

# INTERPOLATION AND SAMPLING IN SMALL BERGMAN SPACES

KRISTIAN SEIP

ABSTRACT. Carleson measures and interpolating and sampling sequences for weighted Bergman spaces on the unit disk are described for weights that are radial and grow faster than the standard weights  $(1 - |z|)^{-\alpha}$ ,  $0 < \alpha < 1$ . These results make the Hardy space  $H^2$  appear naturally as a “degenerate” endpoint case for the class of Bergman spaces under study.

## 1. INTRODUCTION

This work originates in my 1993 paper [5] which concerns interpolation and sampling in Bergman spaces on the unit disk with standard radial weights  $(1 - |z|^2)^{-\alpha}$  and  $\alpha < 1$ . Following [5], A. Borichev, R. Dhuez, and K. Kellay [1] studied the same problem when the weights decay more rapidly than any positive power of  $1 - |z|$  as  $|z| \nearrow 1$ . What remains to be settled is then the case of nontrivial weights growing more rapidly than  $(1 - |z|)^{-\alpha}$  for any  $\alpha$  in  $(0, 1)$ , which should be thought of as dealing with Hilbert spaces of analytic functions lying “between” the classical Hardy and Bergman spaces. In what follows, I will show how this can be done. Somewhat vaguely phrased, the present analysis offers a “smooth” transition from the Hardy space situation and L. Carleson’s theorems (in this context a “degenerate” endpoint case) and the setting of Bergman spaces with standard weights.

Throughout this paper  $w$  will be a positive and continuous function on  $[0, 1)$ , fixed once and for all, such that for a positive constant  $c$

$$(1) \quad w(1 - t) \geq cw(1 - 2t)$$

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whenever  $0 < t \leq 1/2$ . We will assume that  $w$  is integrable and for convenience that

$$\int_0^1 w(x)dx = 1.$$

With our fixed weight  $w$ , we associate the weighted Bergman space  $A_w^2$  consisting of all functions  $f$  analytic in the open unit disk  $\mathbb{D}$  satisfying

$$\int_{z \in \mathbb{D}} |f(z)|^2 w(|z|) d\sigma(z) < \infty,$$

where  $\sigma$  denotes Lebesgue area measure on  $\mathbb{D}$ . The latter integral defines a norm on  $A_w^2$ , but we prefer to use another equivalent norm. Define  $0 \leq r_n < 1$  by the relation

$$\int_{r_n}^1 w(x)dx = 2^{-n}$$

for every nonnegative integer  $n$ , and set

$$(2) \quad \|f\|_w^2 = \sum_{n=1}^{\infty} 2^{-n} \int_0^{2\pi} |f(r_n e^{it})|^2 \frac{dt}{2\pi}.$$

If we had chosen to start from the sequence  $(r_n)$  instead of the weight  $w$ , then we would have needed to replace the condition (1) by the requirement that

$$(3) \quad \inf_{n \geq 0} \frac{1 - r_n}{1 - r_{n+1}} > 1.$$

This alternative approach has the advantage that it permits us to associate the Hardy space  $H^2$  of the unit disk with the “degenerate” case when the sequence of radii  $r_n$  is allowed to be finite and  $\max_n r_n = 1$ .

To see how the scale of Bergman spaces with standard weights fits into this context, we introduce the following scale of weights associated with  $w$ :

$$w_\alpha(x) = (1 - \alpha)w(x) \left( \int_x^1 w(t)dt \right)^{-\alpha}$$

for  $\alpha < 1$ . It is plain that we also have  $w_\alpha(1 - t) \geq c_\alpha w_\alpha(1 - 2t)$  for some constant  $c_\alpha$ , and that  $\int_0^1 w_\alpha(x)dx = 1$ . If we choose  $w \equiv 1$ , then the family of weights  $w_\alpha$  corresponds to the standard weighted Bergman spaces. Note that substituting  $w$  by  $w_\alpha$  corresponds to replacing  $2^{-n}$  in (2) by  $2^{-(1-\alpha)n}$ . It may be verified that this implies that the Carleson measures are described in the same way for all the spaces  $A_{w_\alpha}^2$  and that the notion of density that we will use for  $A_w^2$ , also applies to describe interpolating and sampling sequences for each of the spaces  $A_{w_\alpha}^2$ .

The next sections contain two theorems, the first describing the Carleson measures for our space  $A_w^2$  and the second the interpolating and sampling sequences for the same space.

