# GENERALIZED INDUCTIVE LIMITS OF QUASIDIAGONAL C*-ALGEBRAS

NATHANIAL P. BROWN

Abstract. If $A$ is a unital quasidiagonal C*-algebra, we construct a generalized inductive limit $B_A$ which is simple, unital and inherits many structural properties from $A$. If $A$ is the unitization of a non-simple purely infinite algebra (e.g., the cone over a Cuntz algebra), then $B_A$ is tracially AF which, among other things, lends support to a conjecture of Toms.

This paper evolved out of conversations with Wilhelm Winter, inspired by the following conjecture of Andrew Toms. (See [13] for a remarkable contribution to this conjecture, as well as the relevant definitions, and [14] for a similar conjecture in a broader context.)

**Conjecture.** For simple, unital, nuclear C*-algebras (except matrix algebras), the following are equivalent:

(1) strict comparison;
(2) $\mathcal{Z}$-stability;
(3) finite decomposition rank.

In 2008, when we spoke, there was very little evidence for this audacious conjecture, so I tried to construct a counterexample – and failed. In fact, the failure was spectacular; I stumbled on somewhat surprising examples that support the conjecture. Hence this paper.

The idea for constructing counterexamples was to modify well-known inductive limit constructions so wild building blocks could be incorporated and exploited. For example, $C_0((0,1],\mathcal{O}_n)$, the cone over a Cuntz algebra, is both purely infinite [7] and quasidiagonal [12], hence stably finite, which is a little crazy. Using matrices over its unitization to construct a unital simple $\mathcal{Z}$-stable inductive limit ought to produce a counterexample with infinite decomposition rank, or so I hoped. But it turns out you get an AF algebra.[1] I have no clue how to construct the dense finite-dimensional subalgebras, though. I only know they exist.

2000 *Mathematics Subject Classification.* Primary 46L35.

*Key words and phrases.* C*-algebras, decomposition rank, tracially AF, generalized inductive limit.

Partially supported by DMS-0856197.

[1]This uses K-theory considerations, in general one gets Lin's tracially AF algebras when starting with the unitization of a quasidiagonal, purely infinite algebra.

Before describing the inductive-limit modifications used in this paper, I'd like to explain why they are necessary. It turns out ordinary inductive limits of quasidiagonal, purely infinite algebras are never simple, at least when the connecting maps are injective. To see this, first note that quasidiagonal, purely infinite algebras are always projectionless, because cutting by a projection would result in a *unital*, quasidiagonal, purely infinite corner, which is impossible since every unital, quasidiagonal algebra has a trace. Hence an ordinary inductive limit of quasidiagonal, purely infinite algebras (with injective connecting maps) is necessarily projectionless, too. However, [7, Proposition 4.18] states that ordinary inductive limits of purely infinite algebras are purely infinite, and since *simple* purely infinite algebras always have projections, this completes the proof.

So, instead of ordinary inductive limits, we'll have to use the *generalized* inductive limits introduced by Blackadar and Kirchberg in [1]. Very roughly, they allow connecting maps to be asymptotically multiplicative and still get a C$^*$-algebra in the limit. This makes a detailed analysis of the limit algebra significantly more difficult, but our construction is explicit and simple enough that we can say something. For example, starting with any unital, quasidiagonal C$^*$-algebra $A$, our construction yields a unital, simple, quasidiagonal C$^*$-algebra $B_A$ with stable rank one. And it's approximately divisible, hence $\mathcal{Z}$-stable, in the main case of interest (and tracially AF in the other case). See Theorem 3.1 for these $A$-independent facts, and Theorem 3.2 for a few properties that pass to $B_A$ whenever they're enjoyed by $A$ (e.g., nuclearity, UCT, etc.). Finally, in Corollary 3.3 we specialize to unitizations of quasidiagonal, purely infinite algebras and prove that $B_A$ is always tracially AF in this case.

*Acknowledgement.* Many thanks to my good friends Francesc Perera (for the hospitality), Andrew Toms and Wilhelm Winter (for stimulating conversations related to this paper). But mostly, just for being them. Also, I'm indebted to the Centre de Recerca Matemàtica, where the final write-up of this note was completed.

## 1. The construction

The construction used in this note is but a small variation on one which appears throughout the classification literature. The only semi-novel aspect is the use of asymptotically multiplicative – but not multiplicative – connecting maps.

Let $A$ be a separable unital QD C$^*$-algebra and $\varphi_n\colon A \to \mathbb{M}_{k(n)}(\mathbb{C})$ be u.c.p. maps such that $\|a\| = \lim_n \|\varphi_n(a)\|$ and $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0$ for all $a, b \in A$. We assume $k(n) \to \infty$ since this can always be arranged, and is necessarily the case for all non-subhomogeneous QD algebras.

