COEFFICIENTS OF DRINFELD MODULAR FORMS AND HECKE OPERATORS

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Abstract. Consider the space of Drinfeld modular forms of fixed weight and type for $\Gamma_0(\mathfrak{n}) \subset \mathrm{GL}_2(\mathbf{F}_q[T])$. It has a linear form b_n, given by the coefficient of $t^{m+n(q-1)}$ in the power series expansion of a type m modular form at the cusp infinity, with respect to the uniformizer t. It also has an action of a Hecke algebra. Our aim is to study the Hecke module spanned by b_1. We give elements in the Hecke annihilator of b_1. Some of them are expected to be nontrivial and such a phenomenon does not occur for classical modular forms. Moreover, we show that the Hecke module considered is spanned by coefficients b_n, where n runs through an infinite set of integers. As a consequence, for any Drinfeld Hecke eigenform, we can compute explicitly certain coefficients in terms of the eigenvalues. We give an application to coefficients of the Drinfeld Hecke eigenform h.

1. Introduction

Drinfeld modular forms are certain analogues over $\mathbf{F}_q[T]$ of classical modular forms, introduced by D. Goss [12, 13]. A Drinfeld modular form f has a power series expansion with respect to a canonical uniformizer t at the cusp infinity. If f has type m, this expansion is $\sum_{n\geq 0} b_n(f)t^{m+n(q-1)}$. On the space of Drinfeld modular forms of fixed weight and type, we have the linear form $b_n\colon f \mapsto b_n(f)$ and an action of a Hecke algebra. In the present work, we investigate the Hecke module spanned by b_1.

Our interest in the problem comes from the torsion of rank-2 Drinfeld modules. In a previous work, we established a uniform bound on the torsion under an assumption on the latter Hecke module in weight 2 and type 1 (see [1, 2]). This condition was required for studying a Drinfeld modular curve at a neighborhood of the cusp infinity, namely for showing that the map from the curve (or rather a symmetric power) to a quotient of its Jacobian variety is a formal immersion at this cusp in a special fiber.

Before stating the main results, we fix some notations. Let $A = \mathbf{F}_q[T]$ be the ring of polynomials over a finite field $\mathbf{F}_q$ in an indeterminate T, $K = \mathbf{F}_q(T)$ the field of rational functions, $K_\infty = \mathbf{F}_q((1/T))$ and $\mathbf{C}_\infty$ the completion of an algebraic closure of K_∞. For an ideal $\mathfrak{n}$ of A, $k \in \mathbf{N}$ and $0 \leq m < q-1$, we consider the $\mathbf{C}_\infty$-vector space $M_{k,m}(\Gamma_0(\mathfrak{n}))$ of Drinfeld modular forms of weight k and type m for the congruence subgroup $\Gamma_0(\mathfrak{n})$ of $\mathrm{GL}_2(A)$ (see Section 4.1

for the definition). These are rigid analytic $\mathbf{C}_\infty$-valued functions on $\mathbf{C}_\infty - K_\infty$ which have an interpretation as multi-differentials on the Drinfeld modular curve attached to $\Gamma_0(\mathfrak{n})$.

Let $\mathbf{T} = \mathbf{T}_{k,m}(\Gamma_0(\mathfrak{n}))$ be the Hecke algebra, that is the commutative subring of $\mathrm{End}_{\mathbf{C}_\infty}(M_{k,m}(\Gamma_0(\mathfrak{n})))$ spanned over $\mathbf{C}_\infty$ by all Hecke operators T_P for P monic polynomial in A (see Section 4.2). Its restriction $\mathbf{T}' = \mathbf{T}'_{k,m}(\Gamma_0(\mathfrak{n}))$ to the subspace $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ of doubly cuspidal forms (with expansion vanishing at order ≥ 2 at all cusps) stabilizes this subspace. As Goss first observed, doubly cuspidal Drinfeld modular forms play a role similar to classical cusp forms.

In this work, we are interested in the pairing between the space $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ and the Hecke algebra $\mathbf{T}'$ given by the coefficient b_1 of the expansion. More precisely, the dual space $\mathrm{Hom}_{\mathbf{C}_\infty}(M_{k,m}(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty)$ has a natural right action of $\mathbf{T}$ (given by composition) and contains the linear form $b_n \colon f \mapsto b_n(f)$. Let $u = u_{k,m,\mathfrak{n}} \colon \mathbf{T}' \to \mathrm{Hom}_{\mathbf{C}_\infty}(M^2_{k,m}(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty)$ be $\mathbf{C}_\infty$-linear map defined by $s \mapsto b_1 s$. Our main results concern the kernel $\mathbf{I}$ and the image $b_1\mathbf{T}'$ of u.

Let A_{d+} be the set of monic polynomials of degree d in A. The first statement gives a family of elements of $\mathbf{I}$.

Theorem 1.1. *The following elements of* $\mathbf{T}'$ *belong to* $\mathbf{I}$*:*

(1) $\sum_{P\in A_{1+}} P^{1-m}T_P + T_1$ *if* $m \in \{0,1\}$.

(2) $\sum_{P\in A_{d+}} C_{P,0}^{i_0}\cdots C_{P,d-1}^{i_{d-1}} T_P$ *if* $d \geq 1$ *and* $(i_0,\dots,i_{d-1}) \in \mathbf{N}^d$ *is such that*

$$0 \leq i_j \leq q-m \qquad \text{for all } j \in \{0,\dots,d-1\} \tag{1}$$

$$i_0 + \ldots + i_{d-1} \leq (d-1)(q-1) - m. \tag{2}$$

Here, $C_{P,j} \in A$ *stands for the* j*th coefficient of the Carlitz module at* P *(see Section 3.1 for its definition).*

(3) $\sum_{P\in A_{d+}} P^l T_P$ *if* $0 \leq l \leq q-m$ *and* $d \geq \frac{l+m}{q-1} + 1$

$\sum_{P\in A_{d+}} T_P$ *if* $d \geq 2$*, or if* $d = 1$ *and* $m = 0$.

These elements actually belong to the span over A of all Hecke operators. Moreover, they are universal in the sense that, for a given type m, they do not depend on the weight k nor on the ideal $\mathfrak{n}$.

In most cases, we believe that $\mathbf{I} \neq 0$, that is at least one element of Theorem 1.1 is a nontrivial endomorphism of $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$, hence the pairing is not perfect. Over the space $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$ with $\mathfrak{n}$ prime, the situation is as follows. If $\mathfrak{n}$ has degree 3, we prove that $\mathbf{I} = 0$ (Theorem 7.7). If $\mathfrak{n}$ has degree ≥ 5, numerical experiments suggest that $\mathbf{I} \neq 0$ (Conjecture 6.9). Moreover, it may happen that some elements of Theorem 1.1 are zero in $\mathbf{T}'_{2,1}(\Gamma_0(\mathfrak{n}))$: examples of such a situation are explored in Section 6.3.

For the rest of the introduction, we restrict our attention to Drinfeld modular forms of type 0 or 1. Our second statement gives an infinite family of coefficients of Drinfeld modular forms in $b_1\mathbf{T}'$.

Theorem 1.2. *Assume q is a prime and $m \in \{0,1\}$. Let $\mathscr{S}$ be the set of natural integers of the form $c/(q-1)$, where $c \in \mathbf{N}$ is such that the sum of its base q digits is $q-1$. For every $n \in \mathscr{S}$, there exists $s_n \in \mathbf{T}'$, independent of k and $\mathfrak{n}$, satisfying*

$$b_n = b_1 s_n \quad \in b_1\mathbf{T}'.$$

Moreover, $b_1\mathbf{T}'$ is the $\mathbf{C}_\infty$-vector space spanned by b_n for all $n \in \mathscr{S}$.

The primality assumption on q is not essential (see Remark 7.3). As for the set $\mathscr{S}$, it is infinite of natural density zero and the first integer not belonging to $\mathscr{S}$ is $q+1$. For example, if $q = 3$, the first elements of $\mathscr{S}$ are

$$1, 2, 3, 5, 6, 9, 14, 15, 18, 27, 41, 42, 45, 54, 81.$$

Theorem 1.2 relies on an explicit version, Theorem 7.2 (the elements s_n that we produce depend on whether the type is 0 or 1). The expression for s_n is rather natural: it is a A-linear combination of Hecke operators T_P, with P of fixed degree, involving Carlitz binomial coefficients in A.

Suppose now that $\mathbf{I} \neq 0$. Then the map u fails to be surjective (see Lemma 6.2). In particular, $b_1\mathbf{T}'$ does not contain all linear forms b_n for $n \geq 1$. It is then natural to ask what is the smallest integer n such that $b_n \notin b_1\mathbf{T}'$. Theorem 1.2 suggests that $n = q+1$ might be a good candidate.

Both theorems bring new insight on Drinfeld Hecke eigenforms. Consider a Drinfeld modular form f which is an eigenform for the Hecke algebra $\mathbf{T}$. Theorem 1.1 translates into linear relations among the eigenvalues of f, provided that $b_n(f) \neq 0$ for some $n \in \mathscr{S}$ (Proposition 6.5 and Corollary 7.5). Similarly, Theorem 7.2 gives explicit formulas for coefficients $b_n(f)$ ($n \in \mathscr{S}$) in terms of eigenvalues of f and $b_1(f)$. From Theorem 7.2, we also derive:

- multiplicity one statements in some spaces of Drinfeld modular forms of small dimension (Theorem 7.7); as far as we know, these are the only known results of this kind for Drinfeld modular forms.
- explicit expressions for some coefficients of the Drinfeld modular form h (Proposition 8.1). This extends previous work of Gekeler.

As a side remark, we give a brief account of the multiplicity one problem for Drinfeld modular forms. Since there exist two Hecke eigenforms for $\mathrm{GL}_2(A)$ with different weights and same system of eigenvalues (Goss [12]), the question of multiplicity one should be stated as: do eigenvalues and weight determine the Hecke eigenform, up to a multiplicative constant? (see Gekeler [8], Section 7). Böckle and Pink showed that this does not hold for doubly cuspidal forms of weight 5 for the group $\Gamma_1(T)$ when $q > 2$ by means of cohomological techniques (Example 15.4 of [4]). Except for Theorem 7.7 mentioned above, the question remains open for $\Gamma_0(\mathfrak{n})$.

We now compare our results with their analogues for classical modular forms. Consider the space $S_k(\Gamma_0(N))$ of cuspidal modular forms of weight k for the subgroup $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbf{Z})$ ($N \geq 1$). Let $(c_n(f))_{n\geq 1}$ be the Fourier coefficients of such a modular form f at the cusp infinity. Computing the action of the nth Hecke operator T_n on the Fourier expansion of f gives the well-known relation, for any $n \geq 1$

$$c_n(f) = c_1(T_n f). \tag{3}$$

In particular, the Hecke module spanned by the linear form c_1, which now contains all coefficients c_n, is the whole dual space of $S_k(\Gamma_0(N))$ and the coefficient c_1 gives rise to a perfect pairing over $\mathbf{C}$ between $S_k(\Gamma_0(N))$ and the Hecke algebra. Conjecture 6.9 and Theorem 1.2 thus suggest a phenomenon specific to the function field setting. For Drinfeld modular forms, the reason for not having straightforward statements about the kernel and image of u is that the action of Hecke operators on the expansion is not well understood. Goss [12, 13, 11] and subsequently Gekeler [8] wrote down this action using Goss polynomials. But such polynomials are difficult to handle (see also Remark 5.3). In particular, a relation as general as (3) is lacking.

We now sketch the proofs of Theorems 1.1 and 1.2, which involve rather elementary techniques.

- We first compute the coefficient $b_1(T_P f)$, for any f and P, using Goss polynomials (Proposition 5.5). Note that the formula we get is more intricate than (3): it is a A-linear combination of several coefficients of f. For the next step, the crucial point is that the index of these coefficients depends only on the degree of P. This already proves that $b_1\mathbf{T}'$ is contained in the $\mathbf{C}_\infty$-vector space spanned by b_n, for $n \in \mathscr{S}$ when $m \in \{0, 1\}$ (Corollary 5.8).
- We take advantage of characteristic p. For power sums of polynomials of a given degree in A, vanishing properties and closed formulas are well-known (see for instance [19, III] for a survey). Here we use a variant consisting of power sums of coefficients of the Carlitz module. Such sums are studied in Section 3 and closed formulas are given in Proposition 3.5. In Section 3.4, we also explain their connection with Carlitz binomial coefficients and special values of Goss zeta function at negative integers.
- By taking adequate linear combinations of $b_1(T_P f)$, for P of fixed degree, and using results of Section 3, we obtain elements in the kernel $\mathbf{I}$ (Theorem 1.1, Section 6) and in the image $b_1\mathbf{T}'$ (Theorems 7.2 and 1.2).