The first theorem is easily proved using Carleson's embedding theorem, while the second requires somewhat delicate technicalities. A main ingredient in the proof of the second theorem is a lemma involving a method of redistribution and atomization of certain Riesz measures.

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## 2. CARLESON MEASURES FOR $A_w^2$

Given a Hilbert space  $\mathcal{H}$  of analytic functions on  $\mathbb{D}$ , we say that a nonnegative Borel measure  $\mu$  on  $\mathbb{D}$  is a Carleson measure for  $\mathcal{H}$  if there exists a positive constant  $C$  such that

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C \|f\|_{\mathcal{H}}^2$$

holds for every  $f$  in  $\mathcal{H}$ . The Carleson constant of  $\mu$  is the smallest possible constant  $C$  for which this holds. In our case,  $\mathcal{H}$  will be either  $A_w^2$  or the Hardy space  $H^2$ , where the latter consists of all analytic functions  $f$  in  $\mathbb{D}$  for which

$$\|f\|_{H^2} = \sup_{r < 1} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi} < \infty.$$

Before stating our theorem on Carleson measures, we introduce the following notations, to be retained for the remainder of this paper. Set

$$\Omega_n = \{z : r_n \leq |z| < r_{n+1}\},$$

and let  $\mu_n$  be the measure such that  $d\mu_n(z) = \chi_{\Omega_n}(z) d\mu(z)$  whenever a nonnegative Borel measure  $\mu$  on  $\mathbb{D}$  is given. The notation  $U(z) \lesssim V(z)$  (or equivalently  $V(z) \gtrsim U(z)$ ) means that there is a constant  $C$  such that  $U(z) \leq CV(z)$  holds for all  $z$  in the set in question, which may be a space of functions or a set of numbers. If both  $U(z) \lesssim V(z)$  and  $V(z) \lesssim U(z)$ , then we write  $U(z) \simeq V(z)$ .

**Theorem 1.** *A nonnegative Borel measure  $\mu$  on  $\mathbb{D}$  is a Carleson measure for  $A_w^2$  if and only if each  $\mu_n$  is a Carleson measure for  $H^2$  with Carleson constant  $\lesssim 2^{-n}$ .*

*Proof.* The proof relies on Carleson's theorem [3]. For the necessity, it suffices to check Carleson "squares"  $Q_\zeta = \{z : |\zeta| < |z| < 1, \arg(z\bar{\zeta}) < 1 - |\zeta|\}$  whose top center  $\zeta$  is in  $\Omega_n$ . We use the test function  $f_\zeta(z) = (1 - \bar{\zeta}z)^{-\gamma}$  with  $\gamma$  so large that

$$\|f_\zeta\|_w^2 \simeq 2^{-n}(1 - |z|)^{-\gamma+1};$$

this can be achieved because of (3). It follows readily from the Carleson measure condition that  $\mu(Q_\zeta \cap \Omega_n) \lesssim 2^{-n}(1 - |\zeta|)$  as required by Carleson's theorem.

To prove the sufficiency, we note that if  $\mu_n$  is a Carleson measure for  $H^2$  with Carleson constant  $\lesssim 2^{-n}$ , then, in view of (3), the same holds for  $H^2$  of the smaller disk  $r_{n+2}\mathbb{D}$ . Given an arbitrary function  $f$  in  $A_w^2$ , we sum the corresponding Carleson measure estimates over  $n$  and get

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \lesssim \sum_{n=0}^{\infty} 2^{-n} \int_0^{2\pi} |f(r_{n+2}e^{it})|^2 \frac{dt}{2\pi}. \quad \square$$

### 3. INTERPOLATION AND SAMPLING IN $A_w^2$

Let  $\mathcal{H}$  be as in the previous section, and let  $K_z$  be the reproducing kernel for  $\mathcal{H}$  at the point  $z$  in  $\mathbb{D}$ . We say that a sequence  $\Lambda = (\lambda_j)$  of distinct points in  $\mathbb{D}$  is an interpolating sequence for  $\mathcal{H}$  if we can solve the interpolation problem  $f(\lambda_j) = a_j$  whenever the sequence  $(a_j)$  satisfies the admissibility condition

$$\sum_j \frac{|a_j|^2}{K_{\lambda_j}(\lambda_j)} < \infty;$$

the sequence  $\Lambda$  is said to be a sampling sequence if there are positive constants  $A$  and  $B$  such that

$$(4) \quad A\|f\|_{\mathcal{H}}^2 \leq \sum_j \frac{|f(\lambda_j)|^2}{K_{\lambda_j}(\lambda_j)} \leq B\|f\|_{\mathcal{H}}^2$$

for every  $f$  in  $\mathcal{H}$ .