Next, choose natural numbers $s(n) > k(n)$ and define unital complete order embeddings (see Definition 11.2.1 and Remark 11.2.2 in [5]) $\Phi_n\colon A \to \mathbb{M}_{s(n)}(A)$

as follows:

$$\Phi_n(a) = \left(\begin{array}{cccc|c} a & & & & \\ & a & & & \\ & & \ddots & & \\ & & & a & \\ \hline & & & & \varphi_n(a) \end{array}\right),$$

where all unspecified entries are zero and the corner $\varphi_n(a)$ is a scalar matrix. In tensor notation we have

$$\Phi_n(a) = Q_n \otimes a + \varphi_n(a) \otimes 1_A,$$

where $Q_n \in \mathbb{M}_{s(n)}(\mathbb{C})$ is a projection of rank $s(n) - k(n)$ and we use an identification $\mathbb{M}_{k(n)}(\mathbb{C}) \cong Q_n^{\perp}\mathbb{M}_{s(n)}(\mathbb{C})Q_n^{\perp}$ to make sense of the term $\varphi_n(a) \otimes 1_A$. Finally, we define an inductive sequence

$$A \xrightarrow{\psi_1} \mathbb{M}_{s(1)} \otimes A \xrightarrow{\psi_2} \mathbb{M}_{s(1)} \otimes \mathbb{M}_{s(2)} \otimes A \xrightarrow{\psi_3} \mathbb{M}_{s(1)} \otimes \mathbb{M}_{s(2)} \otimes \mathbb{M}_{s(3)} \otimes A \xrightarrow{\psi_4} \cdots,$$

where

$$\psi_n \colon \mathbb{M}_{s(1)} \otimes \cdots \otimes \mathbb{M}_{s(n-1)} \otimes A \to \mathbb{M}_{s(1)} \otimes \cdots \otimes \mathbb{M}_{s(n-1)} \otimes \mathbb{M}_{s(n)}(A)$$

is the unital complete order embedding

$$\psi_n = \mathrm{id}_{s(1)} \otimes \cdots \otimes \mathrm{id}_{s(n-1)} \otimes \Phi_n.$$

Checking that our inductive sequence defines a generalized inductive system in the sense of [1] is elementary, but a pain. The key points are the asymptotic multiplicativity of the maps $\{\varphi_n\}$ *and* the special form of our connecting maps. For example, one can check that if we define $\Psi_{n+1,n} = \psi_n$ and $\Psi_{m,n} = \psi_{m-1} \circ \cdots \circ \psi_n$ for $m > n+1$, then for all $a \in A$

$$\begin{aligned} \Psi_{m,1}(a) = {} & Q_1 \otimes \cdots \otimes Q_{m-2} \otimes Q_{m-1} \otimes a \\ & + Q_1 \otimes \cdots \otimes Q_{m-2} \otimes \varphi_{m-1}(a) \otimes 1 \\ & + Q_1 \otimes \cdots \otimes \varphi_{m-2}(a) \otimes 1 \otimes 1 \\ & \vdots \\ & + Q_1 \otimes \varphi_2(a) \otimes 1 \otimes \cdots \otimes 1 \\ & + \varphi_1(a) \otimes 1 \otimes \cdots \otimes 1. \end{aligned}$$

It is important to note that the terms above are pairwise orthogonal. This observation helps one verify that for all $k > m > 1$

$$\|\Psi_{k,m}\big(\Psi_{m,1}(a)\Psi_{m,1}(b)\big) - \Psi_{k,1}(a)\Psi_{k,1}(b)\| \leq \max_{m \leq i \leq k-1} \|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\|$$

for all $a, b \in A$. Similar reasoning applied to later building blocks shows that we have indeed satisfied the definition of a generalized inductive system.

**Definition 1.1.** Let $B_A = \mathrm{g}\varinjlim(\mathbb{M}_{s(1)}\otimes\cdots\otimes\mathbb{M}_{s(n-1)}\otimes A, \Psi_{m,n})$ be the generalized inductive limit $C^*$-algebra associated to the system above. Also, let

$$\Psi_n \colon \mathbb{M}_{s(1)} \otimes \cdots \otimes \mathbb{M}_{s(n-1)} \otimes A \to B_A$$

be the canonical unital complete order embeddings.

## 2. An ordinary inductive limit

The key to analyzing the generalized inductive limit of the preceding section is to rewrite it as an ordinary inductive limit. For this we recycle the techniques and ideas used in [4].

Define a unital complete order embedding

$$\theta = \bigoplus_1^\infty \varphi_n \colon A \to \prod_1^\infty \mathbb{M}_{k(n)}$$

and let

$$R_1 := \theta(A) + \bigoplus_1^\infty \mathbb{M}_{k(n)}.$$

Note that $R_1$ is a C$^*$-algebra (since the $\varphi_n$'s are asymptotically multiplicative) and we have a canonical isomorphism $R_1/(\bigoplus_1^\infty \mathbb{M}_{k(n)}) \cong A$ (since the $\varphi_n$'s are asymptotically isometric).

