PARTIALLY DEFINABLE FORCING AND BOUNDED ARITHMETIC

ALBERT ATSERIAS AND MORITZ MÜLLER

1. Introduction

Various independence results in bounded arithmetic have been obtained using forcing type arguments. We describe a frame for forcing that can be seen as a common generalization of these arguments and Cohen forcing in set theory.

We do not assume familiarity with bounded arithmetic nor with forcing. This introduction informally gives some general motivation, describes the connection to propositional proof complexity, reviews the mentioned forcing type arguments, compares them with Cohen forcing and then describes in some more detail the contents of this paper. More precise information can be found following the references, mainly pointing to surveys. All results are stated and proved in a generally accessible language. Some of their links to bounded arithmetic and propositional proof complexity are made explicit by remarks intended for the informed reader.

1.1. **Foundational questions and complexity.** Basic questions concerning the foundations of mathematics quickly lead to fundamental open problems from computational complexity theory like P vs. NP or NP vs. coNP. Indeed, Krajíček argues that these questions can be understood as "quantitative versions" [24, Section 5] of the central questions of mathematical logic a century ago, namely for the consistency and the decidability of first-order theories. Also Krajíček and Pudlák [26] tie the viability of versions of Hilbert's program to the nondeterministic time complexity of coNP.

Pudlák argues that our understanding of independence is unsatisfactory in that "except for Gödel's theorem which gives only special formulas, no general method is known to prove independence of [arithmetical] Π_1 sentences" [31, Section 3]. Here progress is braked by the fact that already weak arithmetical theories like those in Buss' hierarchy correspond in a certain precise sense to the complexity classes in the polynomial hierarchy. [21, 10] are monographs, [9, 8] surveys on the subject.

Furthermore, establishing independence from bounded arithmetics is roughly equivalent to establishing proof-size lower bounds for propositional logic:

1.2. **Proof complexity.** For a sufficiently general notion of propositional proof system, the conjecture NP $\neq$ coNP means that no propositional proof system has short proofs of all tautologies (i.e. of size polynomial in the length of the tautology) [11]. But today this is open even for the usual textbook systems, called *Frege systems*: Hilbert style calculi given by finitely many inference rules. [34, 32, 44, 4, 37] survey known lower bounds for weaker systems with partly different emphases.

Now, arithmetical theories are simulated by (often natural) propositional proof systems in the sense that theorems of the theory translate to sequences of tautologies with short proofs in the system (see [26] for a general treatment, [7] for a recent survey).

Example 1.1 (Paris-Wilkie translation)**.** The theory $I\Delta_0(R)$ is Peano arithmetic where the induction scheme is adopted only for bounded formulas but in the language augmented by some new, say, binary relation symbol R. If $I\Delta_0(R)$ proves a Δ_0-formula $\varphi(R,x)$, then $\forall R \forall x \varphi(R,x)$ is true (in the standard model). This translates to a sequence of tautologies $\langle\varphi(R,x)\rangle_m, m \in \mathbb{N}$: insert m for x in $\varphi(R,x)$, replace bounded quantifiers by conjunctions or disjunctions, replace atoms not mentioning R by their truth values and keep atoms of the form $Rk\ell$ as propositional variables.

Paris and Wilkie [29] construct from a proof of $\varphi(R,x)$ in $I\Delta_0(R)$ and $m \in \mathbb{N}$ a short (length $m^{O(1)}$) proof of $\langle\varphi(R,x)\rangle_m$ in a *bounded depth* Frege system. This is a Frege system where only formulas of at most some fixed $\wedge/\vee$-alternation rank are allowed. ⌟

This way, independence can be inferred from proof-size lower bounds. A weak converse holds too. It is based on a type of argument invented by Ajtai [1], and it is here where forcing comes in.

1.3. **Forcing in bounded arithmetic.** Cohen's method of forcing cannot be used to prove independence of arithmetical statements because V_ω is not changed in generic extensions. In an informal sense however, forcing has been used to prove independence from weak arithmetics.

Paris and Wilkie used "a simple forcing argument" [29, p.333] to show that the least number principle for existential formulas mentioning R does not suffice to prove the (bijective) pigeonhole principle $\mathrm{PHP}(R,x)$: "R is not a bijection from $\{y \mid y \leq x\}$ onto $\{y \mid y < x\}$".

Riis [35] "proved by forcing" [35, p.1] that even the least number principle for formulas with a certain amount of universal quantification does not suffice (Buss' theory $T^1_2(R)$). Furthermore, Riis generalized this to other principles.

Ajtai [1] proved that $I\Delta_0(R)$ does not prove $\mathrm{PHP}(R,x)$. In fact, he proved that the tautologies $\langle\mathrm{PHP}(R,x)\rangle_n, n \geq 1$, do not have short proofs in bounded depth Frege systems. [30, 27] improved this to an exponential proof-size lower bound, implying independence from Buss' $T_2(R)$.

1.4. **Ajtai's argument.** Ajtai constructs a certain expansion (M,R) of a nonstandard model M of true arithmetic where $\mathrm{PHP}(R,n)$ fails for some nonstandard

$n \in M$. Assume M contains a size n^{100} depth 17 Frege proof π of $\langle \mathrm{PHP}(R, x)\rangle_n$. But this formula is 'false' in (M, R) under the assignment corresponding to R. The art is to construct R in such a way that (M, R) satisfies the least number principle up to n^{100} for the property of being a 'false' line in π. Then π contains a first 'false' line. One argues that this contradicts the soundness of the system and concludes that π cannot exist.

The construction of R is "done according to the general ideas of Cohen's method of forcing " [1, p.348]. However, the argument is "mostly combinatorial and probabilistic" [1, p.347] relying on a complicated version of Hastad's switching lemma [15]. As Ben-Sasson and Harsha put it, it is "extremely difficult to understand and explain" [6, section 1].

Lots of efforts have been made to simplify and reinterpret Ajtai's argument e.g. as a construction of valuations in Boolean algebras [32] or partial Boolean algebras [21, 22] or recently in terms of Buss-Pudlák games [6, 33]. In [23] Krajíček gives some general account, motivated "to understand the combinatorics behind constructions" [23, p.437] like Ajtai's. Conceptually, later improvements [5, 30, 27, 45] of Ajtai's result "eliminate the non-standard model theory" [5, p.367] and the forcing mode of speech. And technically, the mentioned switching lemmas have been improved and simplified (see [3, 43] for surveys).

Despite these efforts, not much is known on how to apply Ajtai's argument to stronger systems or other principles (cf. [21, Chapter 12] for known results). Perhaps one can say the abovementioned efforts did not lead to an understanding of Ajtai's argument as instantiating some general method as Pudlák asks for – but see Section 1.7 for recent progress.

1.5. **Comparison with Cohen forcing.** This sorry state of affairs clearly contrasts with Cohen forcing in set theory. We recall briefly and informally its set-up. With a model M of ZF and a 'generic' set G external to M one associates a model $M[G]$ containing G. Intuitively, G being 'generic' means being 'random' with respect to possible partial information about it. Forcing is a way to reason about $M[G]$ using partial information about G. A piece of partial information p *forces* φ if any generic G 'satisfying' p leads to a model $M[G]$ satisfying φ. Such pieces can be extended in various, possibly incompatible ways, so we think of them as being partially ordered (the *forcing frame*).

The Extension Lemma states that extension preserves forcing. Reasoning about forcing rests on this and, following Shoenfield [38], two more central lemmas: the Truth Lemma asserts that every sentence true in $M[G]$ is already forced by some partial information p about G; the Definability Lemma states that forcing, as a binary relation, is definable in M. In turn, these forcing lemmas rest on the Forcing Completeness Theorem, a characterization of the 'semantic' forcing notion above by a handier 'syntactic' notion that is defined via recursion on logical syntax. This understanding of forcing underlies the "Principal Theorem" [38] stating that $M[G]$ models ZF. This way an independence question is reduced to a combinatorial task of designing an appropriate forcing frame.

In contrast, the mentioned forcing type arguments [29, 35, 1] in bounded arithmetic are not based on some more general background theory of forcing. Ajtai writes "Our terminology will be similar to the terminology of forcing but we actually do not use any result from it" [1, p.348]. Insofar it is not completely clear why one should refer to these arguments as forcing arguments. Technically, the crucial difference is that the Definability Lemma fails. Forcing Completeness is proved neither in the original arguments nor in later presentations [46], [21, Section 12.7] that emphasize the forcing mode of speech. In [29, 1] no 'syntactic' notion is defined, in [35] it is, but one for which Forcing Completeness fails.

1.6. **This work.** We propose a general background theory of forcing as a unifying way to understand the arguments of Paris, Wilkie, Riis and Ajtai [29, 35, 1]. In paragraph 2 we develop forcing generally as a method to construct generic "associates" that may happen to be extensions or expansions or neither, and without a Definability Lemma. It is general enough to naturally accommodate the mentioned forcing arguments [29, 35, 1] as well as Cohen forcing and many others.

In the context of bounded arithmetics, a Principal Theorem would state that generic expansions satisfy the least number principle for a certain fragment of formulas. In paragraph 3 we show this holds true when using a forcing that is in an appropriate sense 'definable' for the fragment in question. Thereby again, independence questions reduce to a combinatorial task of designing forcing frames.

In paragraph 4 we prove the independence results in [29, 35, 1] by this method. The aim is to understand the progress as being constituted by inventing forcings that are 'definable' for larger and larger fragments.

1.7. **Related work.** Forcing has been developed outside set theory in many different settings. We will review some of them, but are unable to give a complete survey (cf. [17, 2]). Forcing against bounded arithmetic has been developed by Takeuti and Yasumoto [41, 42] following not Cohen's original method but its reformulation by Scott and Solovay [36] as a method to construct Boolean valued models (see Remark 2.7). Scott [36] describes such a model for a 3rd order theory of the reals, by interpreting the language over real valued random variables. In his recent book [25] Krajíček develops such *forcing with random variables* in full detail as a method to study bounded arithmetics by using algorithmically restricted random variables. Ajtai's result can be proved using this method.

2. Forcing in general

This paragraph develops a general frame for forcing arguments. In 2.1 we fix notation and establish basic facts concerning 'syntactic' forcing relations. In principle, countless 'syntactic" forcing relations $\Vdash$ may be defined, depending on how $\Vdash$ interacts with the logical symbols. Throughout this paper we assume (first-order) formulas to be written in the logical symbols $\{\forall, \exists, \wedge, \vee, \neg\}$ and we shall restrict attention to two kinds of forcings only, namely, universal and existential forcings[1]. Roughly, the choice depends on whether $\{\forall, \wedge, \neg\}$ or $\{\exists, \vee, \neg\}$ is taken as primitive while the other logical symbols are defined using the usual classical dualities. Existential forcing is often used, but we shall see that it has some disadvantages over universal forcing (Remark 2.33). In 2.2 we define a notion of genericity that is sufficiently general for all our purposes. In 2.3 we define generic associates and prove the Truth Lemma and the Forcing Completeness Theorem. Section 2.4 considers an important type of forcing that we call conservative. Section 2.5 gives examples and, finally, 2.6 discusses weak forcing.

In this paragraph we fix

- a countable *forcing frame* $(P, \leq, D_0, D_1, \ldots)$ (defined below),
- a countable structure M interpreting a countable language L,
- a countable language $L^* \supseteq L$.

The *forcing language* is $L^*(M)$, that is L^* together with all $a \in M$ as constants (we do not distinguish between M and its universe notationally). If not explicitly specified otherwise we let $\varphi, \psi, \ldots$ range over $L^*(M)$-sentences.

2.1. **Forcing relations.**

We recall some elementary forcing terminology. A *forcing frame* is a structure $(P, \leq, D_0, D_1, \ldots)$ such that $\leq$ partially orders P and $D_0, D_1, \ldots$ are subsets of P. We use $p, q, r, \ldots$ to range over elements of P, called *conditions*. If $p \leq q$ we say p *extends* q and call p an *extension* of q. If p, q have a common extension, then they are *compatible*, symbolically $p \| q$; otherwise they are *incompatible*, symbolically $p \perp q$.

A set of conditions $X \subseteq P$ is *downward-closed* if it contains all extensions of its elements; being *upward-closed* is similarly explained. The set X is *consistent* if it contains a common extension of any two of its elements. If X is both upward-closed and consistent, then it is a *filter*. Further, X is *dense below* p if for every $q \leq p$ there is $r \leq q$ such that $r \in X$. Finally, X is *dense* if it is dense below all conditions, or equivalently, if every condition has an extension in X.

Definition 2.1. A *pre-forcing* is a binary relation $\Vdash$ between conditions and $L^*(M)$-sentences. If $p \Vdash \varphi$, we say p *forces* φ.

We use the notation

$$[\varphi] := \{p \mid p \Vdash \varphi\}.$$

[1]See [16, 2] for examples of forcings that are neither universal nor existential.

Definition 2.2. A pre-forcing $\Vdash$ is *universal* or *existential* if it satisfies the following conditions of *universal* respectively *existential forcing recurrence*:

	universal	*existential*
$p \Vdash \neg\varphi$	iff $\forall q \leq p : q \nVdash \varphi$	iff $\forall q \leq p : q \nVdash \varphi$
$p \Vdash (\varphi \wedge \psi)$	iff $p \Vdash \varphi$ and $p \Vdash \psi$	iff $p \Vdash \neg(\neg\varphi \vee \neg\psi)$
$p \Vdash (\varphi \vee \psi)$	iff $p \Vdash \neg(\neg\varphi \wedge \neg\psi)$	iff $p \Vdash \varphi$ or $p \Vdash \psi$
$p \Vdash \forall x\chi(x)$	iff $\forall a \in M : p \Vdash \chi(a)$	iff $p \Vdash \neg\exists x\neg\chi(x)$
$p \Vdash \exists x\chi(x)$	iff $p \Vdash \neg\forall x\neg\chi(x)$	iff $\exists a \in M : p \Vdash \chi(a)$.

Observe that a universal or existential pre-forcing is uniquely determined by its restriction to the atomic sentences of the forcing language.

Solving the recurrence one sees, for universal pre-forcings, that $p \Vdash \exists x\chi(x)$ if and only if $\bigcup_{a\in M}[\chi(a)]$ is dense below p. For existential pre-forcings one sees $p \Vdash \forall x\chi(x)$ if and only if $[\chi(a)]$ is dense below p for all $a \in M$. We collect some further direct consequences:

Lemma 2.3. *If $\Vdash$ is a universal or an existential pre-forcing, then*
(1) $p \Vdash \neg\neg\varphi$ *if and only if* $[\varphi]$ *is dense below* p.
(2) (Consistency) $[\varphi] \cap [\neg\varphi] = \emptyset$.
(3) $[\varphi] \cup [\neg\varphi]$ *is dense.*

Definition 2.4. Let $\Vdash$ be a pre-forcing and Φ be a set of $L^*(M)$-formulas.
(a) $\Vdash$ *satisfies Extension for* Φ if for every $\varphi \in \Phi$, the set $[\varphi]$ is downward-closed.
(b) $\Vdash$ *satisfies Stability for* Φ if for every $\varphi \in \Phi$ and $p \in P$, we have that p forces φ whenever $[\varphi]$ is dense below p.

