

LÉVY AREA FOR GAUSSIAN PROCESSES: A DOUBLE WIENER–ITÔ INTEGRAL APPROACH

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ABSTRACT. Let $\{X_1(t)\}_{0 \leq t \leq 1}$ and $\{X_2(t)\}_{0 \leq t \leq 1}$ be two independent continuous centered Gaussian processes with covariance functions R_1 and R_2 . This paper shows that if the covariance functions are of finite p -variation and q -variation respectively and such that $p^{-1} + q^{-1} > 1$, then the Lévy area can be defined as a double Wiener–Itô integral with respect to an isonormal Gaussian process induced by X_1 and X_2 . Moreover, some properties of the characteristic function of that generalised Lévy area are studied.

1. INTRODUCTION

Let $\{W_1(t) \mid 0 \leq t \leq 1\}$ and $\{W_2(t) \mid 0 \leq t \leq 1\}$ be two independent standard Wiener processes defined in a probability space (Ω, \mathcal{F}, P) , and let A be the area included by the curve

$$x = W_1(t), \quad y = W_2(t) \quad 0 \leq t \leq 1$$

and its chord. This random variable was first introduced by Lévy in [8], where it is described by means of stochastic integrals as

$$A = \int_0^1 W_1(t) dW_2(t) - \int_0^1 W_2(t) dW_1(t).$$

Lévy [8] also computed its characteristic function, which is

$$(1) \quad \varphi(t) := \mathbb{E}[e^{itA}] = \frac{1}{\cosh(t)}, \quad t \in \mathbb{R}.$$

It is easy to show that A has the law of an element of the homogeneous second Wiener chaos generated by a Brownian motion. This can be proved directly due to the fact that the elements of the homogeneous second Wiener chaos have a very particular

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characteristic function (see Janson [7, ch. 6]), and one can realize that (1) has this form thanks to the factorisation

$$(2) \quad \cosh(z) = \prod_{n \geq 0} \left(1 + \frac{4z^2}{\pi^2(2n+1)^2} \right) = \prod'_{n \in \mathbb{Z}} (1 - 2iz\alpha_n)^{1/2} e^{iz\alpha_n},$$

where $\alpha_n = (\pi(2n+1))^{-1}$ and \prod' means that each factor is repeated twice. Alternatively, also from (2), the law of A can be given as the law of a double Wiener–Itô integral

$$I_2^B(f) := \iint_{[0,1]^2} f(s,t) dB(s) dB(t)$$

for an arbitrary Brownian motion $B = \{B_t \mid t \geq 0\}$, where the kernel f is obtained in the following way: consider an orthonormal basis of $L^2([0,1])$, which for convenience we write as $\{\phi_n, \psi_n \mid n \in \mathbb{Z}\}$, and define

$$f(s,t) = \sum_{n \in \mathbb{Z}} \alpha_n \phi_n(s) \phi_n(t) + \sum_{n \in \mathbb{Z}} \alpha_n \psi_n(s) \psi_n(t).$$

Then

$$I_2^B(f) = \sum_{n \in \mathbb{Z}} 2\alpha_n H_2(I_1^B(\phi_n)) + \sum_{n \in \mathbb{Z}} 2\alpha_n H_2(I_1^B(\psi_n)),$$

where $I_1^B(\phi) = \int_0^1 \phi(s) dB(s)$ and $H_2(x) = (x^2 - 1)/2$ is the Hermite polynomial of order two. Observe that $2H_2(I_1^B(\phi_n))$ and $2H_2(I_1^B(\psi_n))$ are all independent centered $\chi^2(1)$ random variables and thus $A \stackrel{\text{law}}{=} I_2^B(f)$.

However, it is more difficult to get a strong representation of A as a double Wiener–Itô integral with respect to the original Brownian motions W_1 and W_2 . To this end, it is necessary to rely on the construction of multiple Itô–Wiener integrals for a general white noise, see for instance Nualart [10, pages 8 and 14]. There, both Brownian motions W_1 and W_2 are embedded in a Gaussian noise W on $[0,1] \times \{1,2\}$. For $h \in L^2([0,1] \times \{1,2\}, dt \otimes \text{Card}) \cong L^2([0,1], \mathbb{R}^2)$, where Card is the counting measure, we have

$$\int_{[0,1] \times \{1,2\}} h dW = \int_0^1 h(s,1) dW_1(s) + \int_0^1 h(s,2) dW_2(s).$$

Moreover, for $f \in L^2(([0,1] \times \{1,2\})^2)$ symmetric,

$$(3) \quad \begin{aligned} I_2(f) &= \sum_{i,j=1}^2 \iint_{[0,1]^2} f((s,i), (t,j)) dW_i(s) dW_j(t) \\ &= 2 \sum_{i,j=1}^2 \int_0^1 \int_0^t f((s,i), (t,j)) dW_i(s) dW_j(t), \end{aligned}$$

see Nualart [10, p. 23]. For the sake of simplicity, we will indistinctly use $f_{ij}(s, t)$ for $f((s, i), (t, j))$ from now on. Define

$$(4) \quad f_{ij}^{\mathcal{L}}(s, t) = \begin{cases} 0, & \text{if } i = j \\ \frac{1}{2}(\mathbf{1}_{T_1}(s, t) - \mathbf{1}_{T_2}(s, t)), & \text{if } i = 1, j = 2 \\ \frac{1}{2}(\mathbf{1}_{T_2}(s, t) - \mathbf{1}_{T_1}(s, t)), & \text{if } i = 2, j = 1, \end{cases}$$

where $\mathbf{1}_C$ is the indicator function of the set C and

$$T_1 := \{(s, t) \in [0, 1]^2 \mid s < t\}, \quad T_2 := \{(s, t) \in [0, 1]^2 \mid s > t\}.$$

Note that $f^{\mathcal{L}}$ is symmetric, and from (3) and (1) it follows that $A \stackrel{\text{a.s.}}{=} I_2(f^{\mathcal{L}})$. We will refer to (4) as the Lévy kernel.

The aim of this paper is to extend the above strong construction in order to define the Lévy area for general Gaussian processes under minimal conditions of their covariance functions, and to study its characteristic function. We will consider two independent continuous centered Gaussian processes $\{X_1(t) \mid 0 \leq t \leq 1\}$ and $\{X_2(t) \mid 0 \leq t \leq 1\}$ with (continuous) covariance functions R_1 and R_2 and we prove that if the covariance functions are of finite p -variation and q -variation respectively and such that $p^{-1} + q^{-1} > 1$, then the Lévy area can be defined as an element of the second Wiener chaos generated by X_1 and X_2 . Such a kind of results have been obtained (for the non-antisymmetrized Lévy area $\int_0^1 X_1(t) dX_2(t)$) in the context of rough path analysis by Fritz and Victoir, [5, 4], but, as far as we know, in such generality they are new for classical Gaussian processes. Our results applied to two fractional Brownian motions of Hurst parameter H and H' states that the Lévy area can be defined if $H + H' > 1/2$. In particular, if $H = H'$, then the condition is $H \in (1/4, 1)$ which is a known result (see Neuenkirch *et al.* [9] and the references therein) but we present an alternative point of view based in the Huang and Cambanis [6] approach to stochastic integration for Gaussian processes. Our results also extends the ones given by Bardina and Tudor [1] where the integral $\int_0^1 X_t^H dX_t^{H'}$ is defined using Malliavin calculus techniques for $H \in (0, 1)$ and $H' > 1/2$.