We are interested in such sequences when  $\mathcal{H} = A_w^2$ , and we therefore need a precise estimate for  $K_z(z)$  in this case. By (3), we have that  $1 - |z| \leq c(r_{n+2} - |z|)$  when  $z$  is in  $\Omega_n$  for some constant  $c$  independent of  $n$ . Thus

$$(5) \quad |f(z)|^2 \leq C2^n(1 - |z|)^{-1}\|f\|_w^2$$

for every  $f$  in  $A_w^2$  and  $z$  in  $\Omega_n$  with  $C$  independent of  $n$ . On the other hand, choosing  $f_z$  as in the proof of Theorem 1, we get that

$$|f_z(z)|^2 \gtrsim 2^n(1 - |z|)^{-1}\|f_z\|_w^2$$

if  $\gamma$  is again chosen sufficiently large; we conclude that

$$(6) \quad K_z(z) \simeq 2^n(1 - |z|)^{-1}$$

for  $z$  in  $\Omega_n$  and all  $n$  when  $K_z$  is the reproducing kernel for  $A_w^2$ .

We denote by  $\varrho(z, \zeta)$  the pseudohyperbolic distance between two points  $z$  and  $\zeta$  in  $\mathbb{D}$ , i.e.,

$$\varrho(z, \zeta) = \left| \frac{z - \zeta}{1 - \bar{\zeta}z} \right|.$$

Let  $\Lambda = (\lambda_j)$  be a separated sequence in  $\mathbb{D}$ , which as usual we take to mean that  $\inf_{j \neq l} \varrho(\lambda_j, \lambda_l) > 0$ . For a given  $z$ , let  $n(z) = n(|z|)$  be the nonnegative integer such that  $r_{n(z)} \leq |z| < r_{n(z)+1}$ . We then define the following densities:

$$D_w^+(\Lambda) = \limsup_{m \rightarrow \infty} \frac{1}{m} \sup_{|z| < 1} \sum_{|\lambda_j| \leq r_{n(z)+m}} (1 - \varrho(z, \lambda_j))$$

and

$$D_w^-(\Lambda) = \liminf_{m \rightarrow \infty} \frac{1}{m} \inf_{|z| < 1} \sum_{|\lambda_j| < r_{n(z)+m}} (1 - \varrho(z, \lambda_j)).$$

It may be checked that in the special case when

$$\mu = \sum_{n=0}^{\infty} 2^{-n} \sum_{r_n < |\lambda_j| < r_{n+1}} (1 - |\lambda_j|) \delta_{\lambda_j},$$

$\mu$  is a Carleson measure for  $A_w^2$  if and only if  $D_w^+(\Lambda) < \infty$ .

Our main result is the following theorem.

**Theorem 2.** (I) A sequence  $\Lambda$  is an interpolating sequence for  $A_w^2$  if and only if it is separated and  $D_w^+(\Lambda) < (\log 2)/2$ . (S) A separated sequence  $\Lambda$  is a sampling sequence for  $A_w^2$  if and only if  $D_w^+(\Lambda) < \infty$  and  $D_w^-(\Lambda) > (\log 2)/2$ .

In the “degenerate” case of  $H^2$  (when the sequence of radii  $r_n$  is allowed to be finite and  $\max_n r_n = 1$ ), H. Shapiro and A. Shield’s  $L^2$  version of Carleson’s interpolation theorem [7, 2] gives that the condition  $D_w^+(\Lambda) < (\log 2)/2$  in part (I) should be replaced by the simpler condition

$$\sup_{|z| < 1} \sum_j (1 - \varrho(z, \lambda_j)) < \infty;$$

it is well-known that there is no counterpart to part (S) when  $A_w^2$  is replaced by  $H^2$ .

The densities used in Theorem 2 are defined somewhat differently from those used in the original paper [5] and in [6]. These densities can also be defined via harmonic measure as shown in [4]. It seems clear that our Theorem 2 can be rephrased in a similar way using harmonic measure. One can of course also prove similar results for Bergman  $L^p$  spaces without any essential changes of the arguments.

The remainder of this paper is devoted to proving respectively the necessity (Section 4) and the sufficiency (Section 5) of the conditions of Theorem 2.

