ON LUNDH'S PERCOLATION DIFFUSION

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Abstract. A collection of spherical obstacles in the unit ball in Euclidean space is said to be *avoidable* for Brownian motion if there is a positive probability that Brownian motion diffusing from some point in the ball will avoid all the obstacles and reach the boundary of the ball. The centres of the spherical obstacles are generated according to a Poisson point process while the radius of an obstacle is a deterministic function. If avoidable configurations are generated with positive probability Lundh calls this *percolation diffusion*. An integral condition for percolation diffusion is derived in terms of the intensity of the point process and the function that determines the radii of the obstacles.

1. Introduction

Lundh proposed in [10] a percolation model in the unit ball $\mathbb{B} = \{x \in \mathbb{R}^d : |x| < 1\}$, $d \geq 3$, involving diffusion through a random collection of spherical obstacles. In Lundh's formulation, the radius of an obstacle is proportional to the distance from its centre to the boundary $\mathbb{S} = \{x \in \mathbb{R}^d : |x| = 1\}$ of the ball. The centres of the obstacles are generated at random by a Poisson point process with a spherically symmetric intensity μ. Lundh called a random collection of obstacles *avoidable* if Brownian motion diffusing from a point in the ball $\mathbb{B}$ has a positive probability of reaching the outer boundary $\mathbb{S}$ without first hitting any of the obstacles. Lundh set himself the task of characterising those Poisson intensities μ which would generate an avoidable collection of obstacles with positive probability, and named this phenomenon *percolation diffusion*. Our main objective herein is to extend Lundh's work by removing some of his assumptions on the Poisson intensity and on the radii of the obstacles.

Deterministic configurations of obstacles in two dimensions are considered in detail by Akeroyd [3] and by Ortega-Cerdà and Seip [13], while O'Donovan [11] and Gardiner and Ghergu [8] consider configurations in higher dimensions. The result below is taken from these articles. First some notation is needed. Let $B(x,r)$ and $S(x,r)$ stand for the Euclidean ball and sphere, respectively, with centre x and radius r and let $\overline{B}(x,r)$ stand for the closed ball with this centre

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and radius. Let Λ be a countable set of points in the ball $\mathbb{B}$ which is *regularly spaced* in that it has the following properties

(a) there is a positive ϵ such that if λ, $\lambda' \in \Lambda$, $\lambda \neq \lambda'$ and $|\lambda| \geq |\lambda'|$ then

$$|\lambda - \lambda'| \geq \epsilon\big(1 - |\lambda|\big). \tag{1}$$

(b) there is an $r < 1$ such that

$$\mathbb{B} = \bigcup_{\lambda\in\Lambda} B\big(\lambda, r(1-|\lambda|)\big). \tag{2}$$

Let ϕ: $[0,1) \to [0,1)$ be a decreasing function such that the closed balls $\big\{\overline{B}\big(\lambda, \phi(|\lambda|)\big)\big\}$, $\lambda \in \Lambda$, are disjoint, and set

$$\mathcal{O} = \bigcup_{\lambda\in\Lambda} \overline{B}\big(\lambda, \phi(|\lambda|)\big).$$

Avoidability of the collection of spherical obstacles $\mathcal{O}$ is equivalent to the harmonic measure condition $\omega(x, \mathbb{S}, \Omega) > 0$, where $\Omega = \mathbb{B} \setminus \mathcal{O}$ and x is some (any) point in the domain Ω.

Theorem A. *The collection of spherical obstacles $\mathcal{O}$ is avoidable if and only if*

$$\int_0^1 \frac{dt}{(1-t)\log\big((1-t)/\phi(t)\big)} < \infty \qquad \text{if } d = 2, \tag{3}$$

$$\int_0^1 \frac{\phi(t)^{d-2}}{(1-t)^{d-1}} < \infty \qquad \text{if } d \geq 3. \tag{4}$$

