

SOBOLEV MAPPINGS, DEGREE, HOMOTOPY CLASSES AND RATIONAL HOMOLOGY SPHERES

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ABSTRACT. In the paper we investigate the degree and the homotopy theory of Orlicz-Sobolev mappings $W^{1,P}(M, N)$ between manifolds, where the Young function P satisfies a divergence condition and forms a slightly larger space than $W^{1,n}$, $n = \dim M$. In particular, we prove that if M and N are compact oriented manifolds without boundary and $\dim M = \dim N = n$, then the degree is well defined in $W^{1,P}(M, N)$ if and only if the universal cover of N is not a rational homology sphere, and in the case $n = 4$, if and only if N is not homeomorphic to S^4 .

1. INTRODUCTION

Let M and N be compact smooth Riemannian manifolds without boundary. We consider the space of Sobolev mappings between manifolds defined in a usual way: we assume that N is isometrically embedded in a Euclidean space \mathbb{R}^k and define $W^{1,p}(M, N)$ to be the class of Sobolev mappings $u \in W^{1,p}(M, \mathbb{R}^k)$ such that $u(x) \in N$ a.e. The space $W^{1,p}(M, N)$ is a subset of a Banach space $W^{1,p}(M, \mathbb{R}^k)$ and it is equipped with a metric inherited from the norm of the Sobolev space. It turns out that smooth mappings $C^\infty(M, N)$ are not always dense in $W^{1,p}(M, N)$ and we denote by $H^{1,p}(M, N)$ the closure of $C^\infty(M, N)$ in the metric of $W^{1,p}(M, N)$. A complete characterization of manifolds M and N for which smooth mappings are dense in $W^{1,p}(M, N)$, i.e. $W^{1,p}(M, N) = H^{1,p}(M, N)$, has recently been obtained by Hang and Lin, [22].

The class of Sobolev mappings between manifolds plays a central role in applications to geometric variational problems and deep connections to algebraic topology have been investigated recently, see e.g. [2], [3], [4], [7], [11], [12], [14], [15], [16], [17], [19], [20], [21], [22], [25], [31], [32].

In the borderline case, $p = n = \dim M$, it was proved by Schoen and Uhlenbeck ([33], [34]) that the smooth mappings $C^\infty(M, N)$ form a dense subset of $W^{1,n}(M, N)$, i.e. $W^{1,n}(M, N) = H^{1,n}(M, N)$, and White [38] proved that the homotopy classes are well defined in $W^{1,n}(M, N)$, see also [8], [9]. Indeed, he proved

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that for every $u \in W^{1,n}(M, N)$, if two smooth mappings $u_1, u_2 \in C^\infty(M, N)$ are sufficiently close to u in the Sobolev norm, then u_1 and u_2 are homotopic.

If $\dim M = \dim N = n$ and both manifolds are oriented, then the degree of smooth mappings is well defined. There are several equivalent definitions of the degree and here we consider the one that involves integration of differential forms. If ω is a volume form on N , then for $f \in C^\infty(M, N)$ we define

$$\deg f = \left(\int_M f^* \omega \right) / \left(\int_N \omega \right).$$

Since $f^* \omega$ equals the Jacobian J_f of f (after identification of n -forms with functions), it easily follows from Hölder's inequality that the degree is continuous in the Sobolev norm $W^{1,n}$. Finally, the density of smooth mappings in $W^{1,n}(M, N)$ allows us to extend the degree continuously and uniquely to $\deg: W^{1,n}(M, N) \rightarrow \mathbb{Z}$.

Results regarding degree and homotopy classes do not, in general, extend to the case of $W^{1,p}(M, N)$ mappings when $p < n$. This stems from the fact that the radial projection mapping $u_0(x) = x/|x|$ belongs to $W^{1,p}(B^n, S^{n-1})$ for all $1 \leq p < n$.

A particularly interesting class of Sobolev mappings to which we can extend the degree and the homotopy theory is the Orlicz-Sobolev space $W^{1,P}(M, N)$, where the Young function P satisfies the so called divergence condition

$$(1.1) \quad \int_1^\infty \frac{P(t)}{t^{n+1}} dx = \infty.$$

In particular, $P(t) = t^n$ satisfies this condition, so the space $W^{1,n}(M, N)$ is an example. However, we want the space to be slightly larger than $W^{1,n}$, and hence we also assume that

$$(1.2) \quad P(t) = o(t^n) \quad \text{as } t \rightarrow \infty.$$

In addition to the conditions described here we require some technical assumptions about P ; see Section 4. Roughly speaking, $u \in W^{1,P}(M, N)$ if $\int_M P(|Du|) < \infty$. A typical Orlicz-Sobolev space $W^{1,P}$ discussed here contains $W^{1,n}$, and it is contained in all spaces $W^{1,p}$ for $p < n$

$$W^{1,n} \subset W^{1,P} \subset \bigcap_{1 \leq p < n} W^{1,p}.$$

A fundamental example of a Young function that satisfies all required conditions, forms a strictly larger space than $W^{1,n}$, and strictly smaller than the intersection of all $W^{1,p}$, $1 \leq p < n$, is

$$P(t) = \frac{t^n}{\log(e+t)}.$$

It is easy to check that $u_0 \notin W^{1,P}(B^n, S^{n-1})$ if and only if the condition (1.1) is satisfied, see [17, p. 2], so the presence of the divergence condition is necessary and sufficient for the exclusion of mappings like $x/|x|$ that can easily cause topological problems.

The class of Orlicz-Sobolev mappings under the divergence condition has been investigated in connections to nonlinear elasticity, mappings of finite distortion [30] and degree theory [13], [17]. Roughly speaking, many results that are true for $W^{1,n}$ mappings have counterparts in $W^{1,P}$ as well. In particular, it was proved in [17] that smooth mappings $C^\infty(M, N)$ are dense in $W^{1,P}(M, N)$ if P satisfies (1.1), see Section 4.

We say that a compact connected manifold N without boundary, $\dim N = n$, is a *rational homology sphere*, if it has the same deRham cohomology groups as the sphere S^n . It has been proved in [17] that if M and N are smooth compact oriented n -dimensional manifolds without boundary and N is *not* a rational homology sphere, then the degree, originally defined on a class of smooth mappings $C^\infty(M, N)$, uniquely extends to a continuous function $\deg: W^{1,P}(M, N) \rightarrow \mathbb{Z}$, see also [13] for earlier results. This is not obvious, because the Jacobian of a $W^{1,P}(M, N)$ mapping is not necessarily integrable and we cannot easily use estimates of the Jacobian, like in the case of $W^{1,n}$ mappings, to prove continuity of the degree. The proof given in [17] (cf. [13]) is not very geometric and it is based on estimates of integrals of differential forms, Hodge decomposition, and the study of the so called Cartan forms. Surprisingly, it has also been shown in [17] that the degree is not continuous in $W^{1,P}(M, S^n)$. Thus the results of [17] provide a good understanding of the situation when N is not a rational homology sphere or when $N = S^n$. There is, however, a large class of rational homology spheres which are not homeomorphic to S^n and it is natural to ask what happens for such manifolds.

