A LOWER BOUND IN NEHARI'S THEOREM ON THE POLYDISC

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ABSTRACT. By theorems of Ferguson and Lacey ($d = 2$) and Lacey and Terwilleger ($d > 2$), Nehari's theorem is known to hold on the polydisc $\mathbb{D}^d$ for $d > 1$, i.e., if H_ψ is a bounded Hankel form on $H^2(\mathbb{D}^d)$ with analytic symbol ψ, then there is a function φ in $L^\infty(\mathbb{T}^d)$ such that ψ is the Riesz projection of φ. A method proposed in Helson's last paper is used to show that the constant C_d in the estimate $\|\varphi\|_\infty \le C_d \|H_\psi\|$ grows at least exponentially with d; it follows that there is no analogue of Nehari's theorem on the infinite-dimensional polydisc.

This note solves the following problem studied by H. Helson [2, 3]: Is there an analogue of Nehari's theorem on the infinite-dimensional polydisc? By using a method proposed in [3], we show that the answer is negative. The proof is of interest also in the finite-dimensional situation because it gives a nontrivial lower bound for the constant appearing in the norm estimate in Nehari's theorem; we choose to present this bound as our main result.

We first introduce some notation and give a brief account of Nehari's theorem. Let d be a positive integer, $\mathbb{D}$ the open unit disc, and $\mathbb{T}$ the unit circle. We let $H^2(\mathbb{D}^d)$ be the Hilbert space of functions analytic in $\mathbb{D}^d$ with square-summable Taylor coefficients. Alternatively, we may view $H^2(\mathbb{D}^d)$ as a subspace of $L^2(\mathbb{T}^d)$ and express the inner product of $H^2(\mathbb{D}^d)$ as $\langle f, g\rangle = \int_{\mathbb{T}^d} f\overline{g}$, where we integrate with respect to normalized Lebesgue measure on $\mathbb{T}^d$. Every function ψ in $H^2(\mathbb{D}^d)$ defines a Hankel form H_ψ by the relation $H_\psi(fg) = \langle fg, \psi\rangle$; this makes sense at least for holomorphic polynomials f and g. Nehari's theorem—a classical result [6] when $d = 1$ and a remarkable and relatively recent achievement of S. Ferguson and M. Lacey [1] ($d = 2$) and M. Lacey and E. Terwilleger [5] ($d > 2$) in the general case—says that H_ψ extends to a bounded form on $H^2(\mathbb{D}^d) \times H^2(\mathbb{D}^d)$ if and only if $\psi = P_+\varphi$ for some bounded function φ on $\mathbb{T}^d$; here P_+ is the Riesz projection on $\mathbb{T}^d$ or, in other words, the orthogonal projection of $L^2(\mathbb{T}^d)$ onto $H^2(\mathbb{D}^d)$. We define C_d as the smallest constant C that can be chosen in the estimate

$$\|\varphi\|_\infty \le C\|H_\psi\|,$$

where it is assumed that φ has minimal L^∞ norm. Nehari's original theorem says that $C_1 = 1$.

Theorem. *For even integers $d \ge 2$, the constant C_d is at least $(\pi^2/8)^{d/4}$.*

The theorem thus shows that the blow-up of the constants observed in [4, 5] is not an artifact resulting from the particular inductive argument used there.

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Since clearly C_d increases with d and, in particular, we would need that $C_d \le C_\infty$ should Nehari's theorem extend to the infinite-dimensional polydisc, our theorem gives a negative solution to Helson's problem.