The paper is organized as follows. We first introduce the general framework of the isonormal Gaussian processes, and following the scheme of Huang and Cambanis [6], we associate an isonormal Gaussian process to a pair of independent Gaussian processes. We also give here a definition of a generalised Lévy area. In the next section we derive the conditions on the covariance functions so that $f^{\mathcal{L}}$ generates a Lévy area. As an example we explore what happens with two fractional Brownian motion (fBm) with the same covariance function, and the case with two different covariance function. This later case allows us to let one of the processes be as irregular as desired, that is no low bounds for its Hurst parameter is assumed, provided that the other one is regular enough. Finally, we discuss about the representation of the characteristic function of a double Wiener-Itô integral in terms of a Carleman–Fredholm determinant, that we apply to compute the characteristic function of A . Under a further condition of symmetry over the stochastic processes we will show that the characteristic function

of a generalised Lévy area has a Carleman-Fredholm determinant with symmetric poles and even multiplicity.

2. ISONORMAL GAUSSIAN PROCESSES

The framework where (Gaussian) multiple integrals are defined is the one of isonormal Gaussian processes. Main reference are Nualart [10], and Peccati and Taqqu [11]. The more general abstract context of Gaussian Hilbert spaces developed by Janson [7] is also very useful and interesting.

Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$. An isonormal Gaussian process $\{X(f) \mid f \in H\}$ is a centered Gaussian family of random variables such that $\mathbb{E}[X(f)X(g)] = \langle f, g \rangle_H$. It is well known that the construction of the multiple Wiener–Itô integrals with respect to a Brownian motion can be transferred to isonormal Gaussian processes; see Nualart [10, pages 7 and 8], or Peccati and Taqqu [11, ch. 8.4]. In that general setup, $H^{\otimes n}$ (resp. $H^{\odot n}$) denotes the n th (Hilbert) tensor power of H (resp. the n th symmetric tensor power), and $I_n(f)$ for $f \in H^{\odot n}$ its n th multiple integral. For detailed constructions of that Hilbert spaces see Janson [7].

2.1. The isonormal Gaussian process associated with two Gaussian processes. In this section we describe how two ordinary Gaussian process can be imbedded into an isonormal Gaussian process. We extend Huang and Cambanis [6] approach, where that construction was done for one Gaussian process. Let $X_1 = \{X_1(t) \mid t \in [0, 1]\}$ and $X_2 = \{X_2(t) \mid t \in [0, 1]\}$ be two independent continuous centered Gaussian processes, both starting at zero, with (continuous) covariance function $R_1(s, t)$ and $R_2(s, t)$ respectively. Following Huang and Cambanis [6], let \mathcal{E} denote the set of step functions on $[0, 1]$

$$\phi(t) = \sum_{i=1}^n a_i \mathbf{1}_{(t_i, t_{i+1}]}(t) \quad a_i \in \mathbb{R}.$$

Associated with R_i , for $i = 1, 2$, we can construct the Hilbert space H_i which is the completion of \mathcal{E} under the inner product (with the convenient identifications):

$$\langle \phi_1, \phi_2 \rangle_{H_i} := \iint_{[0,1]^2} \phi_1(s) \phi_2(t) dR_i(s, t).$$

The above integral is defined so that

$$(5) \quad \iint_{(0,u] \times (0,v]} dR_i(s, t) = R_i(u, v).$$

Remark 2.1. Due to the continuity of the covariance functions the limits of integration in (5) might or might not be included in the integral without changing the result. For instance

$$\iint_{[0,u] \times [0,v]} dR_i(s, t) = \iint_{(0,u] \times (0,v]} dR_i(s, t).$$

In order to define an isonormal Gaussian process associated to both X_1 and X_2 , the set of appropriate elementary functions, \mathcal{E}_2 , are the ones that can be written as $f(t, i) = \delta_{1i}\phi_1(t) + \delta_{2i}\phi_2(t)$ for $\phi_1, \phi_2 \in \mathcal{E}$, where δ_{ij} is the Kronecker's delta. It is clear that on \mathcal{E}_2 we can consider the inner product (with the convenient identifications):

$$\begin{aligned} \langle f, g \rangle_{\mathcal{E}_2} &= \langle f(\cdot, 1), g(\cdot, 1) \rangle_{H_1} + \langle f(\cdot, 2), g(\cdot, 2) \rangle_{H_2} \\ &= \iint_{[0,1]^2} f(s, 1)g(t, 1)dR_1(s, t) + \iint_{[0,1]^2} f(s, 2)g(t, 2)dR_2(s, t). \end{aligned}$$

Let us call H the Hilbert space which is the completion of \mathcal{E}_2 with the above inner product. Next lemma characterises H ; its proof is straightforward.

Lemma 2.2. *Under the above notation $H \cong H_1 \oplus H_2$, where $H_1 \oplus H_2$ is the Hilbertian direct sum of H_1 and H_2 , that is the Hilbert space which consists in all ordered pairs $(x_1, x_2) \in H_1 \times H_2$ equipped with the inner product $\langle (x_1, x_2), (y_1, y_2) \rangle_{H_1 \oplus H_2} = \langle x_1, y_1 \rangle_{H_1} + \langle x_2, y_2 \rangle_{H_2}$.*

Now we are ready to construct the isonormal Gaussian process indexed by H which will represent the 2-dimensional process $\{(X_1(t), X_2(t)) \mid t \in [0, 1]\}$. From the independence of X_1 and X_2 , it turns out that $X : \mathcal{E}_2 \rightarrow L^2(\Omega, \mathcal{F}, P)$ defined by $X(f) := X_1(f(\cdot, 1)) + X_2(f(\cdot, 2))$ is an isometry which can be extended to H . Thus $X = \{X(f) \mid f \in H\}$ is an isonormal Gaussian process.