For the study of the Hecke module $b_1\mathbf{T}'$, our method has reached its limit and improving our results would require new ideas. Our approach might be used to tackle other Hecke modules $b_i\mathbf{T}'$: however, computing $b_i(T_P f)$ for any $i \geq 2$ is a harder combinatorial problem.

2. Notations

A tuple will always be a tuple of nonnegative integers. For such a tuple $\underline{i} = (i_0, \dots, i_s)$, $\binom{i_0+\dots+i_s}{\underline{i}}$ denotes the generalized multinomial coefficient $\frac{(i_0+\dots+i_s)!}{i_0!\cdots i_s!}$.

Let q be a power of a prime p and $\mathbf{F}_q$ (resp. $\mathbf{F}_p$) be a finite field with q (resp. p) elements. We will use repeatedly the following theorem of Lucas: $\binom{i_0+\ldots+i_s}{\underline{i}}$ is nonzero in $\mathbf{F}_p$ if and only if there is no carry over base p in the sum $i_0+\ldots+i_s$.

We keep the same notations as in the introduction. On $A=\mathbf{F}_q[T]$, we have the usual degree deg with the convention $\deg 0=-\infty$. By convention, any ideal of A that we will consider is nonzero. We will often identify an ideal $\mathfrak{p}$ of A with the monic polynomial $P\in A$ generating $\mathfrak{p}$. Accordingly, $\deg\mathfrak{p}$ stands for $\deg P$.

Let $K_\infty=\mathbf{F}_q((1/T))$ be the completion of K at $1/T$ with the natural nonarchimedean absolute value $|\cdot|$ such that $|T|=q$. We write $\mathbf{C}_\infty$ for the completion of an algebraic closure of K_∞: it is an algebraically closed complete field for the canonical extension of $|\cdot|$ to $\mathbf{C}_\infty$.

For P,Q in A, (P) denotes the principal ideal generated by P, $P\mid Q$ means P divides Q and (P,Q) is the g.c.d. of P and Q. The integer part is denoted by $\lfloor\cdot\rfloor$.

3. Power sums of Carlitz coefficients

3.1. **The Carlitz module.** Let $A\{\tau\}$ the noncommutative ring of polynomials in the indeterminate τ with coefficients in A for the multiplication given by $\tau a=a^q\tau$ ($a\in A$). By the map $\tau\mapsto X^q$, $A\{\tau\}$ can be identified with the subring of $\mathrm{End}_{\mathbf{C}_\infty}(\mathbf{G}_a)$ consisting of additive polynomials of the form $\sum a_iX^{q^i}$ (where the multiplication law is given by composition). The Carlitz module is the rank-1 Drinfeld module $C\colon A\to A\{\tau\}$ defined by $C_T=T\tau^0+\tau$. For $a\in A$, we put C_a for the image of a by C, as usual, and $C_a=\sum_{k=0}^{\deg a}C_{a,k}\tau^k$ with $C_{a,k}\in A$. In particular, $C_{a,0}=a$ and $C_{a,d}=1$ if a is monic of degree d.

3.2. **Deformation of the Carlitz module.** We study the dependence of $C_{a,k}$ in the coefficients of a, when a is viewed a polynomial in T. For this purpose, we need a formal version of the Carlitz module. Let $\mathbf{F}_q[T,\boldsymbol{a}]=\mathbf{F}_q[T,\boldsymbol{a}_0,\boldsymbol{a}_1,\ldots]$ be the polynomial ring in T and an infinite set of indeterminates $\{\boldsymbol{a}_i\}_{i\geq 0}$. Consider the ring homomorphism

$$\mathbf{C}\colon \mathbf{F}_q[T,\boldsymbol{a}]\longrightarrow \mathbf{F}_q[T,\boldsymbol{a}]\{\tau\}$$

defined by

$$\mathbf{C}_T=T\tau^0+\tau,\quad \mathbf{C}_{\boldsymbol{a}_i}=\boldsymbol{a}_i\tau^0\quad\text{for all } i\geq 0$$

where the noncommutative ring $\mathbf{F}_q[T,\boldsymbol{a}]\{\tau\}$ is defined in the obvious way. Let P be an element of $\mathbf{F}_q[T,\boldsymbol{a}]$ and d its degree as a polynomial in T. We define $\mathbf{C}_{P,0},\ldots,\mathbf{C}_{P,d}$ in $\mathbf{F}_q[T,\boldsymbol{a}]$ by $\mathbf{C}_P=\sum_{i=0}^{d}\mathbf{C}_{P,i}\tau^i$. These coefficients satisfy the following recursive formulas.

Lemma 3.1. *Let $P \in \mathbf{F}_q[T, \boldsymbol{a}]$ monic of degree d in T. Write $P = Tb + c$, with $c \in \mathbf{F}_q[\boldsymbol{a}]$ and $b \in \mathbf{F}_q[T, \boldsymbol{a}]$ monic of degree $d-1$ in T. Then*

$$\begin{aligned}\mathbf{C}_{P,0} &= T\mathbf{C}_{b,0} + c = P \\ \mathbf{C}_{P,i} &= T\mathbf{C}_{b,i} + \mathbf{C}_{b,i-1}^{q} \quad (1 \leq i \leq d-1) \\ \mathbf{C}_{P,d} &= \mathbf{C}_{b,d-1}^{q} = 1.\end{aligned}$$

Proof. Since $\mathbf{C}$ is additive, we have $\mathbf{C}_{P,i} = \mathbf{C}_{Tb,i} + \mathbf{C}_{c,i}$. Moreover, $\mathbf{C}_{c,i}$ is c if $i = 0$ and 0 otherwise. It remains to compute $\mathbf{C}_{Tb,i}$ in terms of $\mathbf{C}_{b,i}$. We have the following equalities in $\mathbf{F}_q[T, \boldsymbol{a}]\{\tau\}$:

$$\mathbf{C}_{Tb} = \mathbf{C}_T\mathbf{C}_b = (T\tau^0 + \tau)\left(\sum_{i=0}^{d-1} \mathbf{C}_{b,i}\tau^i\right) = T\left(\sum_{i=0}^{d-1} \mathbf{C}_{b,i}\tau^i\right) + \sum_{i=0}^{d-1} \mathbf{C}_{b,i}^{q}\tau^{i+1}.$$

By identification, we get our claim. □

Lemma 3.2. *Let $d \geq 1$ and $P \in \mathbf{F}_q[T, \boldsymbol{a}]$ monic of degree d in T. Write $P = T^d + n_{d-1}T^{d-1} + \ldots + n_0$ with $n_0, \ldots, n_{d-1} \in \mathbf{F}_q[\boldsymbol{a}]$. For all $0 \leq j \leq d-1$, one has*

$$\mathbf{C}_{P,j} = n_j^{q^j} + TQ_j \quad \text{with } Q_j \in \mathbf{F}_q[T, n_k \mid k > j].$$

In particular, if $P = T^d + \boldsymbol{a}_{d-1}T^{d-1} + \ldots + \boldsymbol{a}_0$, the polynomial $\mathbf{C}_{P,j}$ is independent of $\boldsymbol{a}_0$ for $j \geq 1$.

Proof. For $j = 0$, we have $\mathbf{C}_{P,0} = P = n_0 + T(n_1 + \ldots + n_{d-1}T^{d-1})$ which has the expected form. For other coefficients, we proceed by induction on d. The statement is already proven for $d = 1$. Suppose the property satisfied for all monic polynomials of degree $< d$ in T. Let $P = T^d + n_{d-1}T^{d-1} + \ldots + n_0$ and write $P = Tb + n_0$ with $b \in \mathbf{F}_q[T, n_1, \ldots, n_{d-1}]$ monic of degree $< d$ in T. Let $1 \leq j \leq d-1$. By Lemma 3.1, we have

$$\mathbf{C}_{P,j} = T\mathbf{C}_{b,j} + \mathbf{C}_{b,j-1}^{q}. \tag{4}$$

By hypothesis, there exists $R_{j-1} \in \mathbf{F}_q[T, n_k \mid k > j]$ and $R_j \in \mathbf{F}_q[T, n_k \mid k > j+1]$ such that $\mathbf{C}_{b,j} = n_{j+1}^{q^j} + TR_j$ and $\mathbf{C}_{b,j-1} = n_j^{q^{j-1}} + TR_{j-1}$. Substituting in (4), we get $\mathbf{C}_{P,j} = n_j^{q^j} + T(n_{j+1}^{q^j} + TR_j + T^{q-1}R_{j-1}^{q})$. Since $n_{j+1}^{q^j} + TR_j + T^{q-1}R_{j-1}^{q}$ belongs to $\mathbf{F}_q[T, n_k \mid k > j]$, the coefficient $\mathbf{C}_{P,j}$ has the expected form. The property is then established for any monic polynomial P of degree d. □

3.3. Power sums of Carlitz coefficients.

***Notation* 3.3.** Let $d \geq 1$. Recall that the set of monic polynomials of degree d in A is denoted by A_{d+}. For $P \in A_{d+}$ and $\underline{i} = (i_0, \ldots, i_{d-1})$, let

$$C(P)^{\underline{i}} = C_{P,0}^{i_0} \cdots C_{P,d}^{i_d} = C_{P,0}^{i_0} \cdots C_{P,d-1}^{i_{d-1}}$$

(the last equality follows from $C_{P,d} = 1$). By convention, $0^0 = 1$. Let

$$S_d(i_0, \ldots, i_{d-1}) = \sum_{P \in A_{d+}} C(P)^{\underline{i}} \quad \in A.$$

Note that for $d = 1$, the sum is just $S_1(i) = \sum_{P \in A_{1+}} P^i$. We will compute $S_d(i_0, \dots, i_{d-1})$ for small $i_0, \dots, i_{d-1}$.

Lemma 3.4. *Let $0 \leq i \leq 2(q-1)$ and $P \in A$. Then*

$$\sum_{a \in \mathbf{F}_q} (P+a)^i = \begin{cases} -1 & \text{if } i = q-1 \text{ or } 2(q-1) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The vanishing case is merely an application of Lemma 3.1 of Goss [10]. Since we need to compute the remaining cases, we give a full proof. Let $R_i(P) = \sum_{a \in \mathbf{F}_q} (P+a)^i$. Then by the binomial formula,

$$R_i(P) = \sum_{k=0}^{i} \binom{i}{k} P^{i-k} (\sum_{a \in \mathbf{F}_q} a^k).$$

Recall that $\sum_{a \in \mathbf{F}_q} a^k$ equals -1 if $k > 0$ and $k \equiv 0 \mod (q-1)$, and 0 otherwise. Thus $R_{q-1}(P) = -1$ and $R_i(P) = 0$ if $0 \leq i < q-1$. Now let $i = q + j$ with $0 \leq j \leq q-2$. Then

$$R_i(P) = \sum_{a \in \mathbf{F}_q} (P^q + a)(P+a)^j = P^q R_j(P) + \sum_{a \in \mathbf{F}_q} a(P+a)^j.$$

Since $j \leq q-2$, $R_j(P)$ is zero. Moreover, by the binomial formula,

$$\sum_{a \in \mathbf{F}_q} a(P+a)^j = \sum_{k=0}^{j} \binom{j}{k} P^{j-k} (\sum_{a \in \mathbf{F}_q} a^{k+1})$$

which is 0 if $j < q-2$ (resp. -1 if $j = q-2$). □

Proposition 3.5. *Let $i_j \in \{0, \dots, 2(q-1)\}$ for all $j \in \{0, \dots, d-1\}$. Then*

$$S_d(i_0, \dots, i_{d-1}) = \begin{cases} (-1)^d & \text{if } i_j = q-1 \text{ or } 2(q-1) \text{ for all } j \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The sum $S_d(i_0, \dots, i_{d-1})$ is equal to

$$\sum_{a_0, \dots, a_{d-1} \in \mathbf{F}_q} C_{T^d + a_{d-1}T^{d-1} + \dots + a_0, 0}^{i_0} \cdots C_{T^d + a_{d-1}T^{d-1} + \dots + a_0, d-1}^{i_{d-1}}.$$

By Lemma 3.2, the polynomials $C_{T^d + \dots + a_0, 1}, \dots, C_{T^d + \dots + a_0, d-1}$ do not depend on a_0, so we can rewrite the sum as

$$\sum_{a_1, \dots, a_{d-1} \in \mathbf{F}_q} C_{T^d + \dots + a_1 T, 1}^{i_1} \cdots C_{T^d + \dots + a_1 T, d-1}^{i_{d-1}} \left(\sum_{a_0 \in \mathbf{F}_q} (T^d + \dots + a_1 T + a_0)^{i_0} \right).$$

