ON ABSOLUTENESS OF CATEGORICITY IN ABSTRACT ELEMENTARY CLASSES

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Abstract. Shelah has shown in [4] that $\aleph_1$-categoricity for Abstract Elementary Classes (AEC's) is not absolute in the following sense: There is an example K of an AEC (which is actually axiomatizable in the logic $L(Q)$) such that if $2^{\aleph_0} < 2^{\aleph_1}$ (the weak CH holds) then K has the maximum possible number of models of size $\aleph_1$, whereas if Martin's Axiom at $\aleph_1$ (denoted by $\mathrm{MA}_{\aleph_1}$) holds then K is $\aleph_1$-categorical. In this note we extract the properties from Shelah's example which make both parts work resulting in our definitions of condition A and condition B, and then we show that for any AEC satisfying these two conditions, neither of these implications can be reversed.

1. The model theoretic context

In Shelah's paper [4], the notion of *Abstract Elementary Classes* (AEC) was introduced, the idea being to write down basic properties of the first order elementary substructure relation.

Definition 1. *Let K be a class of models of a given similarity type and let $\prec$ be a partial ordering on K refining the ordinary substructure relation. The pair $\mathcal{K} = (K, \prec)$ is an* AEC*, if*

(1) *both K and $\prec$ are closed under isomorphism.*
(2) *$A \prec C$, $B \prec C$ and $A \subset B$ imply $A \prec B$*
(3) *for any continuous $\prec$-chain $(A_\alpha)_{\alpha<\lambda}$,*
   (a) $A = \bigcup_{\alpha<\lambda} A_\alpha \in K$
   (b) *for all $\alpha < \lambda$, $A_\alpha \prec A$*
   (c) *if $A_\alpha \prec B$ for some B and all $\alpha < \lambda$, then $A \prec B$*

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(4) *There is a cardinal* LS(K) *such that for all* $A \in K$ *and any* subset $A_0 \subset A$, *there is* $B \prec A$ *containing* A_0 *with* $|B| \leq |A_0| + \mathrm{LS}(K)$.

Many non-elementary classes can be made an AEC with appropriate relations $\prec$, such as classes axiomatized using an additional quantifier Q saying "there are uncountably many" (we will see an example of this later), or classes axiomatized by $L_{\omega_1,\omega}$-sentences (first order with infinite countable conjunctions and disjunctions) with $\prec$ being elementary substructure with respect to some countable fragment of $L_{\omega_1,\omega}$.

It becomes an interesting question to what extent results of first order model theory such as Morley's categoricity theorem extend to arbitrary AEC, or perhaps to AEC with some special properties. Some work in this direction is exposed in Baldwin's book [2], which has a particular emphasis on $L_{\omega_1,\omega}$.

2. Model theoretic properties: condition A and B

We now introduce two properties AEC can have. First, we have to fix some notation.

Notation 2. *Let* $(M_\alpha)_{\alpha<\beta}$ *and* $(N_\alpha)_{\alpha<\beta}$ *be continous, strictly increasing (with respect to inclusion) sequences of structures.*

- *We write* $(M_\alpha)_{\alpha<\beta} \cong (N_\alpha)_{\alpha<\beta}$ *if there exists a function* $f\colon \bigcup_{\alpha<\beta} M_\alpha \to \bigcup_{\alpha<\beta} N_\alpha$ *such that for all* $\alpha < \beta$, $f \restriction M_\alpha$ *is an isomorphism between* M_α *and* N_α. *We call such an* f *a* filtration automorphism *if* $M_\alpha = N_\alpha$ *for all* $\alpha < \beta$.
- *Define* rank: $\bigcup_{\alpha<\beta} M_\alpha \to \beta$ *by* $\mathrm{rank}(a) = \min\{\alpha | a \in M_\alpha\}$. *Note that, by continuity of the chain, the range of* rank *is precisely the set of countable successor ordinals together with zero.*
- *For any finite tuple* $\bar{a}$ *in* $\bigcup_{\alpha<\beta} \mathrm{M}_\alpha$ *and* $\alpha < \beta$, *let* $\bar{a}_\alpha$ *be the subtuple of* $\bar{a}$ *of elements of rank* α.
- *Considering a tuple* $\bar{a} = (a_0, a_1, \ldots, a_{n-1})$ *as a function with domain* $n = \{0, 1, \ldots, n-1\}$ *(via* $\bar{a}(i) = a_i$*), let* $s_{\bar{a}} = \mathrm{rank} \circ \bar{a}$ *(i.e.* $s_{\bar{a}}(i) = \mathrm{rank}(a_i)$ *for all* $i < n$*).*
- *Let* $\mathrm{tp}_{\mathrm{qf}}(\bar{a})$ *denote the quantifier free type of* $\bar{a}$ *(over the empty set).*