Our goal is is to obtain a counterpart of this result for a random configuration of obstacles. We work with a Poisson random point process on the Borel subsets of the ball $\mathbb{B}$ with mean measure $d\mu(x) = \nu(x)\,dx$ which is absolutely continuous relative to Lebesgue measure. (Itô presents a complete, concise treatment of this topic in Section 1.9 of his book [9]). The radius function ϕ and the intensity function ν are assumed to satisfy, for some $C > 1$ and any $x \in \mathbb{B}$,

$$\begin{cases} \frac{1}{C}\phi(x) \leq \phi(y) \leq C\phi(x) \\ \frac{1}{C}\nu(x) \leq \nu(y) \leq C\nu(x) \end{cases} \qquad \text{if } y \in B\left(x, \frac{1-|x|}{2}\right). \tag{5}$$

It is also assumed that

$$\frac{\phi(x)}{1-|x|} \leq c < 1 \quad \text{for} \quad x \in \mathbb{B}. \tag{6}$$

and that

$$(1-|x|)\phi(x)^{d-2}\nu(x) = O\left(\frac{1}{1-|x|}\right) \text{ as } |x| \to 1^-. \tag{7}$$

Let $\mathcal{P}$ be a realisation of points from this Poisson random point process and let

$$A_{\mathcal{P}} = \bigcup_{p\in\mathcal{P}} \overline{B}\big(p,\phi(p)\big), \qquad \Omega_{\mathcal{P}} = \mathbb{B}\setminus A_{\mathcal{P}}, \tag{8}$$

so that $\Omega_{\mathcal{P}}$ is an open, though not necessarily connected, subset of $\mathbb{B}$. The archipelago of spherical obstacles $A_{\mathcal{P}}$ is said to be *avoidable* if there is a positive probability that Brownian motion diffusing from some point in $\Omega_{\mathcal{P}}$ reaches the unit sphere $\mathbb{S}$ before hitting the obstacles $A_{\mathcal{P}}$, that is if the harmonic measure of $A_{\mathcal{P}}$ relative to $\Omega_{\mathcal{P}}$ satisfies $\omega(x, A_{\mathcal{P}}, \Omega_{\mathcal{P}}) < 1$ for some x in $\Omega_{\mathcal{P}}$. If $\Omega_{\mathcal{P}}$ is connected then, by the maximum principle, this condition does not depend on $x \in \Omega_{\mathcal{P}}$. We do not insist, however, on the configuration being avoidable for Brownian motion diffusing from the origin.

We have *percolation diffusion* if there is a positive probability that the realisation of points from the Poisson random point process results in an avoidable configuration. Our main result is

Theorem 1. *Suppose that (5), (6) and (7) hold. Percolation diffusion occurs if and only if there is a set of points τ of positive measure on the sphere such that*

$$\int_{\mathbb{B}} \frac{(1-|x|^2)^2}{|x-\tau|^d}\,\phi(x)^{d-2}\nu(x)\,dx < \infty. \tag{9}$$

Thus the random archipelago $A_{\mathcal{P}}$ is avoidable with positive probability if and only if the Poisson balayage of the measure $(1-|x|^2)\phi(x)^{d-2}\nu(x)\,dx$ is bounded on a set of positive measure on the boundary of the unit ball.

Furthermore, in the case of percolation diffusion the random archipelago $A_{\mathcal{P}}$ is avoidable with probability one.

In the radial case the following corollary follows directly from Theorem 1.

Corollary 1. *Suppose that, in addition to (5), (6) and (7), the intensity ν and the radius function ϕ are radial in that they depend only on $|x|$. Then percolation diffusion occurs if and only if*

$$\int_0^1 (1-t)\,\phi(t)^{d-2}\,\nu(t)\,dt < \infty. \tag{10}$$

Lundh's result [10, Theorem 3.1] is the case $\phi(t) = c(1-t)$ of this corollary, in which case (10) becomes

$$\int_0^1 (1-t)^{d-1}\nu(t)\,dt < \infty. \tag{11}$$

This corresponds to the condition stated by Lundh that the radial intensity function should be integrable on $(0,\infty)$ when allowance is made for the fact that he works in the hyperbolic unit ball. As pointed out in [12], Lundh's deduction from (11) (see [10, Remark 3.2]) that percolation diffusion can only occur when the expected number of obstacles in a configuration is finite isn't correct. In fact,