Let ϵ_j be -1 if $i_j = q-1$ or $2(q-1)$, and 0 otherwise. Since $0 \leq i_0 \leq 2(q-1)$, Lemma 3.4 gives $\sum_{a_0 \in \mathbf{F}_q} (T^d + \dots + a_1 T + a_0)^{i_0} = \epsilon_0$. Then, again by Lemma 3.2,

$S_d(i_0,\dots,i_{d-1})$ is equal to

$$\epsilon_0 \sum_{a_2,\dots,a_{d-1}\in\mathbf{F}_q} C^{i_2}_{T^d+\dots+a_2T^2,2}\cdots C^{i_{d-1}}_{T^d+\dots+a_2T^2,d-1}\left(\sum_{a_1\in\mathbf{F}_q}(TQ_1+a_1^q)^{i_1}\right).$$

Since $0 \leq i_1 \leq 2(q-1)$, Lemma 3.4 yields $\sum_{a_1\in\mathbf{F}_q}(TQ_1+a_1^q)^{i_1} = \sum_{a_1\in\mathbf{F}_q}(TQ_1+a_1)^{i_1} = \epsilon_1$. Continuing in this fashion, we obtain $S_d(i_0,,\dots,i_{d-1}) = \epsilon_0\cdots\epsilon_{d-1}$. □

3.4. **Connection with Carlitz binomial coefficients and special zeta values.** We recall Carlitz's analogue $\left\{{a \atop k}\right\}$ in $\mathbf{F}_q[T]$ of the binomial coefficient $\binom{n}{k}$ (the reader may consult Thakur's article [19] for examples of such analogies). Let $a\in A$ and $k\in\mathbf{N}$ with base q expansion $\sum_{i=0}^{w}k_iq^i$ $(0\leq k_i<q)$. We put $\left\{{a \atop k}\right\} = \prod_{i=0}^{w}C_{a,i}^{k_i}$ (if $i>\deg a$, $C_{a,i}=0$ by convention). In particular, $\left\{{a \atop q^i}\right\} = C_{a,i}$. Note that if $0\leq i_j<q$, then

$$C(P)^{\underline{i}} = C^{i_0}_{P,0}\dots C^{i_{d-1}}_{P,d-1} = \left\{{P \atop i_0+i_1+\ldots+i_{d-1}q^{d-1}}\right\}.$$

In general ($i_j\geq q$), it is still possible to write $C^{i_0}_{P,0}\dots C^{i_{d-1}}_{P,d-1}$ in terms of several Carlitz binomials. We now explain how Proposition 3.5 might be proved using this formalism.

If x is an indeterminate, $\left\{{x \atop k}\right\}$ is a polynomial in $K[x]$ with degree k (because $\left\{{x \atop k}\right\}$ is also the exponential function of a finite lattice, see Equation 2.5 of [19] or [14]). Any polynomial f in $K[x]$ may therefore be written as a linear combination of $\left\{{x \atop k}\right\}$. Moreover, the coefficients of this combination can be recovered, in terms of $\left\{{x \atop k}\right\}$, by a Mahler inversion type formula due to Carlitz (Theorem 6 in [5], Lemma 3.2.14 in [14] or Theorem XIV in [19]). For $f=1$, the coefficients in the binomial basis are easily computable and, by the inversion, we obtain for $d\geq 0$ and $0\leq i<q^d$ with base q expansion $\sum_{j=0}^{d-1}i_jq^j$,

$$S_d(i_0,\dots,i_{d-1}) = \sum_{P\in A_{d+}}\left\{{P \atop i}\right\} = \begin{cases}(-1)^d & \text{if } i=q^d-1\\ 0 & \text{otherwise.}\end{cases}$$

This is precisely a special case of Proposition 3.5 (see also [19] p. 14 and Theorem 3.2.16 in [14] for similar statements). It seems likely that Proposition 3.5 can be proved by Mahler inversion.

Finally, we explain how, by the previous observations, $S_d(i_0,\dots,i_{d-1})$ is related to special zeta values of Goss zeta function at negative integers. Consider the Carlitz zeta function $\zeta\colon\mathbf{N}\to K_\infty$ defined by $\zeta(k)=\sum_{P\in A, P\text{ monic}}P^{-k}$. In [10] Goss proved that ζ can be extended to $\mathbf{Z}$ by summing over fixed degree: $\zeta(-k) = \sum_{i=0}^{\infty}(\sum_{P\in A_{i+}}P^k)\in A$ for $k\geq 0$. Now, let $\mathfrak{p}$ be a prime ideal of A and $A_{\mathfrak{p}}$ the ring of integers of the completion of K at $\mathfrak{p}$. Following Thakur [19], one can attach to ζ an $A_{\mathfrak{p}}$-valued zeta measure μ determined by its kth moment:

$$\int_{A_{\mathfrak{p}}} x^k d\mu = \begin{cases} \zeta(-k) & \text{if } k > 0 \\ 0 & \text{if } k = 0. \end{cases}$$

By Wagner's Mahler-inversion formula for continous functions on $A_{\mathfrak{p}}$ ([14] or Theorem VI in [19]), the measure μ is uniquely determined by the coefficients of its divided power series i.e. the sequence $\mu_k = \int_{A_{\mathfrak{p}}} \left\{ \begin{smallmatrix} x \\ k \end{smallmatrix} \right\} d\mu$ $(k \geq 0)$. Thakur has computed explicitly μ_k ([19], Theorem VII). It follows from his proof that, when $0 \leq i_j < q$ and $i = i_0 + \ldots + i_{d-1} q^{d-1}$,

$$S_d(i_0, \ldots, i_{d-1}) = \mu_{i+q^d}.$$

4. Drinfeld modular forms and Hecke operators

We collect some basic facts, and set up notation and terminology as well, for Drinfeld modular forms and Hecke operators.

4.1. **Drinfeld modular forms.** The first occurrence of Drinfeld modular forms goes back to the seminal work of D. Goss [12, 13]. Subsequent developments in the 1980s are due to Gekeler [6, 8].

The so-called Drinfeld upper-half plane is $\Omega = \mathbf{C}_\infty - K_\infty$, which has a rigid analytic structure. For an ideal $\mathfrak{n}$ of A, the Hecke congruence subgroup $\Gamma_0(\mathfrak{n})$ is the subgroup of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(A)$ such that $c \in \mathfrak{n}$. Fix an integer $k \geq 0$ and a class m in $\mathbf{Z}/(q-1)\mathbf{Z}$. A *Drinfeld modular form* (for $\Gamma_0(\mathfrak{n})$) of weight k and type m is a rigid holomorphic function $f \colon \Omega \to \mathbf{C}_\infty$ such that

$$f\left(\frac{az+b}{cz+d}\right) = (ad-bc)^{-m}(cz+d)^k f(z) \quad \text{for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{n}) \tag{5}$$

and f is holomorphic at the cusps of $\Gamma_0(\mathfrak{n})$. We will not detail the second assumption and rather refer to [6] (V, Section 3) and [15] (Section 2). For our purpose, we need only the behaviour at the cusp infinity, which we now recall.

Let $\overline{\pi}$ be the period of the Carlitz module (well-defined up to an element in $\mathbf{F}_q^\times$). The Carlitz exponential e is the holomorphic function $\mathbf{C}_\infty \to \mathbf{C}_\infty$ defined by

$$e(z) = z \prod_{\lambda \in \overline{\pi} A - \{0\}} \left(1 - \frac{z}{\lambda}\right).$$

It is surjective and $\mathbf{F}_q$-linear with kernel $\overline{\pi} A$. For $z \in \mathbf{C}_\infty - A$, let

$$t(z) = \frac{1}{e(\overline{\pi} z)} = \frac{1}{\overline{\pi}} \sum_{\lambda \in A} \frac{1}{z - \lambda}.$$

The function t, invariant by translations $z \mapsto z + a$ $(a \in A)$, is then a uniformizer at the cusp infinity. Since any f satisfying (5) is invariant under such translations, it has a Laurent series expansion $f(z) = \sum_{i \geq i_0} a_i(f) t(z)^i$ with $i_0 \in \mathbf{Z}$ (the series does not converge on all Ω, but only for $|t(z)|$ small enough). Such a function is said to be *holomorphic at the cusp infinity* if the expansion has the form $\sum_{i\geq 0} a_i(f)t^i$. We call it the *t-expansion of f* (at infinity). Since Ω is a connected rigid analytic space, any Drinfeld modular form is uniquely determined by its t-expansion.

Let $M_{k,m}(\Gamma_0(\mathfrak{n}))$ be the space of Drinfeld modular forms of weight k and type m for $\Gamma_0(\mathfrak{n})$. It is a finite-dimensional vector space over $\mathbf{C}_\infty$ whose dimension may be calculated explicitly thanks to Gekeler [6]. If $a_0(f) = 0$ (resp. $a_0(f) = a_1(f) = 0$) and similar conditions at other cusps, f is *cuspidal* (resp. *doubly cuspidal*) and the subspace of such functions is denoted by $M^1_{k,m}(\Gamma_0(\mathfrak{n}))$ (resp. $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$). Goss observed that doubly cuspidal forms play a role similar to classical cusp forms. For an interpretation of Drinfeld modular forms as differentials on a Drinfeld modular curve, one may refer to Section V.5 in [6].

Type and weight are not independent: namely, if $k - 2m \not\equiv 0 \mod (q-1)$, the space $M_{k,m}(\Gamma_0(\mathfrak{n}))$ is zero. Therefore, from now on we assume $k \equiv 2m \mod (q-1)$. Moreover, we choose the representative m in the class with $0 \leq m < q-1$.

Since $\Gamma_0(\mathfrak{n})$ contains matrices of the form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $\lambda \in \mathbf{F}_q^\times$, (5) implies $a_i(f) = 0$ when $i \not\equiv m \mod (q-1)$. Thus any $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ has t-expansion of the form

$$\sum_{j\geq 0} a_{m+j(q-1)}(f) t^{m+j(q-1)}.$$

For $j \geq 0$, let

$$b_j(f) = a_{m+j(q-1)}(f).$$

Later on, we will use both notations for coefficients. A Drinfeld modular form of type > 0 (resp. > 1) is automatically cuspidal (resp. doubly cuspidal). If f is doubly cuspidal, the coefficient $b_0(f)$ may not vanish in general (it does when $m \in \{0, 1\}$).

4.2. **Hecke algebra.** We define a formal Hecke algebra $\mathbf{R}_\mathfrak{n}$ which acts on the different spaces $M_{k,m}(\Gamma_0(\mathfrak{n}))$. In this section, we adopt the notation $\Gamma = \Gamma_0(\mathfrak{n})$.

Let $\Delta = \Delta_0(\mathfrak{n})$ be the set of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries in A such that $ad - bc$ is monic, $c \in \mathfrak{n}$ and $(a) + \mathfrak{n} = A$. Let $\mathbf{R}_\mathfrak{n}$ be the $\mathbf{C}_\infty$-vector space spanned by formal sums of double cosets $\Gamma g \Gamma$ for $g \in \Delta$. One can make $\mathbf{R}_\mathfrak{n}$ a commutative algebra over $\mathbf{C}_\infty$ (see Section 3.1 of [16] for a general treatment or Section 6.1 of [4] for Drinfeld modular forms).

For an ideal $\mathfrak{p}$ of A, let $\Delta^\mathfrak{p} = \{g \in \Delta \mid (\det g) = \mathfrak{p}\}$. The Hecke operator $T_\mathfrak{p}$ is then defined as the formal sum of all double cosets $\Gamma g \Gamma$ with $g \in \Delta^\mathfrak{p}$ in $\mathbf{R}_\mathfrak{n}$. For example, when $\mathfrak{p}$ is prime, $T_\mathfrak{p} = \Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & P \end{smallmatrix}\right)\Gamma$ where P is the monic generator of $\mathfrak{p}$.

As elements of $\mathbf{R}_\mathfrak{n}$ have coefficients in a field of characteristic p, the usual relation for the product gives

$$T_\mathfrak{p} T_{\mathfrak{p}'} = T_{\mathfrak{p}\mathfrak{p}'} \quad \text{for } \textit{any} \text{ ideals } \mathfrak{p}, \mathfrak{p}'$$

(see [11]). This is very different from Hecke operators for classical modular forms, where the above relation only holds for relatively prime ideals. One can check that $\mathbf{R}_\mathfrak{n}$ is the polynomial ring over $\mathbf{C}_\infty$ spanned by $T_\mathfrak{p}$ for $\mathfrak{p}$ prime (such elements are algebraically independent over $\mathbf{C}_\infty$).

As for the notation, $T_\mathfrak{p}$ depends on the subgroup $\Gamma_0(\mathfrak{n})$ but from the context, there will be no confusion on which Hecke algebra (or space of endomorphisms of Drinfeld modular forms) it belongs to.