Now we mimic the construction of the previous section – with one twist. Rather than use matrices over $R_1$ for our building blocks, we'll use matrices over canonical quotients of $R_1$. To be more precise, let $e_n \in R_1$ be the central projection corresponding to the unit of $\mathbb{M}_{k(n)}$ and $e_n^\perp$ be the orthogonal complement. Next, for $n \geq 1$ let $R_{n+1} = e_n^\perp R_n$ and consider the (ordinary) inductive system

$$R_1 \xrightarrow{\pi_1} \mathbb{M}_{s(1)} \otimes R_2 \xrightarrow{\pi_2} \mathbb{M}_{s(1)} \otimes \mathbb{M}_{s(2)} \otimes R_3 \xrightarrow{\pi_3} \mathbb{M}_{s(1)} \otimes \mathbb{M}_{s(2)} \otimes \mathbb{M}_{s(3)} \otimes R_4 \xrightarrow{\pi_4} \cdots,$$

where the connecting maps are defined exactly like the $\psi_n$'s – except we replace the diagonal $a$'s with $e_n^\perp x$'s and the scalar corners $\varphi_n(a)$ with the matrices $e_n x$. For example, the first map looks like this:

$$\pi_1(x) = \left(\begin{array}{cccc|c} e_1^\perp x & & & & \\ & e_1^\perp x & & & \\ & & \ddots & & \\ & & & e_1^\perp x & \\ \hline & & & & e_1 x \end{array}\right),$$

for all $x \in R_1$; or, in tensor notation, $\pi_1(x) = Q_1 \otimes e_1^\perp x + e_1 x \otimes 1_{R_2}$. (We're using the same projections $Q_n$ and identifications $\mathbb{M}_{k(n)}(\mathbb{C}) \cong Q_n^\perp \mathbb{M}_{s(n)}(\mathbb{C}) Q_n^\perp$ as before.)

Consider the diagram

$$\begin{array}{ccccccc} R_1 & \xrightarrow{\pi_1} & \mathbb{M}_{s(1)}\otimes R_2 & \xrightarrow{\pi_2} & \mathbb{M}_{s(1)}\otimes\mathbb{M}_{s(2)}\otimes R_3 & \xrightarrow{\pi_3} & \cdots \\ \downarrow{\scriptstyle\sigma_1} & & \downarrow{\scriptstyle\sigma_2} & & \downarrow{\scriptstyle\sigma_3} & & \\ A & \xrightarrow{\psi_1} & \mathbb{M}_{s(1)}\otimes A & \xrightarrow{\psi_2} & \mathbb{M}_{s(1)}\otimes\mathbb{M}_{s(2)}\otimes A & \xrightarrow{\psi_3} & \cdots, \end{array}$$

where the $\sigma_s$'s are the canonical quotient maps (coming from the canonical isomorphisms $R_s/(\bigoplus_{n=s}^{\infty}\mathbb{M}_{k(n)})\cong A$). Note that the diagram above is *not* commutative on all elements, but it is commutative on elements of the form $\theta(a)\in R_1$ (and on those of the form $T_1\otimes\cdots\otimes T_{n-1}\otimes e_{n-1}^{\perp}\theta(a)\in\mathbb{M}_{s(1)}\otimes\cdots\otimes\mathbb{M}_{s(n-1)}\otimes R_n$). To see this, let's write out what the maps look like on $R_1$. Letting $\pi_{m,1}=\pi_{m-1}\circ\cdots\circ\pi_1$, one finds that for elements $\theta(a)+(T_n)_1^{\infty}\in\theta(A)+\bigoplus_1^{\infty}\mathbb{M}_{k(n)}=R_1$, we have

$$\begin{aligned}\sigma_m\circ\pi_{m,1}\big(\theta(a)+(T_n)\big) = {} & Q_1\otimes\cdots\otimes Q_{m-2}\otimes Q_{m-1}\otimes a \\ & + Q_1\otimes\cdots\otimes Q_{m-2}\otimes(\varphi_{m-1}(a)+T_{m-1})\otimes 1 \\ & + Q_1\otimes\cdots\otimes(\varphi_{m-2}(a)+T_{m-2})\otimes 1\otimes 1 \\ & \vdots \\ & + Q_1\otimes(\varphi_2(a)+T_2)\otimes 1\otimes\cdots\otimes 1 \\ & + (\varphi_1(a)+T_1)\otimes 1\otimes\cdots\otimes 1.\end{aligned}$$

**Proposition 2.1.** *If* $C_A=\varinjlim(\mathbb{M}_{s(1)}\otimes\cdots\otimes\mathbb{M}_{s(n-1)}\otimes R_n,\pi_n)$, *then* $C_A\cong B_A$.

*Proof.* We'll construct a u.c.p. map $\alpha\colon C_A\to B_A$, then show it's a $*$-isomorphism.

Let's begin with an element $x=\theta(a)+(T_n)_1^{\infty}\in\theta(A)+\bigoplus_1^{\infty}\mathbb{M}_{k(n)}=R_1$. The computation preceding this proposition implies that for $m'>m$,

$$\|\Psi_{m',m}\circ\sigma_m\circ\pi_{m,1}(x)-\sigma_{m'}\circ\pi_{m',1}(x)\|=\max_{m\le i\le m'-1}\|T_i\|.$$

This implies that $\{\Psi_m\circ\sigma_m\circ\pi_{m,1}(x)\}$ is a Cauchy sequence in $B_A$, so we get a u.c.p. map $\alpha\colon R_1\to B_A$ by declaring $\alpha(x)=\lim_m\Psi_m\circ\sigma_m\circ\pi_{m,1}(x)$. In fact, $\alpha$ is completely isometric. Indeed,

$$\|\theta(a)+(T_n)_1^{\infty}\|=\sup_{i\in\mathbb{N}}\|\varphi_i(a)+T_i\|$$

while

$$\begin{aligned}\|\alpha(\theta(a)+(T_n)_1^{\infty})\| &= \lim_{m\to\infty}\|\sigma_m\circ\pi_{m,1}(\theta(a)+(T_n)_1^{\infty})\| \\ &= \lim_{m\to\infty}\max\{\|a\|,\max_{1\le i\le m-1}\|\varphi_i(a)+T_i\|\} \\ &= \lim_{m\to\infty}\max_{1\le i\le m-1}\|\varphi_i(a)+T_i\|,\end{aligned}$$

where the last equality follows from two facts: $\|T_i\| \to 0$ and $\|a\| = \lim_i \|\varphi_i(a)\|$. Thus we see that $\|\theta(a) + (T_n)_1^\infty\| = \|\alpha(\theta(a) + (T_n)_1^\infty)\|$ and a similar argument shows that $\alpha$ is in fact completely isometric.