(11) holds in the case $\nu(t) = (1-t)^{1-d}$ and we have percolation diffusion. At the same time, the expected number of obstacles $N(\mathbb{B})$ in the ball is

$$\mathbb{E}[N(\mathbb{B})] = \int_{\mathbb{B}} d\mu(x) = \int_{\mathbb{B}} \frac{dx}{(1-|x|)^{d-1}} = \infty. \tag{12}$$

Lundh's remark erroneously undervalues his work since it gives the impression that, in his original setting, percolation diffusion can only occur if the number of obstacles in a configuration is finite almost surely.

The intensity $\nu(t) = 1/(1-t)^d$ corresponds, in principle, to a regularly spaced collection of points since the expected number of points in a Whitney cube Q of sidelength $\ell(Q)$ and centre $c(Q)$ is, in the case of this intensity,

$$\mathbb{E}[N(Q)] = \int_Q d\mu(x) \sim \nu\big(c(Q)\big)\mathrm{Vol}(Q) = \frac{\ell(Q)^d}{(1-|c(Q)|)^d} \sim \text{constant}.$$

We note that there is agreement in principle between the integral condition (4) for the deterministic setting and the integral condition (10) with $\nu(t) = 1/(1-t)^d$ for the random setting.

2. Avoidability, minimal thinness and a Wiener-type criterion

Avoidability of a realised configuration of obstacles $A_{\mathcal{P}}$ may be reinterpreted in terms of minimal thinness of $A_{\mathcal{P}}$ at points on the boundary of the unit ball (see [4] for a thorough account of minimal thinness). This is Lundh's original approach, and is also the approach adopted by the authors of [11, 12, 8].

For a positive superharmonic function u on $\mathbb{B}$ and a closed subset A of $\mathbb{B}$, the reduced function R_u^A is defined by

$$R_u^A = \inf\big\{v\colon v \text{ is positive and superharmonic on } \mathbb{B} \text{ and } v \geq u \text{ on } A\big\}.$$

The set A is minimally thin at $\tau \in \mathbb{S}$ if there is an x in $\mathbb{B}$ at which the reduced function of the Poisson kernel $P(\cdot,\tau)$ for $\mathbb{B}$ with pole at τ satisfies $R^A_{P(\cdot,\tau)}(x) < P(x,\tau)$. Minimal thinness in this context has been characterised in terms of capacity by Essén [7] in dimension 2 and by Aikawa [1] in higher dimensions. Let $\{Q_k\}_{k=1}^\infty$ be a Whitney decomposition of the ball $\mathbb{B}$ into cubes so that, in particular,

$$\mathrm{diam}(Q_k) \leq \mathrm{dist}(Q_k,\mathbb{S}) \leq 4\,\mathrm{diam}(Q_k).$$

Let $\ell(Q_k)$ be the sidelength of Q_k. Let $\mathrm{cap}\,(E)$ denote the Newtonian capacity of a Borel set E. Aikawa's criterion for minimal thinness of A at a boundary point τ of $\mathbb{B}$ is that the series $W(A,\tau)$ is convergent, where

$$W(A,\tau) = \sum_k \frac{\ell(Q_k)^2}{\rho_k(\tau)^d}\,\mathrm{cap}\,(A\cap Q_k), \tag{13}$$

$\rho_k(\tau)$ being the distance from Q_k to the boundary point τ. A proof of the following proposition can be found in [8, Page 323]. The proof goes through with only very minor modifications even though we do not insist on evaluating harmonic measure at the origin and the open set $\mathbb{B} \setminus A$ may not be connected.

Lemma 1. *Let A be a closed subset of $\mathbb{B}$. Let*

$$\mathcal{M} = \{\tau \in \mathbb{S} : A \text{ is minimally thin at } \tau\}. \tag{14}$$

Then A is avoidable if and only if $\mathcal{M}$ has positive measure on $\mathbb{S}$, that is if and only if $W(A, \tau) < \infty$ for a set of τ of positive measure on $\mathbb{S}$.