For $\Phi = L^*(M)$ we omit the reference to it.
(c) $\Vdash$ is a *forcing* if it satisfies Extension and Stability for $L^*(M)$-atoms.

Lemma 2.5. *The following hold:*
(1) (Extension) *Universal and existential forcings satisfy Extension.*
(2) (Stability) *Universal forcings satisfy Stability.*
(3) *For a universal forcing* $\Vdash$ *it holds that* $p \Vdash \varphi$ *if and only if* $[\varphi]$ *is dense below* p.
(4) *For a universal forcing* $\Vdash$ *it holds that* $p \nVdash \varphi$ *if and only if* $q \Vdash \neg\varphi$ *for some* $q \leq p$.

Proof. Extension can be shown by a straightforward induction using forcing recurrence. We prove Stability by induction on (the number of logical symbols) in φ. Having a universal forcing we can assume that φ is written in the logical base $\{\wedge, \neg, \forall\}$.

For atomic φ Stability is part of the definition of being a forcing.

For the $\neg$-step argue indirectly: if $p \not\Vdash \neg\varphi$, then by forcing recurrence some $q \leq p$ forces φ, so by Extension and Consistency no extension of q forces $\neg\varphi$. Hence $[\neg\varphi]$ is not dense below p.

For the $\wedge$-step, note $[(\varphi \wedge \psi)] = [\varphi] \cap [\psi]$ by universal recurrence. If this set is dense below p then so are both $[\varphi]$ and $[\psi]$. By induction p forces both φ and ψ, and hence $p \Vdash (\varphi \wedge \psi)$ by universal recurrence.

The $\forall$-step is similar.

(3) is immediate by (1) and (2), and (4) follows from (3): $p \not\Vdash \varphi$ if and only if $[\varphi]$ is not dense below p if and only if there is $q \leq p$ such that for all $r \leq q$, $r \not\Vdash \varphi$, if and only if (by forcing recurrence) there is $q \leq p$ such that $q \Vdash \neg\varphi$. $\square$

Example 2.6. Let $\Vdash$ be a universal forcing. A pre-forcing of obvious interest is:

$$p \| \varphi \text{ if and only if } p \not\Vdash \neg\varphi, \text{ that is, } q \Vdash \varphi \text{ for some } q \leq p.$$

We have $\Vdash \subseteq \|$ by Consistency. Stability of $\Vdash$ implies: $p\|\neg\neg\varphi$ if and only if $p\|\varphi$. Further

$$\begin{array}{l|lll} p\|\neg\varphi & \text{iff} & \exists q \leq p : q \not\| \varphi & \\ p\|(\varphi \vee \psi) & \text{iff} & p\|\varphi \text{ or } p\|\psi & \text{(as existential pre-forcing)} \\ p\|\exists x \chi(x) & \text{iff} & \exists a \in M : p\|\chi(a) & \text{(as existential pre-forcing).} \end{array}$$

Remark 2.7 (Boolean valued models)**.** The last lemma has a natural topological reading. Namely, $(P, \leq)$ carries the topology whose open sets are the downward-closed sets. A set $X \subseteq P$ has interior $\mathring{X} = \{p \mid \forall q \leq p : q \in X\}$ and closure $\overline{X} = \{p \mid \exists q \in X : q \leq p\}$. For example, $\{p \mid p\|\varphi\} = \overline{[\varphi]}$. The sets equal to the interior of their closure are the regularly open ones. Note $\mathring{\overline{X}} = \{p \mid X \text{ is dense below } p\}$.

Thus Extension means that the sets $[\varphi]$ are open and Stability means that they are even regularly open.

The regularly open sets form a complete Boolean algebra in such a way that, for universal forcings, the map $\varphi \mapsto [\varphi]$ is a Boolean valuation of $L^*(M)$ in this algebra.

2.2. **Genericity.** Let $\Vdash$ be an arbitrary pre-forcing. Ideally one would like to call a set generic if it intersects every dense set. As in general such sets do not exist, one has to restrict attention to those dense sets coming from a certain 'sufficiently rich' but countable Boolean algebra $\mathcal{B}(\Vdash)$.

In set theory usually the forcing frame is a set in M and one simply takes the algebra of sets definable in M (cf. Example 2.24). As M models ZF it is not surprising that this algebra is sufficiently rich. For some purposes (cf. Examples 2.26, 2.28, 2.29) already the algebra generated by the $[\varphi]$s is sufficiently rich, but not so in forcing against bounded arithmetic. One needs the family to contain certain sets as e.g. $\bigcup_{a \in M}[\varphi(a)]$ that we do construct in proofs. [29, 1, 35] all define suitable algebras ad hoc for their respective situations and a canonical choice does not seem to exist in general.

It is therefore that we padded the forcing frame by the sets $D_0, D_1, \ldots$: these sets will determine the algebra $\mathcal{B}(\Vdash)$.

Definition 2.8. A set $G \subseteq P$ is *generic* if it is a filter and intersects every dense (in P) set in $\mathcal{B}(\Vdash)$.

Our definition of $\mathcal{B}(\Vdash)$ is based on the following logic. We call it the *Stern formalism* because a similar one has been used by Stern in [40].

First we define its syntax. Consider the two-sorted first-order structure (P, M) consisting of one sort carrying the forcing frame $(P, \leq, D_0, D_1, \ldots)$ and a second sort carrying the structure M. Let $\nu, \mu, \xi, \ldots$ be individual variables ranging over the first sort and $x, y, z, \ldots$ range over the second sort. The syntax of the Stern formalism is given by the syntax of the two-sorted first-order language of (P, M) plus the following syntactical rule:

> if $\varphi(\bar{x})$ is an L^*-formula, then $(\xi \vdash \varphi(\bar{x}))$ is a formula of the Stern formalism.

Note this does not allow for nested occurrences of $\vdash$.

We now explain the semantics of the Stern formalism. Generally, it is interpreted in 'forcing structures' $(Q, N, \Vdash)$ given by a two-sorted first-order structure (Q, N) as above (the first sort Q carries a forcing frame and the second sort N an L-structure) together with $\Vdash \subseteq Q \times L^*(N)$.

The definition of the satisfaction relation is straightforward. We give it for our 'standard' structure $(P, M, \Vdash)$. Formulas not involving $\vdash$ are evaluated as usual in (P, M). Further we say that $p\bar{a}$ *satisfies* $(\xi \vdash \varphi(\bar{x}))$ in $(P, M, \Vdash)$ if and only if $p \Vdash \varphi(\bar{a})$, where $p\bar{a}$ is any assignment for $\xi\bar{x}$, i.e. $p \in P$ and $\bar{a}$ is a suitable tuple from M; note that here $\varphi(\bar{a})$ is a sentence of the forcing language $L^*(M)$.

Definition 2.9. The *forcing algebra* $\mathcal{B}(\Vdash)$ is the set of all subsets of P definable in $(P, M, \Vdash)$ by a formula $\varphi(\xi)$ of the Stern formalism.

Here and in the following, *definable* (in a certain structure) always means definable *with parameters* (from the structure). Clearly, the forcing algebra $\mathcal{B}(\Vdash)$ is countable. Then

Lemma 2.10. *Every condition is contained in some generic set.*

Sketch of Proof. Given $p \in P$, choose $p_1 \leq p$ in the first dense set, then $p_2 \leq p_1$ in the second dense set and so on. The filter generated by the sequence $p, p_1, p_2, \ldots$ is generic. □

Lemma 2.11. *If G is generic and $D \in \mathcal{B}(\Vdash)$ is dense below $p \in G$, then there is $q \in G \cap D$ with $q \leq p$.*

Proof. Let $D \in \mathcal{B}(\Vdash)$ be dense below p. It is routine to verify that

$$D(p) := (D \cap \{q \mid q \leq p\}) \cup \{q \mid p \perp q\}$$

is dense. Further $D(p) \in \mathcal{B}(\Vdash)$: if D is defined by $\varphi_D(\xi)$, then $D(p)$ is defined by

$$(\varphi_D(\xi) \wedge \xi \leq p) \vee \neg\exists\nu(\nu \leq \xi \wedge \nu \leq p),$$

a formula (with parameters) of the Stern formalism. By genericity there exists an $r \in G \cap D(p)$. As $p \in G$ and G is consistent, $r \notin \{q \mid p \perp q\}$, so $r \in D \cap \{q \mid q \leq p\}$. $\square$

2.3. **Generic associates.** Let $\Vdash$ be a universal or existential forcing.

The aim is to define for suitable $G \subseteq P$ (and our fixed structure M) an $L^*(M)$-structure $M[G]$ in such a way, that it models the following theory in the forcing language $L^*(M)$:

$$\mathrm{Th}(G) := \{\varphi \in L^*(M) \mid \exists p \in G : p \Vdash \varphi\}.$$

Obviously this cannot work in general, e.g. $\mathrm{Th}(G)$ may contradict usual first-order equality axioms. But we shall see that this is the only obstacle provided we stick to the idea that the constants from M "name" all the elements of $M[G]$.

First observe that for generic G, the theory $\mathrm{Th}(G)$ is complete and formally consistent in the following sense:

Lemma 2.12. *Let G be generic. For every $L^*(M)$-sentence φ either $\varphi \in \mathrm{Th}(G)$ or $\neg\varphi \in \mathrm{Th}(G)$, but not both.*

Proof. By Lemma 2.3 (3), G intersects $[\varphi] \cup [\neg\varphi] \in \mathcal{B}(\Vdash)$. Hence $\varphi \in \mathrm{Th}(G)$ or $\neg\varphi \in \mathrm{Th}(G)$ – but not both: assume there would be $p \in G$ forcing φ and $q \in G$ forcing $\neg\varphi$. Since G is a filter and filters are consistent, there would exist r extending both p and q; by Extension r would force both φ and $\neg\varphi$ contradicting Consistency (of forcing). $\square$

To define $M[G]$ we rely on some elementary facts about factorizations: for a theory T in a language L containing some constant symbol, the *Herbrand term structure* $\mathfrak{T}(T)$ for T has as universe all closed L-terms, interprets a function symbol $f \in L$ by $\bar{t} \mapsto f(\bar{t})$ and interprets a relation symbol $R \in L$ by $\{\bar{t} \mid R\bar{t} \in T\}$. Note that in $\mathfrak{T}(T)$ every closed term denotes itself. A *congruence* $\sim$ on $\mathfrak{T}(T)$ is an equivalence relation on $\mathfrak{T}(T)$ such that functions in $\mathfrak{T}(T)$ (i.e. interpretations of function symbols of L) map equivalent arguments (i.e. componentwise equivalent argument tuples) to equivalent values and every relation of $\mathfrak{T}(T)$ is a union of equivalence classes of tuples. In this case, let $\mathfrak{T}(T)/\sim$ denote the L-structure induced by $\mathfrak{T}(T)$ on the $\sim$-classes in the natural way. In $\mathfrak{T}(T)/\sim$ every closed term t denotes its $\sim$-class $t/\sim$.

Fact 2.13. If $\sim_T := \{(s,t) \mid s = t \in T\}$ is a congruence on $\mathfrak{T}(T)$, then the atomic sentences true in $\mathfrak{T}(T)/\sim_T$ are precisely those contained in T.

Definition 2.14. Let $G \subseteq P$. If $\sim_{\mathrm{Th}(G)}$ is a congruence on $\mathfrak{T}(\mathrm{Th}(G))$ and every closed term of the forcing language is $\sim_{\mathrm{Th}(G)}$-congruent to a constant $a \in M$, then we say $M[G]$ *is defined* and set

$$M[G] := \mathfrak{T}(\mathrm{Th}(G))/{\sim_{\mathrm{Th}(G)}} .$$

If G is generic and $M[G]$ defined, then $M[G]$ is a *generic associate of* M.

We call $M[G]$ a *generic extension of* M, if $L = L^*$ and there is an embedding of M into $M[G]$.

We call $M[G]$ a *generic expansion of* M, if

$$a \mapsto a/{\sim_{\mathrm{Th}(G)}}\colon M \cong M[G] \restriction L,$$

that is, if the map that sends each $a \in M$ to its $\sim_{\mathrm{Th}(G)}$-congruence class $a/{\sim_{\mathrm{Th}(G)}}$ is an isomorphism of M onto the restriction of $M[G]$ to L.

Remark 2.15. Sometimes we shall need the assumption that $M[G]$ is defined for every generic G. Because this assumption is trivially satisfied in all applications we are aware of, we consider it as a mere technicality and make no efforts to avoid it.

Lemma 2.16. *Let G be generic.*

(1) *$M[G]$ is defined if for all closed $L^*(M)$-terms t, t', all $L^*(M)$-atoms $\varphi(x)$ and all $p \in P$*
   (a) *if $p \Vdash t = t'$, then $q \Vdash t' = t$ for some $q \leq p$,*
   (b) *if $p \Vdash \varphi(t)$ and $p \Vdash t = t'$, then $q \Vdash \varphi(t')$ for some $q \leq p$,*
   (c) *$q \Vdash t = a$ for some $q \leq p$ and $a \in M$.*

(2) *If $M[G]$ is defined, then it has universe $\{a/{\sim_{\mathrm{Th}(G)}} \mid a \in M\}$.*

We omit the proof.

Theorem 2.17 (Truth Lemma)**.** *Let G be generic. If $M[G]$ is defined, then $\mathrm{Th}(M[G]) = \mathrm{Th}(G)$.*

Proof. We have to show: $M[G] \models \varphi$ if and only if $p \Vdash \varphi$ for some $p \in G$. We have two cases depending of whether $\Vdash$ is universal or existential. In both cases we proceed by induction on φ.

The case where $\Vdash$ is existential is easy. The base case follows by construction (Fact 2.13). Both the $\vee$-step and the $\exists$-step are trivial. Finally, $\neg\varphi \in \mathrm{Th}(M[G])$, i.e. $\varphi \notin \mathrm{Th}(M[G])$, is equivalent to $\varphi \notin \mathrm{Th}(G)$ by induction and thus to $\neg\varphi \in \mathrm{Th}(G)$ by Lemma 2.12.

The case where $\Vdash$ is universal is more complicated. The base case and the $\neg$-step follow exactly as in the existential case. The $\wedge$-step is straightforward using the consistency of G.

For the $\forall$-step, first assume that some $p \in G$ forces $\forall x \varphi(x)$, i.e. $p \Vdash \varphi(a)$ for every $a \in M$ by universal recurrence. By induction $M[G] \models \varphi(a)$ for every $a \in M$. Hence $M[G] \models \forall x \varphi(x)$ by Lemma 2.16 (2).