Showing that $\alpha$ extends to a complete isometry $C_A \to B_A$ is a notational nightmare, but otherwise it's identical to the argument above. It is clear that the range of $\alpha\colon C_A \to B_A$ is dense. And since the range is closed, $\alpha$ is a complete isometry *onto* $B_A$; hence it's necessarily multiplicative, i.e., a $*$-isomorphism (cf. [5, Remark 11.2.2]).[2] □

Understanding traces on $C_A$ is important, but requires a few general facts about traces on non-simple algebras (cf. [10, Definition 2.6] and the paragraph following it). If $I \lhd D$ is a closed two-sided ideal and $\tau$ is a tracial state on $D$, then we can write

$$\tau = (1-c)\tau_{D/I} + c\tau_I$$

for some tracial states $\tau_{D/I}$ on $D/I$ and $\tau_I$ on $I$ and some $0 \le c \le 1$. To make sense of this, we let $\{f_n\}$ be a quasicentral approximate unit for $I$ and $c = \lim_n \tau(f_n)$. Then one can prove that $\tau_{D/I}(\dot{d}) = \frac{1}{1-c}\lim_n \tau((1-f_n)d)$ defines a tracial state on $D/I$ (where $\dot{d} \in D/I$ denotes the image of $d \in D$). Similarly, one checks that a tracial state $\tau_I$ on $I$ extends to a trace on $D$ via the formula $\tau_I(d) = \lim_n \tau(f_n d)$. With these facts in hand, it is clear that if one starts with a tracial state $\tau$ on $D$, then defining $\tau_I := \frac{1}{c}\tau|_I$ we have $\tau = (1-c)\tau_{D/I} + c\tau_I$.

We will apply these remarks to the short exact sequences

$$0 \to \mathbb{M}_{s(1)s(2)\cdots s(m-1)}\Big(\bigoplus_{n=m}^{\infty} \mathbb{M}_{k(n)}\Big) \to \mathbb{M}_{s(1)s(2)\cdots s(m-1)}(R_m) \stackrel{\sigma_m}{\to} \mathbb{M}_{s(1)s(2)\cdots s(m-1)}(A) \to 0.$$

However, for notational convenience we first set $I_m = \bigoplus_{n=m}^{\infty} \mathbb{M}_{k(n)}$,

$$S(m) = s(1)s(2)\cdots s(m-1)$$

and let $\mathrm{tr}_n$ denote the unique tracial state on $\mathbb{M}_n$. Then each (not necessarily normalized) trace $\gamma$ on $\mathbb{M}_{S(m)} \otimes I_m$ has a unique decomposition

$$\gamma = \mathrm{tr}_{S(m)} \otimes \Big(\sum_{n=m}^{\infty} \alpha_n \,\mathrm{tr}_{k(n)}\Big),$$

where $\alpha_n = \gamma(1 \otimes e_n)$.

**Lemma 2.2.** *Via the canonical inclusions $\mathbb{M}_{S(m)} \otimes I_m \subset \mathbb{M}_{S(m)} \otimes R_m \subset C_A$, every trace on $C_A$ restricts to the same thing on $\mathbb{M}_{S(m)} \otimes I_m$. More precisely, if*

[2] It can be shown directly that $\alpha$ is multiplicative, but the proof is no easier. On the other hand, it forces one to fully absorb the constructions of $B_A$ and $C_A$, so the interested reader may want to work it out.

*$\tau \in \mathrm{T}(C_A)$, then*

$$\tau|_{\mathbb{M}_{S(m)}\otimes I_m} = \mathrm{tr}_{S(m)} \otimes \left( \frac{k(m)}{s(m)} \mathrm{tr}_{k(m)} + \sum_{i=m+1}^{\infty} \left( \frac{k(i)}{s(i)} \prod_{j=m}^{i-1} \left( 1 - \frac{k(j)}{s(j)} \right) \right) \mathrm{tr}_{k(i)} \right).$$

*Moreover, if*

$$c_m = \frac{k(m)}{s(m)} + \sum_{i=m+1}^{\infty} \left( \frac{k(i)}{s(i)} \prod_{j=m}^{i-1} \left( 1 - \frac{k(j)}{s(j)} \right) \right),$$

*then $\tau_m := \frac{1}{c_m}\tau|_{\mathbb{M}_{S(m)}\otimes I_m}$ is a tracial state on $\mathbb{M}_{S(m)} \otimes I_m$ and there exists a tracial state $\tau_A$ on $A$ (depending only on $\tau$) such that*

$$\tau|_{\mathbb{M}_{S(m)}\otimes R_m} = (1 - c_m)(\mathrm{tr}_{S(m)} \otimes \tau_A) + c_m \tau_m$$

*for all $m \in \mathbb{N}$.*

*Proof.* The first assertion is an unpleasant, but otherwise straightforward calculation which we leave to the reader.