The question of whether a given set A is avoidable for Brownian motion is thereby reduced to an estimation of capacity.

The following zero-one law simplifies the subsequent analysis, and will imply that the random archipelago is avoidable with probability zero or probability one, as stated in Theorem 1. Again, τ is used to denote points on the sphere $\mathbb{S}$ and $A_{\mathcal{P}}$ denotes an archipelago constructed as in (8) from a random realisation $\mathcal{P}$ of points taken from the Poisson point process.

Lemma 2. *The event that $A_{\mathcal{P}}$ is minimally thin at τ has probability 0 or 1.*

Proof. Whether or not the set $A_{\mathcal{P}}$ is minimally thin at τ depends on the convergence of the series $W(A_{\mathcal{P}}, \tau)$. Partition the cubes $\{Q_k\}_1^\infty$ into finitely many disjoint groups $\{Q_k^i\}_{k=1}^\infty$, $i = 1, 2, \ldots, n$, so that any ball in $A_{\mathcal{P}}$ can meet at most one cube in each group. Then break the summation $W(A_{\mathcal{P}}, \tau)$ into corresponding summations

$$W^i(A_{\mathcal{P}}, \tau) = \sum_{k=1}^\infty X_k^i \quad \text{where } X_k = \frac{\ell(Q_k)^2}{\rho_k(\tau)^d} \operatorname{cap}(A_{\mathcal{P}} \cap Q_k) \tag{15}$$

The random variables X_k^i in each resulting summation are independent. The event $W^i(A_{\mathcal{P}}, \tau) < \infty$ belongs to the tail field of the corresponding X_k^i's, hence this event has probability 0 or 1. It follows that the event $W(A_{\mathcal{P}}, \tau) < \infty$ has probability 0 or 1. □

3. The expected value of the Wiener-type criterion and the Poisson balayage

The proof of Theorem 1 follows the outline of Lundh's argument [10] and the second author's thesis [12].

We work with a Poisson point process in the ball. Each realisation $\mathcal{P}$ of this process gives rise to an archipelago $A_{\mathcal{P}}$ via (8), which is avoidable for Brownian motion if and only if the associated Wiener-type series $W(A_{\mathcal{P}}, \tau)$ is finite for a set of τ of positive measure on the sphere $\mathbb{S}$. For a fixed τ on the sphere $\mathbb{S}$, the series $W(A_{\mathcal{P}}, \tau)$ is a random variable. Proposition 1 states that its expected value is comparable to the Poisson balayage (9). We denote by c and C any positive finite numbers whose values depend only on dimension and are immaterial to the main argument.

Proposition 1. *Fix a point τ on the sphere $\mathbb{S}$. Then*

$$\mathbb{E}\big[W(A_{\mathcal{P}},\tau)\big] \sim \int_{\mathbb{B}} \frac{(1-|x|^2)^2}{|\tau-x|^d}\,\phi(x)^{d-2}\,\nu(x)\,dx. \tag{16}$$

The proof of Proposition 1 depends on a two-sided estimate for the expected value of the capacity of the intersection of a Whitney cube Q_k with the set of obstacles $A_{\mathcal{P}}$ in terms of the mean measure $\mu(Q_k)$ of the cube and a typical value of the radius function ϕ on the cube.

Lemma 3. *For a Whitney cube Q and any point $x \in Q$,*

$$\mathbb{E}[\operatorname{cap}(A_{\mathcal{P}} \cap Q)] \sim \phi(x)^{d-2}\,\mu(Q). \tag{17}$$

Lundh did not require an estimate of this type as the size of one of his obstacles was comparable to the size of the Whitney cube containing its centre. The capacity of $A_{\mathcal{P}} \cap Q$ therefore depended only on the probability of whether of not the cube Q contained a point from the Poisson point process. We first deduce Proposition 1 from Lemma 3 and then prove Lemma 3.