For $\mathfrak{n} = A$, let us consider the formal Hecke algebra $\mathbf{R}_A$ attached to the sets $\mathrm{GL}_2(A)$ and $\Delta_0(A)$. Let $\tilde{T}_\mathfrak{p}$ temporarily denotes the $\mathfrak{p}$th Hecke operator in $\mathbf{R}_A$. The map $\tilde{T}_\mathfrak{p} \mapsto T_\mathfrak{p}$, for $\mathfrak{p}$ prime, defines an algebra homomorphism $\mathbf{R}_A \to \mathbf{R}_\mathfrak{n}$. We regard $\mathbf{R}_A$ as a universal formal Hecke algebra, independent of $\mathfrak{n}$. Any algebraic relation among the Hecke operators in $\mathbf{R}_A$ can be translated to the corresponding relation in $\mathbf{R}_\mathfrak{n}$ for any $\mathfrak{n}$.

4.3. **Hecke operators on Drinfeld modular forms.** For $v = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries in A and $f\colon \Omega \to \mathbf{C}_\infty$, let

$$f_{|[v]_k} : z \longmapsto (ad-bc)^{k-1}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Fix $g \in \Delta$. The group Γ acts on the left on the double coset $\Gamma g \Gamma$. Let $\{g_i\}_i$ be a finite system of representatives such that g_i has monic determinant. We define an action of $\Gamma g \Gamma$ on $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ by

$$f_{|[\Gamma g \Gamma]_k} = \sum_i f_{|[g_i]_k}$$

(independently of the choice of $\{g_i\}_i$). It extends, in a unique way, to a nonfaithful action of $\mathbf{R}_\mathfrak{n}$ on $M_{k,m}(\Gamma_0(\mathfrak{n}))$. Let $\mathbf{T} = \mathbf{T}_{k,m}(\Gamma_0(\mathfrak{n}))$ be the commutative sub-$\mathbf{C}_\infty$-algebra of $\mathrm{End}_{\mathbf{C}_\infty}(M_{k,m}(\Gamma_0(\mathfrak{n})))$ induced by $\mathbf{R}_\mathfrak{n}$.

For any $g \in \Delta^\mathfrak{p}$, a set of representatives of $\Gamma \backslash \Gamma g \Gamma$ with monic determinant is given by

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}, \quad \alpha, \delta \text{ monic in } A, (\alpha\delta) = \mathfrak{p}, (\alpha) + A = \mathfrak{n}, \beta \in A/(\delta).$$

Therefore, the action of $T_\mathfrak{p}$ on the Drinfeld modular form f can be written more concretely as

$$T_\mathfrak{p}(f)(z) = P^{k-1} \sum_{\substack{\alpha,\delta \text{ monic } \in A \\ \beta \in A,\ \deg\beta < \deg\delta \\ \alpha\delta = P, (\alpha)+\mathfrak{n}=A}} \delta^{-k} f\left(\frac{\alpha z+\beta}{\delta}\right) = \frac{1}{P} \sum_{\alpha,\beta,\delta} \alpha^k f\left(\frac{\alpha z+\beta}{\delta}\right) \tag{6}$$

where P is the monic generator of $\mathfrak{p}$. This formula slightly differs from other references. Gekeler [8] (resp. Böckle [4], Section 6) considered $PT_\mathfrak{p}$ (resp. $P^{m+1-k}T_\mathfrak{p}$). In particular, our operator coincides with Böckle's when $k = m-1$ (for instance, when $k = 2$ and $m = 1$). In general, these variously normalized Hecke operators have the same eigenforms, however with different eigenvalues.

The Hecke algebra $\mathbf{T}$ stabilizes the subspaces $M^1_{k,m}(\Gamma_0(\mathfrak{n}))$ et $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ (see for example Proposition 6.9 of [4]). We denote by $\mathbf{T}' = \mathbf{T}'_{k,m}(\Gamma_0(\mathfrak{n}))$ the restriction of $\mathbf{T}$ to $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$.

5. Hecke action on the first coefficient of Drinfeld modular forms

We recall some results on Goss polynomials for finite lattices and their role in the t-expansion of Drinfeld modular forms. Then we give an explicit formula for the action on the first coefficient of this expansion.

5.1. **Goss polynomials.** Let Λ be a $\mathbf{F}_q$-lattice in $\mathbf{C}_\infty$, i.e. a $\mathbf{F}_q$-submodule of $\mathbf{C}_\infty$ having finite intersection with each ball of $\mathbf{C}_\infty$ of finite radius. We assume Λ to be *finite* of dimension d over $\mathbf{F}_q$. The exponential corresponding to Λ

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda - \{0\}} \left(1 - \frac{z}{\lambda}\right) \qquad (z \in \mathbf{C}_\infty)$$

is an entire Λ-periodic $\mathbf{F}_q$-linear function. Since Λ is finite, it is a polynomial in z of the form

$$e_\Lambda(z) = \sum_{i=0}^{d} \lambda_i z^{q^i}$$

with coefficients $\lambda_i \in \mathbf{C}_\infty$ depending on Λ. Goss has considered Newton's sums associated to the reciprocal polynomial of $e_\Lambda(X-z) = e_\Lambda(X) - e_\Lambda(z) \in \mathbf{C}_\infty[z][X]$, namely

$$\begin{aligned} &N_0 = 0 \\ &N_j(z) = N_{j,\Lambda}(z) = \textstyle\sum_{\lambda \in \Lambda} \frac{1}{(z+\lambda)^j} \qquad (j \geq 1, z \in \mathbf{C}_\infty - \Lambda). \end{aligned}$$

Let

$$t_\Lambda(z) = e_\Lambda(z)^{-1} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda} \qquad (z \in \mathbf{C}_\infty - \Lambda).$$

Proposition 5.1 (Section 2 of [13], 3.4–3.9 in [8])**.** *Let $j \geq 1$. There exists a unique polynomial $G_j = G_{j,\Lambda}(X) \in \mathbf{C}_\infty[X]$ such that the following equalities hold:*

1. *if $j \leq q$ then $G_j(X) = X^j$*
2. *$G_j(X) = X \sum_{i \geq 0, j - q^i \geq 0} \lambda_i G_{j-q^i}(X)$.*

The polynomial $G_j(X)$ is monic of degree j and satisfies $N_j = G_j(t_\Lambda)$. Moreover,

$$G_j(X) = \sum_{n=0}^{j-1} \sum_{\underline{i}} \binom{n}{\underline{i}} \lambda^{\underline{i}} X^{n+1} \tag{7}$$

for $\underline{i} = (i_0, \ldots, i_d)$ running through $(d+1)$-tuples such that

$$i_0 + \ldots + i_d = n$$
$$i_0 + i_1 q + \ldots + i_d q^d = j - 1$$

and $\lambda^{\underline{i}}$ *denotes* $\lambda_0^{i_0}\cdots\lambda_d^{i_d}$. *The polynomial* $G_j(X)$ *is divisible by* X^u *where* $u = \lfloor j/q^d \rfloor + 1$.

Gekeler provided the explicit formula (7) using a generating function. We further put $G_{0,\Lambda}(X) = 0$.

5.2. **Hecke algebra and Goss polynomials.** Let $\mathfrak{p}$ an ideal of A of degree $d \geq 0$ with monic generator P. Recall that C denotes the Carlitz module over $\mathbf{C}_\infty$ (Section 3.1). As usual, for an indeterminate X, put $C_P(X) = \sum_{i=0}^d C_{P,i}X^{q^i}$. For our purpose, we consider the $\mathbf{F}_q$-lattice of dimension d

$$\Lambda_P = \mathrm{Ker}(C_P) = \{x \in \mathbf{C}_\infty \mid C_P(x) = 0\}$$

whose jth Goss polynomial is denoted by $G_{j,P}$. Let

$$t_P(z) = t(Pz) = \frac{1}{e(\bar{\pi}Pz)} \qquad (z \in \mathbf{C}_\infty - A).$$

Then, if $f_P(X)$ is the Pth inverse cyclotomic polynomial $C_P(X^{-1})X^{q^d}$, one has

$$t_P = \frac{t^{q^d}}{f_P(t)}. \tag{8}$$

The following statement mildly extends Gekeler's formula 7.3 in [8] (which was established for $\mathrm{GL}_2(A)$ and $\mathfrak{p}$ prime) to $\Gamma_0(\mathfrak{n})$ and any $\mathfrak{p}$.

Proposition 5.2. *Let* $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ *with* t*-expansion* $\sum_{i\geq 0} a_i t^i$. *We have*

$$T_{\mathfrak{p}}f = P^{k-1}\sum_{i\geq 0}\sum_{\substack{\delta \text{ monic in } A\\ \delta|P,\ (\frac{P}{\delta})+\mathfrak{n}=A}} \delta^{-k}a_i G_{i,\delta}(\delta t_{\frac{P}{\delta}}) \tag{9}$$

Moreover, for fixed j, *only a finite number of terms in the right-hand side contribute to* t^j *in the* t*-expansion of* $T_{\mathfrak{p}}f$.

Proof. Let δ be a monic divisor of P. Recall that e is the Carlitz exponential. We write $F(z)$ for $\sum_{\beta\in A,\deg\beta<\deg\delta} f\left((Pz/\delta+\beta)/\delta\right)$. For $|t(z)|$ small enough, $F(z)$ is

$$\sum_{\beta\in A,\deg\beta<\deg\delta}\sum_{i\geq 0} a_i t\left(\frac{\frac{P}{\delta}z+\beta}{\delta}\right)^i = \sum_{i\geq 0} a_i \sum_{\beta\in A,\deg\beta<\deg\delta} e\left(\bar{\pi}\frac{\frac{P}{\delta}z+\beta}{\delta}\right)^{-i}$$
$$= \sum_{i\geq 0} a_i \sum_{\beta\in A,\deg\beta<\deg\delta}\left(e\left(\frac{\bar{\pi}Pz}{\delta^2}\right) + e\left(\frac{\bar{\pi}\beta}{\delta}\right)\right)^{-i}$$

by additivity of e. According to the analytic theory of Drinfeld modules, the finite set $\{e(\bar{\pi}\beta/\delta) \mid \beta \in A, \deg\beta < \deg\delta\}$ is in bijection with the lattice $\Lambda_\delta = \mathrm{Ker}(C_\delta)$. Let $w = Pz/\delta^2$. Then, by Proposition 5.1, $F(z)$ is

$$\sum_{i\geq 0} a_i \sum_{\lambda\in\Lambda_\delta}(e(\bar{\pi}w)+\lambda)^{-i} = \sum_{i\geq 0} a_i N_{i,\Lambda_\delta}(e(\bar{\pi}w)) = \sum_{i\geq 0} a_i G_{i,\Lambda_\delta}(e_{\Lambda_\delta}(e(\bar{\pi}w))^{-1}).$$

Observe that $e_{\Lambda_\delta}(z) = C_\delta(z)/\delta$ (both polynomials have the same set of zeros and the multiplicative constant is obtained by comparing the terms in z). By property of the Carlitz exponential, we also have $C_\delta(e(\overline{\pi}w)) = C(\overline{\pi}zP/\delta) = t(zP/\delta)^{-1}$. Substituting, we get

$$F(z) = \sum_{i\geq 0} a_i G_{i,\Lambda_\delta}\left(\delta t\left(\frac{zP}{\delta}\right)\right) = \sum_{i\geq 0} a_i G_{i,\Lambda_\delta}(\delta t_{\frac{P}{\delta}}(z)).$$

Our last claim follows from (6) and the last statement of Proposition 5.1. □

***Remark* 5.3.** To obtain the t-expansion of $T_{\mathfrak{p}}f$ from Equation (9), it would suffice to compose the t-expansions of $t_{P/\delta}$ and Goss polynomials $G_{i,\delta}$. However, making this explicit seems to be a difficult problem. Indeed, a similar question arises when trying to make explicit the t-expansion of Drinfeld-Eisenstein series (see Section 6 of [8]) since it involves the t-expansion of $G_{i,\overline{\pi}A}(t_P)$[1]. This is quite different from the classical situation where coefficients of Eisenstein series are well-known arithmetic functions.

5.3. **Hecke module spanned by b_1.**

***Notation* 5.4.** The dual space of $M_{k,m}(\Gamma_0(\mathfrak{n}))$ has the natural right action of $\mathbf{T}$, given by composition, and contains the following linear forms, for any $n \geq 1$:

$$a_{m+n(q-1)} = b_n \colon f \mapsto a_{m+n(q-1)}(f) = b_n(f).$$

Let $u = u_{k,m,\mathfrak{n}} \colon \mathbf{T}' \to \mathrm{Hom}_{\mathbf{C}_\infty}(M^2_{k,m}(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty)$ be the $\mathbf{C}_\infty$-linear map $s \mapsto b_1 s$. We write $b_1\mathbf{T}'$ for the image of u.