Definition 3. *Let* $(\mathcal{K}, \prec)$ *be an AEC in a relational signature with Löwenheim-Skolem number* $\aleph_0$. *We say that*

(1) $(\mathcal{K}, \prec)$ *satisfies* condition A, *if it is* $\aleph_0$*-categorical and fails amalgamation for countable models (i.e. there is a triple of countable models* $M_0 \prec M_1, M_2$ *such that there are no countable* M_3 *and embeddings* $f_i\colon M_i \to M_3$ *(*$i = 1, 2$*) with* $f[M_i] \prec M_3$ *and* $f_1 \restriction M_0 = f_2 \restriction M_0$*).*

(2) $(\mathcal{K}, \prec)$ *satisfies* condition B, *if there is an increasing and continous $\prec$-chain $(M_\alpha)_{\alpha<\omega_1}$ of countable models such that*
   (i) *(decomposition) any $N \in \mathcal{K}$ of size $\aleph_1$ can be written as $N = \bigcup_{\alpha<\omega_1} N_\alpha$ with $(N_\alpha)_{\alpha<\beta} \cong (M_\alpha)_{\alpha<\beta}$ for all $\beta < \omega_1$*
   (ii) *(triviality) For any $N = \bigcup_{\alpha<\omega_1} N_\alpha$ as in (i), and any finite tuples $\bar{a}, \bar{b}, \bar{c}$ in N with $\max(s_{\bar{c}}) < \min(s_{\bar{a}})$, if $s_{\bar{b}} = s_{\bar{a}}$ and for all α $\mathrm{tp}_{\mathrm{qf}}(\bar{b}_\alpha \bar{c}) = \mathrm{tp}_{\mathrm{qf}}(\bar{a}_\alpha \bar{c})$ then $\mathrm{tp}_{\mathrm{qf}}(\bar{b}\bar{c}) = \mathrm{tp}_{\mathrm{qf}}(\bar{a}\bar{c})$.*
   (iii) *(homogeneity) Suppose $N = \bigcup_{\alpha<\omega_1} N_\alpha$ is as in (i) and $\bar{a}, \bar{b}$ are finite tuples in N such that there is an isomorphism $f : \bar{a} \to \bar{b}$ with $x \in N_\alpha$ if and only if $f(x) \in N_\alpha$ for all $x \in \mathrm{dom}(f)$ and $\alpha < \omega_1$. Then for any $\beta > \max(s_{\bar{a}}), \max(s_{\bar{b}})$, there is a filtration automorphism of $(N_\alpha)_{\alpha<\beta}$ extending f.*

3. How set theory affects the number of models

Theorem 4. *If $2^{\aleph_0} < 2^{\aleph_1}$ and condition A holds, then $\mathcal{K}$ has $2^{\aleph_1}$ many non-isomorphic models of size $\aleph_1$.*

Proof. This result and its proof are exposed in [2], Theorem 17.11. □

The proof of the following result is an abstract version of the proof given for Shelah's specific $L(Q)$-example (Theorem 6.6 in [4]). A simpler version can also be found in [2].

Theorem 5. *Martin's Axiom at $\aleph_1$ and condition B imply that $\mathcal{K}$ is $\aleph_1$-categorical.*

Proof. Let $N^i = \bigcup_{\alpha<\omega_1} N^i_\alpha$ (for $i < 2$) with $(N^i_\alpha)_{\alpha<\beta} \cong (M_\alpha)_{\alpha<\beta}$ for all $\beta < \omega_1$ (by (decomposition)). Let $\mathcal{F}$ be the set of finite partial isomorphisms f from N^0 to N^1 with $x \in N^0_\alpha$ if and only if $f(x) \in N^1_\alpha$ for all $x \in \mathrm{dom}(f)$ and $\alpha < \omega_1$. We show that the partial order $(\mathcal{F}, \supset)$ has the ccc:

Let $\{f_i | i < \omega_1\} \subset \mathcal{F}$. We attempt to find two distinct f_i whose union is an element of $\mathcal{F}$. By simple applications of the delta system lemma and the pigeonhole principle, we can assume the following:

- There is some $n < \omega$ such that for all $i < \omega_1$, $|\mathrm{dom}(f_i)| = |\mathrm{ran}(f_i)| = n$
- The sets $\{\mathrm{dom}(f_i) | i < \omega_1\}$ and $\{\mathrm{ran}(f_i) | i < \omega_1\}$ are delta systems with roots r and r' respectively, and for any $i < \omega_1$, $\max(s_r) < \min(s_{\mathrm{dom}(f_i)\setminus r})$ and $\max(s_{r'}) < \min(s_{\mathrm{ran}(f_i)\setminus r'})$
- For all $i < j < \omega_1$, $f_i \restriction r = f_j \restriction r$ and $\mathrm{ran}(f_i \restriction r) = r'$
- (filtration disjointness) For all $i < j < \omega_1$, $\mathrm{ran}(s_{\mathrm{dom}(f_i)\setminus r})$ is disjoint from $\mathrm{ran}(s_{\mathrm{dom}(f_j)\setminus r})$ (and thus, since the f_i preserve the filtrations, the same holds for the ranges).

Now we claim that actually the union of any two f_i is an element of $\mathcal{F}$. Take $i < j < \omega_1$ and set $g = f_i \cup f_j$. Let $\bar{a} = \mathrm{dom}(f_i)\backslash r$, $\bar{b} = \mathrm{dom}(f_j)\backslash r$. For any relation symbol R in our signature, we want to show that $N^0 \models R(\bar{a}, \bar{b}, r)$ holds if and only if $N^1 \models R(g(\bar{a}), g(\bar{b}), r')$ (not all elements of the tuples may actually occur in R). Let $\gamma < \omega_1$ be greater than $\max(s_{\bar{a}})$ and $\max(s_{\bar{b}})$ and (by (decomposition)) choose any h witnessing $(N^0_\alpha)_{\alpha<\gamma} \cong (N^1_\alpha)_{\alpha<\gamma}$. By (homogeneity), we can assume that $h \restriction r = f_i \restriction r (= f_j \restriction r)$. Because $f_i, f_j \in \mathcal{F}$, $\mathrm{tp}_{\mathrm{qf}}(g(\bar{a}), r') = \mathrm{tp}_{\mathrm{qf}}(\bar{a}, r) = \mathrm{tp}_{\mathrm{qf}}(h(\bar{a}), r')$ and $\mathrm{tp}_{\mathrm{qf}}(g(\bar{b}), r') = \mathrm{tp}_{\mathrm{qf}}(\bar{b}, r) = \mathrm{tp}_{\mathrm{qf}}(h(\bar{b}), r')$ and thus by (triviality) (using (filtration disjointness)),

$$\mathrm{tp}_{\mathrm{qf}}(h(\bar{a}), h(\bar{b}), r') = \mathrm{tp}_{\mathrm{qf}}(g(\bar{a}), g(\bar{b}), r'). \qquad (*)$$

This means that $N^0 \models R(\bar{a}, \bar{b}, r)$ if and only if (h is an isomorphism) $N^1 \models R(h(\bar{a}), h(\bar{b}), r')$ if and only if (by $(*)$) $N^1 \models R(g(\bar{a}), g(\bar{b}), r')$. This finishes the proof of ccc.

Now we prove that the sets $D_a = \{f \in \mathcal{F} | a \in \mathrm{dom}(f)\}$, $R_b = \{f \in \mathcal{F} | b \in \mathrm{ran}(f)\}$ (for $a \in N^0$, $b \in N^1$) are dense in $(\mathcal{F}, \supset)$. Take any $g \in \mathcal{F}$, $a \in N^0$ and, using (decomposition), an h witnessing $(N^0_\alpha)_{\alpha<\beta} \cong (N^1_\alpha)_{\alpha<\beta}$ for some β greater than $\max(s_{\mathrm{dom}(g)})$ and $\max(s_a)$. By (homogeneity), there is a filtration automorphism k of $(N^1_\alpha)_{\alpha<\beta}$ mapping $h[\mathrm{dom}(g)]$ to $\mathrm{ran}(g)$ such that on $\mathrm{dom}(g)$ we have $k \circ h = g$. Now, $g' = k \circ h \restriction (\mathrm{dom}(g) \cup \{a\})$ is an extension of g with $g' \in D_a$. The same argument also works for R_b.