Proof of Proposition 1. The upper bound for $\mathbb{E}[\operatorname{cap}(A_{\mathcal{P}} \cap Q)]$ in Lemma 3 leads to an upper bound for the expected value of Aikawa's series (13) with $A = A_{\mathcal{P}}$ as follows:

$$\begin{aligned}
\mathbb{E}[W(A_{\mathcal{P}},\tau)] &= \mathbb{E}\left[\sum_k \frac{\ell(Q_k)^2}{\rho_k(\tau)^d}\operatorname{cap}(A_{\mathcal{P}} \cap Q_k)\right] \\
&= \sum_k \frac{\ell(Q_k)^2}{\rho_k(\tau)^d}\,\mathbb{E}[\operatorname{cap}(A_{\mathcal{P}} \cap Q_k)] \\
&\le C \sum_k \frac{\ell(Q_k)^2}{\rho_k(\tau)^d}\,\phi(x_k)^{d-2}\,\mu(Q_k)
\end{aligned}$$

where x_k is any point in Q_k. Since the radius function ϕ is approximately constant on each Whitney cube by (5), it follows that

$$\begin{aligned}
\mathbb{E}[W(A_{\mathcal{P}},\tau)] &\le C \sum_k \int_{Q_k} \frac{(1-|x|^2)^2}{|\tau-x|^d}\,\phi(x)^{d-2}\,\nu(x)\,dx \\
&= C \int_{\mathbb{B}} \frac{(1-|x|^2)^2}{|\tau-x|^d}\,\phi(x)^{d-2}\,\nu(x)\,dx.
\end{aligned}$$

In the other direction, first choose a point x_k in each Whitney cube Q_k. Then,

$$\begin{aligned}
\int_{\mathbb{B}} \frac{(1-|x|^2)^2}{|\tau-x|^d}\,\phi(x)^{d-2}\,\nu(x)\,dx &\le C \sum_k \frac{\ell(Q_k)^2}{\rho_k(\tau)^d}\,\phi(x_k)^{d-2}\,\mu(Q_k) \\
&\le C \sum_k \frac{\ell(Q_k)^2}{\rho_k(\tau)^d}\,\mathbb{E}[\operatorname{cap}(A_{\mathcal{P}} \cap Q_k)] \\
&= C\,\mathbb{E}[W(A_{\mathcal{P}},\tau)],
\end{aligned}$$

where the second inequality comes from the lower bound for $\mathbb{E}[\mathrm{cap}\,(A_{\mathcal{P}} \cap Q_k)]$ in Lemma 3. $\square$

Proof of Lemma 3. The assumption (6) implies that if an obstacle meets a Whitney cube Q then its centre can lie in at most some fixed number N of Whitney cubes neighbouring the cube Q. We label these cubes Q^i, where the index i varies from 1 to at most N, and write Q' for their union. Both the distance to the boundary, and the distance to a specific boundary point, are comparable in Q and in Q'. Analogously, an obstacle with a centre in a specified cube can intersect at most some fixed number of neighbouring cubes.

Consider a random realisation of points $\mathcal{P}$ and a Whitney cube Q_k. By (5) the radius function ϕ is roughly constant on the cubes Q_k^i, say $\phi(x) \sim \phi(x_k)$, $x \in Q_k'$, where x_k is any point chosen in Q_k. Therefore, by the subadditivity property of capacity,

$$\mathrm{cap}\,(A_{\mathcal{P}} \cap Q_k) \le C\,\phi(x_k)^{d-2}\,N(Q_k'),$$

where $N(Q_k')$ is the number of centres from the realised point process $\mathcal{P}$ that lie in the union of cubes Q_k'. Taking the expectation leads to

$$\mathbb{E}[\mathrm{cap}\,(A_{\mathcal{P}} \cap Q_k)] \le C\,\phi(x_k)^{d-2}\mathbb{E}\,[N(Q_k')] = C\,\phi(x_k)^{d-2}\mu(Q_k').$$

By (5), $\mu(Q_k') \le C\mu(Q_k)$ and the upper bound for $\mathbb{E}[\mathrm{cap}\,(A_{\mathcal{P}} \cap Q_k)]$ in Lemma 3 follows.