2.2. Generalised Lévy area. In the previous context, the generalised Lévy area will be an element of the second Wiener chaos with respect to the process X . Therefore we need to identify the elements of $H^{\otimes 2}$. Note that

$$(6) \quad H^{\otimes 2} \cong H_1^{\otimes 2} \oplus (H_1 \otimes H_2) \oplus (H_2 \otimes H_1) \oplus H_2^{\otimes 2},$$

which gives a very appropriate interpretation of the elements $f \in H^{\otimes 2}$ as 2 by 2 matrices with entries $f_{ij} \in H_i \otimes H_j$ for $i, j = 1, 2$. The above isometry also induces the decomposition

$$\langle f, g \rangle_{H^{\otimes 2}} = \sum_{i,j=1}^2 \langle f_{ij}, g_{ij} \rangle_{H_i \otimes H_j}.$$

The double Itô–Wiener integral, $I_2(\cdot)$, is an isometry between $H^{\odot 2}$ and the second Wiener chaos. Therefore, the desirable definition of a generalised Lévy area would be $I_2(f^{\mathcal{L}})$ whenever $f^{\mathcal{L}} \in H^{\odot 2}$, where $f^{\mathcal{L}}$ was defined in (4). Unfortunately this is very difficult, if possible at all, to prove. We will circumvent this problem by finding an element in $H^{\odot 2}$ which is indistinguishable from $f^{\mathcal{L}}$ and to which we will apply the isometry $I_2(\cdot)$. In other words, we will say that a function $f \in L^2(\{[0, 1] \times \{1, 2\}\}^2)$ (symmetric) belongs to $H^{\odot 2}$ as long as there is an element $\hat{f} \in H^{\odot 2}$ such that

$$\langle \hat{f}, g \rangle_{H^{\odot 2}} = \sum_{i,j=1}^2 \iiint_{[0,1]^4} f_{ij}(s, t)g_{ij}(u, v)dR_i(s, u)dR_j(t, v) \quad \forall g \in \mathcal{E}_2,$$

where the above integral is an iterated Riemann–Stieltjes integral, see Corollary 2.5 below. Note that we are not enlarging the space $H^{\odot 2}$ but renaming the element \hat{f} by f , since under the inner product in $H^{\odot 2}$ they are indistinguishable. Thus the map I_2 is well defined for \hat{f} and we put $I_2(f) := I_2(\hat{f})$. This is a common procedure to ease the identification of the elements of Hilbert spaces which have been constructed by completion, see Huang and Cambanis [6]. Therefore we will define the generalised Lévy area in the following way:

Definition 2.3 (Generalised Lévy area). *We will say that $I_2(f^{\mathcal{L}})$ is a generalised Lévy area if there exists $\hat{f}^{\mathcal{L}} \in H^{\odot 2}$ such that*

$$(7) \quad \langle \hat{f}^{\mathcal{L}}, g \rangle_{H^{\odot 2}} = \sum_{i,j=1}^2 \iiint\limits_{[0,1]^4} f_{ij}^{\mathcal{L}}(s, u) g_{ij}(t, v) dR_i(s, t) dR_j(u, v) ,$$

for all step functions $g \in \mathcal{E}_2^{\otimes 2}$. Then $I_2(f^{\mathcal{L}}) := I_2(\hat{f}^{\mathcal{L}})$.

Another problem we have to face in order to make this definition tractable is that we do not know how to compute the inner product $\langle \hat{f}^{\mathcal{L}}, g \rangle_{H^{\odot 2}}$. We were very careful to only write the integral form of the inner product in $H^{\otimes 2}$ for step functions, indeed we only know how to calculate the inner product for step functions since the rest of the space was constructed by completion. Hence we will need to approximate $\hat{f}^{\mathcal{L}}$ by step functions and check equation (7) as a limit equality. Before that, let us explicit the inner product in $H^{\otimes 2}$ for step functions as integrals with respect to the covariance functions R_1 and R_2 .

Lemma 2.4. *Let $f, g \in \mathcal{E}^{\otimes 2}$, then*

$$\langle f, g \rangle_{H_i \otimes H_j} = \iiint\limits_{[0,1]^4} f(s, t) g(u, v) dR_i(s, u) dR_j(t, v) .$$

Proof. Let $f = f_1 \otimes f_2$ and $g = g_1 \otimes g_2$, then

$$\begin{aligned} \langle f, g \rangle_{H_i \otimes H_j} &= \langle f_1, g_1 \rangle_{H_i} \langle f_2, g_2 \rangle_{H_j} \\ &= \iint\limits_{[0,1]^2} f_1(s) g_1(t) dR_1(s, t) \iint\limits_{[0,1]^2} f_2(u) g_2(v) dR_2(u, v) \\ &= \iiint\limits_{[0,1]^4} f_1(s) g_1(t) f_2(u) g_2(v) dR_i(s, t) dR_j(u, v) , \end{aligned}$$

and we get the result since a realization of the tensor product for functions is just the plain product. ■

Corollary 2.5. *Let $f, g \in H^{\otimes 2}$, such that $f_{ij}, g_{ij} \in \mathcal{E}^{\otimes 2}$. Then*

$$\begin{aligned} \langle f, g \rangle_{H^{\otimes 2}} &= \sum_{i,j=1}^2 \iiint \int_{[0,1]^4} f_{ij}(s, u) g_{ij}(t, v) dR_i(s, t) dR_i(u, v) \\ &= \sum_{i,j=1}^2 \iint_{[0,1]^2} \left(\iint_{[0,1]^2} f_{ij}(s, u) g_{ij}(t, v) dR_i(s, t) \right) dR_i(u, v). \end{aligned}$$

3. EXISTENCE OF A GENERALISED LÉVY AREA

This section will give the sufficient conditions on the processes $\{X_1(t) \mid 0 \leq t \leq 1\}$ and $\{X_2(t) \mid 0 \leq t \leq 1\}$ so a generalised Lévy area exists (see Definition 2.3). In fact the conditions on the processes will be constraints on their covariance functions, indeed this is what the previous sections suggest as the Hilbert space of the domain of $I_2(\cdot)$ is characterised by the covariance function of X_1 and X_2 . Friz and Victoir [4] claim the p -variation of the covariance function of a Gaussian process to be a fundamental quantity related to the process. Therefore, we first recall some definitions on the p -variation of a function.

3.1. Functions of finite p -variation and Young's inequality. For the sake of completeness and to introduce notation, we here give some definitions on the p -variation of a function, what is meant by a control map and state Young's inequality. For further reading see Dudley and Norvaiša [2] and Friz and Victoir [4, 5].

For a given interval of the real line $[s, t]$ such that $s \leq t$, we will denote the set of all partitions of $[s, t]$ by

$$\mathcal{P}([s, t]) := \{\{t_0, \dots, t_n\} \mid s = t_0 < t_1 < \dots < t_n = t, n \in \mathbb{N}\}.$$

If $D \in \mathcal{P}([s, t])$, then write $|D| := \max_{t_i \in D} \{t_i - t_{i-1}\}$.

Definition 3.1. *Let $f : [s, t] \rightarrow \mathbb{R}$ be a function and $p \geq 1$. We say that f has finite p -variation if $V_p^1(f, [s, t]) < \infty$, where*

$$V_p^1(f, [s, t]) := \sup_{D \in \mathcal{P}([s, t])} \left(\sum_{t_i \in D} |f(t_{i+1}) - f(t_i)|^p \right)^{1/p}.$$

The superscript on V_p^1 is to emphasise that f is 1-dimensional in contrast to the p -variation of a 2-dimensional function, which it is defined below.