In this paper we give a complete answer to this problem. In our proof we do not rely on methods of [17] and we provide a new geometric argument that avoids most of the machinery of differential forms developed in [17].

Theorem 1.1. *Let M and N be compact oriented n -dimensional Riemannian manifolds, $n \geq 2$, without boundary, and let P be a Young function satisfying conditions (1.1), (1.2), (4.1) and (4.2). Then the degree is well defined, integer valued and continuous in the space $W^{1,P}(M, N)$ if and only if the universal cover of N is not a rational homology sphere.*

The theorem should be understood as follows: if the universal cover of N is not a rational homotopy sphere, then $\deg: C^\infty(M, N) \rightarrow \mathbb{Z}$ is continuous in the norm of $W^{1,P}$ and since smooth mappings are dense in $W^{1,P}(M, N)$ the degree uniquely and continuously extends to $\deg: W^{1,P}(M, N) \rightarrow \mathbb{Z}$. On the other hand, if the universal cover of N is a rational homology sphere, then there is a sequence of

smooth mappings $u_k \in C^\infty(M, N)$ with $\deg u_k = d > 0$ such that u_k converges to a constant mapping (and hence of degree zero) in the norm of $W^{1,P}$.

If $n = 4$, Theorem 1.1 together with Proposition 2.6 give

Corollary 1.2. *Let M and N be compact oriented 4-dimensional Riemannian manifolds without boundary and let P be a Young function satisfying conditions (1.1), (1.2), (4.1) and (4.2) with $n = 4$. Then the degree is well defined, integer valued and continuous in the space $W^{1,P}(M, N)$ if and only if N is not homeomorphic to S^4 .*

The next result concerns the definition of homotopy classes in $W^{1,P}(M, N)$.

Theorem 1.3. *Let M and N be two compact Riemannian manifolds without boundary, $n = \dim M \geq 2$, and let P be a Young function satisfying the conditions (1.1), (1.2), (4.1) and (4.2). If $\pi_n(N) = 0$, then the homotopy classes are well defined in $W^{1,P}(M, N)$. If $\pi_n(N) \neq 0$, the homotopy classes cannot be well defined in $W^{1,P}(S^n, N)$.*

Theorem 1.3 has been announced in [17, p.5], but no details of the proof have been provided. It should be understood in a similar way as Theorem 1.1. Let $\pi_n(N) = 0$. Then for every $u \in W^{1,P}(M, N)$ there is $\varepsilon > 0$ such that if $u_1, u_2 \in C^\infty(M, N)$, $\|u - u_i\|_{1,P} < \varepsilon$, $i = 1, 2$, then u_1 and u_2 are homotopic. Since smooth mappings are dense in $W^{1,P}(M, N)$, homotopy classes of smooth mappings can be uniquely extended to homotopy classes of $W^{1,P}(M, N)$ mappings. On the other hand, if $\pi_n(N) \neq 0$, there is a sequence of smooth mappings $u_k \in C^\infty(S^n, N)$ that converges to a constant mapping in the norm of $W^{1,P}$ and such that the mappings u_k are not homotopic to a constant mapping.

The condition $\pi_n(N) = 0$ is not necessary: any two continuous mappings from \mathbb{CP}^2 to \mathbb{CP}^1 are homotopic (in particular – homotopic to a constant mapping), even though $\pi_4(\mathbb{CP}^1) = \pi_4(S^2) = \mathbb{Z}_2$.

Theorems 1.1 and 1.3 will be proved from corresponding results for $W^{1,p}$ mappings.

Theorem 1.4. *Let M and N be two compact Riemannian manifolds without boundary, $\dim M = n$, $n-1 \leq p < n$. Then the homotopy classes are well defined in $H^{1,p}(M, N)$ if $\pi_n(N) = 0$, and they cannot be well defined in $H^{1,p}(S^n, N)$, if $\pi_n(N) \neq 0$. If $\pi_{n-1}(N) = \pi_n(N) = 0$, then the homotopy classes are well defined in $W^{1,p}(M, N)$.*

The theorem should be understood in a similar way as Theorem 1.3. The last part of the theorem follows from the results of Bethuel [2] and Hang and Lin [22] according to which the condition $\pi_{n-1}(N) = 0$ implies density of smooth mappings in $W^{1,p}(M, N)$, so $H^{1,p}(M, N) = W^{1,p}(M, N)$.

Theorem 1.5. *Let M and N be compact connected oriented n -dimensional Riemannian manifolds without boundary, $n - 1 \leq p < n$, $n \geq 2$. Then the degree is well defined, integer valued and continuous in the space $H^{1,p}(M, N)$ if and only if the universal cover of N is not a rational homology sphere. If the universal cover of N is not a rational homology sphere and $\pi_{n-1}(N) = 0$, then degree is well defined, integer valued and continuous in $W^{1,p}(M, N)$.*

Again, Theorem 1.5 should be understood in a similar way as Theorem 1.1 and the last part of the theorem follows from density of smooth mappings in $W^{1,p}$, just like in the case of Theorem 1.4.

As a direct consequence of Theorem 1.5 and Proposition 2.6 we have

Corollary 1.6. *Let M and N be compact connected oriented 4-dimensional Riemannian manifolds without boundary, $3 \leq p < 4$. Then the degree is well defined, integer valued and continuous in the space $H^{1,p}(M, N)$ if and only if N is not homeomorphic to S^4 . If N is not homeomorphic to S^4 and $\pi_3(N) = 0$, then the degree is well defined, integer valued and continuous in $W^{1,p}(M, N)$.*

The paper should be interesting mainly for people working in geometric analysis. Since the proofs employ quite a lot of algebraic topology, we made some effort to present our arguments from different points of view whenever it was possible. For example, some proofs were presented both from the perspective of algebraic topology and the perspective of differential forms.

The main results in the paper are Theorems 1.1, 1.3, 1.4, 1.5, Corollaries 1.2, 1.6 and also Theorem 2.1.

The paper is organized as follows. In Section 2 we study basic examples and properties of rational homology spheres. In Section 3 we prove Theorems 1.4 and 1.5. The final Section 4 provides a definition of the Orlicz-Sobolev space, its basic properties, and the proofs of Theorems 1.1, and 1.3.

2. RATIONAL HOMOLOGY SPHERES

In what follows, $H^k(M)$ will stand for deRham cohomology groups. We say that a smooth n -dimensional manifold M without boundary is a *rational homology sphere* if it has the same deRham cohomology as S^n , that is

$$H^k(M) = \begin{cases} \mathbb{R} & \text{for } k = 0 \text{ or } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly M must be compact, connected and orientable.

