TENSOR PRODUCTS OF LEAVITT PATH ALGEBRAS

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Abstract. We compute the Hochschild homology of Leavitt path algebras over a field k. As an application, we show that L_2 and $L_2 \otimes L_2$ have different Hochschild homologies, and so they are not Morita equivalent; in particular they are not isomorphic. Similarly, L_∞ and $L_\infty \otimes L_\infty$ are distinguished by their Hochschild homologies and so they are not Morita equivalent either. By contrast, we show that K-theory cannot distinguish these algebras; we have $K_*(L_2) = K_*(L_2 \otimes L_2) = 0$ and $K_*(L_\infty) = K_*(L_\infty \otimes L_\infty) = K_*(k)$.

1. Introduction

Elliott's theorem [21] stating that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ plays an important role in the proof of the celebrated classification theorem of Kirchberg algebras in the UCT class, due to Kirchberg [14] and Phillips [19]. Recall that a Kirchberg algebra is a purely infinite, simple, nuclear and separable C*-algebra. The Kirchberg-Phillips theorem states that this class of simple C*-algebras is completely classified by its topological K-theory. The analogous question whether the algebras L_2 and $L_2 \otimes L_2$ are isomorphic has remained open for some time. Here L_2 is the Leavitt algebra of type $(1,2)$ over a field k (see [17]), that is, the k-algebra with generators x_1, x_2, x_1^*, x_2^* and relations given by $x_i^* x_j = \delta_{i,j}$ and $\sum_{i=1}^2 x_i x_i^* = 1$.

In this paper we obtain a negative answer to this question. Indeed, we analyze a much larger class of algebras, namely the tensor products of Leavitt path algebras of finite quivers, in terms of their Hochschild homology, and we prove that, for $1 \le n < m \le \infty$, the tensor products $E = \bigotimes_{i=1}^n L(E_i)$ and $F = \bigotimes_{j=1}^m L(F_j)$ of Leavitt path algebras of non-acyclic finite quivers E_i, F_j, are distinguished by their Hochschild homologies (Theorem 5.1). Because Hochschild homology is Morita invariant, we conclude that E and F are not Morita equivalent for $n < m$. Since L_2 is the Leavitt path algebra of the graph with one vertex and two arrows, we obtain that $L_2 \otimes L_2$ and L_2 are not Morita equivalent; in particular they are not isomorphic.

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Recall that, by a theorem of Kirchberg [15], a simple, nuclear and separable C^*-algebra A is purely infinite if and only if $A \otimes \mathcal{O}_\infty \cong A$. We also show that the analogue of Kirchberg's result is not true for Leavitt algebras. We prove in Proposition 5.3 that if E is a non-acyclic quiver, then $L_\infty \otimes L(E)$ and $L(E)$ are not Morita equivalent, and also that $L_\infty \otimes L_\infty$ and L_∞ are not Morita equivalent.

Using the results in [6] we prove that the algebras L_2 and $L_2 \otimes L(F)$, for F an arbitrary finite quiver, have trivial K-theory: all algebraic K-theory groups K_i, $i \in \mathbb{Z}$, vanish on them (this follows from Lemma 6.1 and Proposition 6.2). We also compute $K_*(L(F)) = K_*(L_\infty \otimes L(F))$ and that $K_*(L_\infty) = K_*(L_\infty \otimes L_\infty) = K_*(k)$ is the K-theory of the ground field (see Proposition 6.3 and Corollary 6.4). This implies in particular that, in contrast with the analytic situation, no classification result, in terms solely of K-theory, can be expected for a class of central, simple k-algebras, containing all purely infinite simple unital Leavitt path algebras, and closed under tensor products. It is worth mentioning that an important step towards a K-theoretic classification of purely infinite simple Leavitt path algebras of finite quivers has been achieved in [2].

We refer the reader to [4], [8] and [20] for the basics on Leavitt algebras, Leavitt path algebras and graph C*-algebras, and to [22] for a nice survey on the Kirchberg-Phillips Theorem.