Conversely, assume $\forall x \varphi(x) \notin \mathrm{Th}(G)$. We aim to show $\varphi(a) \notin \mathrm{Th}(M[G])$ for some $a \in M$.

By Lemma 2.12, $\neg\forall x\varphi(x) \in \mathrm{Th}(G)$, i.e. some $p \in G$ forces $\neg\forall x\varphi(x)$. By universal recurrence this means that for every $q \leq p$ there is $a \in M$ such that $q \not\Vdash \varphi(a)$. By Lemma 2.5 (4) this means: for every $q \leq p$ there is $a \in M$ and there is $r \leq q$ such that $r \Vdash \neg\varphi(a)$. In other words, the set

$$D := \bigcup_{a\in M} [\neg\varphi(a)]$$

is dense below p.

Clearly, $D \in \mathcal{B}(\Vdash)$: it is defined by $\exists x(\xi \vdash \neg\varphi(x))$, a formula (with parameters) of the Stern formalism.

As $p \in G$, G intersects D by Lemma 2.11, i.e. there is some $a \in M$ such that $\neg\varphi(a) \in \mathrm{Th}(G)$. Then $\varphi(a) \notin \mathrm{Th}(G)$ by Lemma 2.12, so $\varphi(a) \notin \mathrm{Th}(M[G])$ by induction. □

Corollary 2.18. *Assume $M[G]$ is defined for every generic G. Then*

(1) *if $\Vdash$ is existential, then $p \Vdash \varphi$ implies $M[G] \models \varphi$ for every generic G containing p.*
(2) (Forcing Completeness) *if $\Vdash$ is universal, then $p \Vdash \varphi$ if and only if $M[G] \models \varphi$ for every generic G containing p.*

Proof. By the Truth Lemma $p \Vdash \varphi$ implies $M[G] \models \varphi$ for every generic G containing p. This shows (1) and the forward direction of (2). The backward direction of (2) relies on Lemma 2.5 (4) for universal forcings: if $p \not\Vdash \varphi$, there is $q \leq p$ such that $q \Vdash \neg\varphi$. By Lemma 2.10 there is a generic G containing q. By the Truth Lemma $M[G] \models \neg\varphi$, i.e. $M[G] \not\models \varphi$. Being a filter, G contains p. □

Corollary 2.19. *Assume that $M[G]$ is defined for every generic G and that $\Vdash$ is universal. Then for every condition $p \in P$ the set $\{\varphi \mid p \Vdash \varphi\}$ is closed under logical consequence.*

Proof. For every $p \in P$, the set of φ satisfying the right hand side of Forcing Completeness is obviously closed under logical consequence. □

Example 2.20. Let $\Vdash$ be a universal forcing and recall the associated pre-forcing $\|$ from Example 2.6. Assume $M[G]$ is defined for every generic G. Then

(1) $p\|\varphi$ if and only if $M[G] \models \varphi$ for some generic G containing p;
(2) for every condition $p \in P$, the set $\{\varphi \mid p\|\varphi\}$ is closed under logical consequence. ⌟

We have the following preservation result.

Theorem 2.21. *Let T be a universal L^*-theory. If both*

(i) *for every condition p, the theory T is consistent with*

$$\mathrm{Lit}(p) := \{\varphi \mid p \Vdash \varphi, \varphi \textit{ is an } L^*(M)\textit{-Literal}\},$$

(ii) *and for every closed $L^*(M)$-term t, the set $\bigcup_{a\in M}[t = a]$ is dense,*

then $M[G]$ is defined for every generic G and satisfies T.

Proof. Let G be generic. To show $M[G]$ is defined we verify the three conditions (a), (b), (c) in Lemma 2.16 (1). For (a), if $p \Vdash t = t'$ but $q \not\Vdash t' = t$ for every $q \leq p$, then $p \Vdash \neg t' = t$ by forcing recurrence. But then $\mathrm{Lit}(p)$ and hence $\mathrm{Lit}(p) \cup T$ is inconsistent, contradicting (i). Condition (b) is similarly verfied and (c) is the same as (ii).

To show $M[G] \models T$ is suffices to show that $M[G]$ embeds into a model of T (since T is universal). For this it suffices to show that $T \cup \mathrm{Diag}(M[G])$ is consistent. So let Δ be a finite subset of $\mathrm{Diag}(M[G])$. Then $\Delta \subseteq \mathrm{Th}(G)$ by the Truth Lemma, that is, every literal $\lambda \in \Delta$ is forced by some $p_\lambda \in G$. Since G is consistent it contains a common extension p of all the p_λ's. Then $\Delta \subseteq \mathrm{Lit}(p)$ by Extension and $T \cup \Delta$ is consistent by (i). □

2.4. **Conservative forcing.** Let $\Vdash$ be an existential or universal forcing. Which forcings produce generic expansions? We characterize these as follows.

Definition 2.22. The forcing $\Vdash$ is *conservative* if for every condition p and every atomic $L(M)$-sentence φ (i.e. without a symbol from $L^* \setminus L$)

$$p \Vdash \varphi \text{ if and only if } M \models \varphi.$$

Proposition 2.23. *If* $\Vdash$ *is conservative, then every generic associates is a generic expansion. The converse holds true in case* $\Vdash$ *is universal and* $M[G]$ *is defined for every generic* G.

Proof. For the first statement, let $M[G]$ be a generic associate of M. By Lemma 2.16 the map $a \mapsto a/\sim_{\mathrm{Th}(G)} : M \to M[G] \upharpoonright L$ is surjective. If it is not an isomorphism, then $\mathrm{Th}(M)$ and $\mathrm{Th}(M[G])$ disagree on some atomic $L(M)$-sentence. As $\mathrm{Th}(M[G]) = \mathrm{Th}(G)$ by the Truth Lemma, this contradicts conservativity.

For the second statement, argue indirectly and assume $\Vdash$ is not conservative. Choose an atomic $L(M)$-sentence φ and a condition p witnessing this. Then $p \Vdash \varphi$ if and only if $M \not\models \varphi$. By Forcing Completeness we find a generic associate $M[G]$ of M such that $M[G] \models \varphi$ if and only if $p \Vdash \varphi$. Hence $\varphi \in \mathrm{Th}(M) \triangle \mathrm{Th}(M[G])$, so $\mathrm{Th}(M) \neq \mathrm{Th}(M[G] \upharpoonright L)$ and $M[G]$ cannot be an expansion of M. □

2.5. **Some examples.** Cohen forcing from set theory can be viewed as a special case of our general set-up:

Example 2.24 (Cohen forcing)**.** Cohen forcing starts with a countable transitive standard model M of, say, $\mathrm{ZF} + \mathrm{GCH}$ and wants $M[G]$ to be an extension of M. In particular $L^* = L = \{\in\}$. Different forcing extensions are obtained by different choices of $(P, \leq)$, typically a set in M, while the forcing $\Vdash_{\mathrm{Co}}$ is kept fix.

Following e.g. [14] one can define this forcing by universal forcing recurrence stipulating for atoms:

$$\begin{aligned} p \Vdash_{\mathrm{Co}} a \in b &\iff \big\{q \mid \exists r \exists c\ ((c,r) \in b \wedge q \leq r \wedge q \Vdash_{\mathrm{Co}} a = c)\big\} \\ &\qquad \text{is dense below } p, \\ p \Vdash_{\mathrm{Co}} a = b &\iff \forall c \in \mathrm{dom}(a \cup b) : p \Vdash_{\mathrm{Co}} (c \in a \leftrightarrow c \in b). \end{aligned}$$

It is not hard to show that this uniquely determines a universal pre-forcing. The technicality of the definition is to ensure that it is a forcing. Genericity is defined to mean: intersect every dense set that is definable in M. This coincides with our notion for $\emptyset = D_0 = D_1 = \ldots$.

In set theory one defines $M[G]$ as follows: the membership symbol $\in$ is interpreted by membership and the constants $a \in M$ are interpreted by $a_G := \{b_G \mid \exists p \in G : (b,p) \in a\}$. It is easily seen that $M[G]$ is an extension of M for every generic G.

Under this definition of $M[G]$, one can show the Truth Lemma for atoms, that is: for every generic G, $a_G = b_G$ if and only if $a \sim_{\mathrm{Th}(G)} b$ and $a_G \in b_G$ if and only if $\mathrm{Th}(G)$ contains the atom $a \in b$. It follows that $M[G]$ in our sense is defined for every generic G. Second it follows that $M[G]$ in our sense is isomorphic to $M[G]$ in the sense of set theory. Indeed, $\{(a/{\sim_{\mathrm{Th}(G)}}, a_G) \mid a \in M\}$ is such an isomorphism. ⌟

Feferman was the first explicitly using forcing outside set-theory, namely to adress questions in computability theory. But already Cantor's back and forth method can be seen as a forcing argument. Both are examples of conservative forcing:

Example 2.25 (Cantor's Theorem)**.** We give this simple example in some detail, because it reappears in similar form in paragraph 4.

Let $M = (A, A')$ be a countable two-sorted structure where the two sorts A and A' carry dense linear orders without endpoints $\preceq$ and $\preceq'$ respectively (i.e. $L = \{\preceq, \preceq'\}$). Set $L^* := L \cup \{R\}$ for a new binary relation symbol R.

Define the forcing frame $(P, \leq, D_0, D_1, \ldots)$ as follows: P is the set of all finite partial isomorphisms between A and A'; take $p \leq q$ to mean $p \supseteq q$; finally the sets $D_0, D_1, \ldots$ enumerate the sets $\{p \mid a \in \mathrm{dom}(p)\}, \{p \mid a' \in \mathrm{im}(p)\}$ for $a \in A, a' \in A'$. Each of these sets is dense.

To define a conservative universal pre-forcing $\Vdash_{\mathrm{Ca}}$ it suffices to define $p \Vdash_{\mathrm{Ca}} \varphi$ for φ an atom of the form Rab. Take this to mean $(a,b) \in p$.

Then $\Vdash_{\mathrm{Ca}}$ is a forcing: that $\Vdash_{\mathrm{Ca}}$ satisfies Extension for atoms is obvious. Because $\Vdash_{\mathrm{Ca}}$ is conservative we only have to show that it satifies Stability for atoms of the form Raa' for $a \in A, a' \in A'$. Argue indirectly: if $p \nVdash_{\mathrm{Ca}} Raa'$, then $(a,a') \notin p$. Choose $b' \neq a'$ such that $q := p \cup \{(a,b')\}$ is a condition ($\mathrm{im}(p)$ is finite). Then $q \leq p$ and no extension of q contains (a,a'), so no extension of q forces Raa'. Hence $[Raa']$ is not dense below p.

It is easy to see that $M[G]$ is defined for every generic G (e.g. by Lemma 2.16 (1)). By Proposition 2.23 every generic associate $M[G]$ is a generic expansion of M, that is, $a \mapsto a/{\sim_{\mathrm{Th}(G)}}: M \cong M[G] \upharpoonright L$. By definition $M[G]$ interprets R by

$$\begin{aligned}&\big\{(a/{\sim_{\mathrm{Th}(G)}}, b/{\sim_{\mathrm{Th}(G)}}) \mid \exists p \in G : p \Vdash_{\mathrm{Ca}} Rab\big\}\\ &= \big\{(a/{\sim_{\mathrm{Th}(G)}}, b/{\sim_{\mathrm{Th}(G)}}) \mid (a,b) \in \bigcup G\big\}\end{aligned}$$

Thus $a \mapsto a/{\sim_{\mathrm{Th}(G)}}: (M, \bigcup G) \cong M[G]$. From this and the fact that G intersects all the sets $D_0, D_1, \ldots$, it easily follows that $\bigcup G$ is an isomorphism from $(A, \preceq)$ onto $(A', \preceq')$. ⌟

Example 2.26 (Feferman forcing)**.** In [12] Feferman considers $M = \mathbb{N}$ interpreting the language L that has relation symbols for the graphs of successor, addition and multiplication. L^* expands L by at most countably many unary predicate symbols. A condition p is a finite consistent set of literals in the new predicates $L^* \setminus L$ and constants from $\mathbb{N}$. A condition p extends another q if $p \supseteq q$. For the sets $D_0, D_1, \ldots$ choose, say, always $\emptyset$. Feferman defines a conservative existential pre-forcing $\Vdash_{\mathrm{Fe}}$ by letting p force an atom involving a new predicate if and only if the atom belongs to p. It is not hard to see that $\Vdash_{\mathrm{Fe}}$ is a forcing and that $M[G]$ is defined for every generic G. Applications of Feferman forcing in computability theory are surveyed in [28]. ⌟

Variations and generalizations of Feferman forcing have been studied in complexity theory:

Example 2.27 (Generic oracles)**.** In [13] Fenner et al. generalize Feferman forcing for the case where $L^* = L \cup \{R\}$ for one new unary predicate R. View a Feferman condition p as the set of functions in $\{0,1\}^{\mathbb{N}}$ that map $n \in \mathbb{N}$ to 0 or 1 whenever $Rn \in p$ or $\neg Rn \in p$ respectively. Now, instead of using these basic clopen sets as conditions, [13] use perfect sets in $\{0,1\}^{\mathbb{N}}$. Forcing frames considered in [13] are certain subframes of this forcing frame (cf. [13, Definition 3.3]). Straightforwardly, Fenner et al. let a perfect set p force an atom Rn if and only if every function in p maps n to 1. This determines a conservative existential pre-forcing, that is actually a forcing on the frames considered. For various frames, Fenner et al. study complexity classes relativized by R in generic expansions. ⌟

Finally Robinson developed forcing in model theory:

Example 2.28 (Finite Robinson forcing)**.** We degrade M to a set of constants, i.e. we let $L = \emptyset$. Further let L^* be a countable language and T be a consistent L^*-theory; $T_\forall$ is the set of universal consequences of T.

Define the following forcing frame $(P, \leq, D_0, D_1, \ldots)$. A condition p is a finite set of $L^*(M)$-literals such that $T_\forall \cup p$ is consistent. Define $p \leq q$ to mean $p \supseteq q$. Finally, let $D_0, D_1, \ldots$ enumerate the sets $\bigcup_{a \in M}[t = a]$ for closed $L^*(M)$-terms t. It is easy to see that these sets are dense in P.

To define an existential pre-forcing $\Vdash_{\mathrm{Ro}}$, it suffices to define $p \Vdash_{\mathrm{Ro}} \varphi$ for atomic φ. Take this to mean $T_\forall \cup p \vdash \varphi$.

Then $\Vdash_{\mathrm{Ro}}$ is a forcing: Extension for atoms is obvious. To verify Stability for atoms, argue indirectly and assume $p \nVdash_{\mathrm{Ro}} \varphi$ where φ is an atom. Then $q := p \cup \{\neg\varphi\}$ is a condition. Clearly no extension of q forces φ, so $[\varphi]$ is not dense below p.