If M and N are smooth compact connected oriented n -dimensional manifolds without boundary, then the *degree* of a smooth mapping $f: M \rightarrow N$ is defined by

$$\deg f = \frac{\int_M f^* \omega}{\int_N \omega},$$

where ω is any n -form on N with $\int_N \omega \neq 0$ (ω is not exact by Stokes' theorem and hence it defines a non-trivial element in $H^n(N)$). It is well known that $\deg f \in \mathbb{Z}$, that it does not depend on the choice of ω and that it is a homotopy invariant.

The reason why rational homology spheres play such an important role in the degree theory of Sobolev mappings stems from the following result.

Theorem 2.1. *Let M be a smooth compact connected oriented n -dimensional manifold without boundary, $n \geq 2$. Then there is a smooth mapping $f: S^n \rightarrow M$ of nonzero degree if and only if the universal cover of M is a rational homology sphere.*

Proof. Let us notice first that if a mapping $f: S^n \rightarrow M$ of nonzero degree exists, then M is a rational homology sphere. Indeed, clearly $H^0(M) = H^n(M) = \mathbb{R}$, because M is compact, connected and oriented. Suppose that M has non-trivial cohomology in dimension $0 < k < n$, that is – there is a closed k form α that is not exact. According to the Hodge Decomposition Theorem, [37], we may also assume that α is coclosed, so $*\alpha$ is closed. Then, $\omega = \alpha \wedge *\alpha$ is an n -form on M such that

$$\int_M \omega = \int_M |\alpha|^2 > 0.$$

We have

$$\deg f = \frac{\int_{S^n} f^* \omega}{\int_M \omega} = \frac{\int_{S^n} f^* \alpha \wedge f^* (*\alpha)}{\int_M \omega} = 0$$

which is a contradiction. The last equality follows from the fact that $H^k(S^n) = 0$, hence the form $f^* \alpha$ is exact, $f^* \alpha = d\eta$. Since $f^* (*\alpha)$ is closed, we have

$$\int_{S^n} f^* \alpha \wedge f^* (*\alpha) = \int_{S^n} d(\eta \wedge f^* (*\alpha)) = 0$$

by Stokes' theorem.

The contradiction proves that $H^k(M) = 0$ for all $0 < k < n$, so M is a rational homology sphere.

On the other hand, any such mapping f factors through the universal cover \widetilde{M} of M , because S^n is simply connected, see [24, Proposition 1.33].

$$\begin{array}{ccc} & & \widetilde{M} \\ & \nearrow \tilde{f} & \downarrow p \\ S^n & \xrightarrow{f} & M \end{array}$$

Clearly, \widetilde{M} is orientable and we can choose an orientation in such a way that p be orientation preserving.

Since $\deg f \neq 0$, $f^*: H^n(M) \rightarrow H^n(S^n)$ is an isomorphism. The factorization gives $f^* = \tilde{f}^* \circ p^*$, and hence $\tilde{f}^*: H^n(\widetilde{M}) \rightarrow H^n(S^n) = \mathbb{R}$ is an isomorphism, so \widetilde{M} is compact. This implies that if ω is a volume form on \widetilde{M} , then $\int_{S^n} \tilde{f}^* \omega \neq 0$ and hence $\deg \tilde{f} \neq 0$. Indeed, ω defines a generator (i.e. a non-zero element) in $H^n(\widetilde{M})$ and hence $\tilde{f}^* \omega$ defines a generator in $H^n(S^n)$. If η is a volume form on S^n , then $\tilde{f}^* \omega = c\eta + d\lambda$, $c \neq 0$, because $H^n(S^n) = \mathbb{R}$ and the elements $\tilde{f}^* \omega$ and η are proportional in $H^n(S^n)$. Thus $\int_{S^n} \tilde{f}^* \omega = c \int_{S^n} \eta \neq 0$.

The last argument can be expressed differently. Since \widetilde{M} is compact, the number γ of sheets of the covering is finite (it equals $|\pi_n(M)|$, see [24, Proposition 1.32]). It is easy to see that $\deg f = \gamma \deg \tilde{f}$, so $\deg \tilde{f} \neq 0$.

Thus we proved that the mapping $\tilde{f}: S^n \rightarrow \widetilde{M}$ has nonzero degree and hence \widetilde{M} is a rational homology sphere, by the fact obtained at the beginning of our proof.

We are now left with the proof that if \widetilde{M} is a rational homology sphere, then there is a mapping from S^n to M of nonzero degree. Clearly, it suffices to prove that there is a mapping $f: S^n \rightarrow \widetilde{M}$ of nonzero degree, since the composition with the covering map multiplies degree by the (finite) order γ of the covering.

To this end we shall employ the Hurewicz theorem mod Serre class \mathcal{C} of all finite abelian groups (see [26, Chapter X, Theorem 8.1], also [29], [23, Chapter 1, Theorem 1.8] and [35, Ch. 9, Sec. 6, Theorem 15]) that we state as a lemma.

Lemma 2.2. *Let \mathcal{C} denote the class of all finite abelian groups. If X is a simply connected space and $n \geq 2$ is an integer such that $\pi_m(X) \in \mathcal{C}$ whenever $1 < m < n$, then the natural Hurewicz homomorphism*

$$h_m: \pi_m(X) \rightarrow H_m(X, \mathbb{Z})$$

is a \mathcal{C} -isomorphism (i.e. both $\ker h_m$ and $\operatorname{coker} h_m$ lie in \mathcal{C}) whenever $0 < m \leq n$.

Since \widetilde{M} is a simply connected rational homology sphere, the integral homology groups in dimensions $2, 3, \dots, n-1$ are finite abelian groups: integral homology groups of compact manifolds are finitely generated, thus of form $\mathbb{Z}^\ell \oplus \mathbb{Z}_{p_1}^{k_1} \oplus \dots \oplus \mathbb{Z}_{p_r}^{k_r}$, and if we calculate $H^m(\widetilde{M}) = H^m(\widetilde{M}, \mathbb{R})$ with the help of the Universal Coefficient Theorem ([24, Theorem 3.2]), we have $0 = H^m(\widetilde{M}) = \operatorname{Hom}(H_m(\widetilde{M}, \mathbb{Z}), \mathbb{R})$. However, $\operatorname{Hom}(\mathbb{Z}^\ell, \mathbb{R}) = \mathbb{R}^\ell$, thus there can be no free abelian summand \mathbb{Z}^ℓ in $H_m(\widetilde{M}, \mathbb{Z})$ – and all that is possibly left is a finite abelian group.