We collect some remarks on the dimension of the $\mathbf{C}_\infty$-algebra $\mathbf{T}'$. The map u is not necessarily an isomorphism, therefore the dimension of $\mathbf{T}'$ is unknown *a priori*. In the case $\mathbf{T}' = \mathbf{T}'_{2,1}(\Gamma_0(\mathfrak{n}))$, one can prove that its dimension coincides with $\dim_{\mathbf{C}_\infty} M^2_{2,1}(\Gamma_0(\mathfrak{n}))$, using results from automorphic forms and work of Gekeler and Reversat [15].

We keep Notation 3.3. The next statement gives a first description of $b_1\mathbf{T}'$.

Proposition 5.5. *Let $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ with t-expansion $\sum_{i\geq 0} a_i(f)t^i$. Let $\mathfrak{p}$ an ideal of A of degree d with monic generator P. Then $a_{m+(q-1)}(T_{\mathfrak{p}}f)$ is*

$$\sum_{\underline{n}} \binom{m+q-2}{\underline{n}} C(P)^{\underline{n}} a_{1+n_0+n_1q+\ldots+n_dq^d}(f) + \varepsilon \sum_{\substack{Q|P, Q\in A_{1+}\\ (Q)+\mathfrak{n}=A}} Q^{k-1}a_1(f) \tag{10}$$

where $\underline{n} = (n_0, \ldots, n_d)$ is such that $n_0 + \ldots + n_d = m + q - 2$ and ε is defined by $\varepsilon = \begin{cases} 1 & \textit{if } m = 1 \\ 0 & \textit{otherwise.}\end{cases}$

[1]The lattice $\overline{\pi}A$ is not finite but Goss polynomials can be defined in that more general setting (see [13, 8]).

***Remark* 5.6.** (1) In Example 7.4 of [8], Gekeler treated $a_i(T_{\mathfrak{p}}f)$ for $\mathfrak{p}$ of degree 1, $i \geq 0$, and f modular for $\mathrm{GL}_2(A)$. Proposition 5.5 supplements Gekeler's statement.

(2) Actually, the proof only uses the subgroup $\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in A\}$ of $\Gamma_0(\mathfrak{n})$. Let $\mathcal{O}$ be the ring of holomorphic functions $f\colon \Omega \to \mathbf{C}_\infty$ which are A-periodic ($f(z+a) = f(z)$, $a \in A$) and holomorphic at the cusp infinity (in particular, this ring contains the space $M_{k,m}(\Gamma_0(\mathfrak{n}))$). As recalled in Section 4, such functions have a t-expansion. Equation (6) still defines a function $T_{\mathfrak{p}}f\colon \Omega \to \mathbf{C}_\infty$. Then Propositions 5.2 and 5.5 are more generally valid for $f \in \mathcal{O}$ such that $T_{\mathfrak{p}}f \in \mathcal{O}$.

Proof. By Proposition 5.2, we have to find the coefficient of $t^{m+(q-1)}$ in the t-expansion of $G_{i,\delta}(\delta t_{P/\delta})$. First, if $i = 0$, then $G_{0,\delta}(X) = 0$ so the expansion of $G_{0,\delta}(\delta t_{P/\delta})$ has no $t^{m+(q-1)}$-term.

Assume $i > 0$. By (8) the t-expansion of $t_{P/\delta}$ is divisible by $t^{q^{d-\deg\delta}}$. Moreover, it follows from the definition of Goss polynomials that $G_{i,\delta}(X)$ has X as a factor. Hence, the t-expansion of $G_{i,\delta}(\delta t_{P/\delta})$ is divisible by $t^{q^{d-\deg\delta}}$. Since $m < q-1$, $G_{i,\delta}(\delta t_{P/\delta})$ has no $t^{m+(q-1)}$-term when $d - \deg\delta \geq 2$. Now assume $0 \leq d - \deg\delta \leq 1$. Put $s = \deg\delta$. Recall that $e_{\Lambda_\delta}(z) = C_\delta(z)/\delta = \sum_{i=0}^{s} C_{\delta,i} z^{q^i}/\delta$. The explicit formula for Goss polynomials gives

$$G_{i,\delta}(X) = \sum_{j=0}^{i-1} \delta^{-j} \sum_{\underline{n}} \binom{j}{\underline{n}} C(\delta)^{\underline{n}} X^{j+1}$$

where $\underline{n} = (n_0, \ldots, n_s)$ are such that $n_0 + \ldots + n_s = j$ and $n_0 + n_1 q + \ldots + n_s q^s = i-1$.

Suppose that $s = d$, i.e. $\delta = P$. Then the corresponding partial sum in (9) is

$$\frac{1}{P}\sum_{i\geq 0} a_i G_{i,P}(Pt) = \frac{1}{P}\sum_{i\geq 0} a_i \sum_{j=0}^{i-1} P^{-j} \sum_{\underline{n}} \binom{j}{\underline{n}} C(P)^{\underline{n}} (Pt)^{j+1}.$$

The $t^{m+(q-1)}$-term corresponds to $j = m+q-2$; namely, it is

$$\sum_{\underline{n}} \binom{m+q-2}{\underline{n}} C(P)^{\underline{n}} a_{1+n_0+n_1q+\ldots+n_dq^d}(f)$$

with $\underline{n} = (n_0, \ldots, n_d)$ such that $n_0 + \ldots + n_d = m+q-2$.

Next, suppose that $s = d-1$. Using Equation (8), we get

$$G_{i,\delta}(\delta t_{\frac{P}{\delta}}) = \sum_{j=0}^{i-1} \delta^{-j} \sum_{\underline{n}} \binom{j}{\underline{n}} C(\delta)^{\underline{n}} \left(\delta \frac{t^q}{1+\frac{P}{\delta}t^{q-1}}\right)^{j+1} \tag{11}$$

where $(n_0, \ldots, n_{d-1})$ with $n_0 + \ldots + n_{d-1} = j$ and $n_0 + n_1 q + \ldots + n_{d-1}q^{d-1} = i-1$. If $j \geq 1$, then $q(j+1) \geq 2q > m+q-1$, thus there is no $t^{m+(q-1)}$-term in the expansion of (11). Finally, we assume $j = 0$, in other words $n_0 = \ldots = n_{d-1} = 0$

and $i=1$. We have

$$G_{1,\delta}(\delta t_{\frac{P}{\delta}}) = \delta \frac{t^q}{1+\frac{P}{\delta}t^{q-1}} = \delta t^q \sum_{n=0}^{+\infty} (-1)^n \frac{P^n}{\delta^n} t^{n(q-1)}.$$

This series has a $t^{m+(q-1)}$-term if and only if $q-1$ divides $m-1$. This happens only if $m=1$, and in that case the corresponding coefficient is δ. To summarize, we obtain (10) where $\underline{n}=(n_0,\dots,n_d)$ satisfies $n_0+n_1+\ldots+n_d=m+q-2$. □

Assume $m\in\{0,1\}$. By (10), the linear form $b_1 T_{\mathfrak{p}} = a_{m+(q-1)} T_{\mathfrak{p}}$ is a A-linear combination of a_i, where i runs through the set of natural integers satisfying the condition: the expansion of i in base q has at most $d+1$ digits, whose sum is equal to $m+q-1$. In particular, the set of such i's only depends on the degree d of $\mathfrak{p}$. This observation, also communicated to the author by D. Goss, will be used in Section 7. For the moment, we derive the following statement for $b_1\mathbf{T}'$.

***Notation* 5.7.** Let $\mathscr{S}$ be the set of natural integers of the form $c/(q-1)$ where $c\in\mathbf{N}$ is such that the sum of its base q digits is $q-1$.

Corollary 5.8. *If $m\in\{0,1\}$ then $b_1\mathbf{T}'$ is contained in the $\mathbf{C}_\infty$-vector space spanned by b_n for $n\in\mathscr{S}$.*

The reverse inclusion will be proved in Section 7. Finally, we state another straightforward application of Proposition 5.5.

***Notation* 5.9.** For $d\geq 1$ and $\underline{i}=(i_0,\dots,i_{d-1})$, let

$$\Theta_d(i_0,\dots,i_{d-1}) = \sum_{P\in A_{d+}} C(P)^{\underline{i}} T_P = \sum_{P\in A_{d+}} C_{P,0}^{i_0}\cdots C_{P,d-1}^{i_{d-1}} T_P \quad \in \mathbf{R}_A.$$

Corollary 5.10. *Let $f\in M_{k,m}(\Gamma_0(\mathfrak{n}))$. With the notations of Proposition 5.5 and Section 3, the coefficient $a_{m+(q-1)}(\Theta_d(i_0,\dots,i_{d-1})f)$ is*

$$\sum_{\substack{\underline{n}=(n_0,\dots,n_d)\\ n_0+\ldots+n_d=m+q-2}} \binom{m+q-2}{\underline{n}} S_d(n_0+i_0,\dots,n_{d-1}+i_{d-1}) a_{1+n_0+n_1q+\ldots+n_dq^d}(f)$$
$$+\varepsilon \sum_{P\in A_{d+}} C(P)^{\underline{i}} \sum_{\substack{Q|P, Q\in A_{1+}\\ (Q)+\mathfrak{n}=A}} Q^{k-1} a_1(f).$$

6. Annihilator of b_1 for the Hecke action

***Notation* 6.1.** Let $\mathbf{I}=\mathbf{I}_{k,m,\mathfrak{n}}$ be the kernel of u i.e. the ideal of elements $s\in\mathbf{T}'$ such that $b_1 s=0$ in the dual space of $M_{k,m}^2(\Gamma_0(\mathfrak{n}))$.

In particular, $\mathbf{I}$ is a sub-$\mathbf{C}_\infty$-algebra of $\mathbf{T}'$ which maps doubly cuspidal forms to Drinfeld modular forms f satisfying $a_0(f)=b_0(f)=b_1(f)=0$.

Lemma 6.2. *If the map $u\colon \mathbf{T}'\to \mathrm{Hom}_{\mathbf{C}_\infty}(M_{k,m}^2(\Gamma_0(\mathfrak{n})),\mathbf{C}_\infty)$ is surjective, then it is an isomorphism.*

Proof. Since u is surjective, we take an element t_n in the preimage of b_n for any $n \geq 1$. If s belongs to the ideal $\mathbf{I}$, so does $t_n s$. Hence, for any $f \in M^2_{k,m}(\Gamma_0(\mathfrak{n}))$, the nth coefficient $b_n(sf)$ is zero for any $n \geq 1$. As the t-expansion uniquely determines a Drinfeld modular form, sf must be zero. Therefore s is zero as an endomorphism of $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$. □

6.1. **Proof of Theorem 1.1.**

Proof of Theorem 1.1. Actually we prove a slightly more general statement: all the following equalities of linear forms will take place in the dual of $M_{k,m}(\Gamma_0(\mathfrak{n}))$ if $m \neq 1$ (resp. of $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ if $m = 1$).

(1) Without any assumption on m, we apply Corollary 5.10 to $d = 1$. For $i \geq 0$ we get
$$b_1\left(\sum_{P\in A_{1+}} P^i T_P\right) = \sum_{n=0}^{m+q-2} \binom{m+q-2}{n} S_1(n+i) a_{1+n+q(m+q-2-n)}.$$
This follows also from Gekeler's example 7.4 in [8], although stated there for $\mathrm{GL}_2(A)$ and with a different normalization of Hecke operators.

Assume $m = 0$. The sum $S_1(n+1) = \sum_{Q\in A_{1+}} Q^{n+1}$ is nonzero if and only if $n = q-2$, and $S_1(q-1) = -1$ (by Lemma 3.4 for instance). Taking $i = 1$, our expression simplifies as $b_1\left(\sum_{P\in A_{1+}} PT_P\right) = -b_1$.

Assume $m = 1$. Since the sum $S_1(n)$ is nonzero if and only if $n = q-1$, taking $i = 0$, we get $b_1\left(\sum_{P\in A_{1+}} T_P\right) = -b_1$.

(2) Consider $(i_0, \ldots, i_{d-1})$ as in the statement. By Corollary 5.10, we get that $b_1(\Theta_d(i_0, \ldots, i_{d-1}))$ is
$$\sum_{\substack{\underline{n} \\ n_0+\ldots+n_d=m+q-2}} \binom{m+q-2}{\underline{n}} S_d(n_0+i_0, \ldots, n_{d-1}+i_{d-1}) a_{1+n_0+n_1q+\ldots+n_dq^d}.$$
We have $0 \leq n_j + i_j \leq 2(q-1)$, hence we can evaluate $S_d(n_0+i_0, \ldots, n_{d-1}+i_{d-1})$ thanks to Proposition 3.5. This sum is nonzero if and only if $n_j + i_j = q-1$ or $2(q-1)$ for all $j \in \{0, \ldots, d-1\}$. If this happens, we have
$$d(q-1) \leq \sum_{l=0}^{d-1} (n_l + i_l) \leq i_0 + \ldots + i_{d-1} + m + q - 2$$
which contradicts $i_0 + \ldots + i_{d-1} \leq (d-1)(q-1) - m$. Accordingly, the sum always vanishes and $b_1(\Theta_d(i_0, \ldots, i_{d-1})) = 0$.