By Theorem 2.21, $M[G]$ is defined for every generic G and satisfies $T_\forall$. Note $\bigcup G$ is roughly the same as $\mathrm{Diag}(M[G])$. Hence the Truth Lemma essentially[2] says, that generic associates are *finitely generic for* T, so in particular such structures exist ([16, Theorem 5.11]). Their theory can be seen as a generalized model-companion for T. We refer to the book [16] for more information.

Keisler [19] gives more model-theoretic and algebraic applications of Robinson forcing and some variants of it. ⌟

2.6. **Weak forcing.** This section is not needed in the following. In [38] Shoenfield develops Cohen forcing in an indirect way: as an intermediate step he introduces an existential forcing $\Vdash$ and verifies the Truth Lemma for it. The actual forcing wanted, namely one satisfying Forcing Completeness, is then obtained as the *weak forcing* $\Vdash^*$:

$$p \Vdash^* \varphi \text{ if and only if } p \Vdash \neg\neg\varphi.$$

Cohen forcing, and more generally, any universal forcing coincides with its weak version (Lemmas 2.3 (1) and 2.5 (3)).

In other contexts weak forcings play a more important role [19, 40, 28, 2, 16, 13]. Often, starting with a particular existential forcing one verifies certain desired properties for the corresponding weak forcing. We use the opportunity of having a more general set-up and include a short discussion of the two notions.

Example 2.29 (Keisler forcing)**.** Keisler studies generally existential pre-forcings that satisfy Extension for atoms and the conditions in Lemma 2.16 (1), and proves Forcing Completeness for $\Vdash^*$ [19, Corollary 1.6]. In a similar context, Stern notes universal recurrence for $\Vdash^*$ [40, Proposition 1-1]. ⌟

Proposition 2.30. *Assume* $\Vdash$ *is a universal or existential pre-forcing satisfying Extension for atoms. Then*

(1) $\Vdash \subseteq \Vdash^*$.
(2) $\Vdash^*$ *satisfies Stability, i.e.* $(\Vdash^*)^* = \Vdash^*$.
(3) *If* $M[G]$ *is defined for every generic* G*, then* $\Vdash^*$ *is a universal forcing.*

Proof. Recall Remark 2.7. By Lemma 2.3 (1)

$$\{p \mid p \Vdash^* \varphi\} = \{p \mid [\varphi] \text{ is dense below } p\} = \overline{[\varphi]}^{\circ}.$$

[2]The forcing used in [16] is slightly different from $\Vdash_{\mathrm{Ro}}$ as defined here.

(1) is clear as the sets $[\varphi]$ for $\varphi \in L^*(M)$, are open and trivially $X \subseteq \overset{\circ}{\overline{X}}$ for open X. (2) then follows from

$$\overset{\circ}{\overline{\overset{\circ}{\overline{X}}}} \subseteq \overset{\circ}{\overline{X}}.$$

For (3) we have to show that $\Vdash^*$ satisfies Extension and Stability for atoms and satisfies universal forcing recurrence. But $\Vdash^*$ satisfies Extension (for all sentences of the forcing language) as sets of the form $\overset{\circ}{\overline{X}}$ are open. Further $\Vdash^*$ satisfies Stability by (2). So $\Vdash^*$ is a forcing. To show $\Vdash^*$ satisfies universal recurrence we first observe:

Claim. $\Vdash^*$ satisfies Forcing Completeness, i.e. $p \Vdash^* \varphi$ if and only if $M[G] \models \varphi$ for every generic G containing p.

Proof of the claim: (2) implies that $\Vdash^*$ satisfies Lemma 2.5 (4) as seen in the proof there. The claim then follows as in the proof of Corollary 2.18 (2). □

The claim implies that $\Vdash^*$ satisfies the $\forall$-clause and the $\wedge$-clause of universal recurrence. The $\neg$-clause for $\Vdash^*$ follows immediately from the $\neg$-clause for $\Vdash$. □

Corollary 2.31. *Assume $\Vdash$ is a universal forcing and $\Vvdash$ is an existential forcing such that $\Vdash$ and $\Vvdash$ agree on atoms of the forcing language. If further $M[G]$ is defined for every generic G, then $\Vdash = \Vvdash^*$.*

Proof. As $\Vvdash$ is a forcing, we have for atomic φ: $\{p \mid p \Vvdash \varphi\} = \{p \mid \{q \mid q \Vvdash \varphi\}$ is dense below $p\} = \{p \mid p \Vvdash \neg\neg\varphi\} = \{p \mid p \Vvdash^* \varphi\}$. Thus $\Vvdash$ and $\Vvdash^*$ agree on atoms and hence so do $\Vvdash^*$ and $\Vdash$. By Proposition 2.30 (3), $\Vvdash^*$ satisfies universal recurrence, so $\Vvdash^* = \Vdash$. □

Proposition 2.32. *Assume $\Vdash$ is an existential pre-forcing such that there are p_0, φ_0 such that $p_0 \nVdash \varphi_0$ and $p_0 \nVdash \neg\varphi_0$. Then $\{\varphi \mid p_0 \Vdash \varphi\}$ is not closed under logical consequence. If $\Vdash$ satisfies Extension for atoms and $M[G]$ is defined for every generic G, then $\Vdash$ does not satisfy Stability.*

Proof. By assumption, p_0 does not force $(\varphi_0 \vee \neg\varphi_0)$. This is valid, so $\{\varphi \mid p_0 \Vdash \varphi\}$ is not closed under logical consequence. If $\Vdash$ satisfies Extension for atoms and $M[G]$ is defined for every generic G, then $\Vdash^*$ is a universal forcing by Proposition 2.30 (3), so by Corollary 2.19 every valid sentence is weakly forced by every condition. Hence $\Vdash \neq \Vdash^*$ and $\Vdash$ does not satisfy Stability . □

Remark 2.33 (Universal versus existential forcing)**.** Intuitively, Corollary 2.19 says that universal forcing refers to the meaning of a sentence, not to its syntax. In contrast existential forcing is syntax sensible, if not trivial (Proposition 2.32), and Forcing Completeness fails. Informally, existential forcing has defects and these defects may be repaired when moving to the weak forcing (Proposition 2.30).

2.7. **Summary.** To sum up, given an L-model M and $L^* \supseteq L$, one specifies a forcing frame $(P, \leq, D_0, D_1, \ldots)$, a relation $\Vdash$ between conditions and *atoms* of the forcing language that satisfies Extension and Stability (for atoms, cf. Definition 2.4).

Then (universal or existential) forcing recurrence determines a (universal or existential) forcing $\Vdash$. For every generic G the generic associate $M[G]$, if defined, satisfies the Truth Lemma, i.e. in $M[G]$ is true exactly what is forced by some condition in G.

Moreover, to get a conservative forcing (Definition 2.22) it suffices to specify $\Vdash$ only for $L^*(M) \setminus L(M)$-atoms. In this case, $M[G]$ is isomorphic to an L^*-expansion of M.

3. Principal theorems

In set theory one usually considers the case where $M[G]$ is an extension of a model M of ZF (Example 2.24). Independence results are based on the "Principal Theorem" [38] stating that every generic extension $M[G]$ models ZF.

In weak theories of arithmetic one is often interested in constructing generic expansions of a countable nonstandard model M of true arithmetic (cf. section 1.3). To get relativized independence results one needs the generic expansions to model some weak arithmetic. This boils down to the question of when generic expansions satisfy certain least number principles.

In this paragraph we fix

- a countable forcing frame $(P, \leq, D_0, D_1, \ldots)$
- a conservative universal forcing $\Vdash$,
- an ordered countable L-structure M satisfying the least number principle (defined below).
- a countable language $L^* \supset L$.

A model is *ordered* if it interprets the symbol $<$ by some linear order on its universe. Given an ordered model N and $b_0 \in N$, the quantifiers $\forall x < b_0$ and $\exists x < b_0$ are called *b_0-bounded.*

Remark 3.1. Due to conservativity, forcing recurrence works for bounded quantifiers as it does for unbounded quantifiers:

$$\begin{array}{c|l} p \Vdash \forall x < b_0 \chi(x) & \text{iff} \quad \forall a <^M b_0 : p \Vdash \chi(a) \\ p \Vdash \exists x < b_0 \chi(x) & \text{iff} \quad p \Vdash \neg \forall x < b_0 \neg \chi(x) \end{array}$$

Note, $p \Vdash \exists x < b_0 \chi(x)$ if and only if $\bigcup_{a <^M b_0} [\chi(a)]$ is dense below p.

Definition 3.2. Let N be an ordered model, $b_0 \in N$ and Φ be a set of formulas in the language of N with parameters from N.

(a) N satisfies the *least number principle for* Φ if every nonempty subset of its universe that is definable by a formula in Φ has a $<^N$-least element.

(b) N satisfies the least number principle for Φ *up to* b_0 if it satisfies the least number principle for $\{(\varphi(x) \wedge x < b_0) \mid \varphi(x) \in \Phi\}$.

We omit reference to Φ, if it is the set of all formulas in the language of N with parameters from N.

3.1. **Partial definability.** Recall Examples 2.6, 2.20: a condition p is compatible with φ, written $p \| \varphi$, if p does not force $\neg\varphi$, or equivalently, if some extension of p forces φ.

Definition 3.3. Let $b_0 \in M$ and $\varphi = \varphi(\bar{x})$ be an $L^*(M)$-formula.

(a) $\Vdash$ is *definable for* φ if for every $p \in P$ the set $\{\bar{a} \mid p \| \varphi(\bar{a})\}$ is definable in M.

(b) $\Vdash$ is *densely definable for* φ *up to* b_0 if for every $p \in P$ there is $q \leq p$ such that $\{\bar{c} <^M b_0 \mid q \| \varphi(\bar{c})\}$ is definable in M.

We say $\Vdash$ is (densely) definable (up to b_0) for a set Φ of $L^*(M)$-formulas if $\Vdash$ is (densely) definable (up to b_0) for every $\varphi \in \Phi$.

Here, for $\bar{c} = c_1 \cdots c_k$ by $\bar{c} <^M b_0$ we mean $c_i <^M b_0$ for every $1 \leq i \leq k$.

Lemma 3.4. *Let $b_0 \in M$ and Φ be a set of $L^*(M)$-formulas that is closed under negations. Then*

(1) *$\Vdash$ is definable for Φ if and only if for every $\varphi(x) \in \Phi$ and $p \in P$ the set $\{\bar{c} \mid p \Vdash \varphi(\bar{c})\}$ is definable in M.*

(2) *$\Vdash$ is densely definable for Φ up to b_0 if and only if for every $\varphi(x) \in \Phi$ and $p \in P$ there is $q \leq p$ such that $\{\bar{c} <^M b_0 \mid q \Vdash \varphi(\bar{c})\}$ is definable in M.*

Proof. For the forward directions note $p \Vdash \varphi$ if and only if $p \Vdash \neg\neg\varphi$ (by Stability) if and only if $p \not\| \neg\varphi$.

For the backward directions note $p \| \varphi$ if and only if $p \not\Vdash \neg\varphi$. □

Recall that, by conservativity, every generic associate is a generic expansion (Proposition 2.23).

Theorem 3.5 (Principal). *Let $b_0 \in M$ and Φ be a set of $L^*(M)$-formulas. If $\Vdash$ is densely definable for Φ up to b_0, then every generic expansion of M satisfies the least number principle for Φ up to b_0.*

In particular, if $\Vdash$ is definable for Φ, then every generic expansion of M satisfies the least number principle for Φ.

Proof. The second statement follows from the first noting that definability implies dense definability up to any $b_0 \in M$. To prove the first, let $M[G]$ be a generic expansion of M and $\varphi(x) \in \Phi$ be such that $M[G] \models \exists x < b_0 \varphi(x)$. We look for a least element in the set defined by $\varphi(x)$ in $M[G]$. It suffices to find $a <^M b_0$ such that $M[G] \models \varphi(a)$ and $M[G] \not\models \varphi(b)$ for every $b <^M a$.

Define

$$D_\varphi := \bigcup_{a <^M b_0} \bigcap_{b <^M a} [(\varphi(a) \wedge \neg\varphi(b))].$$

Claim. D_φ is dense below every condition forcing $\exists x < b_0\varphi(x)$.

Proof of the claim. Given p forcing $\exists x < b_0\varphi(x)$ we are looking for some $q \leq p$ in D_φ. By universal recurrence $\bigcup_{a\in M}[a < b_0 \wedge \varphi(a)]$ is dense below p. By conservativity each set $[a < b_0 \wedge \varphi(a)]$ equals $[\varphi(a)]$ or $\emptyset$ depending on whether $a <^M b_0$ or not. Hence $\bigcup_{a<^M b_0}[\varphi(a)]$ is dense below p, so for some $b <^M b_0$ there is an extension $q_b \leq p$ forcing $\varphi(b)$.

By dense definability applied to $\varphi \in \Phi$ and $q_b \in P$ we find some $\tilde{q} \leq q_b$ such that

$$C := \{c <^M b_0 \mid \tilde{q} \not\Vdash \neg\varphi(c)\}$$

is definable in M. By Extension $\tilde{q} \Vdash \varphi(b)$, so $\tilde{q} \not\Vdash \neg\varphi(b)$ by Consistency. Hence $b \in C$, so $C \neq \emptyset$. Because M satisfies the least number principle, C has a least element $a \leq^M b <^M b_0$.

As $a \in C$ we have $\tilde{q} \not\Vdash \neg\varphi(a)$, so by forcing recurrence we find $q_a \leq \tilde{q}$ forcing $\varphi(a)$. Then $q_a \leq \tilde{q} \leq q_b \leq p$.

To show $q_a \in D_\varphi$, it suffices to show $q_a \Vdash \neg\varphi(b')$ for every $b' <^M a$. But any $b' <^M a \leq^M b <^M b_0$ is not in C by minimality of a, so $\tilde{q} \Vdash \neg\varphi(b')$ and hence also $q_a \Vdash \neg\varphi(b')$ by Extension. □

Choose $p_0 \in G$ forcing $\exists x < b_0\varphi(x)$ by the Truth Lemma. First note $D_\varphi \in \mathcal{B}(\Vdash)$ as it is defined by the following formula (with parameters) of the Stern formalism:

$$\exists x\big(x < b_0 \wedge \forall y\big(y < x \to \big(\xi \vdash (\varphi(x) \wedge \neg\varphi(y))\big)\big)\big).$$

The claim and Lemma 2.11 imply that there is a condition $p \in G \cap D_\varphi$. Hence there is $a <^M b_0$ such that for every $b <^M a$ we have $p \Vdash (\varphi(a) \wedge \neg\varphi(b))$. By the Truth Lemma $M[G] \models \varphi(a)$ and $M[G] \models \neg\varphi(b)$ for every $b <^M a$. Thus a is a least element as we are looking for. □

Here is a dual formulation of the Principal Theorem:

Corollary 3.6. *Let $b_0 \in M$ and Φ be a set of $L^*(M)$-formulas. If for every $\varphi(\bar{x}) \in \Phi$ and $p \in P$ there is $q \leq p$ such that*

$$\{\bar{c} <^M b_0 \mid q \Vdash \varphi(\bar{c})\}$$

is definable in M, then every generic expansion of M satisfies transfinite induction for Φ up to b_0, *that is, for every $\varphi(x) \in \Phi$ the sentence*

$$\forall y < b_0(\forall z < y\varphi(z) \to \varphi(y)) \to \forall x < b_0\varphi(x).$$

Proof. The assumption implies that $\Vdash$ is densely definable for $\neg\Phi$ up to b_0 (see the proof of Lemma 3.4). Now observe that the least number principle for $\neg\Phi$ up to b_0 is equivalent to transfinite induction for Φ up to b_0. □

Lemma 3.7. *The following hold:*

(1) *Let Ψ be the set of $L^*(M)$-formulas φ such that $\Vdash$ is definable for φ. Then Ψ is closed under disjunctions and existential quantification.*

(2) *Let* $b_0 \in M$ *and* Ψ *be the set of* $L^*(M)$*-formulas* φ *such that* $\Vdash$ *is densely definable for* φ *up to* b_0*. Then* Ψ *is closed under disjunctions and* b_0*-bounded existential quantification.*

Proof. (1) and closure under disjunction in (2) follow easily from the recurrence in Example 2.6. We show closure under b_0-bounded existential quantification in (2).