Definition 3.2. *Let $f : [s, t] \times [u, v] \rightarrow \mathbb{R}$ be a function and $p \geq 1$. We say that f has finite p -variation if $V_p^2(f, [s, t] \times [u, v]) < \infty$, where*

$$V_p^2(f, [s, t] \times [u, v]) := \sup_{\substack{D \in \mathcal{P}([s, t]) \\ D' \in \mathcal{P}([u, v])}} \left(\sum_{(t_i, t'_j) \in D \times D'} \left| f \left(\begin{array}{c} t_i \\ t_{i+1} \end{array}, \begin{array}{c} t'_j \\ t'_{j+1} \end{array} \right) \right|^p \right)^{1/p}$$

and

$$f \left(\begin{array}{c} t_i \\ t_{i+1} \end{array}, \begin{array}{c} t'_j \\ t'_{j+1} \end{array} \right) := f(t_{i+1}, t'_{j+1}) - f(t_{i+1}, t'_j) - f(t_i, t'_{j+1}) + f(t_i, t'_j).$$

Another important concept related to the p -variation is the *control map* (see Friz and Victoir [5, ch. 5.1, 5.3]).

Definition 3.3. A 2-dimensional control is a map ω from $[s, t] \times [u, v]$ to $[0, \infty)$ where $0 \leq s \leq t \leq 1$, $0 \leq u \leq v \leq 1$ and such that for all $r \leq s \leq t$, $u \leq v$,

$$\begin{aligned} \omega([r, s] \times [u, v]) + \omega([s, t] \times [u, v]) &\leq \omega([r, t] \times [u, v]) \\ \omega([u, v] \times [r, s]) + \omega([u, v] \times [s, t]) &\leq \omega([u, v] \times [r, t]), \end{aligned}$$

and $\lim_{s \rightarrow t} \omega([s, t] \times [u, v]) = \lim_{s \rightarrow t} \omega([u, v] \times [s, t]) = 0$.

It is just for convenience that we set the variables to be in $[0, 1]$. The relationship between the control and the p -variation is given by the following lemma (Friz and Victoir [5, p. 106]):

Lemma 3.4. Let f be a continuous function of finite p -variation – $V_p^2(f, [s, t] \times [u, v]) < \infty$ –, then there is a 2-dimensional control map, ω , such that

$$V_p^2(f, [s, t] \times [u, v]) \leq \omega^{1/p}([s, t] \times [u, v]).$$

We will need the following technical result about the product of control maps; see Friz and Victoir [4]; its proof is a consequence of a discrete Hölder type inequality proved by Young [13, p. 252].

Lemma 3.5. Let ω_1 and ω_2 be 2-dimensional control maps over the same rectangle and $p, q > 0$ such that $p^{-1} + q^{-1} \geq 1$, then $\omega_1^{1/p} \omega_2^{1/q}$ is also a 2-dimensional control map.

Finally, we recall the statement of Young's inequality for a 2-dimensional function (see Towghi [12]):

Theorem 3.6. Let f and g be functions such that

- i) $V_p^2(f, [0, 1] \times [0, 1]) < \infty$,
- ii) $V_p^1(f(0, \cdot), [0, 1]) < \infty$,
- iii) $V_p^1(f(\cdot, 0), [0, 1]) < \infty$,
- iv) $|f(0, 0)| < \infty$,
- v) $V_p^2(g, [0, 1] \times [0, 1]) < \infty$

and $p^{-1} + q^{-1} > 1$, and define

$$\|f\|_{W_p^2([0,1]^2)} := V_p^2(f, [0, 1] \times [0, 1]) + V_p^1(f(0, \cdot), [0, 1]) + V_p^1(f(\cdot, 0), [0, 1]) + |f(0, 0)|.$$

If f and g do not have any common jump points then the Young–Stieltjes integral of f with respect to g exists, and

$$\left| \int_0^1 \int_0^1 f(x, y) dg(x, y) \right| \leq c(p, q) \|f\|_{W_p^2([0,1]^2)} V_p^2(g, [0, 1] \times [0, 1]) ,$$

where $c(p, q)$ is a constant independent of f and g .

In our setting, the functions to which we are going to apply Young’s inequality are continuous and thus do not have jump points.

Remark 3.7. The definition of finite p -variation could be stated for $p > 0$ both in the 1-dimensional and in the 2-dimensional case, but we restrict ourselves to $p \geq 1$. This is because a 1-dimensional continuous function of finite p -variation for $p < 1$ is constant (see Friz and Victoir [4, p. 78]). This is not true for the 2-dimensional case, for example the function $f(x, y) = x + y$ has finite p -variation for all $p > 0$. However, for continuous covariance functions coming from processes that start at a point rather than from a distribution it is true (see next result). We will see in the next section that the hypotheses related with the finite variation are always with respect to continuous covariance functions. Therefore, without loss of generality, we consider p -variations for $p \geq 1$.

Lemma 3.8. A continuous function f on $[0, 1]^2$ such that $f(0, 0) = f(s, 0) = f(0, t)$ for all $t, s \in [0, 1]$ and of finite p -variation with $p < 1$ is constant.

Proof. For a fixed $a \in [0, 1]$, the function $y \rightarrow f(a, y) - f(0, y)$ is a 1-dimensional continuous function of finite p -variation and hence constant. Indeed, it is zero since $f(a, 0) = f(0, 0)$, and the result follows. ■

3.2. Main result. The main result of the paper is proved in this section. We construct a sequence of step functions which converge almost sure to the Lévy kernel and show that it is a Cauchy sequence in $H^{\otimes 2}$, and finally, we also show that its limit satisfies Definition 2.3.

We start by a technical lemma which will ease the proof of the main result, but, before that, let us introduce some notation which will be used extensively in this section. According to definition (5) we have that

$$\int_s^t \int_u^v dR_i(x, y) = R_i \left(\begin{array}{c} s & u \\ t & v \end{array} \right) .$$

Let $\{t_i^n = i2^{-n} \mid i = 0, \dots, 2^n - 1\}$ be the dyadic partition of the interval $[0, 1]$ for a given n , and consider the dyadic partition of the triangles T_1 and T_2

$$T_1^n := \bigcup_{i < j} I_i^n \times I_j^n \quad T_2^n := \bigcup_{i > j} I_i^n \times I_j^n ,$$

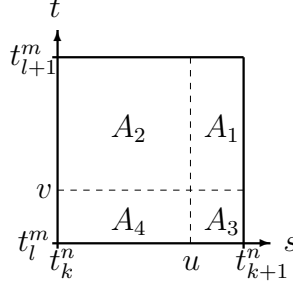


Figure 1. Integration of the $(\star\star)$ -term over the single rectangle $J_{k,l}^{n,m}$ for a fixed $(u, v) \in J_{k,l}^{n,m}$.

where $I_i^n := (t_i^n, t_{i+1}^n]$. Then a natural approximation of the Lévy kernel will be

$$f_n((s, i), (t, j)) := \begin{cases} 0, & \text{if } i = j \\ \frac{1}{2}(\mathbf{1}_{T_1^n}(s, t) - \mathbf{1}_{T_2^n}(s, t)), & \text{if } i = 1, j = 2 \\ \frac{1}{2}(\mathbf{1}_{T_2^n}(s, t) - \mathbf{1}_{T_1^n}(s, t)), & \text{if } i = 2, j = 1 \end{cases} .$$