(3) Apply the statement proved before to $i_0 = l$ and $i_1 = \ldots = i_{d-1} = 0$. □

It is worth pointing out that the elements of $\mathbf{I}$ given in Theorem 1.1 are universal in the sense that, for a given type, they do not depend on the weight k nor the ideal $\mathfrak{n}$. Some of them, as $\sum_{P\in A_{d+}} T_P$ for $d \geq 2$ for instance, are also independent of the type m. This means that, in the universal formal Hecke algebra $\mathbf{R}_A$, such an element is independent of k, m and $\mathfrak{n}$.

***Remark* 6.3.** This phenomenon does not occur for classical modular forms of weight 2 as we now explain. Let $S_2(\Gamma_0(N))$ be the complex space of weight-2 cusp forms for $\Gamma_0(N)$ ($N \geq 1$). We write $(c_n)_{n\geq 1}$ for the linear forms given by Fourier coefficients of such modular forms at the cusp infinity. The Hecke algebra $\mathbf{T}_c$ of weight 2 for $\Gamma_0(N)$ is the subring of $\mathrm{End}(S_2(\Gamma_0(N))$ spanned over $\mathbf{C}$ by all Hecke operators T_n for $n \in \mathbf{N}$. Let u_c be the $\mathbf{C}$-linear map $\mathbf{T}_c \to \mathrm{Hom}_{\mathbf{C}}(S_2(\Gamma_0(N), \mathbf{C})$ given by $s \mapsto c_1 s$. Relation (3) gives $c_n = u_c(T_n)$ for all $n \geq 1$, thus u_c is bijective. We claim that if there exists a $\mathbf{C}$-linear combination $s = \lambda_1 T_{i_1} + \ldots + \lambda_j T_{i_j}$, with j, $\lambda_1, \ldots, \lambda_j, i_1, \ldots, i_j$ independent of N, such that $s = 0$ as an endomorphism of $S_2(\Gamma_0(N))$, then the coefficients $\lambda_1, \ldots, \lambda_s$ must be zero. In fact, when N is prime, the Hecke operators $T_1, \ldots, T_{g(N)}$ are $\mathbf{C}$-linearly independent in $\mathrm{End}(S_2(\Gamma_0(N))$ for $g(N) = \dim S_2(\Gamma_0(N))$ (this follows from the cusp infinity not being a Weierstrass point on the modular curve $X_0(N)$). Choosing N prime such that $g(N)$ is large enough yields $\lambda_1 = \ldots = \lambda_j = 0$ and proves our claim.

In Section 7.2, we will further our investigation of the ideal $\mathbf{I}$ and prove that it vanishes in some cases (Theorem 7.7).

6.2. **Linear relations for eigenvalues.**

***Notation* 6.4.** Let $\mathfrak{p}$ an ideal of A with monic generator P. A *Hecke eigenform* f is a Drinfeld modular form which is an eigenform for all Hecke operators. We write $\lambda_P(f)$ for its eigenvalue for $T_P = T_{\mathfrak{p}}$.

For a Hecke eigenform f such that $b_1(f) \neq 0$, Theorem 1.1 yields linear relations among its eigenvalues. It seems rather remarkable that these relations are universal in the sense that, for a fixed type, they do not depend on the weight k nor on the level $\mathfrak{n}$.

Proposition 6.5. *Let $f \in M_{k,m}(\Gamma_0(\mathfrak{n}))$ be a Hecke eigenform with $b_1(f) \neq 0$. If $m = 1$, we assume further $f \in M^2_{k,m}(\Gamma_0(\mathfrak{n}))$.*

(1) *If $m \in \{0, 1\}$, then*

$$\sum_{P \in A_{1+}} P^{1-m} \lambda_P(f) + 1 = 0.$$

(2) *Let $d \geq 1$ and $i_0, \ldots, i_{d-1}$ satisfying* (1) *and* (2)*. Then*

$$\sum_{P \in A_{d+}} C(P)^{\underline{i}} \lambda_P(f) = 0.$$

(3) *Let l and d be integers such that $0 \leq l \leq q-m$ and $d \geq (l+m)/(q-1)+1$. Then*

$$\sum_{P \in A_{d+}} P^l \lambda_P(f) = 0.$$

In particular, if $d \geq 2$, or f has type 0 and $d = 1$, then

$$\sum_{P \in A_{d+}} \lambda_P(f) = 0.$$

6.3. **Linear relations for Hecke operators.** We explain how some relations of Proposition 6.5 follow from a more general statement, namely linear relations among Hecke operators in characteristic zero or p. In other words, we prove that certain elements of $\mathbf{I}$ given in Theorem 1.1 are zero in $\mathbf{T}'$.

***Notation* 6.6.** For an ideal $\mathfrak{n}$ of A, let $\mathbf{H}_\mathfrak{n}$ be the abelian group of $\mathbf{Z}$-valued cuspidal harmonic cochains for $\Gamma_0(\mathfrak{n})$ on the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}(2, K_\infty)$ (we refer to Section 3 of [15] for the relevant definitions and properties). The group $\mathrm{GL}_2(K)$ acts on the left on the set of oriented edges $Y(\mathcal{T})$ of $\mathcal{T}$. We define an endomorphism $\theta_\mathfrak{p}$ of $\mathbf{H}_\mathfrak{n}$ by

$$(\theta_\mathfrak{p} F)(e) = \sum_{\substack{\alpha,\delta \text{ monic } \in A \\ \beta \in A,\ \deg\beta < \deg\delta \\ (\alpha\delta) = \mathfrak{p},\ (\alpha)+\mathfrak{n} = A}} F\left(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} e\right)$$

for $F \in \mathbf{H}_\mathfrak{n}$ and $e \in Y(\mathcal{T})$.

After scalar extension to the complex numbers $\mathbf{C}$, $\mathbf{H}_\mathfrak{n}$ is identified with a space of cuspidal automorphic forms on $\mathrm{GL}(2)$ of the adeles of K (by the strong approximation theorem). Moreover, using Teitelbaum's residue map [17], Gekeler and Reversat [15] gave an isomorphism between $\mathbf{H}_\mathfrak{n}/p\mathbf{H}_\mathfrak{n}$ and a subspace of Drinfeld modular forms, namely the subspace $M^2_{2,1}(\Gamma_0(\mathfrak{n}), \mathbf{F}_p)$ of $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$ consisting of such forms with residues in $\mathbf{F}_p$. It turns out that this isomorphism is Hecke-equivariant, with the normalizations we have adopted here for $T_\mathfrak{p}$ and $\theta_\mathfrak{p}$. Finally, $M^2_{2,1}(\Gamma_0(\mathfrak{n}), \mathbf{F}_p)$ is an $\mathbf{F}_p$-vector space which, after scalar extension to $\mathbf{C}_\infty$, gives the whole space $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$. Put differently, the Hecke operator $T_\mathfrak{p}$ acting on $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$ can be thought of as the mod p reduction of $\theta_\mathfrak{p}$.

Lemma 6.7. *Let $\mathfrak{n}$ be a prime. Assume $d \geq \deg(\mathfrak{n}) - 1$. Then $\sum_{\deg \mathfrak{p} = d} \theta_\mathfrak{p} = 0$. In particular, $\sum_{\deg \mathfrak{p} = d} T_\mathfrak{p} = 0$ on $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$.*

Proof. Let $F \in \mathbf{H}_\mathfrak{n}(\mathbf{C}) = \mathbf{H}_\mathfrak{n} \otimes_\mathbf{Z} \mathbf{C}$ be an eigenform for $(\theta_\mathfrak{p})_\mathfrak{p}$ with eigenvalues $(\lambda_\mathfrak{p})_\mathfrak{p}$. It is well-known, and follows from elementary estimates on the L-function of F, that $\sum_{\deg \mathfrak{p} \leq d} \lambda_\mathfrak{p} = 0$ if $d > \deg(\mathfrak{n}) - 3$ (details can be found in Section 2.1.2 of [1] for instance). Since $\mathfrak{n}$ is prime, there exists a basis of $\mathbf{H}_\mathfrak{n}(\mathbf{C})$ consisting of normalized eigenforms for $(\theta_\mathfrak{p})_\mathfrak{p}$. Hence we have $\sum_{\deg \mathfrak{p} \leq d} \theta_\mathfrak{p} = 0$ if $d > \deg(\mathfrak{n}) - 3$. An equivalent formulation is: $\sum_{\deg \mathfrak{p} \leq \deg(\mathfrak{n})-2} \theta_\mathfrak{p} = 0$ and $\sum_{\deg \mathfrak{p} = d} \theta_\mathfrak{p} = 0$ if $d \geq \deg(\mathfrak{n}) - 1$. This completes the proof. □

Therefore, from the theory of automorphic forms, we know that certain elements of $\mathbf{I}$ given in Theorem 1.1 are zero on $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$, because so they are on $\mathbf{H}_\mathfrak{n}$: this is the case for $\sum_{\deg \mathfrak{p} = d} T_\mathfrak{p}$ if $\mathfrak{n}$ is prime and $d \geq \deg(\mathfrak{n}) - 1$.

It is now natural to ask whether some elements of $\mathbf{I}$ in Theorem 1.1 can act nontrivially on $\mathbf{H}_{\mathfrak{n}}$ and be zero in $\mathbf{T}'$ (i.e. in characteristic p). We suggest that this happens.

Conjecture 6.8. *Assume $\mathfrak{n}$ is prime. We have the following relations among Hecke operators on $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$:*

(1) *If $\mathfrak{n}$ has degree 4, then $\sum_{\deg \mathfrak{p}\leq 1} T_{\mathfrak{p}} = 0$.*
(2) *If $\mathfrak{n}$ has degree ≥ 4, then $\sum_{\deg \mathfrak{p}=\deg(\mathfrak{n})-2} T_{\mathfrak{p}} = 0$.*

We have numerical evidences supporting the conjecture. We computed Hecke operators on $\mathbf{H}_{\mathfrak{n}}/p\mathbf{H}_{\mathfrak{n}}$, for $\mathfrak{n}$ prime, using Teitelbaum's modular symbols for $\mathbf{F}_q(T)$ [18, 3]. The first relation has been checked for $q \in \{2, 3, 5\}$ and the second one for all primes $\mathfrak{n}$ of degree 5 and 6 in $\mathbf{F}_2[T]$. Note that, when $\deg \mathfrak{n} = 4$, both relations are equivalent: indeed, we have $\sum_{\deg \mathfrak{p}\leq 2} \theta_{\mathfrak{p}} = 0$ (see proof of Lemma 6.7).

Conjecture 6.8 predicts that some elements of $\mathbf{I}$ would be zero in $\mathbf{T}'$ but may be nonzero on the automorphic level, more precisely:

- $\sum_{\deg \mathfrak{p}\leq 1} T_{\mathfrak{p}} = 0$ in $\mathbf{T}'_{2,1}(\Gamma_0(\mathfrak{n}))$ for $\mathfrak{n}$ prime of degree 4;
- $\sum_{\deg \mathfrak{p}=\deg(\mathfrak{n})-2} T_{\mathfrak{p}} = 0$ in $\mathbf{T}'_{2,1}(\Gamma_0(\mathfrak{n}))$ for $\mathfrak{n}$ prime of degree ≥ 4.

In the next paragraph, we are interested in the reverse problem: finding nonzero elements in the ideal $\mathbf{I}$.

6.4. **Nonzero elements in the annihilator.** The following conjecture suggests that, in general, the Hecke annihilator $\mathbf{I}$ of b_1 is nonzero.

Conjecture 6.9. *Assume $\mathfrak{n}$ is prime of degree ≥ 5. Then $\sum_{\deg \mathfrak{p}\leq 1} T_{\mathfrak{p}} \in \mathbf{I}$ is nonzero as an endomorphism of $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$. In particular, the map*

$$\begin{array}{rccl} u\colon \mathbf{T}' & \longrightarrow & \mathrm{Hom}_{\mathbf{C}_\infty}(M^2_{2,1}(\Gamma_0(\mathfrak{n})), \mathbf{C}_\infty) \\ s & \longmapsto & b_1 s \end{array}$$

is not surjective.