Finally we apply Martin's Axiom to the partial order $(\mathcal{F}, \supset)$ to get a $\{D_a | a \in N^0\} \cup \{R_b | b \in N^1\}$-generic filter G. $\bigcup G$ is a total isomorphism between N^0 and N^1. Because the N^i were arbitrary models in $\mathcal{K}$ of size $\aleph_1$, $\aleph_1$-categoricity of $\mathcal{K}$ follows. □

The example given in the proof of the following Theorem is due to Shelah an can be found in [4].

Theorem 6. *There is an AEC satisfying both condition A and condition B.*

Proof. Let ψ be the $L_{\omega_1,\omega}(Q)$-sentence in the signature $L = \{P, Q, R, E\}$ (P, Q unary predicates, R, E binary relations) stating:

(1) P, Q partition the universe and P is infinite, countable.
(2) E is an equivalence relation on Q with infinitely many classes, each countably infinite.
(3) $R \subset P \times Q$ has the following properties:
  (3a) For any finite disjoint $F, G \subset Q$, there is some $a \in P$ such that for all $b \in F \cup G$, $R(a, b)$ if and only if $b \in F$.
  (3b) For any finite disjoint $F, G \subset P$, there is some $b \in Q$ *in each E-class* such that for all $a \in F \cup G$, $R(a, b)$ if and only if $a \in F$.

It is easy to see that $\mathcal{K} = \mathrm{mod}(\psi)$ together with the substructure relation $\prec$ defined by

$$M \prec N \text{ if and only if } M \subset N, P^M = P^N \text{ and no element of } N \setminus M$$
$$\text{is } E\text{-equivalent to an element of } M$$

is an AEC with $\mathrm{LS}(K) = \aleph_0$. Note that by (3a), in any model of ψ, the collection of all sets $A_q = \{p \in P | R(p,q\}$ $(q \in Q)$ is an *independent family* in the sense that any intersection of finitely many distinct sets or their complements is non-empty.

Amalgamation fails for countable models: take for M_0 any countable model and let M_1, M_2 be extensions where we add one E-class B_1, B_2 respectively to M_0 such that there are $b_1 \in B_1$ and $b_2 \in B_2$ with $R(a, b_1)$ if and only if $\neg R(a, b_2)$ for all $a \in P$. Such extensions exist by the facts that countable independent families are not maximal (even with the additional requirement of (3b)), and that an independent family stays independent if we replace some set with its complement.

Clearly, M_1 and M_2 do not amalgamate over M_0 because the amalgam would fail property (3a).

Now let M_0 be any countable model of ψ and define M_α for $\alpha < \omega_1$ by induction: at limits take unions and let $M_{\alpha+1}$ be such that $M_{\alpha+1} \setminus M_\alpha$ consists of exactly one E-class. We first show that the sequence $(M_\alpha)_{\alpha<\omega_1}$ witnesses (decomposition):

Let N be any model of ψ of size $\aleph_1$, let $N_0 \prec N$ be countable and define inductively a continuous $\prec$-chain in N of models N_α such that $N_{\alpha+1} \setminus N_\alpha$ consists of exactly one E-class and such that $N = \bigcup_{\alpha<\omega_1} N_\alpha$. Let $\beta < \omega_1$ and f be a *finite partial* isomorphism $f\colon (N_\alpha)_{\alpha<\beta} \to (M_\alpha)_{\alpha<\beta}$. We want to extend f to a (still filtration preserving) partial isomorphism with domain $\mathrm{dom}(f) \cup \{a\}$ for any given $a \in N_\beta$. If $P(a)$, this is possible by (3b), if $Q(a)$, we use (3a).

This "filtration preserving extension property" for finite partial isomorphisms shows not only (decomposition), but also $\aleph_0$-categoricity (since the models are countable; thereby also finishing the proof of condition A) and (homogeneity) (apply the argument with $N_\alpha = M_\alpha$).