Notations. We fix a field k; all vectorspaces, tensor products and algebras are over k. If R and S are unital k-algebras, then by an (R, S)-bimodule we understand a left module over $R \otimes S^{op}$. By an R-bimodule we shall mean an (R, R) bimodule, that is, a left module over the enveloping algebra $R^e = R \otimes R^{op}$. Hochschild homology of k-algebras is always taken over k; if M is an R-bimodule, we write

$$HH_n(R, M) = \operatorname{Tor}_n^{R^e}(R, M)$$

for the Hochschild homology of R with coefficients in M; we abbreviate $HH_n(R) = HH_n(R, R)$.

2. Hochschild homology

Let k be a field, R a k-algebra and M an R-bimodule. The Hochschild homology $HH_*(R, M)$ of R with coefficients in M was defined in the introduction; it is computed by the *Hochschild complex* $HH(R, M)$ which is given in degree n by

$$HH(R, M)_n = M \otimes R^{\otimes n}.$$

It is equipped with the Hochschild boundary map b defined by

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}.$$

If R and M happen to be $\mathbb{Z}$-graded, then $HH(R,M)$ splits into a direct sum of subcomplexes

$$HH(R,M)=\bigoplus_{m\in\mathbb{Z}} {}_mHH(R,M).$$

The homogeneous component of degree m of $HH(R,M)_n$ is the linear subspace of $HH(R,M)_n$ generated by all elementary tensors $a_0\otimes\cdots\otimes a_n$ with a_i homogeneous and $\sum_i |a_i| = m$. One of the first basic properties of the Hochschild complex is that it commutes with filtering colimits. Thus we have

Lemma 2.1. *Let I be a filtered ordered set and let $\{(R_i, M_i) : i \in I\}$ be a directed system of pairs (R_i, M_i) consisting of an algebra R_i and and an R_i-bimodule M_i, with algebra maps $R_i \to R_j$ and R_i-bimodule maps $M_i \to M_j$ for each $i \le j$. Let $(R, M) = \operatorname{colim}_i(R_i, M_i)$. Then $HH_n(R, M) = \operatorname{colim}_i HH_n(R_i, M_i)$ $(n \ge 0)$.*

Let R_i be a k-algebra and M_i an R_i- bimodule $(i = 1, 2)$. The Künneth formula establishes a natural isomorphism ([23, 9.4.1])

$$HH_n(R_1\otimes R_2, M_1\otimes M_2)\cong\bigoplus_{p=0}^{n} HH_p(R_1,M_1)\otimes HH_{n-p}(R_2,M_2).$$

Another fundamental fact about Hochschild homology which we shall need is Morita invariance. Let R and S be Morita equivalent algebras, and let $P \in R\otimes S^{op}-\ \operatorname{mod}$ and $Q \in S\otimes R^{op}-\ \operatorname{mod}$ implement the Morita equivalence. Then ([23, Thm. 9.5.6])

$$HH_n(R,M)=HH_n(S,Q\otimes_R M\otimes_R P). \tag{2.2}$$

Lemma 2.3. *Let $R_1,\dots,R_n$ and $S_1,\dots,S_m,\dots$ be a finite and an infinite sequence of algebras, and let $R=\bigotimes_{i=1}^n R_i$, $S_{\le m}=\bigotimes_{j=1}^m S_j$ and $S=\bigotimes_{j=1}^\infty S_j$. Assume:*

1. *$HH_q(R_i)\neq 0\neq HH_q(S_j)$ $\quad (q = 0, 1)$, $(1 \le i \le n)$, $(1 \le j)$.*
2. *$HH_p(R_i) = HH_p(S_j) = 0$ for $p \ge 2$, $1 \le i \le n$, $1 \le j$.*
3. *$n \neq m$.*

Then no two of R, $S_{\le m}$ and S are Morita equivalent.