The last statement follows from Lemma 6.2. As in Section 6.3, we were able to compute the action of $\sum_{\deg \mathfrak{p}\leq 1} T_{\mathfrak{p}}$ on $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$ on some examples. We checked Conjecture 6.9 for all primes $\mathfrak{n}$ of degree 5, 6, 7 and 8 in $\mathbf{F}_2[T]$ and $\mathbf{F}_3[T]$.

7. Proof and applications of Theorem 1.2

7.1. Explicit version of Theorem 1.2.

***Notation* 7.1.** We call a *decomposition of* $c \in \mathbf{N}$ a tuple $\underline{c} = (c_0, \dots, c_d)$ such that $c = \sum_{j=0}^{d} c_j q^j$ and $0 \leq c_j < q$ for any $j \in \{0, \dots, d\}$, for some $d \geq 0$. The *length of* $\underline{c}$ is $d + 1$. Note that we do not require $c_d \neq 0$. The base q expansion gives a decomposition of c. By putting zeros at the end of any decomposition of c, we obtain decompositions of larger length.

If $\underline{i} = (i_0, \ldots, i_d)$ is a decomposition of $i \geq 0$, let

$$l(i) = \sum_{P \in A_{d+}} C(P)^{\underline{i}} \sum_{\substack{Q|P, Q \in A_{1+} \\ (Q)+\mathfrak{n}=A}} Q^{k-1} \quad \in A.$$

We prove Theorem 1.2 by establishing the following explicit version.

Theorem 7.2. *Assume q is a prime.*

(1) *Suppose $m = 0$. Let $n = c/(q-1) \in \mathscr{S}$. We fix a decomposition $(c_0, \ldots, c_d)$ of c of length $d+1$ (therefore $c_0 + \ldots + c_d = q-1$). Let*

$$t_{c_0,\ldots,c_d} = (-1)^d \binom{q-2}{c_0-1, c_1, \ldots, c_d}^{-1} \sum_{P \in A_{d+}} \left\{ {P \atop q^{d+1}-c} \right\} T_P \quad \in \mathbf{R}_A.$$

Then, for any k and $\mathfrak{n}$, we have $b_n = b_1 t_{c_0,\ldots,c_d}$ in the dual space of $M_{k,0}(\Gamma_0(\mathfrak{n}))$.

(2) *Suppose $m = 1$. Let $n = c/(q-1) \in \mathscr{S}$. We fix a decomposition $(c_0, \ldots, c_d)$ of c of length $d+1$ (therefore $c_0 + \ldots + c_d = q-1$) with $c_d \neq q-1$. Let*

$$t'_{c_0,\ldots,c_d} = (-1)^d \binom{q-1}{c_0, \ldots, c_d}^{-1} \sum_{P \in A_{d+}} \left\{ {P \atop q^{d+1}-1-c} \right\} T_P \quad \in \mathbf{R}_A.$$

Then, for any k and $\mathfrak{n}$, we have

$$b_n = b_1 t'_{c_0,\ldots,c_d} + (-1)^{d+1} \binom{q-1}{c_0, \ldots, c_d}^{-1} l(q^{d+1}-1-c) a_1$$

in the dual space of $M_{k,1}(\Gamma_0(\mathfrak{n}))$.

(3) *Assume $m = 1$. Let $d \geq 1$*

$$t_d = (-1)^d \sum_{P \in A_{d+}} \left(\left\{ {P \atop q^d-1} \right\} - \sum_{i=0}^{d-1} \left\{ {P \atop q^d-1-(q-1)q^i} \right\} \right) T_P \quad \in \mathbf{R}_A.$$

Then, for any k and $\mathfrak{n}$, we have

$$b_{q^d} = b_1 t_d + (-1)^d \left(-l(q^d-1) + \sum_{i=0}^{d-1} l(q^d-1-(q-1)q^i) \right) a_1 \tag{12}$$

in the dual space of $M_{k,1}(\Gamma_0(\mathfrak{n}))$.

***Remark* 7.3.** (1) Since q is prime and $\sum_{j=0}^d c_j = q-1$, the multinomial coefficients $\binom{q-1}{c_0,\ldots,c_d}$ and $\binom{q-2}{c_0-1,c_1,\ldots,c_d}$ are nonzero in $\mathbf{F}_p$, by Lucas's theorem, hence invertible.

(2) On doubly cuspidal forms, a_1 vanishes and the expressions of Theorem 7.2 simplify and provide Theorem 1.2. Moreover, since $b_1\mathbf{T}'$ is contained in the $\mathbf{C}_\infty$-vector space spanned by b_n for $n \in \mathscr{S}$ (Corollary 5.8), we get the equality provided that q is prime and $m \in \{0, 1\}$.

(3) For a given $n \in \mathscr{S}$, we get infinitely many expressions $s_n \in \mathbf{T}'$ such that $b_n = b_1 s_n$. The reason is that, in the first two items of Theorem 7.2, *any* decomposition of $c = (q-1)n$ gives rise to a formula for $s_n \in \mathbf{T}'$ satisfying the desired property. More generally, any element of $s_n + \mathbf{I}$ would satisfy the same property.

(4) The primality assumption on q is not essential: it is required to ensure that the multinomial coefficient $\binom{q-1}{c_0,\dots,c_d}$ for $m = 1$ (resp. $\binom{q-2}{c_0-1,c_1,\dots,c_d}$ for $m = 0$) is nonzero in $\mathbf{F}_p$. Hence, the assumption is unnecessary in (12). Moreover, if q is not a prime, the statement of Theorem 7.2 remains true for n in a subset of $\mathscr{S}$.

Before proving Theorem 7.2, we give an example.

Example 7.4 ($d = 1$)**.** We put

$$s_n = -\binom{q-1}{n-1}^{-1} \sum_{P \in A_{1+}} P^{n-1} T_P \qquad \text{for} \qquad 1 \le n \le q-1$$
$$s_q = -\sum_{P \in A_{1+}} (P^{q-1} - 1) T_P.$$

Then $b_n(f) = b_1(s_n(f))$ for all $f \in M^2_{k,1}(\Gamma_0(\mathfrak{n}))$ and $1 \le n \le q$. This is valid for q a power of a prime, by Remark 7.3 and Lucas's theorem. Using these formulas, we can recover the first q coefficients of any Hecke eigenform f in $M^2_{k,1}(\Gamma_0(\mathfrak{n}))$ in terms of $b_1(f)$ and the eigenvalues.

Proof of Theorem 7.2. (1) Assume that the type m is 0. We put $n_0 = c_0 - 1$, $n_1 = c_1, \dots, n_d = c_d$, so that $n_0 + \ldots + n_d = q - 2$. By Corollary 5.10, $a_{q-1}(\Theta_d(q-1-n_0, \dots, q-1-n_{d-1}))$ is

$$\sum_{\underline{r}} \binom{q-2}{\underline{r}} S_d(r_0 + q - 1 - n_0, \dots, r_{d-1} + q - 1 - n_{d-1}) a_{1+r_0+r_1 q + \ldots + r_d q^d}$$

where $\underline{r} = (r_0, \dots, r_d)$ satisfies $r_0 + \ldots + r_d = q - 2$. From $n_i \ge -1$, we get $0 \le r_i + q - 1 - n_i \le 2(q-1)$ for all i. We can thus evaluate the sum $S_d(r_0 + q - 1 - n_0, \dots, r_{d-1} + q - 1 - n_{d-1})$ by Proposition 3.5: it is nonzero only if $\underline{r}$ is such that $r_i = n_i$ or $q - 1 + n_i$, for all $i \in \{0, \dots d-1\}$. Since $r_i \le q - 2$, we have $r_0 = n_0, \dots, r_{d-1} = n_{d-1}$ and by Proposition 3.5,

$$a_{q-1}(\Theta_d(q-1-n_0, \dots, q-1-n_{d-1})) = \binom{q-2}{\underline{n}} (-1)^d a_{1+n_0+n_1 q + \ldots + n_d q^d}.$$

Finally, $a_{1+n_0+\ldots+n_d q^d} = a_{n(q-1)} = b_n$ and the conclusion follows.

(2) Assume that the type m is 1. Since $q^{d+1} - 1 - c$ has base q expansion $\sum_{j=0}^d (q-1-c_j) q^j$, we have

$$\sum_{P \in A_{d+}} \left\{ {P \atop q^{d+1}-1-c} \right\} T_P = \sum_{P \in A_{d+}} C_{P,0}^{q-1-c_0} \ldots C_{P,d-1}^{q-1-c_{d-1}} T_P$$
$$= \Theta_d(q-1-c_0, \dots, q-1-c_{d-1}).$$

By Corollary 5.10, $b_1(\Theta_d(q-1-c_0,\dots,q-1-c_{d-1}))$ is

$$\sum_{\underline{r}} \binom{q-1}{\underline{r}} S_d(r_0+q-1-c_0,\dots,r_{d-1}+q-1-c_{d-1}) a_{1+r_0+r_1q+\dots+r_dq^d} + l(q^{d+1}-1-c)a_1$$

with $\underline{r}=(r_0,\dots,r_d)$ such that $r_0+\dots+r_d=q-1$. From $c_i\geq 0$ and $0\leq r_i\leq q-1$, we get $0\leq r_i+q-1-c_i\leq 2(q-1)$. Thus the sum $S_d(r_0+q-1-c_0,\dots,r_{d-1}+q-1-c_{d-1})$ can be evaluated thanks to Proposition 3.5: it is nonzero if and only if $r_i=c_i$ or $q-1+c_i$ for all $0\leq i\leq d-1$.

Suppose there exists $k\in\{0,\dots,d-1\}$ with $r_k=q-1+c_k$. Then, according to the previous remarks, we have

$$q-1-r_d=\sum_{j=0}^{d-1} r_j = q-1+c_k+\sum_{j=0,j\neq k}^{d-1} r_j \geq q-1+\sum_{j=0}^{d-1} c_j = 2(q-1)-c_d$$

hence $0\leq q-1-c_d\leq -r_d$. This implies $r_d=0$, thus $c_d=q-1$, which is impossible. Therefore, we have $r_j=c_j$ for $0\leq j\leq d-1$ and $r_d=c_d$ as a consequence. Proposition 3.5 then provides

$$b_1(\Theta_d(q-1-c_0,\dots,q-1-c_{d-1})) = a_{1+c_0+c_1q+\dots+c_dq^d} + (-1)^d \binom{q-1}{c_0,\dots,c_d}^{-1} l(q^{d+1}-1-c)a_1.$$

Finally, $a_{1+c_0+c_1q+\dots+c_dq^d}=a_{1+n(q-1)}=b_n$, thus the statement is proved.

(3) Assume that the type m is 1. We first compute $b_1(\Theta_d(q-1,\dots,q-1))$. According to Corollary 5.10, it is

$$\sum_{\substack{\underline{r}=(r_0,\dots,r_d)\\ r_0+\dots+r_d=q-1}} \binom{q-1}{\underline{r}} S_d(r_0+q-1,\dots,r_{d-1}+q-1) a_{1+r_0+r_1q+\dots+r_dq^d} + l(q^{d+1}-1)a_1.$$

By Proposition 3.5, the sum $S_d(r_0+q-1,\dots,r_{d-1}+q-1)$ is nonzero if and only if $r_i=0$ or $q-1$ for all $0\leq i\leq d-1$. This means that $(r_0,\dots,r_{d-1})$ is one of the following:

$$(q-1,0,\dots,0)\,,(0,q-1,0,\dots,0)\,,\dots\,,(0,\dots,0,q-1)\,,(0,\dots,0).$$

Thus $b_1(\Theta_d(q-1,\dots,q-1))$ equals

$$(-1)^d\left(a_{1+(q-1)}+\dots+a_{1+(q-1)q^{d-1}}+a_{1+(q-1)q^d}\right)+l(q^{d+1}-1)a_1. \tag{13}$$

Next, we compute $b_1(\Theta(q-1,\ldots,0,\ldots,q-1))$, the only zero term being at the $(j+1)$th position $(0\leq j\leq d-1)$. From Corollary 5.10, it is

$$\sum_{\substack{\underline{r}=(r_0,\ldots,r_d)\\ r_0+\ldots+r_d=q-1}}\binom{q-1}{\underline{r}}S_d(r_0+q-1,\ldots,r_j,\ldots,r_{d-1}+q-1)a_{1+r_0+r_1q+\ldots+r_dq^d}$$
$$+\,l(q^{d+1}-1-(q-1)q^j)a_1.$$

Again by Proposition 3.5, the sum is only over $\underline{r}$ satisfying the following two properties:

$$r_i=0 \text{ or } q-1 \quad \text{for all } i\in\{0,\ldots,d-1\}, i\neq j$$
$$r_j=q-1 \text{ or } 2(q-1).$$

Since $r_0+\ldots+r_d=q-1$, we have necessarily $r_j=q-1$, $r_i=0$ for all $i\neq j$ and $r_d=0$. Then

$$b_1(\Theta(q-1,\ldots,0,\ldots,q-1))=(-1)^d a_{1+(q-1)q^j}+l(q^{d+1}-1-(q-1)q^j)a_1 \tag{14}$$

Combining (13) and (14), we get the claim. □

7.2. Applications.

Theorem 1.2 has the following straightforward consequence.