Now we proceed by induction to show that the hypotheses of Lemma 2.2 are satisfied. We have $\pi_1(\widetilde{M}) = 0$, thus $h_2: \pi_2(\widetilde{M}) \rightarrow H_2(\widetilde{M}, \mathbb{Z})$ is a \mathcal{C} -isomorphism. Since $H_2(\widetilde{M}, \mathbb{Z})$ is, as we have shown, in \mathcal{C} , the group $\pi_2(\widetilde{M})$ is a finite abelian group as well, and we can apply Theorem 2.2 to show that h_3 is a \mathcal{C} -isomorphism and thus that $\pi_3(\widetilde{M}) \in \mathcal{C}$.

We proceed likewise by induction until we have that h_n is a \mathcal{C} -isomorphism between $\pi_n(\widetilde{M})$ and $H_n(\widetilde{M}, \mathbb{Z}) = \mathbb{Z}$. Since $\text{coker } h_n = \mathbb{Z}/h_n(\pi_n(\widetilde{M}))$ is a finite group, there exists a non-zero element k in the image of h_n , and $[f] \in \pi_n(\widetilde{M})$ such that $h_n([f]) = k \neq 0$.

The generator of $H_n(\widetilde{M}, \mathbb{Z}) = \mathbb{Z}$ is the cycle class given by the whole manifold \widetilde{M} , i.e. $h_n([f]) = k[\widetilde{M}]$; in other words – having fixed a volume form ω on \widetilde{M} , we identify a cycle class $[C]$ with an integer by

$$[C] \mapsto \left(\int_C \omega \right) / \left(\int_{\widetilde{M}} \omega \right),$$

We also recall that the Hurewicz homomorphism h_n attributes to an $[f] \in \pi_n(\widetilde{M})$ a cycle class $h_n([f]) = f_*[S^n]$, where $[S^n]$ is the cycle that generates $H_n(S^n, \mathbb{Z}) = \mathbb{Z}$. Altogether,

$$\deg f = \frac{\int_{S^n} f^* \omega}{\int_{\widetilde{M}} \omega} = \frac{\int_{f_*[S^n]} \omega}{\int_{\widetilde{M}} \omega} = k \neq 0,$$

since $f_*[S^n] = k[\widetilde{M}]$. □

To see the scope of applications of Theorem 1.1 and 1.5 it is important to understand how large the class of rational homology spheres is, or, more precisely, the class of manifolds whose universal cover is a rational homology sphere.

One can conclude from the proof presented above that if \widetilde{M} is a rational homology sphere then M is a rational homology sphere as well. We will provide two different proofs of this fact, the second one being related to the argument presented above.

Theorem 2.3. *Let M be an n -dimensional compact orientable manifold without boundary such that its universal cover \widetilde{M} is a rational homology n -sphere. Then*

- a) *M is a rational homology n -sphere as well.*
- b) *If n is even, then M is simply connected and $\widetilde{M} = M$.*

Proof. We start with two trivial observations: the fact that \widetilde{M} is a rational homology sphere implies immediately that \widetilde{M} is compact ($H^n(\widetilde{M}) = \mathbb{R}$) and connected ($H^0(\widetilde{M}) = \mathbb{R}$). This, in turn, shows that M is connected and that the number of sheets in the covering is finite.

For any finite covering $p: \tilde{N} \rightarrow N$ with number of sheets γ there is not only the induced *lifting* homomorphism $p^*: H^k(N) \rightarrow H^k(\tilde{N})$, but also the so-called *transfer* or *pushforward* homomorphism

$$\tau_*: H^k(\tilde{N}) \rightarrow H^k(N)$$

defined on differential forms as follows. For any $x \in N$ there is a neighborhood U such that $p^{-1}(U) = U_1 \cup \dots \cup U_\gamma$ is a disjoint sum of open sets such that

$$p_i = p|_{U_i}: U_i \rightarrow U$$

is a diffeomorphism. Then for a k -form ω on \tilde{N} we define

$$\tau_*\omega|_U = \sum_{i=1}^{\gamma} (p_i^{-1})^*(\omega|_{U_i}).$$

If U and W are two different neighborhoods in N , then $\tau_*\omega|_U$ coincides with $\tau_*\omega|_W$ on $U \cap W$ and hence $\tau_*\omega$ is globally defined on N . Since

$$\tau_*(d\omega)|_U = \sum_{i=1}^{\gamma} (p_i^{-1})^*(d\omega|_{U_i}) = \sum_{i=1}^{\gamma} d(p_i^{-1})^*(\omega|_{U_i}) = d(\tau_*\omega|_U)$$

we see that $\tau_* \circ d = d \circ \tau_*$ and hence τ_* is a homomorphism

$$\tau_*: H^k(\tilde{N}) \rightarrow H^k(N).$$

Clearly $\tau_* \circ p^* = \gamma \text{id}$ on $H^k(N)$ and hence τ_* is a surjection.

In our case $\tilde{N} = \tilde{M}$ is a rational homology sphere and all cohomology groups $H^k(\tilde{M})$ vanish for $0 < k < n$. Since τ_* is a surjection, $H^k(M) = 0$. The remaining $H^0(M)$ and $H^n(M)$ are equal \mathbb{R} by compactness, orientability and connectedness of M .

Another proof is related to the argument used in the proof of Theorem 2.1. By contradiction, suppose that $H^k(M) \neq 0$ for some $0 < k < n$. Hence there is a non-trivial closed and coclosed k -form α on M . Since α is coclosed, $*\alpha$ is closed. Then

$$\int_M \alpha \wedge *\alpha = \int_M |\alpha|^2 > 0,$$

thus

$$0 \neq \gamma \int_M \alpha \wedge *\alpha = \int_{\tilde{M}} p^*(\alpha \wedge *\alpha) = \int_{\tilde{M}} p^*\alpha \wedge p^*(*\alpha) = 0,$$

because $p^*\alpha$ is exact on \tilde{M} (we have $H^k(\tilde{M}) = 0$), $p^*(*\alpha)$ is closed, and hence $p^*\alpha \wedge p^*(*\alpha)$ is exact.

To prove b), we recall that if $p: \tilde{N} \rightarrow N$ is a covering of order γ , then the Euler characteristic $\chi_{\tilde{N}}$ of \tilde{N} is equal to $\gamma\chi_N$. This can be easily seen through the triangulation definition of χ_N : to any sufficiently small simplex in N there correspond γ distinct simplices in \tilde{N} – in such a way we may lift a sufficiently

fine triangulation of N to \widetilde{N} ; clearly, the alternating sum of number of simplices in every dimension calculated for \widetilde{N} is γ times that for N .

In our case, the Euler characteristic of M and \widetilde{M} is either 0 (for n odd) or 2 (for n even) – this follows immediately from the fact that

$$\chi_N = \sum_{i=0}^{\dim N} (-1)^i \dim H^i(N).$$

Therefore, for $n = \dim M$ even, the order of the universal covering $\widetilde{M} \rightarrow M$ must be 1, and b) follows. \square

Remark 2.4. We actually proved a stronger version of part a). If a cover (not necessarily universal) of a manifold M is a rational homology sphere, then M is a rational homology sphere, too – in the proof we never used the fact that the covering space is simply connected. However, we stated the result for the universal cover only, because this is exactly what we need in our applications.