Lemma 3.9. *Let R_1 and R_2 be two continuous covariance functions in $[0, 1]^2$. Let R_1 be of finite p -variation and R_2 of finite q -variation and assume that $p^{-1} + q^{-1} > 1$, then*

$$\lim_{\substack{n \rightarrow \infty \\ m \rightarrow \infty}} \underbrace{\sum_{i,j=1}^2 \iiint \int_{[0,1]^4} (f^\mathcal{L} - f_n)_{ij}(s, u) \cdot (f^\mathcal{L} - f_m)_{ij}(t, v) dR_i(s, t) dR_j(u, v)}_{(\star)} = 0 .$$

Proof. Write $J_{k,l}^{n,m} := I_k^n \times I_l^m$ and note that

$$\begin{aligned} & (f^\mathcal{L} - f_n)_{ij}(s, u) \cdot (f^\mathcal{L} - f_m)_{ij}(t, v) \\ &= \frac{(1 - \delta_{ij})}{4} \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^m-1} \mathbf{1}_{(J_{k,l}^{n,m})^2}(s, t, u, v) (\mathbf{1}_{\{(v-t)(u-s) > 0\}} - \mathbf{1}_{\{(v-t)(u-s) < 0\}}) . \end{aligned}$$

Therefore the quadruple integral of the (\star) -term is split into a sum of quadruple integrals over $(J_{k,l}^{n,m})^2$. These integrals are iterated integrals and they can be further reduced, according to Figure 1, to

$$(\star) = \frac{1}{4} \sum_{\substack{i,j=1 \\ i \neq j}}^2 \underbrace{\sum_{k=0}^{2^n-1} \sum_{l=0}^{2^m-1} \iint_{J_{k,l}^{n,m}} F_{k,l}^{n,m,i}(u, v) dR_j(u, v)}_{(\star\star)} ,$$

where

$$\begin{aligned}
F_{k,l}^{n,m,i}(u, v) &:= \iint_{A_1} dR_i(s, t) - \iint_{A_2} dR_i(s, t) \\
&\quad - \iint_{A_3} dR_i(s, t) + \iint_{A_4} dR_i(s, t) \\
&= R_i \left(\begin{array}{cc} u & v \\ t_{k+1}^n & t_{l+1}^m \end{array} \right) - R_i \left(\begin{array}{cc} t_k^n & v \\ u & t_{l+1}^m \end{array} \right) \\
&\quad - R_i \left(\begin{array}{cc} u & t_l^m \\ t_{k+1}^n & v \end{array} \right) + R_i \left(\begin{array}{cc} t_k^n & t_l^m \\ u & v \end{array} \right).
\end{aligned}$$

Note that in the above definition we have used Remark 2.1.

It is enough to prove that the $(\star\star)$ -term goes to zero as $n, m \rightarrow \infty$ for $i = 1$ and $j = 2$. The key point is to apply Young's inequality to each summand of the $(\star\star)$ -term, in that way we will prove the existence of the integrals and get a bound for them. In order to do so note the following identities which relate the function $F_{k,l}^{n,m,1}$ with the function R_1

$$\begin{aligned}
F_{k,l}^{n,m,1} \left(\begin{array}{cc} u & v \\ u' & v' \end{array} \right) &= 4R_1 \left(\begin{array}{cc} u & v \\ u' & v' \end{array} \right), \\
F_{k,l}^{n,m,1}(t_k^n, v) - F_{k,l}^{n,m,1}(t_k^n, v') &= 2R_1 \left(\begin{array}{cc} t_k^n & v \\ t_{k+1}^n & v' \end{array} \right), \\
F_{k,l}^{n,m,1}(t_k^n, t_l^m) &= R_1 \left(\begin{array}{cc} t_k^n & t_l^m \\ t_{k+1}^n & t_{l+1}^m \end{array} \right), \\
F_{k,l}^{n,m,1}(u, t_l^m) - F_{k,l}^{n,m,1}(u', t_l^m) &= 2R_1 \left(\begin{array}{cc} u & t_l^m \\ u' & t_{l+1}^m \end{array} \right).
\end{aligned}$$

Now, since $p^{-1} + q^{-1} > 1$, there exists $r, p' > 0$ such that $r + (p')^{-1} = p^{-1}$ and $(p')^{-1} + q^{-1} > 1$. Thus, from $p' > p \geq 1$ we have that R_1 is also of finite p' -variation and use the above equalities to obtain

$$\begin{aligned}
V_{p'}^2(F_{k,l}^{n,m,1}, \overline{J_{k,l}^{n,m}}) &= 4V_{p'}^2(R_1, \overline{J_{k,l}^{n,m}}), \\
V_{p'}^1(F_{k,l}^{n,m,1}(t_k^n, \cdot), [t_l^m, t_{l+1}^m]) &\leq 2V_{p'}^2(R_1, \overline{J_{k,l}^{n,m}}), \\
|F_{k,l}^{n,m,1}(t_k^n, t_l^m)| &\leq V_{p'}^2(R_1, \overline{J_{k,l}^{n,m}}), \\
V_{p'}^1(F_{k,l}^{n,m,1}(\cdot, t_l^m), [t_k^n, t_{k+1}^n]) &\leq 2V_{p'}^2(R_1, \overline{J_{k,l}^{n,m}}),
\end{aligned}$$

where $\overline{J_{k,l}^{n,m}}$ is the closure of $J_{k,l}^{n,m}$. This consideration is a technicality required by the definition of the finite variation. According to the notation of Theorem 3.6 the above inequalities imply that $\|F_{k,l}^{n,m,1}\|_{W_{p'}^2(\overline{J_{k,l}^{n,m}})} \leq 9V_{p'}^2(R_1, \overline{J_{k,l}^{n,m}})$, and thus we can apply

Young's inequality to every integral of the $(\star\star)$ -term to get

$$\begin{aligned} |(\star\star)| &\leq c(p', q) \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^m-1} \|F_{k,l}^{n,m,1}\|_{W_{p'}^2(\overline{J_{k,l}^{n,m}})} V_q^2(R_2, \overline{J_{k,l}^{n,m}}) \\ &\leq 9c(p', q) \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^m-1} V_{p'}^2(R_1, \overline{J_{k,l}^{n,m}}) V_q^2(R_2, \overline{J_{k,l}^{n,m}}). \end{aligned}$$