In the other direction we proceed, as did Gardiner and Ghergu [8], by employing the following super-additivity property of capacity due to Aikawa and Borichev [2]. Let σ_d be the volume of the unit ball. Let $F = \bigcup B(y_k, \rho_k)$ be a union of balls which lie inside some ball of unit radius. Suppose also that $\rho_k \le 1/\sqrt{\sigma_d 2^d}$ for each k and that the larger balls $B(y_k, \sigma_d^{-1/d}\rho_k^{1-2/d})$ are disjoint. Then

$$\mathrm{cap}\,(F) \ge c\sum_k \mathrm{cap}\,\big(B(y_k,\rho_k)\big) = c\sum_k \rho_k^{d-2}. \tag{18}$$

Let ϕ_0 be the minimum of $\phi(x)$ for x in Q. By (5), ϕ_0 is comparable to $\phi(x)$ for any x in Q. We only consider obstacles with centres in Q and suppose that all such obstacles have radius ϕ_0, since in so doing the capacity of $A_{\mathcal{P}} \cap Q$ decreases. Set

$$\alpha = \min\left\{\big(\ell(Q)\sqrt{d}\big)^{-1}, \big(\sqrt{\sigma_d\,2^d}\,\phi_0\big)^{-1}\right\}$$

and set

$$N = \lfloor 4^{-1}\,\ell(Q)\,\sigma_d^{1/d}\,\alpha^{2/d}\,\phi_0^{2/d-1}\rfloor.$$

By (6), we have $\alpha \ge c/\ell(Q)$. The cube Q is divided into N^d smaller cubes each of sidelength $\ell(Q)/N$: we write Q' for a typical sub-cube. Inside each cube Q' consider a smaller concentric cube Q'' of sidelength $\ell(Q)/(4N)$. If a cube Q'' happens to contain points from the realisation $\mathcal{P}$ of the random point process, we choose one such point. This results in points λ_1, λ_2, …, λ_m, say, where $m \le N^d$.

By the choice of α and N, each ball $B(\lambda_k, \phi_0)$ is contained within the sub-cube Q' that contains its centre. We set

$$A_{\mathcal{P},Q} = \bigcup_{k=1}^{m} \overline{B}(\lambda_k, \phi_0).$$

Since $A_{\mathcal{P},Q} \subset A_{\mathcal{P}} \cap Q$, if follows from monotonicity of capacity that $\mathrm{cap}\,\big(A_{\mathcal{P}} \cap Q\big) \geq \mathrm{cap}\,\big(A_{\mathcal{P},Q}\big)$.

To estimate the capacity of $A_{\mathcal{P},Q}$, we scale the cube Q by α. By the choice of α, the cube αQ lies inside a ball of unit radius and the radius of each scaled ball from $A_{\mathcal{P},Q}$ satisfies $\alpha\phi_0 \leq (\sigma_d 2^d)^{-1/2}$. The only condition that remains to be checked before applying Borichev and Aikawa's estimate (18) to the union of balls $\alpha A_{\mathcal{P},Q}$ is that the balls with centre $\alpha\lambda_k$ and radius $\sigma_d^{-1/d}(\alpha\phi_0)^{1-2/d}$ are disjoint. They are if

$$2\sigma_d^{-1/d}(\alpha\,\phi_0)^{1-2/d} \leq \frac{\alpha\,\ell(Q)}{2N}$$

since the centres of the balls are at least a distance $\alpha\,\ell(Q)/(2N)$ apart. This inequality follows from the choice of N. Applying (18) and the scaling law for capacity yields

$$\mathrm{cap}\,\big(A_{\mathcal{P},Q}\big) = \alpha^{2-d}\mathrm{cap}\,\big(\alpha A_{\mathcal{P},Q}\big) \geq \alpha^{2-d}\,c\,X\,(\alpha\phi_0)^{d-2} = c\,X\,\phi_0^{d-2},$$

where $X = m$ is the number of sub-cubes Q'' of Q in our construction that contain at least one point of $\mathcal{P}$. Hence,

$$\mathbb{E}[\mathrm{cap}\,(A_{\mathcal{P}} \cap Q)] \geq c\,\phi_0^{d-2}\,\mathbb{E}[X]. \tag{19}$$