Proof. By the Künneth formula, we have

$$HH_n(R)=\bigotimes_{i=1}^n HH_1(R_i)\neq 0,\qquad HH_p(R)=0\qquad p>n.$$

By the same argument, $HH_p(S_{\le m})$ is nonzero for $p = m$, and zero for $p > m$. Hence if $n \neq m$, R and $S_{\le m}$ do not have the same Hochschild homology and therefore they cannot be Morita equivalent, by (2.2). Similarly, by Lemma 2.1, we have

$$HH_n(S)=\bigoplus_{J\subset\mathbb{N},|J|=n}\Big(\bigotimes_{j\in J}HH_1(S_j)\Big)\otimes\Big(\bigotimes_{j\notin J}HH_0(S_j)\Big),$$

so that $HH_n(S)$ is nonzero for all $n \geq 1$, and thus it cannot be Morita equivalent to either R or $S_{\leq m}$. $\square$

3. Hochschild homology of crossed products

Let R be a unital algebra and G a group acting on R by algebra automorphisms. Form the crossed-product algebra $S = R \rtimes G$, and consider the Hochschild complex $HH(S)$. For each conjugacy class ξ of G, the graded submodule $HH^{\xi}(S) \subset HH(S)$ generated in degree n by the elementary tensors $a_0 \rtimes g_0 \otimes \cdots \otimes a_n \rtimes g_n$ with $g_0 \cdots g_n \in \xi$ is a subcomplex, and we have a direct sum decomposition $HH(S) = \bigoplus_{\xi} HH^{\xi}(S)$. The following theorem of Lorenz describes the complex $HH^{\xi}(S)$ corresponding to the conjugacy class $\xi = \langle g \rangle$ of an element $g \in G$ as hyperhomology over the centralizer subgroup $Z_g \subset G$.

Theorem 3.1. [16]. *Let R be a unital k-algebra, G a group acting on R by automorphisms, $g \in G$ and $Z_g \subset G$ the centralizer subgoup. Let $S = R \rtimes G$ be the crossed product algebra, and $HH^{\langle g \rangle}(S) \subset HH(S)$ the subcomplex described above. Consider the R-submodule $S_g = R \rtimes g \subset S$. Then there is a quasi-isomorphism*

$$HH^{\langle g \rangle}(S) \overset{\sim}{\to} \mathbb{H}(Z_g, HH(R, S_g)).$$

In particular we have a spectral sequence

$$E^2_{p,q} = H_p(Z_g, HH_q(R, S_g)) \Rightarrow HH^{\langle g \rangle}_{p+q}(S).$$

Remark 3.2. Lorenz formulates his result in terms of the spectral sequence alone, but his proof shows that there is a quasi-isomorphism as stated above.

Let A be a not necessarily unital k-algebra, write $\tilde{A}$ for its unitalization. Recall from [24] that A is called *H-unital* if the groups $\mathrm{Tor}^{\tilde{A}}_n(k, A)$ vanish for all $n \geq 0$. Wodzicki proved in [24] that A is H-unital if and only if for every embedding $A \lhd R$ of A as a two-sided ideal of a unital ring R, the map

$$HH(A) \to HH(R : A) = \ker\left(HH(R) \to HH(R/A)\right)$$

is a quasi-isomorphism.

Lemma 3.3. *Theorem 3.1 still holds if the condition that R be unital is replaced by the condition that it be H-unital.*

Proof. Follows from Theorem 3.1 and the fact, proved in [11, Prop. A.6.5], that $R \rtimes G$ is H-unital if R is. $\square$

Let R be a unital algebra, and $\phi : R \to pRp$ a corner isomorphism. As in [7], we consider the skew Laurent polynomial algebra $R[t_+, t_-, \phi]$; this is the R-algebra generated by elements t_+ and t_- subject to the following relations.

$$\begin{aligned} t_+a &= \phi(a)t_+ \\ at_- &= t_-\phi(a) \\ t_-t_+ &= 1 \\ t_+t_- &= p. \end{aligned}$$