Recall Lemma 3.4 to associate the finite p -variation of R_1 to a control map ω_1 , and denote by ω_2 the control map with respect to the q -variation of R_2 . Finally we use Lemma 3.5 to deduce that $\hat{\omega} := \omega_1^{1/p'} \omega_1^{1/q}$ is a control map which bounds the $(\star\star)$ -term as

$$\begin{aligned} |(\star\star)| &\leq C \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^m-1} V_q^2(R_2, \overline{J_{k,l}^{n,m}}) \sup_{\substack{D \in \mathcal{P}(\overline{I_k^n}) \\ D' \in \mathcal{P}(\overline{I_l^m})}} \left(\sum_{(t_i, t'_j) \in D \times D'} \left| R_1 \left(\begin{array}{c} t_i \\ t_{i+1} \end{array}, \begin{array}{c} t'_j \\ t'_{j+1} \end{array} \right) \right|^{p'} \right)^{1/p'} \\ &\leq C \sup_{\substack{|u-u'| \leq 2^{-n} \\ |v-v'| \leq 2^{-m}}} \left| R_1 \left(\begin{array}{c} u \\ u' \end{array}, \begin{array}{c} v \\ v' \end{array} \right) \right|^{\frac{p'-p}{p'}} \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^m-1} V_q^2(R_2, \overline{J_{k,l}^{n,m}}) (V_{p'}^2(R_1, \overline{J_{k,l}^{n,m}}))^{p/p'} \\ &\leq C \sup_{\substack{|u-u'| \leq 2^{-n} \\ |v-v'| \leq 2^{-m}}} \left| R_1 \left(\begin{array}{c} u \\ u' \end{array}, \begin{array}{c} v \\ v' \end{array} \right) \right|^{\frac{p'-p}{p'}} \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^m-1} \omega_2^{1/q}(\overline{J_{k,l}^{n,m}}) \omega_1^{1/p'}(\overline{J_{k,l}^{n,m}}) \\ &\leq C \sup_{\substack{|u-u'| \leq 2^{-n} \\ |v-v'| \leq 2^{-m}}} \left| R_1 \left(\begin{array}{c} u \\ u' \end{array}, \begin{array}{c} v \\ v' \end{array} \right) \right|^{\frac{p'-p}{p'}} \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^m-1} \hat{\omega}(\overline{J_{k,l}^{n,m}}) \\ &\leq C \hat{\omega}([0, 1]^2) \sup_{\substack{|u-u'| \leq 2^{-n} \\ |v-v'| \leq 2^{-m}}} \left| R_1 \left(\begin{array}{c} u \\ u' \end{array}, \begin{array}{c} v \\ v' \end{array} \right) \right|^{\frac{p'-p}{p'}}, \end{aligned}$$

where C is a constant which is renamed when necessary. Finally note that the last supremum goes to zero as $n, m \rightarrow \infty$ by the uniform continuity of R_1 and the result follows. ■

Now we are ready to finalize the construction of the generalized Lévy area, we need to prove that the sequence f_n in Lemma 3.9 is a Cauchy sequence and its limit, $\hat{f}^\mathcal{L}$, satisfies Definition 2.3.

Theorem 3.10. *Let $\{X_1(t) \mid 0 \leq t \leq 1\}$ and $\{X_2(t) \mid 0 \leq t \leq 1\}$ be two continuous centered Gaussian processes, such that $X_1(0) = X_2(0) = 0$, independent, and with covariance functions R_1 and R_2 respectively. Let R_1 be of finite p -variation and R_2 be of finite q -variation and assume that $p^{-1} + q^{-1} > 1$, then the sequence $\{f_n\}_{n \geq 1}$ is a*

Cauchy sequence in $H^{\otimes 2}$. We will denote its limit by $\hat{f}^{\mathcal{L}}$. Moreover, we have that

$$(8) \quad \langle \hat{f}^{\mathcal{L}}, g \rangle_{H^{\otimes 2}} = \sum_{i,j=1}^2 \iiint \int_{[0,1]^4} f_{ij}^{\mathcal{L}}(s, u) g_{ij}(t, v) dR_i(s, t) dR_j(u, v)$$

for all step functions $g \in H^{\otimes 2}$.

Proof. Note that $f_n - f_m \in H^{\otimes 2}$ is a difference of two step functions and hence it is a step function itself. Therefore by Lemma 2.5 we have that

$$\begin{aligned} \|f_n - f_m\|_{H^{\otimes 2}} &= \sum_{\substack{i,j=1 \\ i \neq j}}^2 \iiint \int_{[0,1]^4} (f_n - f_m)_{ij}(s, u) \\ &\quad (f_n - f_m)_{ij}(t, v) dR_i(s, t) dR_i(u, v) \\ &= \sum_{\substack{i,j=1 \\ i \neq j}}^2 \iiint \int_{[0,1]^4} (f_n - f^{\mathcal{L}} + f^{\mathcal{L}} - f_m)_{ij}(s, u) \\ &\quad (f_n - f^{\mathcal{L}} + f^{\mathcal{L}} - f_m)_{ij}(t, v) dR_i(s, t) dR_i(u, v). \end{aligned}$$

Each term of the above product was denoted as a (\star) -term in Lemma 3.9 and thus goes to zero as $n, m \rightarrow \infty$.

For the second part of the proof it suffices to prove the equality for a function g such that $g_{12}(s, t) = \mathbf{1}_{[a,b] \times [c,d]}(s, t)$ where $[a, b] \times [c, d] \subseteq [0, 1]^2$ and $g_{ij}(s, t) \equiv 0$ for $i \neq 1$ or $j \neq 2$. Since $\hat{f}^{\mathcal{L}}$ is the limit of $\{f_n\}_{n \geq 1}$ in $H^{\otimes 2}$ then

$$\lim_{n \rightarrow \infty} \langle f_n, g \rangle_{H^{\otimes 2}} = \langle \hat{f}^{\mathcal{L}}, g \rangle_{H^{\otimes 2}}.$$

Our objective is to prove that $\lim_{n \rightarrow \infty} \langle f_n, g \rangle_{H^{\otimes 2}}$ equals the left hand side of equation (8). From the definition of the Lévy kernel we have that

$$\begin{aligned} &\sum_{i,j=1}^2 \iiint \int_{[0,1]^4} f_{ij}^{\mathcal{L}}(s, u) g_{ij}(t, v) dR_i(s, t) dR_j(u, v) \\ &= \frac{1}{2} \iiint \int_{[0,1]^4} (\mathbf{1}_{s>u}(s, u) - \mathbf{1}_{s<u}(s, u)) \mathbf{1}_{[a,b] \times [c,d]}(t, v) dR_1(s, t) dR_2(u, v) \\ &= \frac{1}{2} \iint_{[0,1] \times [c,d]} dR_2(u, v) \iint_{[0,1] \times [a,b]} (\mathbf{1}_{s>u}(s, u) - \mathbf{1}_{s<u}(s, u)) dR_1(s, t) \\ (9) \quad &= \frac{1}{2} \iint_{[0,1] \times [c,d]} \left[R_1 \left(\begin{array}{cc} u & a \\ 1 & b \end{array} \right) - R_1 \left(\begin{array}{cc} 0 & a \\ u & b \end{array} \right) \right] dR_2(u, v). \end{aligned}$$