Nehari's theorem can be rephrased as saying that functions in $H^1(\mathbb{D}^d)$ (the subspace of holomorphic functions in $L^1(\mathbb{T}^d)$) admit weak factorizations, i.e., every f in $H^1(\mathbb{D}^d)$ can be written as $f = \sum_j g_j h_j$ with f_j, g_j in $H^2(\mathbb{D}^d)$ and $\sum_j \|g_j\|_2 \|h_j\|_2 \le A\|f\|_1$ for some constant A. Taking the infimum of the latter sum when g_j, h_j vary over all weak factorizations of f, we get an alternate norm (a projective tensor product norm) on $H^1(\mathbb{D}^d)$ for which we write $\|f\|_{1,w}$. We let A_d denote the smallest constant A allowed in the norm estimate $\|f\|_{1,w} \le A\|f\|_1$. Our proof shows that we also have $A_d \ge (\pi^2/8)^{d/2}$ when d is an even integer.

Proof of the theorem. We will follow Helson's approach [3] and also use his multiplicative notation. Thus we define a Hankel form on $\mathbb{T}^\infty$ as

$$H_\psi(fg) = \sum_{j,k=1}^{\infty} \rho_{jk} a_j b_k;$$

here (a_j), (b_j), and (ρ_j) are the sequences of coefficients of the power series of the functions f, g, and ψ, respectively. More precisely, we let p_1, p_2, p_3, ... denote the prime numbers; if $j = p_1^{\nu_1} \cdots p_k^{\nu_k}$, then a_j (respectively b_j and ρ_j) is the coefficient of f (respectively of g and ψ) with respect to the monomial $z_1^{\nu_1} \cdots z_k^{\nu_k}$. We will only consider the finite-dimensional case, which means that the coefficients will be nonzero only for indices j of the form $p_1^{\nu_1} \cdots p_d^{\nu_d}$. The prime numbers will play no role in the proof except serving as a convenient tool for bookkeeping.

We now assume that d is an even integer and introduce the set

$$I = \left\{ n \in \mathbb{N} : \ n = \prod_{j=1}^{d/2} q_j \ \text{ and } \ q_j = p_{2j-1} \ \text{ or } \ q_j = p_{2j} \right\}.$$

We define a Hankel form H_ψ on $\mathbb{D}^d$ by setting $\rho_n = 1$ if n is in I and $\rho_n = 0$ otherwise.

We follow [3, pp. 81–82] and use the Schur test to estimate the norm of H_ψ. It suffices to choose a suitable finite sequence of positive numbers c_j with j ranging over those positive integers that divide some number in I; for such j we choose

$$c_j = 2^{-\Omega(j)/2},$$

where $\Omega(j)$ is the number of prime factors in j. We then get

$$\sum_k \rho_{jk} c_k = 2^{d/2-\Omega(j)} \cdot 2^{-(d/2-\Omega(j))/2} = 2^{d/4} c_j,$$

so that $\|H_\psi\| \le 2^{d/4}$ by the Schur test.

If f is a function in $H^1(\mathbb{D}^d)$ with associated Taylor coefficients a_n, then

$$H_\psi(f) = \sum_n a_n \rho_n.$$

We choose

$$f(z) = \prod_{j=1}^{d/2} (z_{2j-1} + z_{2j}) \tag{1}$$

for which $a_n = \rho_n$ and thus $H_\psi(f) = 2^{d/2}$. On the other hand, an explicit computation shows that

$$\|f\|_1 = (4/\pi)^{d/2}$$

so that H_ψ, viewed as a linear functional on $H^1(\mathbb{D}^d)$, has norm at least $(\pi/2)^{d/2}$. This concludes the proof since it follows that we must have $(\pi/2)^{d/2} \le \|\varphi\|_\infty$ and we know from above that $\|H_\psi\| \le 2^{d/4}$. □

It is worth noting that our application of the Schur test shows that in fact $\|H_\psi\| = 2^{d/4}$ since $\|f\|_2 = 2^{d/4}$. The fact that $|H_\psi(f)| = \|H_\psi\| \|f\|_2$ implies that

$$\|f\|_{1,w} = \|f\|_2.$$

In other words, the trivial factorization $f \cdot 1$ is an optimal weak factorization of the function f defined in (1).

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