It remains to show (triviality). Let $\bar{a}, \bar{c}$ be in M_β for some $\beta < \omega_1$ with $\max(s_{\bar{c}}) < \min(s_{\bar{a}})$ and let $\bar{b}$ be such that $s_{\bar{b}} = s_{\bar{a}}$ and $\mathrm{tp}_{\mathrm{qf}}(\bar{b}_\alpha \bar{c}) = \mathrm{tp}_{\mathrm{qf}}(\bar{a}_\alpha \bar{c})$ for all α. Since M_0 must contain all of P, $\max(s_{\bar{c}}) < \min(s_{\bar{a}})$ implies that all components of $\bar{a}$ lie in Q and then $s_{\bar{b}} = s_{\bar{a}}$ implies that $\bar{b}\bar{c}$ and $\bar{a}\bar{c}$ satisfy the same quantifier-free type with respect to formulas only involving E (here we use the fact that the M_α have been chosen to add exactly *one* E-class each time). But also with respect to the relation R, $\bar{b}\bar{c}$ and $\bar{a}\bar{c}$ have the same quantifier-free type because of $\mathrm{tp}_{\mathrm{qf}}(\bar{b}_\alpha \bar{c}) = \mathrm{tp}_{\mathrm{qf}}(\bar{a}_\alpha \bar{c})$, so we can conclude $\bar{b}\bar{c} \models \mathrm{tp}_{\mathrm{qf}}(\bar{a}\bar{c})$ as required. □

Shelah provides a second example of an AEC in [4] which is a modification of the presented $L(Q)$-example, axiomatizable in $L_{\omega_1,\omega}$. The basic idea is to make P countable by making it the countable union of finite definable sets. However, as Chris Laskowski proves in an unpublished note, this AEC has the maximum number of models in $\aleph_1$ under ZFC. In our terminology, that AEC satisfies condition A as well as (decomposition) and (homogeneity), but it fails (triviality). It remains an important open question if categoricity (in $\aleph_1$) is absolute for $L_{\omega_1,\omega}$-sentences.

4. Martin's axiom and WCH are sufficient but not necessary

Our main Theorem is:

Theorem 7. *Let K be an AEC with* $\mathrm{LS}(K) = \aleph_0$.
(a) Suppose condition A holds. If $2^{\aleph_0} < 2^{\aleph_1}$, then K has $2^{\aleph_1}$ models of size $\aleph_1$. However it is consistent that $2^{\aleph_0} = 2^{\aleph_1}$ and the same conclusion holds.
(b) Suppose condition B holds. Assuming Martin's Axiom at $\aleph_1$, K is $\aleph_1$-categorical. However it is consistent that $MA_{\aleph_1}$ fails and the same conclusion holds.

The first statements in (a) and (b) are the contents of the Theorems 4 and 5. We now turn to proofs of the second statements.

A model of ZFC where $2^{\aleph_0} = 2^{\aleph_1}$ yet K has $2^{\aleph_1}$ models of size $\aleph_1$.

There are models M of ZFC in which $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = \aleph_3$. (In fact, Easton [3] showed that any reasonable behaviour of the generalised continuum function $\kappa \mapsto 2^\kappa$ for regular κ is possible.) Now over this model M apply $\aleph_2$-Cohen forcing P. This is the forcing whose conditions are of the form $p\colon |p| \to 2$, $|p| < \omega_2$, ordered by extension. This forcing is $\aleph_2$-closed, i.e., any descending ω_1-sequence of conditions has a lower bound. As a consequence, if G is P-generic over M, any subset of ω_1 in $M[G]$ already belongs to M. It follows that M and $M[G]$ have the same structures with universe ω_1 and the same isomorphisms between such structures; by the first statement of Theorem 7 (a), K has $\aleph_3^M$ many models of size $\aleph_1$ in M. As $\aleph_2$ is the same in M and $M[G]$, it follows that K has at least $\aleph_2^{M[G]}$ many models in $M[G]$ and $2^{\aleph_0}$ is $\aleph_2$ in $M[G]$.

But $2^{\aleph_1}$ equals $\aleph_2$ in $M[G]$: Each subset of ω_1 in $M[G]$ can be described in $M[G]$ by an ω_1-sequence of subsets of P that belongs to M (a "canonical name" for it), and there are $\aleph_3^M$ many such sequences. If $g\colon \omega_2 \to 2$ is the union of the conditions in G, then every subset of ω_1 in M occurs as $\{i < \omega_1 \mid g(\alpha + i) = 1\}$ for some $\alpha < \omega_2$, and therefore $\aleph_3^M = |\mathcal{P}^M(\omega_1)| \leq \aleph_2$ in $M[G]$ (where $\mathcal{P}^M$ denotes the powerset operation of M).