## 4. PROOF OF THE NECESSITY OF THE CONDITIONS OF THEOREM 2

We begin with part (I). To see that an interpolating sequence is separated, we can argue similarly as in the proof of Theorem 1. Namely, if  $\Lambda$  is an interpolating sequence for  $A_w^2$ , then the sequence  $\Lambda \cap \Omega_n$  is an interpolating sequence for  $H^2$  of the smaller disk  $r_{n+2}\mathbb{D}$ , and then, in view of (3), we may argue as we do in the classical  $H^2$  case. We omit the details of this routine argument.

Just as on pages 57–58 in [6] we may modify the definition of the upper density:

$$D_w^+(\Lambda) = \limsup_{m \rightarrow \infty} \frac{1}{m} \sup_{\lambda_l} \sum_{|\lambda_j| \leq r_{n(\lambda_l)+m}} (1 - \varrho(\lambda_l, \lambda_j)).$$

We also repeat the argument on pages 58–59 in [6]. This means that we only need to verify that

$$\sum_{|\lambda_j| \leq r_{n(\lambda_l)+m}} (1 - \varrho(\lambda_l, \lambda_j)) \leq m(\log 2)/2 + C$$

holds with  $C$  depending only on the constant of interpolation for  $\Lambda$ . Assuming  $\Lambda$  is an interpolating sequence and in view of (6), we may solve the problem  $f_l(\lambda_l) = 2^{n(\lambda_l)/2}/\sqrt{1-|\lambda_l|}$  and  $f_l(\lambda_j) = 0$  for  $j \neq l$  with uniform control of norms. Let us for simplicity set  $r = r_{n(\lambda_l)+m}$ . The function  $\tilde{f}_l(z) = f_l(rz)$  has  $H^2$  norm  $\lesssim 2^{(n(\lambda_j)+m)/2}$ . Now set

$$\varphi(z) = \frac{\lambda_j - rz}{r - \bar{\lambda}_j z}.$$

We apply Jensen's formula to  $\tilde{f}_l \circ \varphi$  and get

$$\left( n(\lambda_j) \log 2 + \log \frac{1}{1-|\lambda_l|} \right) / 2 + \sum_{j \neq l, |\lambda_j| < r} \log \frac{|\lambda_j - \lambda_l|}{|r - \bar{\lambda}_l \lambda_j / r|} = \int_0^{2\pi} \log |\tilde{f}_l \circ \varphi(e^{it})| \frac{dt}{2\pi}.$$

By the arithmetic–geometric mean inequality,

$$\int_0^{2\pi} \log |\tilde{f}_l \circ \varphi(e^{it})| \frac{dt}{2\pi} \leq \log \|\tilde{f}_l \circ \varphi\|_{H^2}.$$

Since the norm of the composition operator  $\lesssim (1-|\lambda_l|)^{-1/2}$ , we obtain

$$\int_0^{2\pi} \log |\tilde{f}_l \circ \varphi(e^{it})| dt \leq C + ((n(\lambda_j) + m) \log 2 + \log \frac{1}{1-|\lambda_l|}) / 2.$$

What remains is to prove that

$$\sum_{j \neq l, |\lambda_j| < r} \log \frac{1}{|r - \bar{\lambda}_l \lambda_j / r|} \geq \sum_{j \neq l, |\lambda_j| < r} \log \frac{1}{|1 - \bar{\lambda}_l \lambda_j|} + C$$

for some constant  $C$  independent of  $l$  and  $m$ . This is a consequence of the inequality  $|1 - z/a| \leq |1 - z|/a$  which holds whenever  $|z| \leq a \leq 1$ .

We turn to part (S). We obtain the condition  $D_w^+(\Lambda) < \infty$  from the right inequality of (4), cf. Theorem 1. Following the reasoning on page 59 of [6], we find that it suffices to show that  $D_w^-(\Lambda) \geq (\log 2)/2$ . To prove this, we pick a point  $\lambda_l$  and look at the function

$$f_{l,m}(z) = \frac{1}{1 - \bar{\lambda}_l z} B_{l,m}(z),$$

where  $B_{l,m}$  is the finite Blaschke product with zeros at the points  $\lambda_j$  for which  $\lambda_j \neq \lambda_l$  and  $|\lambda_j| \leq r_{n(\lambda_l)+m}$ . By the (5), we have

$$|f_{l,m}(\lambda_l)|^2 (1 - |\lambda_l|) 2^{-n(\lambda_l)} \lesssim \|f_{l,m}\|_w^2,$$

which implies that

$$e^{-2m(D^-(\Lambda)+o(1))} (1 - |\lambda_l|)^{-1} 2^{-n(\lambda_l)} \lesssim \|f_{l,m}\|_w^2$$

when  $m \rightarrow \infty$  and  $\lambda_l$  is chosen appropriately. We now use the fact that the operators of multiplication by a single Blaschke factor are uniformly bounded below on  $A_w^2$ . Thus applying the left inequality of (4) to the function

$$f_{l,m}(z) \frac{z - \lambda_l}{1 - \bar{\lambda}_l z}$$

and using (6), we get

$$\|f_{l,m}\|_w^2 \lesssim \sum_{|\lambda_j| > r_{n(\lambda_l)+m}} |f(\lambda_j)|^2 (1 - |\lambda_j|) 2^{-n(\lambda_j)}.$$