Observe that the algebra $S = R[t_+, t_-, \phi]$ is $\mathbb{Z}$-graded by $\deg(r) = 0$, $\deg(t_\pm) = \pm 1$. The homogeneous component of degree n is given by

$$R[t_+, t_-, \phi]_n = \begin{cases} t_-^{-n}R & n < 0 \\ R & n = 0 \\ Rt_+^n & n > 0. \end{cases}$$

Proposition 3.4. *Let R be a unital ring, $\phi\colon R \to pRp$ a corner isomorphism, and $S = R[t_+, t_-, \phi]$. Consider the weight decomposition $HH(S) = \bigoplus_{m\in\mathbb{Z}} {}_mHH(S)$. There is a quasi-isomorphism*

$$ {}_mHH(S) \xrightarrow{\sim} \mathrm{Cone}(1 - \phi\colon HH(R, S_m) \to HH(R, S_m)). \tag{3.5}$$

Proof. If ϕ is an automorphism, then $S = R \rtimes_\phi \mathbb{Z}$, the right hand side of (3.5) computes $\mathbb{H}(\mathbb{Z}, HH(R, S_m))$, and the proposition becomes the particular case $G = \mathbb{Z}$ of Theorem 3.1. In the general case, let A be the colimit of the inductive system

$$R \xrightarrow{\phi} R \xrightarrow{\phi} R \xrightarrow{\phi} \cdots$$

Note that ϕ induces an automorphism $\hat{\phi}\colon A \to A$. Now A is H-unital, since it is a filtering colimit of unital algebras, and thus the assertion of the proposition is true for the pair $(A, \hat{\phi})$, by Lemma 3.3. Hence it suffices to show that for $B = A \rtimes_{\hat{\phi}} \mathbb{Z}$ the maps $HH(S) \to HH(B)$ and $\mathrm{Cone}(1 - \phi\colon HH(R, S_m) \to HH(R, S_m)) \to \mathrm{Cone}(1 - \phi\colon HH(A, B_m) \to HH(A, B_m))$ ($m \in \mathbb{Z}$) are quasi-isomorphisms. The analogous property for K-theory is shown in the course of the third step of the proof of [6, Thm. 3.6]. Since the proof in *loc. cit.* uses only that K-theory commutes with filtering colimits and is matrix invariant on those rings for which it satisfies excision, it applies verbatim to Hochschild homology. This concludes the proof. □

4. Hochschild homology of the Leavitt path algebra

Let $E = (E_0, E_1, r, s)$ be a finite quiver and let $\hat{E} = (E_0, E_1 \sqcup E_1^*, r, s)$ be the double of E, which is the quiver obtained from E by adding an arrow α^* for each arrow $\alpha \in E^1$, going in the opposite direction. The *Leavitt path algebra* of E is the algebra $L(E)$ with one generator for each arrow $\alpha \in \hat{E}_1$ and one generator p_i

for each vertex $i \in E_0$, subject to the following relations

$$p_i p_j = \delta_{i,j} p_i \,, \qquad (i,j \in E_0)$$
$$p_{s(\alpha)}\alpha = \alpha = \alpha p_{r(\alpha)} \,, \qquad (\alpha \in \hat{E}_1)$$
$$\alpha^*\beta = \delta_{\alpha,\beta} p_{r(\alpha)} \,, \qquad (\alpha,\beta \in E^1)$$
$$p_i = \sum_{\alpha \in E^1, s(\alpha)=i} \alpha\alpha^* \,, \qquad (i \in E_0 \setminus \mathrm{Sink}(E)).$$