So $M[G]$ is a model of ZFC in which $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and K has the maximum number of models of size $\aleph_1$, as claimed.

We now turn to the second statement of Theorem 7 (b).

A model of ZFC in which $MA_{\aleph_1}$ *fails yet* K *is* $\aleph_1$*-categorical.*

We use iterated forcing with countable support to construct the desired model of ZFC. We first review the argument that $\mathrm{MA}_{\aleph_1}$ yields $\aleph_1$ categoricity. Given two models $\mathcal{A}, \mathcal{B}$ in K of size $\aleph_1$, we write each as the union of an increasing, continuous ω_1-chain of countable models: $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$, $\mathcal{B} = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$, as in (decomposition) of condition B. Then we consider the forcing $P(\vec{\mathcal{A}}, \vec{\mathcal{B}})$ whose conditions are finite partial isomorphisms p from $\mathcal{A}$ to $\mathcal{B}$ which preserve rank, i.e., such that for x in the domain of p, x belongs to A_α iff $p(x)$ belongs to B_α, for each $\alpha < \omega_1$. This forcing has the countable chain condition, and therefore by $\mathrm{MA}_{\aleph_1}$ there is a compatible set H of conditions in it which meets the $\aleph_1$-many dense sets which require that each element of A belongs to the domain and each element of B belongs to the range of some condition in H. Then the union of the conditions in H is an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$.

The key observation is the following. We say that a forcing P is *almost bounding* iff whenever G is P-generic and $f\colon \omega \to \omega$ belongs to $V[G]$ there is $g\colon \omega \to \omega$ in V such that for every infinite $X \subseteq \omega$ in V, $g(n) > f(n)$ for infinitely many n in X.

Lemma 8. *For any* $\mathcal{A}$*,* $\mathcal{B}$ *of size* $\aleph_1$*, the forcing* $P(\vec{\mathcal{A}}, \vec{\mathcal{B}})$ *is almost bounding.*

Proof. Suppose that G is $P(\vec{\mathcal{A}}, \vec{\mathcal{B}})$-generic and $f\colon \omega \to \omega$ is a function in $V[G]$. For any countable α let P_α denote the suborder of $P(\vec{\mathcal{A}}, \vec{\mathcal{B}})$ consisting of conditions with domain in A_α. Then $G_\alpha = G \cap P_\alpha$ is P_α-generic over V, as by (triviality) any condition p is compatible with any extension of $p \restriction A_\alpha$ in P_α and therefore any maximal antichain in P_α is also a maximal antichain in $P(\vec{\mathcal{A}}, \vec{\mathcal{B}})$. And as $P(\vec{\mathcal{A}}, \vec{\mathcal{B}})$ has the countable chain condition, f in fact belongs to $V[G_\alpha]$ for some countable α and therefore it suffices to prove that P_α is almost bounding for each countable α. But P_α is a countable forcing and is therefore equivalent to the forcing that adds one Cohen real. It is easy to check that the latter forcing is almost bounding (see [1]). $\square$

We now use the following general lemma, which can be found in [1]. A forcing P is *weakly bounding* iff whenever G is P-generic and $f\colon \omega \to \omega$ belongs to $V[G]$ there is $g\colon \omega \to \omega$ in V such that $g(n) > f(n)$ for infinitely many n.

Lemma 9. *The countable support iteration of proper, almost bounding forcings is weakly bounding.*

Now to finish our proof, perform a countable support iteration of length ω_2 over L, at each stage forcing with $P(\vec{\mathcal{A}}, \vec{\mathcal{B}})$ for some choice of $\vec{\mathcal{A}}$, $\vec{\mathcal{B}}$. Using a bookkeeping function we can ensure that if G is generic for this iteration, then every pair $\vec{\mathcal{A}}$, $\vec{\mathcal{B}}$ that exists in $V[G]$ will have been considered at some stage of the iteration. The result is a model in which K is $\aleph_1$-categorical. By Lemma 9, the iteration is weakly bounding, and therefore there is no $f\colon \omega \to \omega$ in $V[G]$ which *eventually dominates* each $g\colon \omega \to \omega$ in L, i.e., such that for each $g\colon \omega \to \omega$ in L, $f(n) > g(n)$ for sufficiently large n. Therefore $\mathrm{MA}_{\aleph_1}$ fails in $V[G]$, by the following observation.