Now applying Theorem 1 and Carleson's embedding theorem for  $H^2$ , we get

$$\|f_{l,m}\|_w^2 \lesssim (1 - |\lambda_l|)^{-1} 2^{-n(\lambda_l)-m},$$

and the desired estimate for  $D_w^-(\Lambda)$  thus follows.

## 5. PROOF OF THE SUFFICIENCY OF THE CONDITIONS OF THEOREM 2

The main technical ingredient is the following lemma.

**Lemma 1.** *Let  $\Lambda$  be a separated sequence in the unit disk  $\mathbb{D}$ . (I) When  $D_w^+(\Lambda) < (\log 2)/2$ , there are  $\varepsilon > 0$  and an analytic function  $G(z)$  in  $\mathbb{D}$  with zero set  $\Lambda' \supseteq \Lambda$  and*

$$|G(z)|^2 \simeq 2^{(1-\varepsilon)n(z)} \varrho^2(z, \Lambda').$$

(S) When  $D_w^-(\Lambda) > (\log 2)/2$  and  $D_w^+(\Lambda) < \infty$ , there are  $\varepsilon > 0$  and a meromorphic function  $G(z)$  in  $\mathbb{D}$  with zero set  $\Lambda$  and pole set  $\Lambda'$  that is pseudohyperbolically separated from  $\Lambda$  and such that

$$|G(z)|^2 \simeq 2^{(1+\varepsilon)n(z)} \varrho^2(z, \Lambda) \varrho^{-2}(z, \Lambda').$$

*Proof.* In either case we begin by letting  $F$  be an analytic function having  $\Lambda$  as its zero set. We may write

$$\log |F(z)| = \sum_{n=0}^{\infty} (\log |B_n(z)| + h_n(z)),$$

where  $B_n$  is the Blaschke product with zeros at the  $\lambda_j$  in  $\Omega_n$  and  $h_n$  is an appropriate harmonic function that makes the sum converge. The basic idea is to approximate the subharmonic function

$$U_j(z) = \sum_{n=m_j}^{mj+m-1} \log |B_n(z)|$$

by another subharmonic function  $V_j(z)$  with Riesz measure supported by the circle  $|z| = r_{mj}$ ; the point of this redistribution of the Riesz measure is that the latter measure is more easily atomized.

We choose  $m$  so large that either (I)  $U_j(z) \leq (1 - \varepsilon)m$  for  $|z| = r_{mj}$ , where  $2\varepsilon = (\log 2)/2 - D_w^+(\Lambda)$  or (S)  $U_j(z) \geq (1 + \varepsilon)m$  for  $|z| = r_{mj}$ , where  $2\varepsilon = D_w^-(\Lambda) - (\log 2)/2$ . We claim that the function

$$V_j(z) = \frac{1}{\pi(1 - r_{mj}^2)} \int_0^{2\pi} \log \left| \frac{r_{mj}e^{it} - z}{1 - \bar{z}r_{mj}e^{it}} \right| U_j(r_{mj}e^{it}) dt$$

does the job in the sense that

$$\left| \sum_{j=1}^{\infty} (V_j(z) - U_j(z)) \right| \leq C$$

whenever  $\varrho(\Lambda, z) \geq \delta > 0$  with  $C$  depending on  $\delta$ . It is plain that we have

$$\left| \sum_{j: r_{mj} \leq |z|} (V_j(z) - U_j(z)) \right| \leq C$$

whenever  $\varrho(\Lambda, z) \geq \delta > 0$ . To deal with the case when  $|z| < r_{mj}$ , we note that then, by harmonicity, we may write

$$U_j(z) = \int_0^{2\pi} \frac{1 - |z|^2/r_{mj}^2}{|1 - \bar{z}e^{it}/r_{jm}|^2} U_j(r_{mj}e^{it}) \frac{dt}{2\pi}.$$