Let $\varphi(y\bar{x}) \in \Psi$ and $p \in P$. We are looking for $q \leq p$ such that $\{\bar{a} <^M b_0 \mid q \| \exists y < b_0 \varphi(y\bar{a})\}$ is definable in M. Because $\varphi \in \Psi$ we find $q \leq p$ such that $\{a\bar{a} <^M b_0 \mid q \| \varphi(a\bar{a})\}$ is definable in M. Then also

$$\{\bar{a} <^M b_0 \mid \exists a <^M b_0 : q \| \varphi(a\bar{a})\}$$

is definable in M. By conservativity $a <^M b_0$ is equivalent with $s \Vdash a < b_0$ for any condition s. Hence the above set equals

$$\{\bar{a} <^M b_0 \mid \exists a \in M : q \| (a < b_0 \wedge \varphi(a\bar{a}))\},$$

and this is the set we want (see the recurrence in Example 2.6). □

3.2. **Definable antichains.** We sketch a method to establish dense definability. We are going to apply it in the next paragraph. The method is intended for the typical situation where P is an (in general undefinable) subset of M and there are $L(M)$-formulas $\varphi(x, y), \psi(x, y)$ such that for all $p, q \in P$

$$(p \leq q \Longleftrightarrow M \models \varphi(p, q)) \quad \text{and} \quad (p \| q \Longleftrightarrow M \models \psi(p, q)).$$

In this case, the following two lemmas reduce dense definability of forcing to the definability of predense antichains refining given definable antichains.

We recall some standard forcing terminology: an *antichain* is a set of pairwise incompatible conditions. An antichain A is *maximal in* $X \subseteq P$ if $A \subseteq X$ and every $p \in X$ is compatible with some element of A. A set $X \subseteq P$ is *predense* (*below* p) if every condition (extending p) is compatible with some condition in X. E.g. an antichain is predense if and only if it is maximal in P.

We write

$$X \downarrow q := \{p \in X \mid p \leq q\} \quad \text{and} \quad X \downarrow Y := \textstyle\bigcup_{q \in Y} X \downarrow q.$$

The method is based on the simple observation that in order to define the forcing for some φ it suffices to define a maximal antichain in $[\varphi]$:

Lemma 3.8. *If* $p \leq q$ *and* X *is a maximal antichain in* $[\varphi] \downarrow q$*, then* $p \| \varphi$ *if and only if* p *is compatible with some condition in* X*.*

Proof. If $p \| \varphi$, then there is $r \in [\varphi]$ extending p. Then $r \in [\varphi] \downarrow q$ since $r \leq p \leq q$. By maximality of X, r is compatible with some condition in X, and hence, as $r \leq p$, so is p. The converse is immediate by Extension. □

To find maximal antichains we intend to proceed by induction on φ. How to get, say, a maximal antichain in $[\neg\varphi]$ from a maximal antichain X in $[\varphi]$? The next lemma shows that this can be done via a predense antichain *refining* X in the following sense:

Definition 3.9. For $X, Y \subseteq P$ we say X *refines* Y if every condition in X that is compatible with some condition in Y already extends some condition in Y.

Lemma 3.10. *Let φ, ψ be $L^*(M)$-sentences, $\chi(x)$ an $L^*(M)$-formula, $b_0 \in M$ and $p \in P$.*

(1) *If X is a maximal antichain in $[\varphi] \downarrow p$, and $A \subseteq P \downarrow p$ is an antichain that is predense below p and refines X, then $A \setminus (A \downarrow X)$ is a maximal antichain in $[\neg\varphi] \downarrow p$.*
(2) *If X and Y are maximal antichains in $[\neg\varphi] \downarrow p$ and $[\neg\psi] \downarrow p$ respectively, and $A \subseteq P \downarrow p$ is an antichain that is predense below p and refines $X \cup Y$, then $A \setminus (A \downarrow (X \cup Y))$ is a maximal antichain in $[\varphi \wedge \psi] \downarrow p$.*
(3) *If for every $a <^M b_0$, the set X_a is a maximal antichain in $[\neg\chi(a)] \downarrow p$, and $A \subseteq P \downarrow p$ is an antichain that is predense below p and refines $\bigcup_{a <^M b_0} X_a$, then $A \setminus (A \downarrow \bigcup_{a <^M b_0} X_a)$ is a maximal antichain in $[\forall x < b_0 \chi(x)] \downarrow p$.*

Proof. We only show (3). Obviously $A' := A \setminus (A \downarrow \bigcup_{a <^M b_0} X_a)$ is an antichain in $P \downarrow p$. To see $A' \subseteq [\forall x < b_0 \chi(x)]$, let $q \notin [\forall x < b_0 \chi(x)]$ be given. We claim $q \notin A'$. If $q \notin A$, there is nothing to show, so we assume $q \in A$ and claim $q \in A \downarrow \bigcup_{a <^M b_0} X_a$. Since $q \nVdash \forall x < b_0 \chi(x)$ there is $a_0 <^M b_0$ such that $q \nVdash \chi(a_0)$ (Remark 3.1). By Lemma 2.5 (4) some extension $r \leq q$ forces $\neg\chi(a_0)$. By maximality of X_{a_0}, the condition r, and hence also q, is compatible with some condition in $X_{a_0} \subseteq \bigcup_{a <^M b_0} X_a$. Since $q \in A$ and A refines $\bigcup_{a <^M b_0} X_a$, we get $q \in A \downarrow \bigcup_{a <^M b_0} X_a$.

To see that A' is maximal, let $q \leq p$ force $\forall x < b_0 \chi(x)$. Then q is compatible with some $r \in A$ since A is predense below p. We claim $r \in A'$, i.e. $r \notin A \downarrow \bigcup_{a <^M b_0} X_a$. Otherwise r forces $\neg\chi(a_0)$ for some $a_0 <^M b_0$ by Extension, and thus also $\neg\forall x < b_0 \chi(x)$ (Corollary 2.19). Hence r cannot be compatible with q by Extension and Consistency, a contradiction. □

Corollary 3.11. *Let Φ be a set of $L^*(M)$-sentences and assume P has a maximum 1_P. Assume further that A is a predense antichain such that $A \subseteq [\varphi] \cup [\neg\varphi]$ for every $\varphi \in \Phi$. If ψ, χ are Boolean combinations of sentences from Φ, then*

(1) *$A \cap [\psi]$ is a maximal antichain in $[\psi]$;*
(2) *$A \cap [\neg\psi] = A \setminus (A \cap [\psi])$ is a maximal antichain in $[\neg\psi]$;*
(3) *$A \cap [\psi \wedge \chi] = (A \cap [\psi]) \cap (A \cap [\chi])$ is a maximal antichain in $[\psi \wedge \chi]$.*

Proof. First show by a straightforward induction that $A \subseteq [\psi] \cup [\neg\psi]$ for every Boolean combination ψ of sentences from Φ. This implies (1): to see maximality, observe that any $p \in [\psi]$ must be compatible with some condition in A by predensity, and since such a condition cannot be in $[\neg\psi]$ by Extension and Consistency, it must be in $[\psi]$.

Knowing (1) for ψ and χ, we get (2) and (3) applying Lemma 3.10 for $p := 1_P$: note that, in general, if A is an antichain and $X \subseteq A$, then A refines X, and $A \downarrow X = X$. □

3.3. **Full definability.** The forcing frame P is *definable in* M if there is a first-order interpretation of $(P, \leq)$ in M.

Examples 3.12. In set theory, Cohen forcing (Example 2.24) uses definable forcing frames. Easton forcing extends Cohen forcing in that it allows the forcing frame to be a proper class in M, i.e. instead of being a set in M it is only assumed to be definable in M. In case the class frame is in a certain sense approximable by set frames, one can define a forcing that satisfies the forcing lemmas and the Principal Theorem (cf. Section 1.5).

In arithmetic, Feferman forcing (Example 2.26) uses definable forcing frames. This is due to the fact that it starts with the standard model. Simpson [39] gives an example of a definable forcing frame starting with a nonstandard model of arithmetic. ⌟

An easy induction shows:

Lemma 3.13. *Assume the forcing frame is definable in* M *and* $\Vdash$ *is definable for* $L^*(M)$*-atoms. Then* $\Vdash$ *is definable for all* $L^*(M)$*-formulas.*

Then the Principal Theorem implies:

Corollary 3.14. *Assume the forcing frame is definable in* M *and* $\Vdash$ *is definable for* $L^*(M)$*-atoms. Then every generic expansion of* M *satisfies the least number principle.*

Example 3.15 (Knight's trick)**.** We can use brute force to make a forcing definable by working not with M but with the two-sorted structure (P, M) instead. In [20] Knight does something similar. The trick is to pad M with some other sorts such that the forcing frame becomes definable for atoms. Knight uses a conservative existential forcing (on the padded structure)[3]. Lemma 3.13 (modified for existential forcing, [20, Lemma 2.2]) then gives full definability.

To sample one of Knight's applications, her padding becomes superfluous when M is an ω-model of ZFC and the forcing becomes definable in M. Knight shows that any elementary end extension of M by another ω-model has a generic expansion interpreting a universal choice function that preserves the elementary embedding. ⌟

[3]To be correct, Knight uses a conservative existential pre-forcing that satisfies Extension but not necessarily Stability for atoms.

3.4. **Forcing and propositional proofs.** We give an intuitive summary of the development sofar as a method to establish lower bounds on the size of propositional proofs.

Recall Example 1.1. Let φ be a relational first-order sentence that is true in all finite models and let $\varphi^{<x}$ result from φ by replacing every quantifier Qy by $Qy < x$. Then every $\langle\varphi^{<x}\rangle_m, m \geq 1$, is a tautology. We would like to establish a lower bound on the length of proofs of these tautologies in a given propositional proof system. Assume proofs in our system are sequences of 'lines' with the last line being the formula proved.

Let M be elementary equivalent to some 'standard' L-model $(\mathbb{N}, <, \ldots)$ and contain nonstandard elements. Let L^* extend L by te language of φ. Design a forcing such that $\varphi^{<x}$ is falsified by some nonstandard $n \in M$ in some generic expansion $M[G]$ of M. Define an L^*-formula 'line y in proof z is false' such that any (code of a) proof π of $\langle\varphi^{<x}\rangle_n$ has a 'false' last line. Show in $M[G]$ that the system is sound: if line y in proof z is false, then so is some line $y' < y$.

The art is to construct the forcing frame such that the forcing is densely definable up to b_0 for all sentences 'line i in proof π is false' where $b_0 \in M$ is as large as possible and π is any (code of a) size $<^M b_0$ proof of $\langle\varphi^{<x}\rangle_n$. Typically, the logical complexity of the formula 'line i in proof π is false' will reflect the logical complexity of the propositional formulas the system operates with as well as the bound b_0.

For every L-term $s(x)$ such that $s^M(n) <^M b_0$, one can then conclude that the function $s^{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ cannot upper bound the sizes of proofs of the tautologies $\langle\varphi^{<x}\rangle_m, m \geq 1$.

4. Forcing against bounded arithmetic

We define Paris-Wilkie forcing, Riis forcing and Ajtai forcing, prove a definability result for each and give the corresponding independence results.

In this paragraph we fix

- a countable language L containing $\{+, \cdot, 0, 1, <\}$.
- a countable L-structure M that is elementarily equivalent to an L-expansion of $(\mathbb{N}, +, \cdot, 0, 1, <)$.
- $L^* := L \cup \{R\}$ for a new binary relation symbol $R \notin L$.

We fix some notation. For $n \in M$ we write

$$[n] := \{a \in M \mid a <^M n\}.$$

A relation R over M is *bounded (in M)* if there is $b \in M$ such that any component of any tuple in R is $<^M b$. As $\mathbb{N}$ codes every bounded (in $\mathbb{N}$) relation by an element, M codes every definable bounded (in M) relation by an element. If $m \in \mathbb{N}$ is such a code we let

$$\|m\|$$

denote the cardinality of the coded relation. This is not to be confused with

$$|m|$$

denoting $\log(m+1)$ (rounded down). Using the definitions of these functions in $(\mathbb{N}, +, \cdot, 0, 1, <)$, we get corresponding functions $\|\cdot\|^M$ and $|\cdot|^M$ in M and we shall omit the superscripts.

For arbitrary $n, m \in M$

$$n <^M m^{o(1)}$$

means that $n^\ell <^M m$ for every $\ell \in \mathbb{N}$.

4.1. **Paris-Wilkie forcing.** Paris and Wilkie [29] gave "the first forcing argument in the context of weak arithmetic" [21, p.278] establishing independence of the pigeonhole principle $\forall x \mathrm{PHP}(R, x)$ from the least number principle for existential formulas. Recall $\mathrm{PHP}(R, x)$ expresses "R is not a bijection from $\{y \mid y \leq x\}$ onto $\{y \mid y < x\}$".

Theorem 4.1 (Paris, Wilkie 1985)**.** *Let $n \in M$ be such that $[n]$ is infinite. Then M has an L^*-expansion satisfying both $\neg\mathrm{PHP}(R, n)$ and the least number principle for existential $L^*(M)$-formulas.*

Let $n \in M$ with infinite $[n]$. We define a forcing frame

$$(P, \leq, D_0, D_1, \ldots).$$

Note that every finite bijection from a subset of $[n] \cup \{n\}$ onto a subset of $[n]$ is coded by an element in M. We let P be the set of all these codes. Note that P is not definable in M.

As partial order we use $p \leq q$ if and only if $p \supseteq q$. Here, and below, we blur the distinction between p and the bijection coded.