The above integral is a well defined Young–Stieltjes integral. Then, for $D \in \mathcal{P}([0, 1])$ and $D' \in \mathcal{P}([c, d])$,

$$\begin{aligned}
& \sum_{i,j=1}^2 \iiint\limits_{[0,1]^4} f_{ij}^{\mathcal{L}}(s, u) g_{ij}(t, v) dR_i(s, t) dR_j(u, v) \\
&= \frac{1}{2} \lim_{\substack{|D| \rightarrow 0 \\ |D'| \rightarrow 0}} \sum_{\substack{\xi_i \in D \\ \zeta_j \in D'}} \left[R_1 \left(\begin{array}{c} \nu_i \\ 1 \end{array}, \begin{array}{c} a \\ b \end{array} \right) - R_1 \left(\begin{array}{c} 0 \\ \nu_i \end{array}, \begin{array}{c} a \\ b \end{array} \right) \right] R_2 \left(\begin{array}{c} \xi_i \\ \xi_{i+1} \end{array}, \begin{array}{c} \zeta_j \\ \zeta_{j+1} \end{array} \right) \\
(10) \quad &= \frac{1}{2} \lim_{|D| \rightarrow 0} \sum_{\xi_i \in D} \left[R_1 \left(\begin{array}{c} \nu_i \\ 1 \end{array}, \begin{array}{c} a \\ b \end{array} \right) - R_1 \left(\begin{array}{c} 0 \\ \nu_i \end{array}, \begin{array}{c} a \\ b \end{array} \right) \right] R_2 \left(\begin{array}{c} \xi_i \\ \xi_{i+1} \end{array}, \begin{array}{c} c \\ d \end{array} \right),
\end{aligned}$$

where $\nu_i \in [\xi_i, \xi_{i+1}]$. On the other hand, from Lemma 2.5 we have that

$$\begin{aligned}
\langle f_n, g \rangle_{H^{\otimes 2}} &= \frac{1}{2} \sum_{\substack{k,l=0 \\ k>l}}^{2^n-1} \iiint\limits_{I_k^n \times [a,b] \times I_l^n \times [c,d]} dR_1(s, t) dR_2(u, v) \\
&\quad - \frac{1}{2} \sum_{\substack{k,l=0 \\ k<l}}^{2^n-1} \iiint\limits_{I_k^n \times [a,b] \times I_l^n \times [c,d]} dR_1(s, t) dR_2(u, v) \\
&= \frac{1}{2} \sum_{\substack{k,l=0 \\ k>l}}^{2^n-1} R_1 \left(\begin{array}{c} t_k^n \\ t_{k+1}^n \end{array}, \begin{array}{c} a \\ b \end{array} \right) R_2 \left(\begin{array}{c} t_l^n \\ t_{l+1}^n \end{array}, \begin{array}{c} c \\ d \end{array} \right) \\
&\quad - \frac{1}{2} \sum_{\substack{k,l=0 \\ k<l}}^{2^n-1} R_1 \left(\begin{array}{c} t_k^n \\ t_{k+1}^n \end{array}, \begin{array}{c} a \\ b \end{array} \right) R_2 \left(\begin{array}{c} t_l^n \\ t_{l+1}^n \end{array}, \begin{array}{c} c \\ d \end{array} \right) \\
(11) \quad &= \frac{1}{2} \sum_{l=0}^{2^n-1} \left[R_1 \left(\begin{array}{c} t_{l+1}^n \\ 1 \end{array}, \begin{array}{c} a \\ b \end{array} \right) - R_1 \left(\begin{array}{c} 0 \\ t_l^n \end{array}, \begin{array}{c} a \\ b \end{array} \right) \right] R_2 \left(\begin{array}{c} t_l^n \\ t_{l+1}^n \end{array}, \begin{array}{c} c \\ d \end{array} \right).
\end{aligned}$$

Note that in equation (10) we could replace the first ν_i by ν_i^1 and the second ν_i by ν_i^2 , where $\nu_i^1, \nu_i^2 \in [\xi_i, \xi_{i+1}]$. This is because the integral (9) could be split into two. Indeed you could choose $\nu_i^1 = \xi_i$ and $\nu_i^2 = \xi_{i+1}$. If you do this, then equation (11) becomes a particular election of the partition D in equation (10) and thus

$$\lim_{n \rightarrow \infty} \langle f_n, g \rangle_{H^{\otimes 2}} = \iiint\limits_{[0,1]^4} f_{12}^{\mathcal{L}}(s, u) g_{12}(t, v) dR_1(s, t) dR_2(u, v)$$

from where the result follows. ■

3.3. The case of the fractional Brownian motion. One case of special interest is to explore what happens with the generalised Lévy area for a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$. It is known that when fBm has Hurst parameter $H \leq \frac{1}{2}$ then its covariance function is of finite $\frac{1}{2H}$ -variation, and when

$H > \frac{1}{2}$ then its covariance function is of bounded variation. Then the Lévy area can be defined for two fBm with the same Hurst parameter H as long as $H > \frac{1}{4}$. Moreover we can let one of the processes be as irregular as desired provided that the other one is regular enough. That is, we can let one of the fBm be of Hurst parameter $H < \frac{1}{2}$ as long as the other independent fBm has Hurst parameter $H' > \frac{1}{2} - H$.

4. CARLEMAN-FREDHOLM REPRESENTATION

Consider an isonormal Gaussian process $X = \{X(h) \mid h \in H\}$. If $H = L^2(T, \mathcal{B}, \mu)$, then $H^{\odot 2}$ is the space of symmetric square integrable functions $L^2_{\text{sym}}(T^2)$. To each element $f \in H^{\odot 2}$ corresponds a symmetric Hilbert–Schmidt operator $F : H \rightarrow H$ defined by

$$(F(h))(t) = \int_T f(s, t)h(s)\mu(ds).$$

Denote by $\{\alpha_n \mid n \geq 1\}$ the eigenvalues of F repeated according to its multiplicity. Then

$$\mathbb{E}[e^{zI_2(f)}] = \frac{1}{\sqrt{\prod_{n=1}^{\infty} (1 - 2z\alpha_n)e^{2z\alpha_n}}} \quad \text{for } 2|\Re(z)|\sigma(F) < 1,$$

where $\sigma(F) = \sup_{n \geq 1} \{|\alpha_n|\}$. The infinite product of the above equality is called a generalised determinant or a Carleman–Fredholm determinant of F (see Dunford and Schwartz [3, p. 1036]). There are many works about the characteristic functions of quadratic Wiener functionals. In our case we are interested in a particular functional viewed in different Wiener spaces, therefore the aim of this section is to explicit the procedure to compute the eigenvalues as much as possible. As an example, we will compute the eigenvalues for the particular case where the stochastic processes X_1 and X_2 are of the form

$$X_i = \int_0^1 f(s)dW_i(s) \quad f \in L^2([0, 1]),$$

where W_1 and W_2 are two independent Brownian motion.