We approximate the logarithm in the integral defining  $V_j$  as

$$\log \left| \frac{r_{mj} e^{it} - z}{1 - \bar{z} r_{mj}} \right| = \frac{1}{2} \frac{(1 - r_{mj}^2)(1 - |z|^2)}{|1 - \bar{z} r_{mj} e^{it}|^2} + O \left( \left[ \frac{(1 - r_{mj}^2)(1 - |z|^2)}{|1 - \bar{z} r_{mj} e^{it}|^2} \right]^2 \right)$$

when  $\varrho(z, r_{mj} e^{it}) \rightarrow 1$ . Here the second order term causes no problem, so we only need to estimate the difference

$$D_r(z) = \frac{1 - |z|^2/r^2}{|1 - z/r|^2} - \frac{1 - |z|^2}{|1 - rz|^2}$$

when  $|z| < r$ . It suffices to observe that

$$D_r(z) = \frac{1}{r^2} \left( \frac{(1 - r^2)(1 - |z|^2)^2}{|1 - z/r|^2 |1 - rz|^2} - \frac{1 - r^2}{|1 - z/r|^2} \right)$$

because this identity implies that

$$\sum_{j: r_{mj} > |z|} |D_{r_{mj}}(z)| \lesssim 1.$$

We are now ready to construct the desired functions. To begin with, note that we have

$$2^{n(z)/2} \simeq \exp \left( \sum_{j=0}^{\infty} \frac{\log 2}{4\pi(1 - r_{mj})} \int_0^{2\pi} (\log |z - r_{mj} e^{it}| - \log r_{mj}) dt \right).$$

In other words, the function  $n(z)$  can be approximated by a subharmonic function whose Riesz measure is concentrated on the circles  $|z| = r_{mj}$  with density  $\log 2 / (2\pi(1 - r_{mj}) \log 2)$  with respect to arc length measure on the respective circles. We now employ the counterpart of Lemma 5 on page 50 in [6]. It is plain that the proof in [6] carries over to this situation. (The fact that the total mass of the Riesz measure we want to atomize over the circle  $|z| = r_{mj}$  may be non-integer is of no significance. Just leave the “remainder” untouched; it corresponds to a bounded part of the subharmonic function.)  $\square$

*Proof of the sufficiency of the conditions in part (I) of Theorem 2.* We want to solve the problem  $f(\lambda_j) = a_j$  with  $f$  in  $A_w^2$  when

$$\sum_j |a_j|^2 (1 - |\lambda_j|) 2^{-n(\lambda_j)} < \infty.$$

We will in fact construct a linear operator doing the job:

$$f(z) = \sum_j \frac{a_j}{G'(\lambda_j)} \frac{G(z)}{z - z_j} \frac{1 - |\lambda_j|^2}{1 - \bar{\lambda}_j z},$$

just as formula (53) on page 53 of [6]. It follows from Lemma 1 that

$$|G'(\lambda_j)| \simeq 2^{(1-\varepsilon)n(\lambda_j)/2} (1 - |\lambda_j|)^{-1} \quad \text{and} \quad \frac{|G(z)|}{|z - \lambda_j|} \lesssim \frac{2^{(1-\varepsilon)n(z)/2}}{|1 - \bar{\lambda}_j z|}$$

so that

$$|f(z)| \lesssim 2^{(1-\varepsilon)n(z)/2} \sum_j |a_j| 2^{-(1-\varepsilon)n(\lambda_j)/2} \frac{(1 - |\lambda_j|)^2}{|1 - \overline{\lambda_j}z|^2}.$$

We write

$$h_n(z) = 2^{(1-\varepsilon)n(z)/2} \sum_{j: r_n \leq |\lambda_j| < r_{n+1}} |a_j| 2^{-(1-\varepsilon)n(\lambda_j)/2} \frac{(1 - |\lambda_j|)^2}{|1 - \overline{\lambda_j}z|^2}.$$

Thus we need to show that

$$\sum_{l=1}^{\infty} 2^{-l} \int_0^{2\pi} \left( \sum_{n=0}^{\infty} h_n(r_l e^{it}) dt \right)^2 \lesssim \sum_j |a_j|^2 (1 - |\lambda_j|) 2^{-n(\lambda_j)}.$$

We have

$$\sum_{l=1}^{\infty} 2^{-l} \int_0^{2\pi} \left( \sum_{n=0}^{\infty} h_n(r_l e^{it}) dt \right)^2 \leq I_1 + I_2,$$

where

$$I_1 = \sum_{l=1}^{\infty} 2^{-l+1} \int_0^{2\pi} \left( \sum_{n < l} h_n(r_l e^{it}) dt \right)^2 \quad \text{and} \quad I_2 = \sum_{l=1}^{\infty} 2^{-l+1} \int_0^{2\pi} \left( \sum_{n \geq l} h_n(r_l e^{it}) dt \right)^2.$$