The family $D_0, D_1, \ldots$ enumerates the (countably many) sets

$$\{p \mid b \in \mathrm{dom}(p)\}, \{p \mid c \in \mathrm{im}(p)\} \quad \text{for } b \leq^M n, c <^M n.$$

To determine a universal pre-forcing $\Vdash_{\mathrm{PW}}$ it suffices to define $p \Vdash_{\mathrm{PW}} \varphi$ for atoms φ. Further we want a conservative forcing, so it suffices to define $p \Vdash_{\mathrm{PW}} \varphi$ for φ an $L^*(M)$-atom that is not an $L(M)$-atom. Such an atom has the form Rst for closed $L(M)$-terms s, t. We set

$$p \Vdash_{\mathrm{PW}} Rst \iff (s^M, t^M) \in p.$$

It is easy to check (cf. Example 2.25):

Lemma 4.2. *The following hold:*

(1) $\Vdash_{\mathrm{PW}}$ *is a forcing.*
(2) $M[G]$ *is defined and a generic expansion of M for every generic G.*
(3) $M[G]$ *violates* $\mathrm{PHP}(R, n)$ *for every generic G.*

Lemma 4.3. $\Vdash_{\mathrm{PW}}$ *is definable for quantifier free $L^*(M)$-formulas.*

We give the proof exemplifying the method of definable antichains from Section 3.2. However, a direct proof would be equally easy. Note that we are in the "typical situation" that we have $L(M)$-formulas $\varphi(x, y), \psi(x, y)$ such that for all $p, q \in P$

$$(p \leq q \Longleftrightarrow M \models \varphi(p, q)) \quad \text{and} \quad (p \| q \Longleftrightarrow M \models \psi(p, q)).$$

E.g. $\psi(x, y)$ is a formula expressing that both x and y code partial bijections that agree on arguments on which they are both defined.

Proof of Lemma 4.3. Let $\varphi = \varphi(\bar{x})$ be a quantifier free $L^*(M)$-formula. For $\bar{c}$ from M let $T(\bar{c})$ be the set of those $a \in M$ that are denoted by some closed term in $\varphi(\bar{c})$. Further let $A_{\bar{c}}$ be the set of all inclusively minimal partial bijections p such that both $\mathrm{dom}(p)$ contains $T(\bar{c}) \cap [n+1]$ and $\mathrm{im}(p)$ contains $T(\bar{c}) \cap [n]$. As $T(\bar{c})$ is finite, $A_{\bar{c}} \subseteq P$. It is routine to verify that $A_{\bar{c}}$ is a predense antichain in P and equal to $\alpha(y, \bar{c})(M)$ for a suitable $L(M)$-formula $\alpha(y, \bar{x})$.

For an atom $\psi = \psi(\bar{x})$ occuring in $\varphi(\bar{x})$ we have $A_{\bar{c}} \subseteq [\psi(\bar{c})] \cup [\neg\psi(\bar{c})]$. Further there is an $L(M)$-formula $\xi^{\psi}(y, \bar{x})$ such that $\xi^{\psi}(y, \bar{c})(M)$ defines $A_{\bar{c}} \cap [\psi(\bar{c})]$. We find such a formula $\xi^{\chi}(z, \bar{x})$ for every Boolean combination χ of such atoms following the recursion in Corollary 3.11 (2), (3). In particular, we find an $L(M)$-formula $\xi^{\varphi}(y, \bar{x})$ such that $\xi^{\varphi}(y, \bar{c})$ defines a maximal antichain in $[\varphi(\bar{c})]$.

Lemma 3.8 (for q the empty partial bijection) implies that $\Vdash_{\mathrm{PW}}$ is definable for $\varphi(\bar{x})$. □

Proof of Theorem 4.1. Choose a generic G (Lemma 2.10). Up to isomorphism, then $M[G]$ expands M and violates $\mathrm{PHP}(R, n)$ (Lemma 4.2). By Lemmas 4.3 and 3.7, $\Vdash_{\mathrm{PW}}$ is definable for existential $L^*(M)$-formulas. By the Principal Theorem 3.5, $M[G]$ satisfies the least number principle for these formulas. □

4.2. **Riis forcing.** One may wonder what in the above proof is special about the pigeonhole principle. Riis pointed out that essentially what is needed is that the principle fails in the infinite [35, 35]. He uses existential forcing and allows for certain infinite conditions. The point is that the new forcing frame allows to define the forcing for more formulas.

For an $L(M)$-formula $\varphi_0(x, y)$ let

$$(R : \hat{y}\varphi_0(x, y) \sim [x])$$

be a formula expressing "R is a bijection from $\{y \mid \varphi_0(x, y)\}$ onto $[x]$". This is an $L^*(M)$-formula with free variable x.

Definition 4.4. An $L(M)$-formula $\varphi_0(xy)$ *defines an $n^{\Omega(1)}$ family in M* if there are $\ell \in \mathbb{N}$ and an $L(M)$-formula $\sigma(yz, x)$ such that for every $n \in M$, $\sigma(yz, n)(M)$ is a surjection from $(\varphi_0(n, y)(M))^{\ell}$ onto $[n]$ provided $\varphi_0(n, y)(M)$ is nonempty.

If every $\varphi(ny)(M), n \in M$, is bounded and, say, coded by $c_n \in M$, then defining a $n^{\Omega(1)}$-family means that there is an $\ell \in \mathbb{N}$ such that $n \leq^M \|c_n\|^{\ell}$ for every $n \in M$. For example, $b_0 <^M n^{o(1)}$ if and only if $(x = n \wedge y < b_0)$ does not define a $n^{\Omega(1)}$-family in M.

Examples 4.5. If we choose for $\varphi_0(x, y)$ the formula $y \leq x$, then $(R : \hat{y}\varphi_0(n, y) \sim [n])$ becomes $\neg\text{PHP}(R, n)$. If we choose $y < x \cdot x$, then our formula negates the weak pigeonhole principle with n^2 pigeons and n holes. Choosing $y = y$ we negate the cardinal principle (cf. [18]).

If we assume that the L-structure $(\mathbb{N}, +, \cdot, 0, 1, <, \ldots)$ additionally interprets an infinite unary predicate U and a binary relation $E \subseteq U \times U$, then $(R : \hat{y}\varphi_0(n, y) \sim [n])$ with $\varphi_0(x, y) := Uy$ expresses that R copies the infinite directed graph (U, E) to the new universe $[n]$ ("Finitization" [35]).

These examples define $n^{\Omega(1)}$ families in M. ⌟

Let $\Sigma_1^{b_0}(R)$ denote the closure of the set of quantifier-free $L^*(M)$-formulas by existential and b_0-bounded quantification (i.e. quantifiers of the form $\exists x < b_0$ and $\forall x < b_0$, cf. page 17).

Theorem 4.6 (Riis 1993)**.** *Let $\varphi_0(xy)$ define an $n^{\Omega(1)}$ family in M and let $b_0, n \in M$ be such that $b_0 <^M n^{o(1)}$. Then M has an L^*-expansion satisfying both $(R : \hat{y}\varphi_0(ny) \sim [n])$ and the least number principle for $\Sigma_1^{b_0}(R)$.*

Remark 4.7. The reader familiar with bounded arithmetic will notice the following. Use Buss' language for L and choose n and b_0 such that both $b_0 <^M n^{o(1)}$ and $|n| <^M b_0^{o(1)}$. By the second inequality $M \models |t(n)| < b_0$ for every (parameter free) L-term $t(x)$ and hence $\Sigma_1^{b_0}(R)$ includes all $\Sigma_1^b(R)$ formulas with parameters bounded by some L-term in n. Thus the restriction of the expansion to the corresponding cut is a model of $T_2^1(R)$ and $(R : \hat{y}\varphi_0(ny) \sim [n])$.

Let $\varphi_0(xy)$ and $b_0, n \in M$ accord the assumption of Theorem 4.6. We prove the theorem only for the case where $[b_0]$ is infinite. In case $[b_0]$ is finite, b_0-bounded quantifiers can be eliminated and one can argue as in the last section.

Definition 4.8. A relation R over M is *ℓ-small* if it is empty or there are $\ell \in \mathbb{N}$ and an $L(M)$-definable surjection from $[b_0]^\ell$ onto R. R is *small* if it is ℓ-small for some $\ell \in \mathbb{N}$.

Then $[n]$ is not small and neither is

$$A_0 := \varphi_0(ny)(M).$$

Here, and only here, we use the assumption that $\varphi_0(xy)$ defines a $n^{\Omega(1)}$-family in M.

We define the forcing. An ℓ-small bijection from a subset of A_0 onto a subset of $[n]$ is $L(M)$-definable and bounded in M, and hence coded by an element of M. Let $P_\ell \subseteq M$ be the set of all these codes. The set of conditions is

$$P := \bigcup_{\ell \in \mathbb{N}} P_\ell.$$

Again we set $p \leq q$ if $p \supseteq q$, and let the family $D_0, D_1, \ldots$ enumerates the sets $\{p \mid a \in \mathrm{dom}(p)\}, \{p \mid c \in \mathrm{im}(p)\}$ for $a \in A_0, c \in [n]$.

The forcing relation is defined as in the previous section: $p \Vdash_{\mathrm{Ri}} Rst$ if and only if $(s^M, t^M) \in p$. This uniquely determines a conservative universal pre-forcing, and in fact a forcing (cf. Example 2.25).

Lemma 4.9. *Let $\ell \in \mathbb{N}$.*

(1) $P_\ell \subseteq P_{\ell+1} \subseteq P \subseteq M$.
(2) *P_ℓ is $L(M)$-definable.*
(3) *If $p, q \in P_\ell$, then $p \cup q \in P_{\ell+1}$ and $p \cap q \in P_\ell$,*
(4) *The sets $D_0, D_1, \ldots$ are dense.*

Proof. We only show (4). Observe that both the domain and range of a condition $p \in P$ are small. As neither A_0 nor $[n]$ is small, both $(A_0 \backslash \mathrm{dom}(p))$ and $([n] \backslash \mathrm{im}(p))$ are infinite. Then (4) follows easily. □

Lemma 4.10. *$\Vdash_{\mathrm{Ri}}$ is definable for $\Sigma_1^{b_0}(R)$.*

This implies the theorem:

Proof of Theorem 4.6. Clearly, $M[G]$ is defined for every generic G. By Proposition 2.23 all generic associates of M are generic expansions and it should be clear that they all satisfy $(R : \hat{y}\varphi_0(ny) \sim [n])$. Thus Theorem 4.6 follows from Lemma 4.10 and the Principal Theorem. □

To prove Lemma 4.10 we rely on the following lemma. It can be shown following the proof of Lemma 4.3 in the previous section.

Lemma 4.11. *For every quantifier free $L^*(M)$-formula $\varphi(\bar{x})$ there is an $L(M)$-formula $\Diamond^\varphi(y, \bar{x})$ such that for every $p \in P$*

$$\Diamond^\varphi(p, \bar{x})(M) = \{\bar{c} \mid p \| \varphi(\bar{c})\}.$$

Proof of Lemma 4.10. For variable tuples $\bar{x} = x_1 \cdots x_\ell, \bar{y} = y_1 \cdots y_\ell$ let $Q\bar{x}\bar{y}$ abbreviate the quantifier prefix

$$\forall x_1 < b_0 \exists y_1 \ \forall x_2 < b_0 \exists y_2 \ \cdots \ \forall x_\ell < b_0 \exists y_\ell.$$

It suffices to show that $\Vdash_{\mathrm{Ri}}$ is definable for every formula of the form $Q\bar{x}\bar{y}\varphi$ where φ is a quantifier free $L^*(M)$-formula (by Corollary 2.19 since $M[G]$ is defined for every generic G).

Fix a quantifier free $L^*(M)$-formula $\varphi(\bar{z})$. Define the formula

$$\Box^\varphi(y, \bar{z}) := \neg \Diamond^{\neg\varphi}(y, \bar{z}).$$

By Lemma 4.11 and Stability, we have for every condition $p \in P$

$$\Box^\varphi(p, \bar{z})(M) = \{\bar{c} \mid p \Vdash_{\mathrm{Ri}} \varphi(\bar{c})\}.$$

For a tuple $\bar{c}$ from M let $A_{\bar{c}}$ be the predense antichain as defined in the proof of Lemma 4.3. In particular, $A_{\bar{c}} \subseteq [\varphi(\bar{c})] \cup [\neg\varphi(\bar{c})]$ and $A_{\bar{c}} \cap [\varphi(\bar{c})]$ is a maximal antichain in $[\varphi(\bar{c})]$. Further $A_{\bar{c}} \subseteq P_1$ since every condition in $A_{\bar{c}}$ is finite and we assumed that $[b_0]$ is infinite.

For any two tuples $\bar{x}, \bar{y}$ of variables of the same length ℓ we show the following: for all $p \in P$ and all $\bar{c}'$ there is $q \in P_{\ell+1}, q \| p$ such that

$$(*) \qquad \text{if } p \| Q\bar{x}\bar{y}\ \varphi(\bar{x}, \bar{y}, \bar{c}'), \text{ then } M \models Q\bar{x}\bar{y}\ \Box^{\varphi}(p \cup q, \bar{x}, \bar{y}, \bar{c}'),$$

where $\bar{c}'$ ranges over assignments to the free variables $\bar{z}'$ of $Q\bar{x}\bar{y}\ \varphi$. It is not hard to see that then

$$\exists u(u \in P_{\ell+1} \wedge u \| p \wedge Q\bar{x}\bar{y}\ \Box^{\varphi}(p \cup u, \bar{x}, \bar{y}, \bar{z}'))$$

defines $\{\bar{c}' \mid p \| Q\bar{x}\bar{y}\ \varphi(\bar{x}, \bar{y}, \bar{c}')\}$ in M. Here "$u \in P_{\ell+1}$" is an $L(M)$-formula according Lemma 4.9 (2) and "$x \| y$" is an $L(M)$-formula expressing compatibility as in the previous section.

We prove $(*)$ by induction on ℓ. The base case, $\ell = 0$, is easy: if $p \| \varphi(\bar{c})$, then p is compatible with some $q \in A_{\bar{c}} \cap [\varphi(\bar{c})]$ by Lemma 3.8. But $A_{\bar{c}} \subseteq P_1$ and $p \cup q \Vdash_{\mathrm{Ri}} \varphi(\bar{c})$.

For the inductive step, let $x\bar{x}, y\bar{y}$ be length $\ell+1$ tuples of variables, let $\bar{c}'$ range over assigments to the free variables $\bar{z}'$ in $Qx\bar{x}y\bar{y}\ \varphi$ and write $\varphi = \varphi(x\bar{x}, y\bar{y}, \bar{z}')$.