To get the trace $\tau_A$, recall that $\theta\colon A \to R_1$ is a u.c.p. splitting for the quotient map $R_1 \to A$. Composing with the quotient maps $R_1 \to R_k$ we get u.c.p. splittings $\theta_k\colon A \to R_k$ for the maps $R_k \to A$. Since each $R_k$ is a direct summand of $R_1$, we can regard $\theta_k$ as a non-unital map into $R_1$ and then (via the canonical inclusion $R_1 \subset C_A$) the limit of the maps

$$\tau \circ \theta_k$$

is a tracial functional which we can renormalize and denote by $\tau_A$.

To see that the general quotient procedure applied to $\tau|_{\mathbb{M}_{S(m))}\otimes R_m}$ produces $\mathrm{tr}_{S(m)} \otimes \tau_A$, note that for all $k > m$, $\theta_k\colon A \to R_1$ composed with $\pi_{m,1}\colon R_1 \to \mathbb{M}_{S(m)} \otimes R_m$ yields the formula

$$\pi_{m,1} \circ \theta_k(a) = Q_1 \otimes Q_2 \otimes \cdots \otimes Q_{m-1} \otimes \theta_k(a),$$

where we use the fact that $R_k$ is also a summand of $R_m$ to make sense of the right hand side. Since (for all $k \geq m$) $\mathrm{id}_{S(m)} \otimes \theta_k\colon \mathbb{M}_{S(m)} \otimes A \to \mathbb{M}_{S(m)} \otimes R_k$ is a splitting for the quotient map $\mathbb{M}_{S(m)} \otimes R_k \to \mathbb{M}_{S(m)} \otimes A$, this evidently implies the maps

$$\tau \circ (\mathrm{id}_{S(m)} \otimes \theta_k)$$

converge to a multiple of $\mathrm{tr}_{S(m)} \otimes \tau_A$, so the proof is complete. □

A simple computation establishes the relation

$$c_m = \frac{k(m)}{s(m)} + \left( 1 - \frac{k(m)}{s(m)} \right) c_{m+1},$$

for all $m \in \mathbb{N}$. In other words, $c_m$ is a convex combination of 1 and $c_{m+1}$. We record three useful consequences of this relation.

**Lemma 2.3.** *The following assertions also hold.*

(1) $1 \geq c_1 \geq c_2 \geq c_3 \geq \cdots$.

(2) $c_1 = 1$ *if and only if* $c_m = 1$ *for all* $m \in \mathbb{N}$.
(3) *If* $c_1 < 1$, *then* $\lim \frac{k(m)}{s(m)} = 0$.

*Proof.* The first statement is immediate from the convexity relation. The second follows from convexity, too.

For the third statement, first note that

$$c_m - c_{m+1} = \frac{k(m)}{s(m)}(1 - c_{m+1}).$$

Since $1 \geq c_1 \geq c_{m+1}$, this implies

$$c_m - c_{m+1} \geq \frac{k(m)}{s(m)}(1 - c_1).$$

Thus, if $1 - c_1 > 0$ and $\liminf \frac{k(m)}{s(m)} > 0$, then there is a uniform lower bound on the distance between $c_m$ and $c_{m+1}$, which contradicts the fact that $c_m > 0$, for all $m \in \mathbb{N}$. □

We remind the reader of the freedom one has in choosing the $s(m)$'s (we only require $s(m) > k(m)$). Since we're trying to construct "exotic" algebras, the main case of interest will be when $\lim \frac{k(m)}{s(m)} = 0$, because philosophically one is then tucking $A$ into somewhat singular slices of $B_A$ (meaning slices of tiny trace).

**Proposition 2.4.** *There are two possibilities for the tracial state space of* $C_A$.
(1) *If* $c_1 = 1$, *then* $C_A$ *has a unique tracial state.*
(2) *If* $c_1 < 1$, *then the tracial state space of* $C_A$ *is homeomorphic to that of* $A$.

*Proof.* If $c_m = 1$ for all $m \in \mathbb{N}$, then Lemma 2.2 implies any tracial state on $C_A$ restricts to the same thing on $\mathbb{M}_{S(m)} \otimes R_m$. Since any trace on $C_A$ is determined by the restrictions to $\mathbb{M}_{S(m)} \otimes R_m$, this implies uniqueness.

If $c_1 < 1$, then we define maps $\gamma_m \colon \mathrm{T}(A) \to \mathrm{T}(R_m)$ by

$$\gamma_m(\tau) := (1 - c_m)\tau + c_m\tau_m,$$

where $\tau_m$ is defined in the previous lemma. Note that each $\gamma_m$ is affine, continuous and injective. Roughly, we want to define the desired homeomorphism $\mathrm{T}(A) \cong \mathrm{T}(C_A)$ as a limit of the $\gamma_m$'s, but to make sense of this we must have some compatibility with the connecting maps $\pi_m|_{R_m} \colon R_m \to \mathbb{M}_{s(m)} \otimes R_{m+1}$. To this end, the following computations will be needed (and left to the reader).