We compute the  $L^2$  integral in  $I_1$  by duality. Using the Carleson measure condition, we then get

$$I_1 \lesssim \sum_{l=1}^{\infty} 2^{-l\varepsilon} \left( \sum_{n < l} 2^{n\varepsilon/2} \left( \sum_{j: r_n \leq |\lambda_j| < r_{n+1}} |a_j|^2 (1 - |\lambda_j|) 2^{-n(\lambda_j)} \right)^{\frac{1}{2}} \right)^2.$$

By the Cauchy–Schwarz inequality, we get

$$I_1 \lesssim \sum_{l=1}^{\infty} 2^{-l\varepsilon/2} \sum_{n < l} 2^{n\varepsilon/2} \sum_{j: r_n \leq |\lambda_j| < r_{n+1}} |a_j|^2 (1 - |\lambda_j|) 2^{-n(\lambda_j)},$$

and the desired estimate is obtained if we change the order of summation. To deal with  $I_2$ , we note that

$$\left( \sum_{n \geq l} h_n(z) \right)^2 \leq 2^{l(1-\varepsilon)} \sum_{|\lambda_j| \geq r_l} |a_j|^2 2^{-(1-\varepsilon)n(\lambda_j)} \frac{(1 - |\lambda_j|)^2}{|1 - \overline{\lambda_j}z|^2} \sum_{|\lambda_k| \geq r_l} \frac{(1 - |\lambda_k|)^2}{|1 - \overline{\lambda_k}z|^2}.$$

Thus

$$\left( \sum_{n \geq l} h_n(r_l e^{it}) \right)^2 \lesssim 2^{l(1-\varepsilon)} \sum_{|\lambda_j| \geq r_l} |a_j|^2 2^{-(1-\varepsilon)n(\lambda_j)} \frac{(1 - |\lambda_j|)^2}{|1 - \overline{\lambda_j} r_l e^{ir_l}|^2}$$

from which it follows that

$$\int_0^{2\pi} \left( \sum_{n \geq l} h_n(r_l e^{it}) \right)^2 dt \lesssim 2^{l(1-\varepsilon)} \sum_{|\lambda_j| \geq r_l} |a_j|^2 2^{-(1-\varepsilon)n(\lambda_j)} \frac{(1 - |\lambda_j|)^2}{(1 - r_l)}.$$

Finally, we get

$$I_2 \lesssim \sum_j |a_j|^2 2^{-(1-\varepsilon)n(\lambda_j)} (1 - |\lambda_j|)^2 \sum_{r_l \leq |\lambda_j|} \frac{2^{-\varepsilon l}}{(1 - r_l)}.$$

In the latter sum, we can assume that  $\varepsilon$  is so small (if need be) that the terms grow exponentially, so that we may arrive at the desired estimate.  $\square$

*Proof of the sufficiency of the conditions in part (S) of Theorem 2.* We start from the formula

$$f(z) = \sum_j \frac{f(\lambda_j)}{G'(\lambda_j)} \frac{G(z)}{z - z_j} \frac{1 - |z|^2}{1 - \bar{z}\lambda_j},$$

which holds for every function in  $A_w^2$ , cf. formula (54) on page 53 of [6]. Here  $G$  is the meromorphic function of Lemma 1. We get that

$$|f(z)| \lesssim 2^{(1+\varepsilon)n(z)/2} (1 - |z|) \sum_j |f(\lambda_j)| 2^{-(1+\varepsilon)n(\lambda_j)/2} \frac{(1 - |\lambda_j|)}{|1 - \bar{\lambda}_j z|^2}.$$

We write

$$g_n(z) = 2^{(1+\varepsilon)n(z)/2} (1 - |z|) \sum_{j: r_n \leq |\lambda_j| < r_{n+1}} |f(\lambda_j)| 2^{-(1+\varepsilon)n(\lambda_j)/2} \frac{(1 - |\lambda_j|)}{|1 - \bar{\lambda}_j z|^2}.$$

Thus we need to show that

$$\sum_{l=1}^{\infty} 2^{-l} \int_0^{2\pi} \left( \sum_{n=0}^{\infty} g_n(r_l e^{it}) \right)^2 dt \lesssim \sum_j |f(\lambda_j)|^2 (1 - |\lambda_j|) 2^{-n(\lambda_j)}.$$