Below we provide some examples of rational homology spheres.

Any homology sphere is a rational homology sphere. That includes Poincaré homology sphere (also known as Poincaré dodecahedral space) and Brieskorn manifolds

$$\Sigma(p, q, r) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^p + z_2^q + z_3^r = 0, |(z_1, z_2, z_3)| = 1\},$$

where $1 < p < q < r$ are pairwise relatively prime integers. $\Sigma(2, 3, 5)$ is the Poincaré homology sphere.

Recall that any isometry of \mathbb{R}^n can be described as a composition of an orthogonal linear map and a translation; the group of all such isometries, denoted by $E(n)$, is a semi-direct product $O(n) \ltimes \mathbb{R}^n$. Any discrete subgroup G of $E(n)$ such that $E(n)/G$ is compact is called a *crystallographic group*.

Proposition 2.5 (see [36]). *Let us set $\{e_i\}$ to be the standard basis in \mathbb{R}^{2n+1} and*

$$B_i = \text{Diag}(-1, \dots, -1, \underbrace{1}_i, -1, \dots, -1).$$

Let Γ be a crystallographic group generated by isometries of \mathbb{R}^{2n+1}

$$T_i : v \longmapsto B_i \cdot v + e_i, \quad i = 1, 2, \dots, 2n.$$

Then $M = \mathbb{R}^{2n+1}/\Gamma$ is a rational homology $(2n+1)$ -sphere.

For $n = 1$ this is a well known example of $\mathbb{R}^3/\mathcal{G}_6$, see [39, 3.5.10]. However, $\pi_1(M) = \Gamma$ is infinite and the universal cover of M is \mathbb{R}^{2n+1} , so M is an example of a rational homology sphere whose universal cover is not a rational homology sphere.

The following well known fact illustrates the difficulty of finding non-trivial examples in dimension 4:

Proposition 2.6. *Let N be a compact orientable 4-manifold without boundary. Then the universal cover of N is a rational homology sphere if and only if N is homeomorphic to S^4 .*

Proof. If N is homeomorphic to S^4 , then $\tilde{N} = N$ is a rational homology sphere. Suppose now that the universal cover \tilde{N} is a rational homology sphere. Theorem 2.3 tells us that N must be simply connected – therefore $H_1(N, \mathbb{Z}) = 0$. Standard application of the Universal Coefficients Theorem (see e.g. [6, Corollary 15.14.1]) gives us that $H^2(N, \mathbb{Z})$ has no torsion component (and thus is 0, since N is a rational homology sphere) and $H^1(N, \mathbb{Z}) = 0$. Poincaré duality, in turn, shows that $H^3(N, \mathbb{Z}) \approx H_1(N, \mathbb{Z}) = 0$. The remaining $H^0(N, \mathbb{Z})$ and $H^4(N, \mathbb{Z})$ are \mathbb{Z} by connectedness and orientability of N . Altogether, N is an integral homology sphere, and thus, by the homology Whitehead theorem ([24, Corollary 4.33]), a homotopy sphere. Ultimately, by M. Freedman's celebrated result on Generalized Poincaré Conjecture ([10]) a homotopy 4-sphere is homeomorphic to a S^4 . Whether M is diffeomorphic to S^4 remains, however, an open problem. \square

One can construct lots of examples of non-simply connected rational homology 4-spheres, although not every finite group might arise as a fundamental group of such a manifold ([18, Corollary 4.4], see also [28]).

As for higher even dimensions, A. Borel proved ([5]) that the only rational homology $2n$ -sphere that is a homogeneous G -space for some compact, connected Lie group G is a standard $2n$ -sphere.

Another example is provided by lens spaces [24, Example 2.43]. Given an integer $m > 1$ and integers ℓ_1, \dots, ℓ_n relatively prime to m , the *lens space*, $L_m(\ell_1, \dots, \ell_n)$ is defined as the orbit space S^{2n-1}/\mathbb{Z}_m of $S^{2n-1} \subset \mathbb{C}^n$ with the action of \mathbb{Z}_m generated by rotations

$$\rho(z_1, \dots, z_n) = (e^{2\pi i \ell_1/m} z_1, \dots, e^{2\pi i \ell_n/m} z_n) .$$

Since the action of \mathbb{Z}_m on S^{2n-1} is free, the projection

$$S^{2n-1} \rightarrow L_m(\ell_1, \dots, \ell_n)$$

is a covering and hence lens spaces are manifolds. One can easily prove that the integral homology groups are

$$H_k(L_m(\ell_1, \dots, \ell_n)) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = 2n - 1, \\ \mathbb{Z}_m & \text{if } k \text{ is odd, } 0 < k < 2n - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $L_m(\ell_1, \dots, \ell_n)$ is a rational homology sphere with the universal covering space S^{2n-1} .

In dimensions $n > 4$ there exist numerous simply connected (smooth) rational homology spheres – a particularly interesting set of examples are the exotic spheres, i.e. manifolds homeomorphic, but not diffeomorphic to a sphere (see e.g. a nice survey article of Joachim and Wraith [27]).

3. PROOFS OF THEOREMS 1.4 AND 1.5.

In the proof of these two theorems we shall need the following definition and a theorem of B. White:

Definition 3.1. Two continuous mappings $g_1, g_2: M \rightarrow N$ are ℓ -homotopic if there exists a triangulation of M such that the two mappings restricted to the ℓ -dimensional skeleton M^ℓ are homotopic in N .

It easily follows from the cellular approximation theorem that this definition does not depend on the choice of a triangulation, see [22, Lemma 2.1].

Lemma 3.2 (Theorem 2, [38]). *Let M and N be compact Riemannian manifolds, $f \in W^{1,p}(M, N)$. There exists $\varepsilon > 0$ such that any two Lipschitz mappings g_1 and g_2 satisfying $\|f - g_i\|_{W^{1,p}} < \varepsilon$ are $[p]$ -homotopic.*

Here $[p]$ is the largest integer less than or equal to p . We shall also need the following construction:

Example 3.3. We shall construct a particular sequence of mappings $g_k: S^n \rightarrow S^n$. Consider the spherical coordinates on S^n :

$$\begin{aligned} (z, \theta) &\mapsto (z \sin \theta, \cos \theta), \\ z &\in S^{n-1} \text{ (equatorial coordinate),} \\ \theta &\in [0, \pi] \text{ (latitude angle).} \end{aligned}$$

These coordinates have, clearly, singularities at the north and south pole. Consider the polar cap $C_k = \{(z, \theta) : 0 \leq \theta \leq \frac{1}{k}\}$ and a mapping

$$g_k: S^n \rightarrow S^n, \quad g_k(z, \theta) = \begin{cases} (z, k\pi\theta) & 0 \leq \theta < \frac{1}{k} \\ (z, \pi) \text{ (i.e. south pole)} & \frac{1}{k} \leq \theta \leq \pi. \end{cases}$$

The mapping stretches the polar cap C_k onto the whole sphere, and maps all the rest of the sphere into the south pole. It is clearly homotopic to the identity map – therefore it is of degree 1.

