i.e., there exist

$\bar{x} \in F$ and $\gamma > 0$ such that $a_t'\bar{x} \geq b_t + \gamma$ for all $t \in T$, is fulfilled. Under these weaker conditions, we have shown that our ERM algorithm converges.

In fact, the algorithms in three cases, enter in the scheme of over and under projection with a fixed ratio methods. Agmon names these methods under projection when $\lambda \in (0, 1)$, projection when $\lambda = 1$ and over projection when $\lambda \in (1, 2]$. Further in this presentation, we shall consider a class of ERM algorithms for the whole range of the parameter $\lambda \in (0, 2]$.

3. The ERM algorithms

A formal description of the ERM, depending on λ, is as follows:

Algorithm 1. *(Extended relaxation method)*

(1) *Choose the parameters* $\lambda > 0$, $M > 2$ *and* $\beta > 0$; *choose an arbitrary vector* $x^0 \in \mathbb{R}^n$. *Set the iteration index* $r = 0$.

(2) *Minimize the slack function* $g(t, x)$ *at* x^r, *finding* $u_r = \inf_{t \in T} g(t, x^r)$. *If* $u_r \geq 0$, *stop* ($x^r \in F$). *Otherwise, take the index set* $T_r = \{t \in T | g(t, x^r) < 0\}$ *(indexes of violated inequalities by* x^r*).*

(3) *Set* $\beta_r = \beta$ *and consider the global optimization problem*

$$\sup \left\{ \frac{b_t - a_t' x^r}{\|a_t\|}, t \in T_r \right\} = \mu_r. \tag{2}$$

(4) *Furthermore, find a* β_r *approximation,* ε_r*, of the solution,* μ_r*, of the problem (2)* $(\mu_r - \beta_r < \varepsilon_r \leq \mu_r)$ *. If* $\beta_r < \varepsilon_r(M - 1)$*, then*

$$\frac{\mu_r}{M} < \varepsilon_r := \frac{b_{t_r} - a_{t_r}' x^r}{\|a_{t_r}\|} \leq \mu_r, \text{ for some } t_r \in T_r,$$

and choose $x^{r+1} = x^r + \lambda \varepsilon_r \frac{a_{t_r}}{\|a_{t_r}\|}$. *Replace* r *by* $r + 1$ *and loop to step 2. If not, set* $\beta_r = \beta_r/2$ *and go to the step 4.*

Remark 1. *In* [12]*, we have shown that in the ERM algorithm for all* $r = 1, 2, \ldots,$ ε_r *always exists and it takes finite values different from zero.*

The following convergence theorem holds:

Theorem 2. *Let us have a system,* σ*, and* $\dim F = n$*. Given an initial point,* $x^0 \in \mathbb{R}^n$*, ERM with* $\lambda \in (0, 2]$ *either ends after a finite number of steps, or it generates an infinite sequence,* $\{x^r\}$*, converging to some element of* F*.*

Proof. Let us fix $\lambda \in (0, 2]$ and let us consider the sequence $\{x^r\} \subset \mathbb{R}^n$ generated by the algorithm. We will conclude that $\lim_{r \to \infty} x^r = \hat{x} \in F$.

If the sequence is finite and the last point belongs to F, we are done. So, we assume that $\{x^r\}$ is an infinite sequence of infeasible points. For each $t \in T$ we denote $H_t = \{x \in \mathbb{R}^n \mid a_t' x = b_t\}$. We have $\mu_r > 0$, for all $r \in \mathbb{N}$, i.e., $x^r \notin H_{t_r}$, then the vector x^{r+1} is along the vector a_{t_r} starting from x^r and distance between the two points is $\lambda \varepsilon_r$.

By the hypothesis there exist $z \in \mathbb{R}^n$ and $\delta > 0$ such that the open ball $B_\delta(z)$ of centre z and radius δ satisfies

$$B_\delta(z) \subset F \subset \{x \in \mathbb{R}^n \mid a'_{t_r} x \geq b_{t_r}\}, \quad r = 1, 2, \ldots$$

and $\rho_{t_r} := d(z, H_{t_r}) \geq \delta$.