Let $p \in P$ and $\bar{c}'$ be such that $p \| Qx\bar{x}y\bar{y}\ \varphi(x\bar{x}, y\bar{y}, \bar{c}')$, i.e. there is $\tilde{p} \leq p$ forcing $\forall x < b_0 \exists y\ Q\bar{x}\bar{y}\ \varphi(x\bar{x}, y\bar{y}, \bar{c}')$. Using universal recurrence and Remark 3.1, this is easily seen to be equivalent to: for every $a <^M b_0$ and every $q \leq \tilde{p}$ there is $b \in M$ such that $q \| Q\bar{x}\bar{y}\ \varphi(a\bar{x}, b\bar{y}, \bar{c}')$. By induction we get for every $a <^M b_0$ and every $q \leq \tilde{p}$

$$M \models \exists u(u \in P_{\ell+1} \wedge u \| q \wedge \exists y\ Q\bar{x}\bar{y}\ \Box^{\varphi}(q \cup u, a\bar{x}, y\bar{y}, \bar{c}')).$$

Let $\psi(z)$ be the formula

$$\exists u\big((u = \emptyset \vee \exists v \text{"}v \text{ is a surjection from } [z] \times [b_0]^{\ell+1} \text{ onto } u\text{"}) \wedge u \| \tilde{p} \wedge \forall x < z \exists y\ Q\bar{x}\bar{y}\ \Box^{\varphi}(\tilde{p} \cup u, x\bar{x}, y\bar{y}, \bar{c}')\big).$$

Claim. $M \models \psi(b_0)$.

Proof of the claim. It is straightforward to verify $M \models \psi(0)$ and $M \models (\psi(a) \to \psi(a+1))$ for every $a <^M b_0$. □

By the claim there is $\tilde{q} \in P$ (even in $P_{\ell+2}$) compatible with $\tilde{p}$ such that

$$M \models Qx\bar{x}y\bar{y}\ \Box^{\varphi}(\tilde{p} \cup \tilde{q}, x\bar{x}, y\bar{y}, \bar{c}').$$

As M satisfies the least number principle it defines Skolem functions: there are $L(M)$-definable functions $f\bar{f} = f, f_1, f_2, f_3, \ldots$ such that

$$M \models \forall x\bar{x} < b_0 \Box^{\varphi}\big(\tilde{p} \cup \tilde{q}, x\bar{x}, f(x)\bar{f}(x\bar{x}), \bar{c}'\big).$$

Here, $\bar{f}(x\bar{x})$ is shorthand for $f_1(xx_1)f_2(xx_1x_2)f_3(xx_1x_2x_3)\cdots$.

Recall how the antichains $A_{\bar{c}}$ are defined: they consist in all $\subseteq$-minimal conditions whose domain contains $T(\bar{c}) \cap [n+1]$ and whose image contains $T(\bar{c}) \cap [n]$, where $T(\bar{c})$ is the set of things named by closed terms in $\varphi(\bar{c})$.

Write $T(\bar{z}) = T(x\bar{x}, y\bar{y}, \bar{z}')$ and set

$$S(\bar{c}') := \bigcup_{a\bar{a} <^M b_0} T(a\bar{a}, f(a)\bar{f}(a\bar{a}), \bar{c}').$$

Let $B_{\bar{c}'}$ be defined for $S(\bar{c}')$ as $A_{\bar{c}}$ is for $T(\bar{c})$. Then $B_{\bar{c}'}$ is an $L(M)$-definable set of $(\ell+2)$-small conditions. Furthermore $B_{\bar{c}'}$ is a predense antichain that refines every $A_{\bar{c}}$, where $\bar{c}$ is of the form $a\bar{a}f(a)\bar{f}(a\bar{a})\bar{c}'$ for some $a\bar{a} <^M b_0$.

Claim. If $a\bar{a} <^M b_0$ and $\bar{c} = a\bar{a}f(a)\bar{f}(a\bar{a})\bar{c}'$, then $B_{\bar{c}'} \subseteq [\varphi(\bar{c})] \cup [\neg\varphi(\bar{c})]$.

Proof of the claim. Let $a\bar{a} <^M b_0$ and $\bar{c} = a\bar{a}f(a)\bar{f}(a\bar{a})\bar{c}'$. Every $r \in B_{\bar{c}'}$ is compatible with some condition in $A_{\bar{c}}$ by predensity. Since $B_{\bar{c}'}$ refines $A_{\bar{c}}$, r extends some condition in $A_{\bar{c}}$. Since $A_{\bar{c}} \subseteq [\varphi(\bar{c})] \cup [\neg\varphi(\bar{c})]$, also $r \in [\varphi(\bar{c})] \cup [\neg\varphi(\bar{c})]$ by Extension. □

By predensity there is $r \in B_{\bar{c}'}$ such that $(\tilde{p} \cup \tilde{q}) \| r$. Then $r \| p$ and $r \in P_{\ell+2}$, so we are left to check $M \models \forall x\bar{x} < b_0 \Box^{\varphi}\big(p \cup r, x\bar{x}, f(x)\bar{f}(x\bar{x}), \bar{c}'\big)$. By Extension it suffices to verify

$$M \models \forall x\bar{x} < b_0 \Box^{\varphi}\big(r, x\bar{x}, f(x)\bar{f}(x\bar{x}), \bar{c}'\big).$$

But otherwise there is $a\bar{a} <^M b_0$ such that $r \not\Vdash_{\mathrm{Ri}} \varphi(a\bar{a}, f(a)\bar{f}(a\bar{a}), \bar{c}')$. By the last claim, then $r \Vdash_{\mathrm{Ri}} \neg\varphi(a\bar{a}, f(a)\bar{f}(a\bar{a}), \bar{c}')$. But $\tilde{p} \cup \tilde{q} \Vdash_{\mathrm{Ri}} \varphi(a\bar{a}, f(a)\bar{f}(a\bar{a}), \bar{c}')$ (by the choice of $\tilde{q}$), so r cannot be compatible with $\tilde{p} \cup \tilde{q}$ by Extension and Consistency, a contradiction. □

4.3. **Ajtai forcing.** We prove Ajtai's result [1] including its improvements from [27, 30]. Compared to Riis' Theorem 4.6 it embodies an exponential improvement concerning the bound b_0, but only concerns b_0-bounded formulas. Citing Zambella, any techniques that can allow to handle the case of $\Sigma_1^{b_0}(R)$ for such big b_0 would be extremely interesting [46, p.403].

Let $\Delta_0^{b_0}(R)$ denote the closure of the set of quantifier-free $L^*(M)$-formulas by b_0-bounded quantification (cf. page 17).

Theorem 4.12 (Ajtai 1988)**.** *Let $b_0, n \in M$ be such that $|b_0| <^M n^{o(1)}$. Then M has an L^*-expansion satisfying both $\neg\mathrm{PHP}(R, n)$ and the least number principle for $\Delta_0^{b_0}(R)$ up to b_0.*

Remark 4.13. The reader familiar with bounded arithmetic will notice the following. Use Buss' language for L and choose $b_0, n \in M$ such that both $|b_0| < n^{o(1)}$ and $|n| < |b_0|^{o(1)}$. By the second inequality b_0 bounds $t^M(n)$ for every L-term $t(x)$. Thus the restriction of the expansion to the corresponding cut is a model of $T_2(R)$ that violates $\mathrm{PHP}(R, n)$.

Fix some $d \in \mathbb{N}$. Following Section 3.4 it is not hard to infer from Theorem 4.12 that proofs of $\langle \mathrm{PHP}(R, x)\rangle_m, m \in \mathbb{N}$, in depth d Frege systems must have size at least $2^{m^{\epsilon}}$ for some $\epsilon > 0$ (depending on d).

For $m \in \mathbb{N}$ consider the following finite forcing frame $(P(m), \leq)$ (without a family $D_0, D_1, \ldots$): the conditions are all finite partial bijections from $[m+1]$ to $[m]$ and $p \leq q$ means $p \supseteq q$. Again, we blur the distinction of the bijection and its code in $\mathbb{N}$. The *size* $\|p\|$ of a condition p is its cardinality, i.e. the number of pigeons mapped. The *rank* of a set $X \subseteq P(m)$ is the maximal size of a condition in X (and, say, 0 if X is empty).

Now fix M and $n, b_0 \in M$ according the assumptions of Theorem 4.12. Observe that there are uniform definitions of $P(m)$ in the standard model $(\mathbb{N}, +, \cdot, 0, 1, <)$ meaning that there is a $\{+, \cdot, 0, 1, <\}$-formula $\varphi(x, y)$ such that $P(m) = \varphi(m, y)(\mathbb{N})$ for every $m \in \mathbb{N}$. Applied in M, these definitions give forcing frames $(P(m), \leq)$ with size function $\| \cdot \|$ also for (nonstandard) $m \in M$. Further note that M defines the function $m \mapsto m^\epsilon$ (rounded up) for any (standard) rational $0 < \epsilon < 1$.

We now define the forcing frame P. It is going to be an undefinable subframe of the definable frame $P(n)$. For every standard rational $0 < \epsilon < 1$ the set $\{p \in P(n) \mid \|p\| <^M n - n^\epsilon\}$ is definable in M. We let P be the union of all these sets. As usual $p \leq q$ means $p \supseteq q$, and the family $D_0, D_1, \ldots$ enumerates the sets $\{p \in P \mid b \in \mathrm{dom}(p)\}$ and $\{p \in P \mid c \in \mathrm{im}(p)\}$ for $b \leq^M n, c <^M n$. It is easy to see that these sets are dense (in P).

We define the forcing as in the previous two sections: we let $p \in P$ force an atom Rst if $(s^M, t^M) \in p$ and denote by $\Vdash_{\mathrm{Aj}}$ the resulting conservative universal pre-forcing. It is easy to see that $\Vdash_{\mathrm{Aj}}$ is a forcing and that $M[G]$ is defined for every generic G (cf. Section 4.1).

Lemma 4.14. $\Vdash_{\mathrm{Aj}}$ *is densely definable for* $\Delta_0^{b_0}(R)$ *up to* b_0.

Proof of Theorem 4.12. It is clear that every generic associate of M violates $\mathrm{PHP}(R, n)$ and by conservativity it is a generic expansion of M (Proposition 2.23). Thus Theorem 4.12 follows from the above lemma by the Principal Theorem. □

To prove Lemma 4.14 we follow the method of definable antichains from Section 3.2. Note that Lemma 4.14 follows easily from Lemma 3.8 and:

Lemma 4.15. *Let* $p \in P$. *For every* $\varphi(\bar{x}) \in \Delta_0^{b_0}(R)$ *there is* $r \in P, r \leq p$ *and a sequence of sets* $(X_{\bar{a}})_{\bar{a} <^M b_0}$ *in* M *such that for every* $\bar{a} <^M b_0$, *the set* $X_{\bar{a}}$ *is a maximal antichain in* $[\varphi(\bar{a})] \downarrow r$ *of rank at most* $\|r\| + |b_0|$.

Here, a sequence $(X_{\bar{a}})_{\bar{a} <^M b_0}$ of subsets of M is *in* M if the set $\{(\bar{a}, c) \mid \bar{a} <^M b_0, c \in X_{\bar{a}}\}$ is coded in M.

We are thus left to prove this lemma. To do so we intend to use Lemma 3.10. Therefore we need to define predense antichains refining given sets and it is here where the finite combinatorics enter the argument. The idea is to show that suitable antichains exist in $P(m)$ for $m \in \mathbb{N}$ sufficiently large. Then M codes these antichains for the infinite $P(n)$. As a first problem, predensity does not make much sense in finite frames nor in $P(n)$. Therefore we shall calibrate the notion in the definition below. Second, suitable antichains need not to exist, but they do exist after restricting attention to conditions that extend a suitably chosen condition r. This choice is done according to the Switching Lemma 4.18 below, the combinatorial core of the argument. Details follow.

Definition 4.16. Let $m, k \in \mathbb{N}$, $q \in P(m)$ and $X \subseteq P(m)$. Then X is *k-predense (in $P(m)$) below* q if every condition that extends q and has size at most $m - k$ is compatible with a condition in X.

For $m \in M$, $p \in P(m)$ and $X \subseteq P(m)$ write

$$X^p := \{q \setminus p \mid q \in X, p \| q\} \quad \text{and} \quad X \cup p := \{q \cup p \mid q \in X, p \| q\}.$$

Note that $P(m)^p \cong P(m - \|p\|)$ via a size preserving isomorphism. By saying that an antichain is k-predense in $P(m)^p$ we mean that its image under this isomorphism is k-predense in $P(m - \|p\|)$. In the same way k-predensity is explained in $P(n)^p$.

Lemma 4.17. *Let $X \subseteq P$, $p, q \in P, q \leq p$ and let φ be an $L^*(M)$-sentence. If X is a maximal antichain in $[\varphi] \downarrow p$ and has rank at most $\|p\| + |b_0|$, then $X \cup q$ is a maximal antichain in $[\varphi] \downarrow q$.*

Proof. As $X \subseteq P \downarrow p$, X^p has rank at most $|b_0|$. Then $X \cup q = X^p \cup q$ has rank at most $\|q\| + |b_0|$, so $X \cup q \subseteq P$. Clearly, $X \cup q$ is an antichain.

To show containment in $[\varphi] \downarrow q$, let $r \in X \cup q$ and choose $s \in X, q \| s$ such that $r = s \cup q$. Since $X \subseteq [\varphi]$, we have $s \cup q \in [\varphi] \downarrow q$ by Extension.

To show maximality, let $r \in [\varphi] \downarrow q$. By maximality of X, r is compatible with some $s \in X$. As $r \leq q$, q is compatible with s. Then $s \cup q \in X \cup q$ is compatible with r. □

Lemma 4.18 (Switching)**.** *Let $\ell, m, k, N \in \mathbb{N}, k \leq \ell$ be sufficiently large and let $X_1, \ldots, X_N \subseteq P(m)$ be sets of rank at most k. Assume*

$$(*) \qquad N^{2/k} \ell^{100} < m.$$

Then there is $q \in P(m)$ of size at most $m - \ell$ such that for every $1 \leq i \leq N$ there is an antichain $A_i \subseteq P(m)^q$ refining X_i^q that is k-predense in $P(m)^q$ and has rank at most k.

This lemma can be proved by the probabilistic method or a direct (involved) counting argument. Details can be found in [21, Lemma 12.3.10] or in the references pointed out in Section 1.4.

Applied in M the Switching Lemma provides us with suitable antichains in restrictions of $P(n)$. The following easy lemma allows to move these antichains to P.