Lemma 10. *$MA_{\aleph_1}$ implies that some $f\colon \omega \to \omega$ eventually dominates every $g\colon \omega \to \omega$ in L.*

Proof. Consider *Hechler forcing* in L, whose conditions are pairs (s, g) where $s\colon |s| \to 2$ has domain a natural number and $g\colon \omega \to \omega$ belongs to L. Extension is defined by: $(s^*, g^*) \leq (s, g)$ iff s^* extends s, $g^*(n) > g(n)$ for all n and $s^*(n) > g(n)$ for all n in $|s^*| \setminus |s|$. This forcing is ccc because any two conditions with the same first component are compatible and there are only countably many first components. And for each $h\colon \omega \to \omega$ in L the set $D(s, g)$ of conditions (s, g) such that $g(n) > h(n)$ for all n is dense. It follows that if $f\colon \omega \to \omega$ is the generic function added by Hechler forcing, i.e. the union of the s such that (s, g) belongs to the generic for some g, then f eventually dominates each $g\colon \omega \to \omega$ in L. The latter only requires that the $\aleph_1$ many dense sets $D(s, g)$ are met, so $\mathrm{MA}_{\aleph_1}$ implies that there is a such a function. □

In summary, with a countable support iteration of almost bounding forcings we produce a model where K is $\aleph_1$-categorical yet $\mathrm{MA}_{\aleph_1}$ fails.

Remark 11. *We could do better and actually find a model of ZFC in which $MA_{\aleph_1}$ fails, and in which* all *AEC's satisfying condition B are $\aleph_1$-categorical. The idea would be to apply the described forcings to all pairs of models of size $\aleph_1$ (in all countable signatures) with distinguished filtrations by countable models, for which the corresponding poset of finite partial filtration-preserving isomorphisms has the ccc and for which that forcing is almost bounding. In the procedure of iterating those forcings, we may create new instances of such pairs of models for which we can apply the forcing, but by bookkeeping, we will have taken care of them in an ω_2 long chain of iterated forcings. The resulting universe satisfies our requirement: if (A, B) is a pair of structures of size $\aleph_1$ (with filtrations) of an AEC satisfying condition B, we know by absoluteness of condition B that this instance occurred in our chain of forcings (use Lemma 8) and therefore A and B have been forced to be isomorphic. Thus any AEC satisfying condition B in the resulting universe is $\aleph_1$-categorical.*

On the other hand, it is not clear whether our universe failing WCH in which a particular AEC with condition A has many models in $\aleph_1$ has the property that all *such AEC's have many models in $\aleph_1$. The problem is that although we do not add subsets of $\aleph_1$, we* do *add subsets of the continuum (which is $\aleph_2$) and may create new AEC's satisfying condition A. Still, all AEC's with condition A whose* *restriction to countable models is H_{ω_2} definable will have many models in $\aleph_1$, which is the case for example for AEC's axiomatizable by an $L_{\omega_1,\omega}(Q)$ sentence with a natural notion of substructure.*

Question. Is there an AEC satisfying conditions A and B which is defined by an $L_{\omega_1\omega}$-sentence?

Question. Condition B is sufficient to show $\aleph_1$-categoricity under $\mathrm{MA}_{\aleph_1}$. To what extent is it also a necessary condition? For example, does every potentially (i.e. in some generic extension) $\aleph_1$-categorical AEC have to satisfy (decomposition)? It is not very difficult to show that for a first order theory, (decomposition) is *equivalent* to $\aleph_1$-categoricity ($\aleph_1$-categoricity is an *absolute* property for first-order theories because it is characterized by ω-stability plus "there are no Vaughtian pairs". Both properties follow directly from (decomposition)). Also, clearly, (triviality) is a very strong condition, as it is easy to find $\aleph_1$-categorical first-order theories where it fails (e.g. take an equivalence relation with exactly two classes and a binary relation defining a bijection between those two classes). Is there a way to weaken (triviality) and get the same results?

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