Corollary 7.5. *Under the assumptions of Theorem 1.2, if f is a Hecke eigenform with $b_n(f)\neq 0$ for some $n\in\mathscr{S}$, then $b_1(f)\neq 0$.*

In particular, in Proposition 6.5, one can replace the assumption $b_1(f)\neq 0$ by: there exists $n\in\mathscr{S}$ such that $b_n(f)\neq 0$.

We now provide multiplicity one statements in certains spaces of Drinfeld modular forms.

Lemma 7.6. (1) *Let $d=\dim M_{k,m}(\mathrm{GL}_2(A))$. The $\mathbf{C}_\infty$-linear map*

$$\begin{array}{rcl} M_{k,m}(\mathrm{GL}_2(A)) & \longrightarrow & \mathbf{C}_\infty^d\\ f & \longmapsto & (b_0(f),\ldots,b_{d-1}(f))\end{array}$$

is an isomorphism.

(2) *Let $d=\dim M^2_{2,1}(\Gamma_0(\mathfrak{n}))$. The $\mathbf{C}_\infty$-linear map*

$$\begin{array}{rcl} M^2_{2,1}(\Gamma_0(\mathfrak{n})) & \longrightarrow & \mathbf{C}_\infty^d\\ f & \longmapsto & (b_1(f),\ldots,b_d(f))\end{array}$$

is an isomorphism.

Proof. The first assertion follows readily from a formula relating, for a nonzero $f\in M_{k,m}(\mathrm{GL}_2(A))$, the orders of vanishing of f at elliptic, non-elliptic points and the cusp infinity of $\mathrm{GL}_2(A)$ (see Formula (5.14) in Gekeler's paper [8]). The second assertion is a consequence of the cusp infinity not being a Weierstrass point on the Drinfeld modular curve attached to $\Gamma_0(\mathfrak{n})$ (see Proposition 4.47 of [1] for details). □

Theorem 7.7. *Let M be one of the following spaces of Drinfeld modular forms:*

(1) $M^1_{k,0}(\mathrm{GL}_2(A))$ *with* $k < (q+1)^2(q-1)$
(2) $M^2_{k,1}(\mathrm{GL}_2(A))$ *with* $k < q^2(q+1)$
(3) $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$ *with* $\mathfrak{n}$ *prime of degree* 3.

Then:

- *Any eigenform in M for the operators $(T_\mathfrak{p})_{\deg \mathfrak{p}=1}$ is characterized in the space M by its eigenvalues, up to a multiplicative constant.*
- *The map $u\colon \mathbf{T}' \to \mathrm{Hom}_{\mathbf{C}_\infty}(M, \mathbf{C}_\infty)$ is an isomorphism.*

Proof. Consider the first two cases for M. By the cuspidality (resp. doubly cuspidality) condition and the assumption on the type, we have $b_0(f) = a_m(f) = 0$. Therefore, any function $f \in M$ is determined, in the space M, by its coefficients $b_1(f), \ldots, b_{d-1}(f)$, according to Lemma 7.6. Now, if f is an eigenform for $(T_\mathfrak{p})_{\deg \mathfrak{p}=1}$, we know that $b_1(f), \ldots, b_q(f)$ are determined by the eigenvalues (up to a multiplicative constant), thanks to Example 7.4. Recall that the dimension of $M_{k,m}(\mathrm{GL}_2(A))$ is $d = \lfloor (k-(q+1)m)/(q^2-1) \rfloor + 1$ (this follows from Gekeler's formula (5.14) in [8]). Here, the assumptions on the weight ensure that $d-1 \leq q$. The conclusion follows.

The proof of the third case is similar, except that the dimension of M is q. Indeed, this dimension coincides with the genus of the Drinfeld modular curve attached to $\Gamma_0(\mathfrak{n})$, for which we know closed formulas thanks to Gekeler (see (2.10.3) in [15] and [6, 7]).

For the bijectivity of u, we need only to prove the surjectivity by Lemma 6.2. Consider the first two cases of M. As before, M has dimension $d-1 \leq q$. Moreover, the image of u contains $b_1, \ldots, b_{d-1}$ (by Theorem 7.2) which are linearly independent (by Lemma 7.6), hence the conclusion. The proof of the third case is similar. □

As a corollary, we get that the dimension of the $\mathbf{C}_\infty$-algebra $\mathbf{T}'$ coincides with the dimension of the space of Drinfeld modular forms M, for M as in the statement.

7.3. **Comment on A-structures.** Although we worked with $\mathbf{C}_\infty$-structures, most of the results of this paper could be transferred to the ring A. For instance, one could work with the subspace $M^2_{k,m}(\Gamma_0(\mathfrak{n}); A) \subset M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ consisting of modular forms with expansion in $A[[t]]$ and the Hecke algebra $\mathbf{T}'_A$ spanned over A by Hecke operators. Using Proposition 5.2, one may check that the map

$$\mathbf{T}'_A \to \mathrm{Hom}_A(M^2_{k,m}(\Gamma_0(\mathfrak{n}); A), A)$$

induced by $s \mapsto b_1 s$, is well-defined. We expect that $M^2_{k,m}(\Gamma_0(\mathfrak{n}); A)$ is a A-structure of $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ (i.e. there exists a basis of $M^2_{k,m}(\Gamma_0(\mathfrak{n}))$ consisting of modular forms with coefficients in A). However, a general theory of such algebraic Drinfeld modular forms is still missing in the literature. Some instances of such a theory can be found in [12] (Section 2, for $M_{k,m}(\mathrm{GL}_2(A))$) and [1] (Section 4.2, for $M^2_{2,1}(\Gamma_0(\mathfrak{n}))$).

8. Coefficients of h

We use Theorem 7.2 to compute explicitly some coefficients of Gekeler's Drinfeld modular form h, defined in [8]. Recall that h has weight $q+1$ and type 1 for $\mathrm{GL}_2(A)$. It is defined as a certain Poincaré series and is also a $(q-1)$th root of the Drinfeld discriminant form Δ. Moreover, it is a cuspidal Hecke eigenform with $T_{\mathfrak{p}}h = h$ for any $\mathfrak{p}$ (Corollary 7.6 in [8] with a different normalization of Hecke operators). The first coefficients of h are $a_1(h) = -1$ and $b_1(h) = a_q(h) = \begin{cases} 0 & \text{if } q > 2 \\ 1 & \text{if } q = 2 \end{cases}$.

Proposition 8.1. *For P in A, let $\sigma_P = \sum_{Q|P, Q\in A_{1+}} Q^q$.*

(1) *Assume q is a prime > 2. Let $c \in \mathbf{N}$ such that $c = \sum_{j=0}^d c_j q^j$ with $0 \le c_j < q$, $\sum_{j=0}^d c_j = q-1$ and $c_d \ne q-1$ (we do not necessarily assume $c_d \ne 0$). Then*

$$b_{\frac{c}{q-1}}(h) = (-1)^d \binom{q-1}{c_0,\dots,c_d}^{-1} \sum_{P\in A_{d+}} \left\{ {P \atop q^{d+1}-1-c} \right\} \sigma_P. \tag{15}$$

Moreover, for $d \ge 0$,

$$b_{q^d}(h) = (-1)^{d+1} \sum_{P\in A_{d+}} \left(-\left\{ {P \atop q^d-1} \right\} + \sum_{i=0}^{d-1} \left\{ {P \atop q^d-1-(q-1)q^i} \right\} \right) \sigma_P. \tag{16}$$

(2) *Assume $q = 2$. Then for every $d \ge 0$, one has*

$$b_{2^d}(h) = (-1)^d \sum_{P\in A_{d+}} \left(-\left\{ {P \atop 2^d-1} \right\} + \sum_{i=0}^{d-1} \left\{ {P \atop 2^d-1-2^i} \right\} \right) (1+\sigma_P).$$

***Remark* 8.2.** We recover that the corresponding coefficients of h are polynomials in $T^q - T$ with coefficients in $\mathbf{F}_q$ (indeed, they are elements of A which are invariant under $T \mapsto T + c$ for $c \in \mathbf{F}_q$). More generally, Gekeler proved that this property holds for any coefficient of h (Theorem 2.4 of [9]).

Taking $d = 1$ in Proposition 8.1, one can recover the first q coefficients of h. If q is a prime > 2, then $b_i(h) = 0$ if $1 \le i \le q-2$, $b_{q-1}(h) = -1$ and $b_q(h) = T^q - T$. They can also be obtained from the Taylor series $h = -tU_1^{-1} + o(t^{1+(q-1)(q^3-q^2)})$ with $U_1 = 1 - t^{(q-1)^2} + (T^q - T)t^{(q-1)q}$ (see Corollary 10.4 in [8]).

For $i \in \mathbf{N}$, let $[i] = T^{q^i} - T$. Using congruences and estimates on the degree of coefficients of h, Gekeler proved that for any $d \ge 1$,

$$b_{q^d}(h) = \begin{cases} [d] & \text{if } q > 2 \\ 1 + [d] & \text{if } q = 2 \end{cases} \tag{17}$$

(see Corollary 2.6 of [9]; note that his b_i denotes our $-b_i$). Equation (16) thus provides an alternative formula for $b_{q^d}(h)$. We have not been able to recover Gekeler's formulas from (16). Hence we derive some arithmetic identities in $\mathbf{F}_q[T]$ which may be nontrivial and of some interest.

Corollary 8.3. *Let q a prime > 2 and $d \geq 1$.*

(1)

$$[d] = (-1)^{d+1} \sum_{P \in A_{d+}} \left(-\left\{ {P \atop q^d-1} \right\} + \sum_{i=0}^{d-1} \left\{ {P \atop q^d-1-(q-1)q^i} \right\} \right) \sigma_P.$$

(2) *For $0 \leq i \leq d-1$,*

$$(-1)^d [i] = \sum_{P \in A_{d+}} \left\{ {P \atop q^d-1-(q-1)q^i} \right\} \sigma_P.$$

(3)

$$(-1)^d \sum_{i=1}^{d} [i] = \sum_{P \in A_{d+}} \left\{ {P \atop q^d-1} \right\} \sigma_P.$$

Proof. The first one follows from (16) and (17). For the second one, we first apply (15) to $c = (q-1)q^i$ with $0 \leq i \leq d-1$ and get

$$(-1)^d b_{q^i}(h) = \sum_{P \in A_{d+}} \left\{ {P \atop q^{d+1}-1-(q-1)q^i} \right\} \sigma_P = \sum_{P \in A_{d+}} \left\{ {P \atop q^d-1-(q-1)q^i} \right\} \sigma_P$$

where the last equality follow from $q^{d+1}-1-(q-1)q^i = (q-1)q^d + \sum_{j=0}^{d-1} (q-1)q^j - (q-1)q^i$ and $\deg P = d$. With (17), we get the second claim. The third one is obtained by combining the first two identities. $\square$

In Table 1, we provide further examples of coefficients of h from Proposition 8.1. Observe that when i is even (resp. odd), $b_i(h)$ is an even (resp. odd) polynomial in $[1] = T^q - T$. This is more generally true for any coefficient when $q = 3$: it follows from the coefficients of h being balanced, a property established by Gekeler (Theorem 2.4 of [9]). Note that, in our table, the constant term is -1 when i is even: we wonder if such a statement holds more generally.

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TABLE 1. $q = 3$, $d \leq 4$

i	$b_i(h)$
1	0
2	-1
3	$[1]$
5	$-[1]$
6	$-[1]^2 - 1$
9	$[2] = [1]^3 + [1]$
14	$[1]^4 - 1$
15	$[1]^5 - [1]^3 + [1]$
18	$-[1]^6 + [1]^4 - [1]^2 - 1$
27	$[3] = [1]^9 + [1]^3 + [1]$
41	$-[1]^{13} + [1]^9 - [1]^7 - [1]$
42	$-[1]^{14} + [1]^{12} - [1]^{10} - [1]^8 - [1]^2 - 1$
45	$[1]^{15} - [1]^{13} + [1]^{11} - [1]^9 + [1]^3 + [1]$
54	$-[1]^{18} + [1]^{12} + [1]^{10} - [1]^6 + [1]^4 - [1]^2 - 1$
81	$[4] = [1]^{27} + [1]^9 + [1]^3 + [1]$

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