By construction, the line determined by x^r and x^{r+1} is orthogonal to H_{t_r}. Let h_r be the distance from z to that line. Consider the affine hull of $\{x^r, x^{r+1}, z\}$. We choose a coordinate system in this hyperplane, with abscises axis, the line throughout the points x^r and x^{r+1}, directed in that way, and ordinates axis, the perpendicular to the line throughout the points x^r and x^{r+1}, directed in such a way that z belongs to the first orthant. With this oriented system, the coordinates of the points x^r , x^{r+1} and z are $(-\varepsilon_r, 0)$, $((\lambda-1)\varepsilon_r, 0) = (\xi\varepsilon_r, 0)$, where $\xi \in (-1, 1]$, and (ρ_{t_r}, h_r), respectively, with $h_r \geq 0$ (the case when the dimension of the affine hull is 1 and $h_r = 0$ is trivial). Then

$$\begin{aligned}\|x^r - z\|^2 - \|x^{r+1} - z\|^2 &= \left[(\rho_{t_r} + \varepsilon_r)^2 + h_r^2\right] - \left[(\rho_{t_r} - \xi\varepsilon_r)^2 + h_r^2\right] \\ &= (1 - \xi^2)\varepsilon_r^2 + 2(1 + \xi)\rho_{t_r}\varepsilon_r.\end{aligned}$$

Hence, for $r \in \mathbb{N}$, we have

$$0 \leq \|x^{r+1} - z\|^2 = \|x^r - z\|^2 - (1 - \xi^2)\varepsilon_r^2 - 2(1 + \xi)\rho_{t_r}\varepsilon_r,$$

since $-\rho_{t_r} \leq -\delta$, we have

$$0 \leq \|x^{r+1} - z\|^2 \leq \|x^r - z\|^2 - (1 - \xi^2)\varepsilon_r^2 - 2(1 + \xi)\delta\varepsilon_r.$$

So, we can consider the above relation for the first $r-1$ terms, i.e., for every $k = 0, \ldots, r-1$

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - (1 - \xi^2)\varepsilon_r^2 - 2(1 + \xi)\delta\varepsilon_r.$$

Adding orderly the above inequalities, we have

$$\sum_{k=0}^{r-1} \|x^{k+1} - z\|^2 \leq \sum_{k=0}^{r-1} (\|x^k - z\|^2 - (1 - \xi^2)\varepsilon_r^2 - 2(1 + \xi)\delta\varepsilon_r),$$

then,

$$\begin{aligned}\|x^r - z\|^2 + \sum_{k=1}^{r-1} \|x^k - z\|^2 \leq \|x^0 - z\|^2 + \sum_{k=1}^{r-1} \|x^k - z\|^2 - \\ -(1 - \xi^2) \sum_{k=0}^{r-1} \varepsilon_r^2 - 2(1 + \xi)\delta \sum_{k=0}^{r-1} \varepsilon_k,\end{aligned}$$

removing common terms we have,

$$0 \leq \|x^r - z\|^2 \leq \|x^0 - z\|^2 - (1 - \xi^2) \sum_{k=0}^{r-1} \varepsilon_r^2 - 2(1 + \xi)\delta \sum_{k=0}^{r-1} \varepsilon_k,$$

whereby,

$$2(1+\xi)\delta\sum_{k=0}^{r-1}\delta\varepsilon_k \leq (1-\xi^2)\sum_{k=0}^{r-1}\varepsilon_r^2 + 2(1+\xi)\delta\sum_{k=0}^{r-1}\varepsilon_k \leq \|x^0 - z\|^2,$$

so that,

$$\sum_{k=0}^{r-1}\varepsilon_k \leq \frac{1}{2\lambda\delta}\|x^0 - z\|^2.$$

If we consider the sequence $\eta_{r-1} = \sum_{k=0}^{r-1}\varepsilon_k$, and the constant $K = \frac{1}{2\lambda\delta}\|x^0 - z\|^2$, we have $\eta_{r-1} \geq 0$ for all $r \in \mathbb{N}$, then $0 \leq \lim_r \eta_r \leq K$, i.e., the sequence $\{\eta_r\}$ is bonded and increasing, whereby converges. Hence, $\sum_{r=0}^{\infty}\varepsilon_r$ converges as well (and $\lim_r \varepsilon_r = 0$).

Since, in Step 4 of the ERM algorithm, we have chosen ε_r such that $0 < \frac{\mu_r}{M} < \varepsilon_r$, i.e., $0 < \mu_r < \varepsilon_r M$, we get $\lim_r \mu_r = 0$.