We write

$$\sum_{l=1}^{\infty} 2^{-l} \int_0^{2\pi} \left( \sum_{n=0}^{\infty} g_n(r_l e^{it}) \right)^2 dt \leq J_1 + J_2,$$

where

$$J_1 = \sum_{l=1}^{\infty} 2^{-l+1} \int_0^{2\pi} \left( \sum_{n < l} g_n(r_l e^{it}) \right)^2 dt \quad \text{and} \quad J_2 = \sum_{l=1}^{\infty} 2^{-l+1} \int_0^{2\pi} \left( \sum_{n \geq l} g_n(r_l e^{it}) \right)^2 dt.$$

We compute the  $L^2$  integral in  $J_1$  by duality. Using the Carleson measure condition and the Cauchy–Schwarz inequality, we get

$$J_1 \lesssim \sum_{l=1}^{\infty} (1-r_l)^{\frac{1}{2}} 2^{l\varepsilon} \sum_{n < l} \frac{2^{-n\varepsilon}}{(1-r_n)^{\frac{1}{2}}} \sum_{j: r_n \leq |\lambda_j| < r_{n+1}} |f(\lambda_j)|^2 (1-|\lambda_j|) 2^{-n(\lambda_j)}.$$

Changing the order of summation, we get the desired result. (We need to assume, if need be, that  $\varepsilon$  is so small that  $2^{\varepsilon l} (1-r_l)^{\frac{1}{2}}$  decays exponentially.) To deal with  $J_2$ , we note that

$$\begin{aligned} \left( \sum_{n \geq l} g_n(z) \right)^2 &\leq \\ &\leq 2^{l(1+\varepsilon)} (1-|z|)^2 \sum_{|\lambda_j| \geq r_l} |f(\lambda_j)|^2 2^{-(1+\frac{1}{2}\varepsilon)n(\lambda_j)} \frac{(1-|\lambda_j|)}{|1-\overline{\lambda_j}z|^2} \sum_{|\lambda_k| \geq r_l} \frac{(1-|\lambda_k|)}{|1-\overline{\lambda_k}z|^2} 2^{-\frac{1}{2}\varepsilon n(\lambda_k)}. \end{aligned}$$

Applying the Carleson measure condition to the sum to the right, we get

$$\left( \sum_{n \geq l} g_n(r_l e^{it}) \right)^2 \leq 2^{l(1+\frac{1}{2}\varepsilon)} \sum_{|\lambda_j| \geq r_l} |f(\lambda_j)|^2 (1-|\lambda_j|) 2^{-(1+\frac{1}{2}\varepsilon)n(\lambda_j)} \frac{(1-r_l)}{|1-\overline{\lambda_j}r_l e^{it}|^2}$$

from which it follows that

$$\int_0^{2\pi} \left( \sum_{n \geq l} g_n(r_l e^{it}) \right)^2 dt \lesssim 2^{l\frac{1}{2}\varepsilon} \sum_{|\lambda_j| \geq r_l} |f(\lambda_j)|^2 (1-|\lambda_j|) 2^{-(1+\frac{1}{2}\varepsilon)n(\lambda_j)}.$$

We now sum over  $l$  and get the desired result by changing the order of summation.  $\square$

## REFERENCES

- [1] A. Borichev, R. Dhuez, and K. Kellay, *Sampling and interpolation in large Bergman and Fock spaces*, J. Funct. Anal. **242** (2007), 563–606.
- [2] L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921–930.
- [3] L. Carleson, *Interpolations by bounded analytic functions and the corona problem* Ann. of Math. (2) **76** (1962), 547–559.
- [4] J. Ortega-Cerdà and K. Seip, *Harmonic measure and uniform densities*, Indiana Univ. Math. J. **53** (2004), 905–923.
- [5] K. Seip, *Beurling type density theorems in the unit disk*, Invent. Math. **113** (1993), 21–39.
- [6] K. Seip, *Interpolation and Sampling in Spaces of Analytic Functions*, University Lecture Series **33**, Amer. Math. Soc., Providence, RI, 2004. xii+139 pp. ISBN: 0-8218-3554-8.
- [7] H. S. Shapiro and A. L. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math. **83** (1961), 513–532.

DEPARTMENT OF MATHEMATICAL SCIENCES  
NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY (NTNU)  
NO- 7491 TRONDHEIM  
NORWAY  
*E-mail address:* `seip@math.ntnu.no`