(i) For all $\tau \in \mathrm{T}(A)$ and $x \in R_m$,

$$\gamma_m(\tau)(x) = \big((1 - c_m)\tau + c_m\tau_m\big)(e_m^{\perp}x) + \frac{k(m)}{s(m)}\,\mathrm{tr}_{k(m)}(e_m x).$$

(ii) For all $\tau \in \mathrm{T}(A)$ and $x \in R_m$, $\big(\mathrm{tr}_{s(m)} \otimes \gamma_{m+1}(\tau)\big) \circ \pi_m(x)$ is equal to

$$\frac{s(m) - k(m)}{s(m)}\big((1 - c_{m+1})\tau + c_{m+1}\tau_{m+1}\big)(e_m^{\perp}x) + \frac{k(m)}{s(m)}\,\mathrm{tr}_{k(m)}(e_m x).$$

(iii) For all $m \in \mathbb{N}$,
$$\frac{s(m) - k(m)}{s(m)}(1 - c_{m+1}) = 1 - c_m.$$
(iv) For all $x \in R_m$,
$$\frac{s(m) - k(m)}{s(m)}(c_{m+1}\tau_{m+1})(e_m^{\perp}x) = c_m\tau_m(e_m^{\perp}x).$$
Putting these four facts together, one can check that
$$\mathrm{tr}_{S(m)} \otimes \gamma_m(\tau) = \big(\mathrm{tr}_{S(m+1)} \otimes \gamma_{m+1}(\tau)\big) \circ \pi_m$$
for all $m \in \mathbb{N}$ and $\tau \in \mathrm{T}(A)$. Hence we can define a trace on $C_A$ as the limit – not a cluster point, but an honest limit – of the traces $\mathrm{tr}_{S(m)} \otimes \gamma_m(\tau)$ on $\mathbb{M}_{S(m)} \otimes R_m$. That this correspondence $\mathrm{T}(A) \to \mathrm{T}(C_A)$ is surjective follows from Lemma 2.2. That it is continuous, affine and injective follows from the fact that each $\gamma_m$ has said properties. □

## 3. Structure of $B_A$

Here are some properties of $B_A$ that don't depend on the choice of $A$.

**Theorem 3.1.** *The generalized inductive limit $B_A$ is unital, separable, simple, quasidiagonal and has stable rank one. Moreover,*

(1) *if* $\liminf \frac{k(n)}{s(n)} = 1$*, then $B_A$ is tracially AF in the sense of* [8]*;*
(2) *if* $\liminf \frac{k(n)}{s(n)} < 1$*, then $B_A$ is approximately divisible and (hence) $\mathcal{Z}$-stable.*

*Proof.* It is clear that $B_A$ is unital and separable. Quasidiagonality follows from the isomorphism $C_A \cong B_A$, since $C_A$ is an increasing union of residually finite dimensional algebras. Simplicity also follows from this isomorphism, since $C_A$ is easily seen to be simple. Stable rank one will follow from items (1) and (2), once established, because tracially AF algebras and finite (e.g., QD) approximately divisible C*-algebras always have stable rank one (see [8, Theorem 3.4] and [2, Theorem 1.4 (c)], respectively).

Case (1): Suppose we're given a finite set $\mathcal{F} \subset C_A$, a nonzero positive element $a \in C_A$, $\varepsilon > 0$ and $n \in \mathbb{N}$; we must find a finite-dimensional C*-subalgebra $D \subset C_A$ with unit $p$ such that $\|[x, p]\| < \varepsilon$ and $\mathrm{dist}(pxp, D) < \varepsilon$ for all $x \in \mathcal{F}$, $n[1 - p] \leq [p]$ in the Murray-von Neumann semigroup of projections, and find a projection in the hereditary subalgebra generated by $a$ that is equivalent to $1-p$.

Without loss of generality, we can assume $\mathcal{F} \subset \mathbb{M}_{S(m)} \otimes R_m$ for some $m \in \mathbb{N}$. But for now we'll also assume $a \in R_1$, then explain how to handle the general case in a moment. Since $a \neq 0$, there is $k \in \mathbb{N}$ such that $e_k a \neq 0$. Since $a \geq e_k a$ and
$$\pi_{k+1,1}(e_k a) = Q_1 \otimes \cdots \otimes Q_{k-1} \otimes (e_k a) \otimes 1_{R_{k+1}}$$
it follows that the hereditary subalgebra generated by $a$ contains a non-zero projection of the form $q \otimes 1_{R_{k+1}} \in \mathbb{M}_{S(k)} \otimes R_{k+1}$. Now, choose $i \in \mathbb{N}$ so that

$i > \max\{k, m\}$, $\frac{k(i)}{s(i)} > n(1 - \frac{k(i)}{s(i)})$ and $(1 - \frac{k(i)}{s(i)}) < \mathrm{tr}_{S(k)}(q)$. In this case, we can define

$$D := \mathbb{M}_{s(1)} \otimes \cdots \otimes \mathbb{M}_{s(i-1)} \otimes Q_i^{\perp} \mathbb{M}_{s(i)} Q_i^{\perp} \otimes 1_{R_{i+1}} \subset \mathbb{M}_{S(i+1)} \otimes R_{i+1}.$$