In the Step 4 of the ERM we have

$$\|x^r - x^{r+1}\| = \lambda\varepsilon_r,$$

then the series $\sum_{r=0}^{\infty}\|x^r - x^{r+1}\|$ converges, so $\sum_{r=0}^{\infty}(x^r - x^{r+1})$ is absolutely convergent (see Th. 26.7 [3]), and, $\lim_r x^r = \hat{x}$, for some $\hat{x} \in \mathbb{R}^n$.

Finally, we will show that $\hat{x} \in F$. For any $t \in T$, and for all $r \in \mathbb{N}$ we have

$$\frac{b_t - a_t' x^r}{\|a_t'\|} \leq \begin{cases} \mu_r, & t \in T_r, \\ 0, & \text{otherwise.} \end{cases}$$

Taking limit in the above relation when $r \to \infty$, we get $\frac{b_t - a_t'\hat{x}}{\|a_t'\|} \leq 0$, for all $t \in T$, and this proves that $\hat{x} \in F$. ∎

Remark 2. *We would like to mention that the geometrical proof of the previous theorem strongly requires the full dimensionality of the feasible set ($\delta > 0$).*

4. Test examples

Relaxation method has been implemented in the Matlab software, all test examples were implemented by making appropriate modifications to `fminbnd` Matlab routine, due to the fact that the relaxation method uses the routine to solve the global optimization problem appearing in Step 2. All the three examples were set to stop when $\inf_{t\in T} g(x^r, t) \geq -1 \times 10^{-4}$. In all cases, the initial guess x^0 was randomly generated and we set $\beta = \frac{1}{2}$, and $M = 1000$.

In the Tables 1-3, results of implementation of ERM, just discussed here, are given. The first and third column of the tables are self-explanatory. A description of the remaining columns is as follows. The second column, `ite`, indicates the number of iterations made before the stopping criterion was satisfied (we establish a limit of 15000 iterations as maximum), the last column, `time`, is the time of execution in seconds on a computer of 2Ghz dual processor, 2GB of RAM and

Windows 7 operating system. Finally, Figures 1-3 show the feasible set (region in blank), for each of the systems

Example 3. *Consider the convex set* F *defined by* $F = \{x \in \mathbb{R}^2 : (x_1^2 + x_2^2)^2 \geq \kappa^2(x_1^2 - x_2^2); 0 \leq x_1 \leq \kappa\}$ *(a quarter of the* lemniscate *curve, see Figure 1) and consider its linear representation,* σ*, given by* $\sigma = \{a_t^1 x_1 + a_t^2 x_2 \geq b_t : t \in [0, \frac{\pi}{2}]\}$ *where*

$$a_t^1 := -\kappa \cos 3t,$$
$$a_t^2 := -\kappa \sin 3t,$$
$$b_t := -\kappa^2(\cos 3t \cos t\sqrt{\cos 2t} + \sin 3t \sin t\sqrt{\cos 2t}).$$

The set F *depends on the parameter* κ*, we choose it as* $\kappa = 2$*. For initial guess* $x^0 = (-1564.979244, 2189.253881)$*, the results are presented in Table 1.*

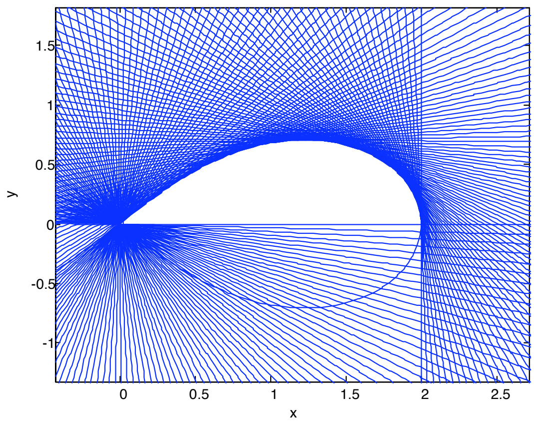

FIGURE 1. The feasible set F of the Example 3.