The algebra $L = L(E)$ is equipped with a a $\mathbb{Z}$-grading. The grading is determined by $|\alpha| = 1$, $|\alpha^*| = -1$, for $\alpha \in E^1$. Let $L_{0,n}$ be the linear span of all the elements of the form $\gamma\nu^*$, where γ and ν are paths with $r(\gamma) = r(\nu)$ and $|\gamma| = |\nu| = n$. By [8, proof of Theorem 5.3], we have $L_0 = \bigcup_{n=0}^{\infty} L_{0,n}$. For each i in E^0, and each $n \in \mathbb{Z}^+$, let us denote by $P(n,i)$ the set of paths γ in E such that $|\gamma| = n$ and $r(\gamma) = i$. The algebra $L_{0,0}$ is isomorphic to $\prod_{i \in E^0} k$. In general the algebra $L_{0,n}$ is isomorphic to

$$\left[\prod_{m=0}^{n-1}\Big(\prod_{i\in \mathrm{Sink}(E)} M_{|P(m,i)|}(k)\Big)\right] \times \left[\prod_{i\in E_0} M_{|P(n,i)|}(k)\right]. \tag{4.1}$$

The transition homomorphism $L_{0,n} \to L_{0,n+1}$ is the identity on the factors

$$\prod_{i\in \mathrm{Sink}(E)} M_{|P(m,i)|}(k),$$

for $0 \le m \le n-1$, and also on the factor

$$\prod_{i\in \mathrm{Sink}(E)} M_{|P(n,i)|}(k)$$

of the last term of the displayed formula. The transition homomorphism

$$\prod_{i\in E_0\setminus \mathrm{Sink}(E)} M_{|P(n,i)|}(k) \longrightarrow \prod_{i\in E_0} M_{|P(n+1,i)|}(k)$$

is a block diagonal map induced by the following identification in $L(E)_0$: A matrix unit in a factor $M_{|P(n,i)|}(k)$, where $i \in E_0 \setminus \mathrm{Sink}(E)$, is a monomial of the form $\gamma\nu^*$, where γ and ν are paths of length n with $r(\gamma) = r(\nu) = i$. Since i is not a sink, we can enlarge the paths γ and ν using the edges that i emits, obtaining paths of length $n+1$, and the last relation in the definition of $L(E)$ gives

$$\gamma\nu^* = \sum_{\{\alpha\in E_1 \mid s(\alpha)=i\}} (\gamma\alpha)(\nu\alpha)^*.$$

Assume E has no sources. For each $i \in E_0$, choose an arrow α_i such that $r(\alpha_i) = i$. Consider the elements

$$t_+ = \sum_{i\in E_0} \alpha_i, \qquad t_- = t_+^*.$$

One checks that $t_-t_+ = 1$. Thus, since $|t_\pm| = \pm 1$, the endomorphism

$$\phi\colon L \longrightarrow L, \qquad \phi(x) = t_+xt_- \tag{4.2}$$

is homogeneous of degree 0 with respect to the $\mathbb{Z}$-grading. In particular it restricts to an endomorphism of L_0. By [7, Lemma 2.4], we have

$$L = L_0[t_+, t_-, \phi]. \tag{4.3}$$

Consider the matrix $N'_E = [n_{i,j}] \in M_{e_0}\mathbb{Z}$ given by

$$n_{i,j} = \#\{\alpha \in E_1 : s(\alpha) = i, \quad r(\alpha) = j\}.$$

Let $e'_0 = |\mathrm{Sink}(E)|$. We assume that E_0 is ordered so that the first e'_0 elements of E_0 correspond to its sinks. Accordingly, the first e'_0 rows of the matrix N'_E are 0. Let N_E be the matrix obtained by deleting these e'_0 rows. The matrix that enters the computation of the Hochschild homology of the Leavitt path algebra is

$$\begin{pmatrix} 0 \\ 1_{e_0-e'_0} \end{pmatrix} - N_E^t\colon \mathbb{Z}^{e_0-e'_0} \longrightarrow \mathbb{Z}^{e_0}.$$

By a slight abuse of notation, we will write $1 - N_E^t$ for this matrix. Note that $1 - N_E^t \in M_{e_0\times(e_0-e'_0)}(\mathbb{Z})$. Of course $N_E = N'_E$ in case E has no sinks.