The probability that a particular sub-cube Q'' contains a point of $\mathcal{P}$ is

$$1 - \mathbb{P}\big(\mathcal{P} \cap Q'' = \emptyset\big) = 1 - e^{-\mu(Q'')},$$

by the Poisson nature of the random point process. For any sub-cube Q'' with centre x, say,

$$\begin{aligned}
\mu(Q'') \sim \nu(x)\left(\frac{\ell(Q)}{N}\right)^d &\sim \nu(x)\frac{\phi_0^{d-2}}{\alpha^2} && \text{(by choice of } N\text{)}\\
&\leq \nu(x)\,\ell(Q)^2\,\phi_0^{d-2} && \text{(since } \alpha \geq c/l(Q)\text{)}\\
&= O(1) && \text{(by (7))}.
\end{aligned}$$

It then follows that

$$\mathbb{E}[X] = \sum_{Q''\subset Q} 1 - e^{-\mu(Q'')} \geq c \sum_{Q''\subset Q} \mu(Q'') \geq c\mu(Q), \tag{20}$$

the last inequality being a consequence of the assumption (5) and the fact that the volume of the union of the cubes Q'' is some fixed fraction of the volume of Q. When combined with (19), the estimate (20) yields the lower bound for $\mathbb{E}[\mathrm{cap}\,(A_{\mathcal{P}} \cap Q)]$. $\square$

4. Proof of Theorem 1

To begin with, we need the following result from Lundh's paper [10].

Lemma 4. *Let $\tau \in \mathbb{S}$. Then $\mathbb{E}[W(A_{\mathcal{P}}, \tau)]$ is finite if and only if the series $W(A_{\mathcal{P}}, \tau)$ is convergent for almost all random configurations $\mathcal{P}$.*

Proof. It is clear that $\mathbb{E}[W(A_{\mathcal{P}}, \tau)]$ being finite implies that $W(A_{\mathcal{P}}, \tau)$ is almost surely convergent. The reverse direction is proved by Lundh [10, p. 241] using Kolmogorov's three series theorem. Indeed, it is a consequence of this result [6, p. 118] that, in the case of a uniformly bounded sequence of non-negative independent random variables, the series $\sum_k X_k$ converges almost surely if and only if $\sum_k \mathbb{E}[X_k]$ is finite. As in the proof of Lemma 2, the series $W(A_{\mathcal{P}}, \tau)$ is split into n series $W^i(A_{\mathcal{P}}, \tau) = \sum_{k=1}^{\infty} X_k^i$, each of which is almost surely convergent by assumption. The random variables X_k in (15) are uniformly bounded. It then follows that $\sum_k \mathbb{E}[X_k^i] = \mathbb{E}\big[\sum_k X_k^i\big]$ is convergent, that is $\mathbb{E}[W^i(A_{\mathcal{P}}, \tau)]$ is finite. Summing over i, we find that $\mathbb{E}[W(A_{\mathcal{P}}, \tau)]$ is finite as claimed. $\square$

Proof of Theorem 1. Let us first assume that the finite Poisson balayage condition (9) holds for all τ in a set T, say, of positive measure $\sigma(T)$ on the boundary of the unit ball and deduce from this that percolation diffusion occurs. In fact, we will show more – we will show that the random archipelago is avoidable with probability one. By Proposition 1, we see that $\mathbb{E}[W(A_{\mathcal{P}}, \tau)]$ is finite for $\tau \in T$, hence the series $W(A_{\mathcal{P}}, \tau)$ is convergent a.s. for each $\tau \in T$. For $\tau \in T$, set

$$F_\tau = \{\mathcal{P} \colon W(A_{\mathcal{P}}, \tau) < \infty\}$$

so that F_τ has probability 1. We have

$$1 = \frac{1}{\sigma(T)} \int_T \mathbb{E}[1_{F_\tau}]\, d\tau = \mathbb{E}\left[\frac{1}{\sigma(T)} \int_T 1_{F_\tau}\, d\tau\right],$$

from which it follows that $\int_{\mathbb{S}} 1_{F_\tau}\, d\tau = \sigma(T)$ with probability one. Equivalently, it is almost surely true that $\mathcal{P} \in F_\tau$ for a.e. $\tau \in T$. In other words, it is almost surely true that $A_{\mathcal{P}}$ is minimally thin at a set of τ of positive measure on the sphere $\mathbb{S}$, hence $A_{\mathcal{P}}$ is almost surely avoidable by Lemma 1.