λ	ite	x^r	time
0.1	179	$(0.127274, 0.113900)$	3.756422
0.3	54	$(0.127840, 0.114339)'$	0.596194
0.5	28	$(0.128261, 0.114649)'$	0.365805
0.7	17	$(0.128210, 0.114510)'$	0.680884
0.9	10	$(0.127417, 0.113514)'$	0.282598
1	7	$(0.187578, 0.140557)'$	0.155591
1.5	23	$(0.242725, 0.015244)'$	0.38461
2	1066	$(0.204403, 0.089469)'$	8.209755

TABLE 1. Results of Example 3

Example 4. *Consider the linear representation of the ellipse,* $F = \{x \in \mathbb{R}^2 : 2x_1^2 + x_2^2 + 2x_1x_2 + 2x_1 \leq 0\}$ *(see Figure 2) given by*

$$\sigma = \{(-t^4 - 2t^3 + 3t^2 + 2t - 1)x_1 - 2t(t^2 - 1)x_2 \geq -2t^2 : t \in [-1, 1]\},$$

(see [11]*). With initial guess,* $x^0 = (3.423734, 14.120922)$*, the relaxation algo-*

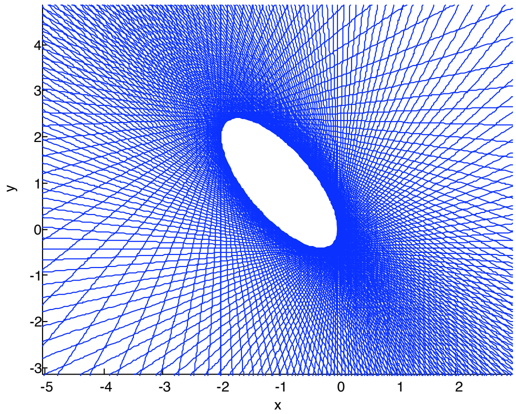

FIGURE 2. The fesible set F of the Example 4.

rithm gives results summarized in Table 2. Note: for the value of the relaxation parameter $\lambda = 0.1$*, the algorithm exceeds the number of iterations permitted and finalize without finding a feasible solution.*

λ	*ite*	x^r	*time*
0.1	-	-	-
0.3	110	$(-1.828697, 2.388394)'$	0.373614
0.5	59	$(-1.827002, 2.39201)'$	0.260062
0.7	38	$(-1.824438, 2.390390)'$	0.210664
0.9	20	$(-1.794767, 2.401682)'$	0.170799
1	9	$(-1.866903, 2.365380)'$	0.129316
1.5	6	$(-1.722008, 2.319358)'$	0.127841
2	12	$(-1.686474, 1.201378)'$	0.141970

TABLE 2. Resullts of Example 4

Example 5. *For* $n = 2$*, we consider the system* $\sigma = \{-te^tx_1 - tx_2 \geq -1 : t \in [0, 1]\}$*,(see* [8], [9]*). In this example for* $t = 0$*,* $a_0 = 0^2$*, so the conditions required at the beginning of the paper are not fulfilled. The feasible set of* σ *is shown in Figure 3. We use the initial guess* $x^0 = (12.353328, 17.188846)'$*. The results are reported in Table 3. Like in Example 4, for some values of the parameter* λ*, f.i.,* $\lambda = 0.1, 0.3, 0.7$ *the algorithm exceeds the iteration number permitted and finalize without finding a feasible solution.*

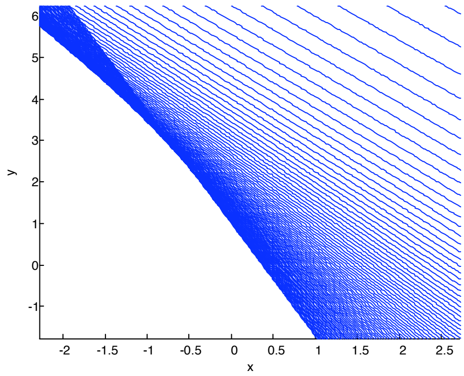

FIGURE 3. The feasible set F of the Example 5.

λ	*ite*	x^r	*time*
0.1	-	-	-
0.3	-	-	-
0.5	*60*	$(-5.269856, 10.330953)'$	*0.256530*
0.7	-	-	-
0.9	*21*	$(-6.046282, 11.435680)$	*0.161449*
1	*8*	$(-5.333119, 10.421842)'$	*0.138474*
1.5	*2*	$(-11.819860, 8.384212)'$	*0.112895*
2	*2*	$(-19.877589, 5.449334)'$	*0.118692*

TABLE 3. Reports for Example 5

At the end, we would like to summarize that, apart of the theoretical converging result obtained in the paper, more numerical experiments should be done in order to get better notion of the role of all parameters, especially λ and M, including in the ERM algorithms. But anyway, the results confirm once again the comment in [1], that the so called over projection methods have better performance.

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