The explicit form of the connecting map $\pi_i \colon \mathbb{M}_{S(i)} \otimes R_i \to \mathbb{M}_{S(i+1)} \otimes R_{i+1}$ ensures that the unit $p$ of $D$ commutes with $\pi_i(\mathbb{M}_{S(i)} \otimes R_i)$ (in particular, with the image of $\mathcal{F}$) and $pxp \in D$ for all $x \in \pi_i(\mathbb{M}_{S(i)} \otimes R_i)$. Finally, the inequality $\frac{k(i)}{s(i)} > n(1 - \frac{k(i)}{s(i)})$ implies $n[1 - p] < [p]$ (since the trace of $(Q_i^{\perp})^{\perp} = Q_i$ is $1 - \frac{k(i)}{s(i)}$), while $(1 - \frac{k(i)}{s(i)}) < \mathrm{tr}_{S(k)}(q)$ guarantees that $1 - p$ is Murray-von Neumann equivalent to a subprojection of (the image of) $q \otimes 1_{R_{k+1}} \in \mathbb{M}_{S(k)} \otimes R_{k+1}$ (since both projections belong to $\mathbb{M}_{S(i+1)} \otimes 1_{R_{i+1}}$).

So that handles the case $a \in R_1$ and it isn't hard to adapt the argument to handle positive elements in $\mathbb{M}_{S(m)} \otimes R_m$. Thus we've verified the tracially AF axioms for positive elements in a set of dense subalgebras of $C_A$, and this is (surely known to be) good enough. However, we're unaware of a proof in the literature, so here's a sketch. If $0 \neq a \in C_A$ is arbitrary, then for all $\delta > 0$ we can find a positive element $b \in \mathbb{M}_{S(m)} \otimes R_m$ such that $\|a - b\| < \delta$. If $\delta$ is sufficiently small, we can find a nonzero positive element $c \in \mathbb{M}_{S(m)} \otimes R_m$ such that $c \precsim a$, meaning $c$ is Cuntz-dominated by $a$. (See [7, Definition 2.1] for $\precsim$ and [7, Lemma 2.5 (ii)] for the construction of $c$.) It follows that any projection in the hereditary subalgebra generated by $c$ is also Cuntz-dominated by $a$. But this implies all such projections are Murray-von Neumann equivalent to projections in the hereditary subalgebra generated by $a$ (cf. [7, Proposition 2.6] and the sentence that follows it). And this implies what we want.

Case (2): Going out far enough in the inductive sequence, we may assume $s(1) - k(1) \geq 2$ for all $n \in \mathbb{N}$. (Note that each step of our inductive system is of the same type, so there is no loss of generality here.) Let's see why $R_1 \subset C_A$ has matrices which almost commute with it. More precisely, given a finite set $\mathcal{F} \subset R_1$ we'll find a finite-dimensional subalgebra $D_1 \oplus D_2 \subset C_A$ which almost commutes with $\mathcal{F}$ and such that neither of the $D_i$'s have a commutative summand. For each element $x \in \mathcal{F}$ we write it as the orthogonal sum $e_1 x + e_1^{\perp} x$ and note that $D_1 := (Q_1 \otimes 1)(\mathbb{M}_{s(1)} \otimes R_1)(Q_1 \otimes 1)$ commutes with $\pi_1(e_1^{\perp} x)$ (and it's noncommutative, since $s(1) - k(1) \geq 2$). In the orthogonal corner, $e_1 C_A e_1$, we note that $e_1 x$ belongs to the unital AF algebra $(\bigoplus_1^{\infty} \mathbb{M}_{k(n)})\tilde{}$ and the restriction of our inductive system to this AF algebra yields a *simple* unital AF algebra that we shall denote by $D$. But now we're done because $e_1 D e_1 (\subset e_1 C_A e_1)$ is also simple unital and AF – hence approximately divisible by [2, Proposition 4.1] – so we can find an appropriate unital finite-dimensional algebra $D_2 \subset D$ that approximately commutes with $e_1 \mathcal{F}$.

A similar argument shows that one can find appropriate finite-dimensional algebras that almost commute with finite sets in any of the algebras $\mathbb{M}_{S(m)} \otimes R_m$, and this completes the proof. □

Here are a few properties of $A$ that propagate to $B_A$.

**Theorem 3.2.** *The following statements are all true:*

1. *if $A$ is nuclear (resp. exact), then $B_A$ is nuclear (resp. exact);*
2. *if $A$ satisfies the Universal Coefficient Theorem (cf.* [11]*), then so does $B_A$;*
3. *if $A$ has a unique tracial state, then so does $B_A$;*
4. *if every tracial state on $A$ is uniformly locally finite dimensional*[3]*, then the same is true for $B_A$.*

*Proof.* (1): Each of the algebras $R_n$ will be nuclear (resp. exact) whenever $A$ is nuclear (resp. exact) (cf. [5, Chapter 10]), which implies the inductive limit $C_A$ is also nuclear (resp. exact).

(2): Since each $R_n$ is an extension of $A$ by an AF algebra, it will satisfy the Universal Coefficient Theorem whenever $A$ does (cf. [11]). Since this property passes to inductive limits, we conclude the same for $C_A$.