Next we prove the reverse implication. For a random configuration $\mathcal{P}$, set

$$M_{\mathcal{P}} = \{\tau \in \mathbb{S} : A_{\mathcal{P}} \text{ is minimally thin at } \tau\},$$

similar to (14). Suppose that percolation diffusion occurs. Then, with positive probability, $A_{\mathcal{P}}$ is minimally thin at each point of some set of positive surface measure on the sphere, so that $\mathbb{E}\left[\int_{\mathbb{S}} 1_{M_{\mathcal{P}}}(\tau)\, d\tau\right] > 0$. Interchanging the order of integration and expectation, we conclude that there is a set T of positive measure on the sphere $\mathbb{S}$ such that $\mathbb{P}\big(\tau \in M_{\mathcal{P}}\big) > 0$ for $\tau \in T$. By Lemma 2, $W(A_{\mathcal{P}}, \tau) < \infty$ a.s. for $\tau \in T$. By Lemma 4, $\mathbb{E}[W(A_{\mathcal{P}}, \tau)]$ is finite for $\tau \in T$. Finally, it follows from Proposition 1 that, for τ in the set T of positive measure on the sphere $\mathbb{S}$, the Poisson balayage (9) is finite. $\square$

Proof of Corollary 1. In the case that both ϕ and ν are radial, the value of the Poisson balayage in (9) is independent of $\tau \in \mathbb{S}$ and equals

$$\int_0^1 (1-t^2)\phi(t)^{d-2} \left(\int_{t\mathbb{S}} \frac{1-|x|^2}{|\tau - x|^d}\, d\sigma(x) \right) \nu(t)\, dt,$$

where $d\sigma$ is surface measure on the sphere $t\mathbb{S}$. Hence (9) is equivalent to (10) in the radial setting. □

5. Percolation diffusion in space

The Wiener criterion for minimal thinness of a set A at ∞ in $\mathbb{R}^d$, $d \geq 3$, is

$$W(A,\infty) = \sum_k \frac{\operatorname{cap}(A \cap Q_k)}{\ell(Q_k)^{d-2}} < \infty \tag{21}$$

(see [8], for example) where the cubes $\{Q_k\}$ are obtained by partitioning the cube of sidelength 3^j (centre 0 and sides parallel to the coordinate axes) into 3^{jd} cubes of sidelength 3^{j-1} and then deleting the central cube. Assuming that the radius function ϕ and the intensity of the Poisson process ν are roughly constant on each cube Q_k, and that $|x|^2\phi(x)^{d-2}\nu(x) = O(1)$ as $|x| \to \infty$, the corresponding version of Lemma 3 is that, for a cube Q_k and any point $x_k \in Q$,

$$\mathbb{E}[\operatorname{cap}(A_{\mathcal{P}} \cap Q_k)] \sim \phi(x_k)^{d-2}\, \mu(Q_k), \tag{22}$$

and the corresponding version of Proposition 1 is

$$\mathbb{E}\big[W(A_{\mathcal{P}},\tau)\big] \sim \int_{\mathbb{R}^d \setminus \mathbb{B}} \left(\frac{\phi(x)}{|x|} \right)^{d-2} \nu(x)\, dx. \tag{23}$$

Since the random archipelago $A_{\mathcal{P}}$ is avoidable in this setting precisely when it is minimally thin at the point at infinity, the criterion for percolation diffusion is that the integral on the right hand side of (23) be finite. Again this agrees in principle with a criterion for avoidability in the deterministic, regularly located setting [5, Theorem 2] (see also [8, Theorem 6]) which corresponds to ν constant and ϕ radial, namely

$$\int_1^\infty r\phi(r)^{d-2}\, dr < \infty.$$

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