Lemma 4.19. *Let $p \in P$ and $X, Y \subseteq P(n)^p$ have rank at most $|b_0|$.*

(1) *If X is an antichain in $P(n)^p$, then $X \cup p$ is an antichain in P.*

(2) *If X is $|b_0|$-predense in $P(n)^p$, then $X \cup p$ is predense in P below p.*
(3) *If X refines Y in $P(n)^p$, then $X \cup p$ refines $Y \cup p$ in P.*

Proof. We only show (2). Note $X \cup p \subseteq P$ because it has rank at most $\|p\| + |b_0|$. Let $q \in P, q \leq p$ and choose $0 < \epsilon < 1$ such that $\|q\| <^M n - n^\epsilon$. Then $\|q \setminus p\| = \|q\| - \|p\| <^M n - n^\epsilon - \|p\| <^M (n - \|p\|) - |b_0|$. Since $(q \setminus p) \in P(n)^p$ and X is $|b_0|$-predense in $P(n)^p$, there is $r \in X$ such that $q \setminus p$ is compatible with r in $P(n)^p$. Then $q \cup r = q \cup (r \cup p)$ extends both q and $r \cup p \in X \cup p$. As $q \cup r$ has size $<^M \|q\| + |b_0|$ it is in P, so q and $r \cup p$ are compatible in P. □

The rest of the argument is straightforward:

Proof of Lemma 4.15. Let $p \in P$. Call a $L^*(M)$-formula *good* if the lemma holds for it. It is easy to verify that atomic formulas are good: for an atom $\varphi(\bar{x})$ of the form Rst with $L(M)$-terms $t = t(\bar{x}), s = s(\bar{x})$ take $r := p$ and define $X_{\bar{a}} := \{r \cup \{(s^M(\bar{a}), t^M(\bar{a}))\}\}$ or $X_{\bar{a}} := \emptyset$ depending on whether $r \cup \{(s^M(\bar{a}), t^M(\bar{a}))\}$ is a partial bijection from $[n+1]$ to $[n]$ or not. Similarly, for an $L(M)$-atom $\varphi(\bar{x})$ set $r := p$ and $X_{\bar{a}} := \{r\}$ or $X_{\bar{a}} := \emptyset$ depending on whether $M \models \varphi(\bar{a})$ or not.

We leave it to the reader to verify that the set of good formulas is closed under conjunctions and negations. As the set of good formulas is closed under logical equivalence (Corollary 2.19), we are thus left to show it is closed under b_0-bounded universal quantification.

So assume $\varphi(x\bar{x})$ is good. Then $\neg\varphi(x\bar{x})$ is good and we can choose $r \in P$, $r \leq p$ and antichains $(X_{a\bar{a}})_{a\bar{a} <^M b_0}$ that satisfy the claim for $\neg\varphi(x\bar{x})$. Since every antichain $X_{a\bar{a}}$ is in $P \downarrow r$ and has rank at most $\|r\| + |b_0|$, we know that $X^r_{a\bar{a}}$ has rank at most $|b_0|$.

Choose $0 < \epsilon < 1$ such that $\|r\| <^M n - n^\epsilon$. Then $n^\epsilon <^M n - \|r\| =: m$. Observe that as partial orders

$$P(n)^r \cong P(m),$$

via an isomorphism that is definable in M and preserves the size $\|\cdot\|$.

For $\bar{a} <^M b_0$ let

$$Y_{\bar{a}} := \bigcup_{a <^M b_0} X^r_{a\bar{a}}.$$

Note $Y_{\bar{a}}$ has rank at most $|b_0|$, and the sequence $(Y_{\bar{a}})_{\bar{a} <^M b_0}$ is in M.

We intend to apply the Switching Lemma in M to get refining antichains for the sequence $(Y_{\bar{a}})_{\bar{a} <^M b_0}$. We check its assumptions:

Calculated in M, the sequence has length $N := b_0^{\ell_0}$ for ℓ_0 the length of $\bar{x}$. Especially $N^{2/|b_0|}$ (calculated in M) i s bounded by a standard number in M (i.e. by a closed $\{+, 1\}$-term). Therefore we can choose a rational $0 < \epsilon' < 1$ (e.g. $\epsilon' := 1/101$) such that the inequality $(*)$ of the Switching Lemma is satisfied for $\ell := m^{\epsilon'}$ and $k := |b_0|$ (and m, N as defined above). Further $k = |b_0| <^M (n^\epsilon)^{\epsilon'} <^M m^{\epsilon'} = \ell$.

Thus the lemma applies: we find $r' \in P(n)^r$ of size at most $m - m^{\epsilon'}$ such that, writing $s := (r \cup r')$, the following holds: for every $\bar{a} <^M b_0$ there is an $A_{\bar{a}} \subseteq (P(n)^r)^{r'} = P(n)^s$ coded in M such that in $P(n)^s$

(i) $A_{\bar{a}}$ is an antichain that is $|b_0|$-predense,
(ii) $A_{\bar{a}}$ refines $Y_{\bar{a}}^{r'} \subseteq P(n)^s$,
(iii) $A_{\bar{a}}$ has rank at most $|b_0|$.

Note that s has size $\|r\| + \|r'\| \leq^M n - m + m - m^{\epsilon'} <^M n - n^{\epsilon\epsilon'}$, so $s \in P$. Further note that with $Y_{\bar{a}}$ also $Y_{\bar{a}}^{r'}$ has rank at most $|b_0|$. By Lemma 4.19 we get in P

(iv) $(A_{\bar{a}} \cup s)$ is an antichain that is predense below s,
(v) $(A_{\bar{a}} \cup s)$ refines $Y_{\bar{a}}^{r'} \cup s = \bigcup_{a <^M b_0} (X_{a\bar{a}} \cup s)$,
(vi) $(A_{\bar{a}} \cup s)$ has rank at most $\|s\| + |b_0|$.

By Lemma 4.17 $(X_{a\bar{a}} \cup s)$ is a maximal antichain in $[\neg\varphi(a\bar{a})] \downarrow s$. By $(A_{\bar{a}} \cup s) \subseteq P \downarrow s$, *(iv)* and *(v)* the assumptions of Lemma 3.10 (3) are satisfied. Thus we get a maximal antichain in $[\forall x < b_0 \varphi(x\bar{a})] \downarrow s$ setting

$$Z_{\bar{a}} := (A_{\bar{a}} \cup s) \setminus \big((A_{\bar{a}} \cup s) \downarrow \bigcup_{a <^M b_0} (X_{\bar{a}} \cup s)\big).$$

Then $(Z_{\bar{a}})_{\bar{a} <^M b_0}$ is in M and has rank at most $\|s\| + |b_0|$ by *(vi)*. □

4.4. **Notes.** Compared to Riis' original argument [35] our proof of Theorem 4.6 relies on the stability of universal forcing (and Corollary 2.19) while Riis uses an existential forcing, and it is simpler in that it sidesteps the analysis of an auxiliary pre-forcing in [35].

Compared with other proofs of Theorem 4.12, roughly, the predense antichains in our argument correspond to the complete systems in [27] and in [46], to branches in shallow decision trees in [45, 25] or to the small covers in [1].

Forcing type arguments for Ajtai's result have been given in [1, 46] and [21, Section 12.7] and recently in [25]. In [21, Section 12.7] Krajíček presents the method of k-evaluations of propositional formulas [27] as a forcing type argument. Our proof constructs for certain φ a predense antichain together with its maximal part in $[\varphi]$. These pairs of sets give rise to a modified notion of $|b_0|$-evaluation. As Zambella's [46] our argument sidesteps a detour through propositional logic like in [1, 27, 45, 21]. Further it avoids the restriction to "internal" generics in [46].

Krajíček's recent proof in [25] (cf. Section 1.7) uses forcing with random variables, developed in [25] as a general method to construct Boolean valued models of bounded arithmetics. This recent argument, the argument given here and in fact all known arguments for Ajtai's result use the Switching Lemma in one or another form. The main obstacle to generalize Ajtai's argument to other principles is the difficulty to find analogues of this lemma. Our interpretation of the role of this lemma is roughly as follows: it states the existence of refining antichains.

The rest of the argument can be taken over by the general machinery, the method of definable antichains and the Principal Theorem as described in paragraph 3.

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References

1. M. Ajtai, *The complexity of the pigeonhole principle*, Proceedings of the 29th Annual Symposion on the Foundations of Computer Science (1988), 346–355.
2. J. Avigad, *Forcing in proof theory*, The Bulletin of Symbolic Logic **10** (2004), no. 3, 305–333.
3. P. Beame, *A switching lemma primer*, Technical Report UW-CSE-95-07-01, University of Washington (1994).
4. P. Beame and T. Pitassi, *Propositional proof complexity: Past, present, and future*, Bulletin of the European Association for Theoretical Computer Science, The Computational Complexity Column (ed. E. Allender). **65** (1998), 66–89.
5. S. Bellantoni, T. Pitassi, and A. Urquhart, *Approximation and small-depth frege proofs*, Journal SIAM Journal on Computing **21** (1992), no. 6.
6. E. Ben-Sasson and P. Harsha, *Lower bounds for bounded depth frege proofs via buss-pudlák games*, ACM Transactions on Computational Logic **11** (2010), no. 3.
7. O. Beyersdorff, *On the correspondence between arithmetic theories and propositional proof systems - a survey*, Mathematical Logic Quarterly **55** (2009), no. 2, 116–137.
8. S. R. Buss, *Bounded arithmetic and propositional proof complexity*, in Logic of Computation, Springer (1995), 67–122.
9. ———, *First-order proof theory of arithmetic*, chapter II in S. R. Buss (ed.): Handbook of Proof Theory (1998), 79–147.
10. S. A. Cook and P. Nguyen, *Logical foundations of proof complexity*, Cambridge University Press, 2010.
11. S. A. Cook and A. R. Reckhow, *The relative efficiency of propositional proof systems*, The Journal of Symbolic Logic **44** (1979), no. 1, 36–50.
12. S. Feferman, *Some applications of forcing and generic sets*, Fundamentae Mathematicae **56** (1965), 325–345.
13. S. Fenner, L. Fortnow, S. A. Kurtz, and L. Li, *An oracle builder's toolkit*, Information and Computation **182** (2003), 95–136.
14. S. D. Friedman, *Topics in set theory, course notes.*
15. J. Hastad, *Almost optimal lower bounds for small depth circuits*, Proceedings of the 18th Annual ACM Symposium on Theory of Computing (1986), 6–20.
16. J. Hirschfeld and W. H. Wheeler, *Forcing, arithmetic, division rings*, vol. 454, Lecture Notes in Mathematics, 1975.
17. W. Hodges, *Building models by games*, Cambridge University Press, 1985.
18. R. Kaye, *The theory of κ-like models of arithmetic*, Notre Dame Journal of Formal Logic **36** (1995), no. 4, 547–559.
19. H. J. Keisler, *Forcing and the omitting types theorem*, in Morley (ed.) Studies in Model Theory, Studies in Mathematics, The Mathematical Association of America **8** (1973), 96–133.

20. J. F. Knight, *Generic expansions of structures*, The Journal of Symbolic Logic **38** (1973), no. 4, 561–570.
21. J. Krajíček, *Bounded arithmetic, propositional logic, and complexity theory*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1995.
22. ______, *On frege and extended frege proof systems*, P. Clote and J. Remmel (eds.), Feasible Mathematics II, Birkhäuser (1995), 284–319.
23. ______, *Combinatorics of first order structures and porpositional proof systems*, Archive of Mathematical Logic **43** (2004), 427–441.
24. ______, *Proof complexity*, in A. Laptev (ed.), European congress of mathematics (ECM), Stockholm, Sweden, Zurich: European Mathematical Society (2005), 221–231.
25. ______, *Forcing with random variables and proof complexity*, vol. 382, London Mathematical Society Lecture Note Series, Cambridge University Press, to appear.
26. J. Krajíček and P. Pudlák, *Propositional proof systems, the consistency of first order theories and the complexity of computations*, The Journal of Symbolic Logic **54** (1989), no. 3, 1063–1079.
27. J. Krajíček, P. Pudlák, and A. Woods, *An exponential lower bound to the size of bounded depth frege proofs of the pigeonhole principle*, Random Structures and Algorithms **7** (1995), no. 1, 15–39.
28. P. Odifreddi, *Forcing and reducibilities*, The Journal of Symbolic Logic **48** (1983), no. 2, 288–310.
29. J. Paris and A. J. Wilkie, *Counting problems in bounded arithmetic*, in: Methods in Mathematical Logic **1130** (1985), 317–340.
30. T. Pitassi, P. Beame, and R. Impagliazzo, *Exponential lower bounds for the pigeonhole principle*, Computational Complexity **3** (1993), 97–108.
31. P. Pudlák, *A bottom-up approach to foundations of mathematics*, Proceedings Gödel'96, Logical Foundations of Mathematics, Computer Science and Physics – Kurt Gödel's Legacy, Springer Lecture Notes in Logic **6** (1996), 81–97.
32. ______, *The lengths of proofs*, chapter VIII in S. R. Buss (ed.): Handbook of Proof Theory (1998), 547–637.
33. P. Pudlák and S.R. Buss, *How to lie without being (easily) convicted and the lengths of proofs in proposotional claculus*, in Pacholski and Tiuryn (eds.), Computer Science Logic'94, Springer Verlag LNCS **933** (1994), 151–162.
34. A. Razborov, *Lower bounds for propositional proofs and independence results in bounded arithmetic*, Proceedings of the 23rd ICALP, Lecture Notes in Computer Science **1099** (1996), 48–62.
35. S. Riis, *Finitisation in bounded arithmetic*, BRICS Report Series RS-94-23 (1994).
36. D. Scott, *A proof of the independence of the continuum hypothesis*, Mathematical Systems Theory **1** (1967), no. 2, 89–111.
37. N. Segerlind, *The complexity of propositional proofs*, The Bulletin of Symbolic Logic **13** (2007), no. 4, 417–481.
38. J. R. Shoenfield, *Unramified forcing*, Axiomatic Set Theory, Proceedings of Symposia in Pure mathematics, American Mathematical Society **VIII** (1971), 357–381.
39. S. G. Simpson, *Forcing and models of arithmetic*, Proceedings of the American Mathematical Society **43** (1974), no. 1, 193–194.
40. J. Stern, *A new look at the interpolation problem*, The Journal of Symbolic Logic **40** (1975), no. 1.
41. G. Takeuti and M. Yasumoto, *Forcing on bounded arithmetic*, in P. Hájek (ed.), Gödel'96, Lecture Notes in Logic **6** (1996), 120–138.
42. ______, *Forcing on bounded arithmetic 2*, The Journal of Symbolic Logic **63** (1998), no. 3, The Journal of Symbolic Logic.

43. N. Thapen, *Notes on switching lemmas, manuscript*, (2009).
44. A. Urquhart, *The complexity of propositional proofs*, The Bulletin of Symbolic Logic **1** (1995), no. 4, 425–467.
45. A. Urquhart and X. Fu, *Simplified lower bounds for propositional proofs*, Notre Dame Journal of Formal Logic **73** (1996), no. 4, 523–544.
46. D. Zambella, *Forcing in finite structures*, Mathematical Logic Quarterly **43** (1997), no. 3, 401–412.

Albert Atserias
Universitat Politècnica de Catalunya (UPC)
Departament de Llenguatges i Sistemes Informàtics (LSI)
c/ Jordi Girona Salgado 1-3, Edif. Omega-S122
08034 Barcelona, Spain
E-mail address: atserias@lsi.upc.edu

Moritz Müller
Centre de Recerca Matemàtica (CRM)
Campus de Bellaterra, Edifici C
08193 Bellaterra (Barcelona), Spain
E-mail address: mmueller@crm.cat