Measure of the polar cap C_k is comparable to k^{-n} , $|C_k| \approx k^{-n}$. It is also easy to see that the derivative of g_k is bounded by Ck .

Let $1 \leq p < n$. Since the mappings g_k are bounded and g_k converges a.e. to the constant mapping into the south pole as $k \rightarrow \infty$, we conclude that g_k converges to a constant map in L^p . On the other hand, the L^p norm of the derivative Dg_k is bounded by

$$\int_{S^n} |Dg_k|^p \leq Ck^p |C_k| \leq C'k^{p-n} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

and hence g_k converges to a constant map in $W^{1,p}$.

Proof of Theorem 1.4.

Assume that $\pi_n(N) = 0$ and $f \in H^{1,p}(M, N)$, with $n - 1 \leq p < n$. By Lemma 3.2 any two smooth mappings g_1 and g_2 sufficiently close to f are $(n - 1)$ -homotopic, that is there exists a triangulation \mathfrak{T} of M such that g_1 and g_2 , restricted to the $(n - 1)$ -dimensional skeleton M^{n-1} of \mathfrak{T} , are homotopic.

Let $H: M^{n-1} \times [0, 1] \rightarrow N$ be a homotopy between g_1 and g_2 and let Δ be an arbitrary n -simplex of \mathfrak{T} . We have defined a mapping \mathcal{H} from the boundary of $\Delta \times [0, 1]$ to N : it is given by g_1 on $\Delta \times \{0\}$, by g_2 on $\Delta \times \{1\}$ and by H on $\partial\Delta \times [0, 1]$. However, $\partial(\Delta \times [0, 1])$ is homeomorphic to S^n . Since $\pi_n(N) = 0$, any such mapping is null-homotopic and extends to the whole $\Delta \times [0, 1]$. In such a way, simplex by simplex, we can extend the homotopy between g_1 and g_2 onto the whole M .

We showed that any two smooth mappings sufficiently close to f in the norm of $W^{1,p}$ are homotopic, and we may define the homotopy class of f as the homotopy class of a sufficiently good approximation of f by a smooth function. Hence homotopy classes can be well defined in $H^{1,p}(M, N)$.

If in addition $\pi_{n-1}(N) = 0$, then according to [22, Corollary 1.7], smooth mappings are dense in $W^{1,p}(M, N)$, so $W^{1,p}(M, N) = H^{1,p}(M, N)$ and thus homotopy classes are well defined in $W^{1,p}(M, N)$.

In order to complete the proof of the theorem we need yet to show that if $\pi_n(N) \neq 0$, we cannot define homotopy classes in $W^{1,p}(S^n, M)$, $n - 1 \leq p < n$, in a continuous way. As advertised in the Introduction, we shall construct a sequence of smooth mappings that converge in $W^{1,p}(S^n, N)$ to a constant mapping, but are homotopically non-trivial.

If $\pi_n(N) \neq 0$, we have a smooth mapping $G: S^n \rightarrow N$ that is not homotopic to a constant one. Hence the mappings $G_k = G \circ g_k$ are not homotopic to a constant mapping, where g_k was constructed in Example 3.3. On the other hand, since the mappings g_k converge to a constant map in $W^{1,p}$, the sequence G_k also converges to a constant map, because composition with G is continuous in the Sobolev norm. \square

The condition $\pi_n(N) = 0$ is not necessary for the mappings g_1 and g_2 to be homotopic, as can be seen from the following well known

Proposition 3.4. *Any two continuous mappings $\mathbb{CP}^2 \rightarrow \mathbb{CP}^1$ are homotopic, while $\pi_4(\mathbb{CP}^1) = \mathbb{Z}_2$.*

Sketch of a proof of Proposition 3.4.

Since $\mathbb{CP}^1 = S^2$, we have $\pi_4(\mathbb{CP}^1) = \pi_4(S^2) = \mathbb{Z}_2$ (see [24, p. 339]).

The space \mathbb{CP}^2 can be envisioned as $\mathbb{CP}^1 = S^2$ with a 4-disk D attached along its boundary by the Hopf mapping $H: S^3 \rightarrow S^2$. Therefore the 2-skeleton $(\mathbb{CP}^2)^{(2)}$ consists of the sphere S^2 .

Suppose now that we have a mapping $\phi: \mathbb{CP}^2 \rightarrow \mathbb{CP}^1$. If ϕ is not null-homotopic (i.e. homotopic to a constant mapping) on $(\mathbb{CP}^2)^{(2)}$, then its composition with the Hopf mapping is not null-homotopic either, since H generates $\pi_3(S^2) = \mathbb{Z}$. Then ϕ restricted to the boundary S^3 of the 4-disk D is not null-homotopic and cannot be extended onto D , and thus onto the whole \mathbb{CP}^2 . Therefore we know that ϕ restricted to the 2-skeleton of \mathbb{CP}^2 is null-homotopic.

This shows that ϕ is homotopic to a composition

$$\mathbb{CP}^2 \xrightarrow{p} \mathbb{CP}^2/(\mathbb{CP}^2)^{(2)} = S^4 \rightarrow \mathbb{CP}^1 = S^2.$$

It is well known that $\pi_4(S^2) = \mathbb{Z}_2$ and that the only non-null-homotopic mapping $S^4 \rightarrow S^2$ is obtained by a composition of the Hopf mapping $H: S^3 \rightarrow S^2$ and of its suspension $\Sigma H: S^4 \rightarrow S^3$ (see [24], Corollary 4J.4 and further remarks on EHP sequence). Then, if ϕ is to be non-null-homotopic, it must be homotopic to a composition

$$\mathbb{CP}^2 \xrightarrow{p} S^4 \xrightarrow{\Sigma H} S^3 \xrightarrow{H} S^2.$$