(3) follows easily from Lemma 2.2.

(4): It suffices to show that if $\tau$ is a trace on $C_A$, then for every $m \in \mathbb{N}$ the restriction $\tau|_{R_m}$ is uniformly locally finite dimensional (cf. [3, Lemma 4.4.1]). This, however, is also a consequence of Lemma 2.2 since all the traces $\mathrm{tr}_{k(i)}$ on $R_m$ have finite-dimensional GNS representations. □

The following corollary holds whenever $A$ is exact and has a unique trace that is uniformly locally finite dimensional, but we'll only state it for the examples that inspired this paper.

**Corollary 3.3.** *If $A$ is the unitization of an exact, QD, purely infinite C*-algebra, then $B_A$ is tracially AF. If $A$ is nuclear and also satisfies the UCT, then $B_A$ is an AH algebra.*

*Proof.* We only have to handle the case that $\liminf \frac{k(n)}{s(n)} < 1$, where Theorem 3.1 tells us $B_A$ is approximately divisible. Among other things, this implies $K_0(B_A)$ is weakly unperforated by [2, Corollary 3.9(b)]. Since purely infinite algebras admit no traces, $A$ has a unique trace (namely, the obvious quotient map $A \to \mathbb{C}$). So part (3) of Theorem 3.2 implies $B_A$ also has a unique trace. Moreover it is uniformly locally finite dimensional, by part (4) of the same theorem. Approximate divisibility plus uniqueness of the trace implies $B_A$ has real rank zero (see [2, Theorem 1.4(f)]) and we established stable rank one in Theorem 3.1.

Recapping, $B_A$ is a simple, unital, C*-algebra with real rank zero, stable rank one, weakly unperforated K-theory, and it has a unique trace that is uniformly locally finite dimensional. This allows us to invoke [3, Proposition 4.5.5] and deduce that $B_A$ is tracially AF.

[3]A trace $\tau$ has this property if for each finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a u.c.p. map $\varphi\colon A \to \mathbb{M}_n(\mathbb{C})$ such that $\|\tau - \mathrm{tr}_n \circ \varphi\|_{A^*} < \varepsilon$ and every element of $\mathcal{F}$ is within $\varepsilon$ (in norm) of some element in the multiplicative domain of $\varphi$ (cf. [3, Definition 3.4.1]).

If $A$ is nuclear and satisfies the UCT, then Theorem 3.2 ensures that $B_A$ has these properties, too. Hence Lin's classification theorem ([9]) applies and $B_A$ must be isomorphic to an AH algebra. $\square$

If $A$ is the unitization of the cone over a nuclear, purely infinite algebra, then K-theory calculations show that we get an AF algebra in the limit. (Note that cones always satisfy the UCT.) Indeed, in this case all the algebras $R_m$ have the K-theory of an AF algebra, which can be seen from the standard six-term exact sequence.

## References


[1] B. Blackadar and E. Kirchberg, *Generalized inductive limits of finite-dimensional $C^*$-algebras*, Math. Ann. **307** (1997), 343–380.

[2] B. Blackadar, A. Kumjian and M. Rørdam, *Approximately central matrix units and the structure of noncommutative tori*, $K$-Theory **6** (1992), 267–284.

[3] N.P. Brown, *Invariant means and finite representation theory of $C^*$-algebras*, Mem. Amer. Math. Soc. **184** (2006), no. 865, viii+105 pp.

[4] N.P. Brown, *Inductive limits, unique traces and tracial rank zero*, Bull. Lond. Math. Soc. **39(3)** (2007), 377–383.

[5] N.P. Brown and N. Ozawa, *$C^*$-algebras and finite-dimensional approximations.* Graduate Studies in Mathematics, 88. American Mathematical Society, Providence, RI, 2008. xvi+509 pp.

[6] M. Dadarlat, *Nonnuclear subalgebras of AF algebras*, Amer. J. Math. **122** (2000), 581–597.

[7] E. Kirchberg and M. Rørdam, *Non-simple purely infinite $C^*$-algebras*, Amer. J. Math. **122** (2000), 637–666.

[8] H. Lin, *Tracially AF $C^*$-algebras*, Trans. Amer. Math. Soc. **353** (2001), 693–722.

[9] H. Lin, *Classification of simple $C^*$-algebras of tracial topological rank zero*, Duke Math. J. **125** (2004), 91–119.

[10] H. Lin, *Traces and simple $C^*$-algebras with tracial topological rank zero*, J. Reine Angew. Math. **568** (2004), 99–137.

[11] J. Rosenberg and C. Schochet, *The Kunneth theorem and the universal coefficient theorem for Kasparov's generalized $K$-functor*, Duke Math. J. **55** (1987), 431–474.

[12] D.V. Voiculescu, *A note on quasi-diagonal $C^*$-algebras and homotopy*, Duke Math. J. **62** (1991), 267–271.

[13] W. Winter *Decomposition rank and $\mathcal{Z}$-stability*, Invent. Math. **179** (2010), 229–301.

[14] W. Winter and J. Zacharias, *The nuclear dimension of $C^*$-algebras*, Adv. Math. **224** (2010), 461–498.



Department of Mathematics
Penn State University
State College, PA 16802
USA

*E-mail address*: nbrown@math.psu.edu