It is worthwhile to state the relationship between elements of the second Wiener chaos and Hilbert–Schmidt operators by working out a particular example and then extending the results to the general case. To this end we study the case of the Lévy area for two standard Wiener processes, as we commented in the introduction. For $R_1(s, t) = R_2(s, t) = s \wedge t$ the Hilbert space H is isometric to $L^2([0, 1] \times \{1, 2\})$. Then the Lévy kernel $f^{\mathcal{L}}$ defines the Hilbert–Smith operator

$$F : L^2([0, 1] \times \{1, 2\}) \rightarrow L^2([0, 1] \times \{1, 2\})$$

$$h \mapsto \int_{[0, 1] \times \{1, 2\}} f_{ij}^{\mathcal{L}}(s, t)h_i(s)ds \otimes \text{Card},$$

which is reduced to the form

(12)

$$F(h)_j(t) = \frac{\delta_{2j}}{2} \left(\int_0^t h_1(s) ds - \int_t^1 h_1(s) ds \right) - \frac{\delta_{1j}}{2} \left(\int_0^t h_2(s) ds - \int_t^1 h_2(s) ds \right).$$

If h is an eigenvector of eigenvalue α , $F(h) = \alpha h$, then it is continuous because it is defined by an integral, and applying again the same argument it is differentiable. Then we differentiate the above expression and obtain the matrix representation

$$(13) \quad \begin{pmatrix} h_1'(t) \\ h_2'(t) \end{pmatrix} = \frac{1}{\alpha} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \frac{1}{\alpha} M \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}$$

with solution given by $h(t) = e^{M \frac{t}{\alpha}} h(0)$. From (12) it is clear that $h(1) + h(0) = 0$ and thus the eigenvalues satisfy the equation

$$e^{M \frac{1}{\alpha}} = \begin{pmatrix} \cos(\alpha^{-1}) & -\sin(\alpha^{-1}) \\ \sin(\alpha^{-1}) & \cos(\alpha^{-1}) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -Id_2.$$

Therefore the eigenvalues, α_n , are $\{\pm(\pi(2n+1))^{-1} \mid n \in \mathbb{N}\}$ with multiplicity 2 since the space of solutions of the ordinary differential equation has dimension 2. Finally we compute the Carleman-Fredholm determinant to obtain $\mathbb{E}[e^{itA}] = \cosh(t)^{-1}$.

Now we use the same sort of ideas into the abstract setting presented in Section 2. Let $f \in H^{\otimes 2}$ and define the operator $F := \Psi \circ \Phi_f$, such that

$$F : H \xrightarrow{\Phi_f} H^* \xrightarrow{\Psi} H,$$

where Ψ is the duality isomorphism and for $g \in H$ we define $\Phi_f(g) : H \rightarrow \mathbb{R}$ as

$$\Phi_f(g)(h) := \langle f, g \otimes h \rangle_{H^{\otimes 2}}, \quad h \in H.$$

It can be proved that F is a Hilbert-Smith operator. Note that $g \in H$ is an eigenvector of the operator F with eigenvalue α if and only if $\langle f, g \otimes h \rangle_{H^{\otimes 2}} = \alpha \langle g, h \rangle_H$ for all $h \in H$. From Lemma 2.2 we can identify $g(t, i) = \delta_{1i} g_1(t) + \delta_{2i} g_2(t)$ where $g_i \in H_i$ and the same sort of identification is valid for h , then $g \in H$ is an eigenvector of eigenvalue α if and only if

$$\sum_{i,j=1}^2 \langle f_{ij}, g_i \otimes h_j \rangle_{H_i \otimes H_j} = \alpha \sum_{i=1}^2 \langle g_i, h_i \rangle_{H_i} \quad \forall h_1 \in H_1, \forall h_2 \in H_2.$$

We will say that two covariance functions R_1 and R_2 are equivalent if the associated Hilbert spaces H_1 and H_2 are the same. Under this symmetry of the processes we recover the spectrum structure of the classical Lévy area.

Proposition 4.1. *Under the notation of the previous sections, let $\{X_1(t) \mid 0 \leq t \leq 1\}$ and $\{X_2(t) \mid 0 \leq t \leq 1\}$ be continuous centered independent Gaussian processes with equivalent covariance functions R_1 and R_2 respectively. Then the corresponding Hilbert-Schmidt operator has eigenvalues with even multiplicity and symmetric with*

respect to zero. As a consequence the characteristic function of the generalised Lévy area is of the form

$$\varphi(t) = \prod_{n \geq 1} \frac{1}{(1 + 4\alpha_n^2 t^2)^{m_n}},$$

where $m_n \geq 1$.

Proof. From the factorisation (6) it is clear that the symmetry of the approximation of the Lévy kernel $\{f_n\}_{n \geq 1}$ is transferred to $\hat{f}^{\mathcal{L}}$. Then, from equation

$$(14) \quad \sum_{\substack{i,j=1 \\ i \neq j}}^2 \langle \hat{f}_{ij}^{\mathcal{L}}, g_i \otimes h_j \rangle_{H_i \otimes H_j} = \alpha \sum_{i=1}^2 \langle g_i, h_i \rangle_{H_i} \quad \forall h_1, h_2 \in H_1 \equiv H_2,$$

it is checked that if $g(t, i) = \delta_{1i}g_1(t) + \delta_{2i}g_2(t)$ is an eigenvector with eigenvalue α , then $\tilde{g}(t, i) = \delta_{1i}g_2(t) - \delta_{2i}g_1(t)$ is an eigenvector with eigenvalue α and $\hat{g}(t, i) = \delta_{1i}g_2(t) + \delta_{2i}g_1(t)$ is an eigenvector with eigenvalue $-\alpha$. If $g = \lambda\tilde{g}$ for $\lambda \in \mathbb{R} \setminus \{0\}$ then $g_1 = \lambda g_2 = -\lambda^2 g_1$ and hence $g \equiv 0$, thus g and \tilde{g} are linear independent. On the other hand, if $\{g_k(t, i), \tilde{g}_k(t, i) \mid k = 1, \dots, K\}$ is a family set of linear independent eigenvectors of eigenvalue α , and

$$h(t, i) = \sum_{k=1}^K \lambda_k g_k(t, i) + \sum_{k=1}^K \mu_k \tilde{g}_k(t, i), \quad \lambda_k, \mu_k \in \mathbb{R},$$

then

$$\tilde{h}(t, i) = \sum_{k=1}^K \lambda_k \tilde{g}_k(t, i) - \sum_{k=1}^K \mu_k g_k(t, i).$$

Therefore α has even multiplicity. Note that this suffices to deduce the same property for the eigenvalue $-\alpha$ and by construction the multiplicity of α and $-\alpha$ is the same. Finally, we recover the same structure for the spectrum of the Hilbert–Schmidt operator that we have in the classical Lévy area. ■

Example. From the explicit calculations made for the classical Lévy area we can easily work out a bit more general case. Let $X_i(t) = \int_0^t f(s) dW_i(s)$, for two independent Brownian motions W_1 and W_2 . Then $R_i(s, t) = \int_0^{s \wedge t} f^2(u) du$, $H_i = L^2([0, 1], f^2(u) du)$ and equation (13) can be written as

$$\begin{pmatrix} h'_1(t) \\ h'_2(t) \end{pmatrix} = \frac{f^2(t)}{\alpha} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}.$$

Therefore the general solution is

$$h(t) = \exp \left(\int_0^t \frac{f^2(u)}{\alpha} du \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) h(0).$$

Finally the characteristic function of A in this setting is $\mathbb{E}[e^{itA}] = \operatorname{sech}(t \|f\|_{L^2}^2)$.

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