The first three elements of this sequence are, however, a part of the cofibration sequence

$$S^2 \hookrightarrow \mathbb{CP}^2 \xrightarrow{p} S^4 \xrightarrow{\Sigma H} S^3 \rightarrow \Sigma \mathbb{CP}^2 \rightarrow \dots,$$

and as such, they are homotopy equivalent to the sequence

$$S^2 \hookrightarrow \mathbb{CP}^2 \rightarrow \mathbb{CP}^2 \cup C(S^2) \hookrightarrow \mathbb{CP}^2 \cup C(S^2) \cup C(\mathbb{CP}^2) \approx \mathbb{CP}^2 \cup C(S^2)/\mathbb{CP}^2,$$

since attaching a cone to a subset is homotopy equivalent to contracting this subset to a point (by $C(A)$ we denote a cone of base A). One can clearly see that in the above sequence the composition of the second and third mapping and the homotopy equivalence at the end, $\mathbb{CP}^2 \rightarrow \mathbb{CP}^2 \cup C(S^2)/\mathbb{CP}^2$, is a constant map; thus the original map $\Sigma H \circ p: \mathbb{CP}^2 \rightarrow S^3$ is homotopic to a constant map, and so is $H \circ \Sigma H \circ p: \mathbb{CP}^2 \rightarrow S^2$, the only candidate for a non-null-homotopic mapping between these spaces. \square

Proof of Theorem 1.5. This proof consist of two parts:

In the first one, suppose N is such that its universal cover \tilde{N} is a rational homology sphere; $n = \dim M = \dim N$. We shall construct an explicit example of a sequence of mappings in $H^{1,p}(M, N)$, $n - 1 \leq p < n$ of a fixed, non-zero degree, that converge in $H^{1,p}$ -norm to a constant mapping.

The manifold M can be smoothly mapped onto an n -dimensional sphere (take a small open ball in M and map its complement into a the south pole). This mapping is clearly of degree 1 – we shall denote it by F . Notice that F , by construction, is a diffeomorphism between an open set in M and $S^n \setminus \{\text{south pole}\}$. Next, let us consider a smooth mapping $G: S^n \rightarrow N$ of non-zero degree, the existence of which we have asserted in Theorem 2.1.

We define a sequence of mappings $F_k: M \rightarrow N$ as a composition of F , mappings g_k given by Example 3.3 and G :

$$(3.1) \quad F_k = G \circ g_k \circ F.$$

The degree of F_k is equal to $\deg G$, thus non-zero and constant. On the other hand, we can, exactly as in the proof of Theorem 1.4, prove that F_k converges in $W^{1,p}$ to a constant map. Since F_k converges a.e. to a constant map, it converges in L^p . As concerns the derivative, observe that $|DF_k| \leq Ck$ and DF_k equals zero outside the set $F^{-1}(C_k)$ whose measure is comparable to k^{-n} . Hence

$$\int_M |DF_k|^p \leq Ck^p |F^{-1}(C_k)| \leq C'k^{p-n} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

and thus the convergence to the constant map is in $W^{1,p}$.

This shows that the mappings F_k can be arbitrarily close to the constant map, so the degree cannot be defined as a continuous function on $H^{1,p}(M, N)$, $n - 1 \leq p < n$.

Now the second part: we shall prove that, as long as the universal cover \tilde{N} of N is not a rational homology sphere, the degree is well defined in $H^{1,p}(M, N)$, $n - 1 \leq p < n$. To this end, we will show that it is continuous on $C^\infty(M, N)$ equipped with the $W^{1,p}$ distance and therefore it extends continuously onto the whole $H^{1,p}(M, N)$. In fact, since the degree mapping takes value in \mathbb{Z} , we need to show that any two smooth mappings that are sufficiently close in $H^{1,p}(M, N)$ have the same degree.

Consider two smooth mappings $g, h: M \rightarrow N$ that are sufficiently close to a given mapping in $H^{1,p}(M, N)$. By Lemma 3.2 of White, g and h are $(n - 1)$ -homotopic, i.e. there exists a triangulation \mathfrak{T} of M such that g and h are homotopic on the $(n - 1)$ -skeleton M^{n-1} . Let $H: M^{n-1} \times [0, 1] \rightarrow N$ be the homotopy, $H(x, 0) = g(x)$, $H(x, 1) = h(x)$ for any $x \in M^{n-1}$.

Let us now look more precisely at the situation over a fixed n -simplex Δ of the triangulation \mathfrak{T} . We have just specified a continuous mapping \mathcal{H} on the boundary of $\Delta \times [0, 1]$: it is given by g on $\Delta \times \{0\}$, by h on $\Delta \times \{1\}$ and by H on $\partial\Delta \times [0, 1]$.

Note that $\partial(\Delta \times [0, 1])$ is homeomorphic to S^n , therefore the mapping $\mathcal{H}: \partial(\Delta \times [0, 1]) \rightarrow N$ is of degree zero, since N is not a rational homology sphere. We thus may fix orientation of $\partial\Delta \times [0, 1]$, so that

$$(3.2) \quad \int_{\Delta} J_g = \int_{\Delta} J_h + \int_{\partial\Delta \times [0, 1]} J_H.$$

Recall that the Jacobian J_f of a function $f: M \rightarrow N$, with fixed volume forms μ on M and ν on N , is given by the relation $f^*\nu = J_f\mu$, and $\deg f = (\int_M J_f d\mu) / (\int_N d\nu) = |N|^{-1} \int_M J_f d\mu$. Therefore, by summing up the relations (3.2) over all the n -simplices of \mathfrak{T} we obtain

$$\begin{aligned} |N| \deg g &= \int_M J_g d\mu \\ &= \sum_{\Delta^n \in \mathfrak{T}} \left(\int_{\Delta^n} J_h d\mu + \int_{\partial\Delta^n \times [0, 1]} J_H d\mu \right) \\ &= \int_M J_h d\mu + \sum_{\Delta^n \in \mathfrak{T}} \int_{\partial\Delta^n \times [0, 1]} J_H d\mu \\ &= |N| \deg h + \sum_{\Delta^n \in \mathfrak{T}} \int_{\partial\Delta^n \times [0, 1]} J_H d\mu. \end{aligned}$$

We observe that every face of $\partial\Delta^n \times [0, 1]$ appears in the above calculation twice, and with opposite orientation, thus $\sum_{\Delta^n \in \mathfrak{T}} \int_{\partial\Delta^n \times [0, 1]} J_H d\mu$ cancels to zero, and $\deg g = \deg h$. Finally, if the universal cover of N is not a rational homology sphere and $\pi_{n-1}(N) = 0$, then $H^{1,p}(M, N) = W^{1,p}(M, N)$ by [22, Corollary 1.7] and hence the degree is well defined in $W^{1,p}(M, N)$. \square

4. ORLICZ-SOBOLEV SPACES AND PROOFS OF THEOREMS 1.1 AND 1.3

We shall begin by recalling some basic definitions of Orlicz and Orlicz-Sobolev spaces; for a more detailed treatment see e.g. [1, Chapter 8] and [17, Chapter 4].

Suppose $P: [0, \infty) \rightarrow [0, \infty)$ is convex, strictly increasing, with $P(0) = 0$. We shall call a function satisfying these conditions a *Young function*. Since we want to deal with Orlicz spaces that are very close to L^n , we will also assume that P satisfies the so-called *doubling* or Δ_2 -condition:

$$(4.1) \quad \text{there exists } K > 0 \text{ such that } P(2t) \leq K P(t) \text{ for all } t \geq 0.$$

This condition is very natural in our situation and it simplifies the theory a great deal. Under the doubling condition the *Orlicz space* $L^P(X)$ on a measure space

(X, μ) is defined as a class of all measurable functions such that $\int_X P(|f|) d\mu < \infty$. It is a Banach space with respect to the so-called *Luxemburg norm*

$$\|f\|_P = \inf \left\{ k > 0 : \int_X P(|f|/k) \leq 1 \right\}.$$

We say that a sequence (f_k) of functions in $L^P(X)$ converges to f *in mean*, if

$$\lim_{k \rightarrow \infty} \int_X P(|f_k - f|) = 0.$$

It is an easy exercise to show that under the doubling condition the convergence in L^P is equivalent to the convergence in mean.

For an open set $\Omega \subset \mathbb{R}^n$ we define the *Orlicz-Sobolev* space $W^{1,P}(\Omega)$ as the space of all the weakly differentiable functions on Ω for which the norm

$$\|f\|_{1,P} = \|f\|_{L^1} + \sum_{i=1}^m \|D_i f\|_P$$

is finite. For example, if $P(t) = t^p$, then $W^{1,P} = W^{1,p}$. In the general case, convexity of P implies that P has at least linear growth and hence $L^P(\Omega) \subset L^1_{\text{loc}}(\Omega)$, $W^{1,P}(\Omega) \subset W^{1,1}_{\text{loc}}(\Omega)$. Using coordinate maps one can then easily extend the definition of the Orlicz-Sobolev space to compact Riemannian manifolds, with the resulting space denoted by $W^{1,P}(M)$.

Let now M and N be compact Riemannian manifolds, $n = \dim M$. We shall be interested in Orlicz-Sobolev spaces that are small enough to exclude, for $M = B^n$, $N = S^{n-1}$, the radial projection $x \mapsto x/|x|$. As shown in [17, p. 2], to exclude such projection it is necessary and sufficient for P to grow fast enough to satisfy the so-called *divergence condition*, already announced (see (1.1)) in the Introduction:

$$\int_1^\infty \frac{P(t)}{t^{n+1}} = \infty.$$

The function $P(t) = t^n$ satisfies this condition, but we are not interested in Orlicz-Sobolev spaces that are too small – that are contained in $W^{1,n}(M, N)$, so we impose an additional growth condition on P , already stated (see (1.2)) in the Introduction:

$$P(t) = o(t^n) \quad \text{as } t \rightarrow \infty.$$

This condition is important. In the case of $P(t) = t^n$ the degree and homotopy results for mappings between manifolds are well known and in this instance the results hold without any topological assumptions about the target manifold N . Our aim is to extend the results beyond the class $W^{1,n}$ and the dependence on the topological structure of N is revealed only when the Orlicz-Sobolev space is larger than $W^{1,n}$, so we really need this condition.

In order to have $C^\infty(M, N)$ functions dense in $L^P(M, N)$, we need yet another technical assumption: that the function P does not ‘slow down’ too much, more precisely, that

$$(4.2) \quad \text{the function } t^{-\alpha}P(t) \text{ is non-decreasing for some } \alpha > n - 1.$$

This condition is also natural for us. We are interested in the Orlicz-Sobolev spaces that are just slightly larger than $W^{1,n}$, so we are mainly interested in the situation when the growth of P is close to that of t^n and the above condition requires less than that. Note that it implies

$$W^{1,P}(M, N) \hookrightarrow W^{1,\alpha}(M, N) \hookrightarrow W^{1,n-1}(M, N).$$

The condition (4.2) plays an important role in the proof of density of smooth mappings.

Lemma 4.1. [17, Theorem 5.2] *If the Young function $P(t)$ satisfies the divergence condition (1.1), doubling condition (4.1) and growth condition (4.2), then $C^\infty(M, N)$ mappings are dense in $W^{1,P}(M, N)$.*

In particular, in order to extend continuously the notions of degree and homotopy classes to $W^{1,P}(M, N)$, it is enough to prove that they are continuous on $C^\infty(M, N)$ endowed with $W^{1,P}$ norm (provided P satisfies all the hypotheses of Lemma 4.1).

Note that density of smooth mappings in $W^{1,P}(M, N)$ and the embedding $W^{1,P}(M, N) \hookrightarrow W^{1,n-1}(M, N)$ implies that $W^{1,P}(M, N) \hookrightarrow H^{1,n-1}(M, N)$

Proof of Theorem 1.3. Let M, N and P be as in the statement of Theorem 1.3. Assume also that $\pi_n(N) = 0$. By the above remark, it is enough to prove that if two smooth mappings $f, g: M \rightarrow N$ are sufficiently close to a $W^{1,P}(M, N)$ mapping in $\|\cdot\|_{1,P}$ -norm then they are homotopic. However, by the inclusion $W^{1,P}(N, M) \hookrightarrow H^{1,n-1}(N, M)$, we know that f and g are close in $H^{1,n-1}(N, M)$, and by Theorem 1.4 they are homotopic.

If $\pi_n(N) \neq 0$, we can construct, like in the proof of Theorem 1.4, a sequence of non-nullhomotopic mappings convergent to a constant mapping in $W^{1,P}(S^n, N)$. Indeed, the mappings G_k constructed as in the proof of Theorem 1.4 converge a.e. to a constant mapping, so they converge to a constant mapping in mean and hence in L^P . On the other hand, the derivative of $G_k = G \circ g_k$ is bounded by Ck and hence

$$\int_{S^n} P(|DG_k|) \leq P(Ck)|C_k| \leq C'P(k)k^{-n} \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

by (1.2) and the doubling condition. Hence DG_k converges to zero in mean and thus in L^P . \square

Proof of Theorem 1.1. If the universal cover of N is not a rational homology sphere, the continuity of the degree mapping with respect to the Orlicz-Sobolev norm, as in the previous proof, is an immediate consequence of Theorem 1.5 – our assumptions on P give us an embedding of $W^{1,P}(M, N)$ into $H^{1,n-1}(M, N)$.

What is left to prove is that if the universal cover of N is a rational homology sphere, the degree cannot be defined continuously in $W^{1,P}(M, N)$. If $F_k = G \circ g_k \circ F$ are the mappings defined as in the proof of Theorem 1.5, then F_k converges a.e. to a constant map, so it converges in L^P . As concerns the derivative, $|DF_k|$ is bounded by Ck and different than zero on a set $F^{-1}(C_k)$ whose measure is comparable to k^{-n} , so

$$\int_M P(|DF_k|) \leq P(Ck)|F^{-1}(C_k)| \leq C'P(k)k^{-n} \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

by (1.2) and the doubling condition. Hence DF_k converges to zero in mean and thus in L^P . \square

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