Theorem 4.4. *Let E be a finite quiver without sources, and let $N = N_E$. For each $i \in E_0 \setminus \mathrm{Sink}(E)$, and $m \geq 1$, let $V_{i,m}$ be the vectorspace generated by all closed paths c of length m with $s(c) = r(c) = i$. Let $\mathbb{Z} =< \sigma >$ act on*

$$V_m = \bigoplus_{i\in E_0\setminus\mathrm{Sink}(E)} V_{i,m}$$

by rotation of closed paths. We have:

$$_mHH_n(L(E)) = \begin{cases} \mathrm{coker}\,(1-\sigma\colon V_{|m|} \to V_{|m|}) & n = 0, m \neq 0 \\ \mathrm{coker}\,(1 - N^t) & n = m = 0 \\ \ker\,(1-\sigma\colon V_{|m|} \to V_{|m|}) & n = 1, m \neq 0 \\ \ker\,(1 - N^t) & n = 1, m = 0 \\ 0 & n \notin \{0, 1\}. \end{cases}$$

Proof. Let $L = L(E)$, $P = P(E) \subset L$ the path algebra of E and $W_m \subset P$ be the subspace generated by all paths of length m. For each fixed $n \geq 1$, and $m \in \mathbb{Z}$, consider the following $L_{0,n}$-bimodule

$$L_{m,n} = \begin{cases} L_{0,n}W_mL_{0,n} & m > 0 \\ L_{0,n}W^*_{-m}L_{0,n} & m < 0. \end{cases}$$

Write $L = L(E)$, and let $_mL$ be the homogeneous part of degree m; we have

$$_mL = \bigcup_{n\geq 1} L_{m,n}.$$

If m is positive, then there is a basis of $L_{m,n}$ consisting of the products $\alpha\theta\beta^*$ where each of α, β and θ is a path in E, $r(\alpha) = s(\theta)$, $r(\beta) = r(\theta)$, $|\alpha| = |\beta| = n$ and $|\theta| = m$. Hence the formula

$$\pi(\alpha\theta\beta) = \begin{cases} \theta & \text{if } \alpha = \beta \\ 0 & \text{else} \end{cases}$$

defines a surjective linear map $L_{m,n} \to V_m$. One checks that π induces an isomorphism

$$HH_0(L_{0,n}, L_{m,n}) \cong V_m \quad (m > 0).$$

Similarly if $m < 0$, then

$$HH_0(L_{0,n}, L_{m,n}) = V_{|m|}^* \cong V_{-m}.$$

Next, by (4.1), we have

$$HH_0(L_{0,n}) = k[E \setminus \mathrm{Sink}(E)] \oplus \bigoplus_{i \in \mathrm{Sink}(E)} k^{r(i,n)}.$$

Here

$$r(i,n) = \max\{r \leq n : P(r,i) \neq \emptyset\}.$$

Now note that, because $L_{0,n}$ is a product of matrix algebras, it is separable, and thus $HH_1(L_{0,n}, M) = 0$ for any bimodule M. As observed in (4.3), for the automorphism (4.2), we have $L = L_0[t_+, t_-, \phi]$. Hence in view of Proposition 3.4 and Lemma 2.1, it only remains to identify the maps $HH_0(L_{0,n}, L_{m,n}) \to HH_0(L_{0,n+1}, L_{m,n+1})$ induced by inclusion and by the homomorphism ϕ. One checks that for $m \neq 0$, these are respectively the cyclic permutation and the identity $V_{|m|} \to V_{|m|}$. The case $m = 0$ is dealt with in the same way as in [6, Proof of Theorem 5.10]. □

Corollary 4.5. *Let E be a finite quiver with at least one non-trivial closed path.*
i) *$HH_n(L(E)) = 0$ for $n \notin \{0, 1\}$.*
ii) *${}_mHH_*(L(E)) \cong {}_{-m}HH_*(L(E))$ $(m \in \mathbb{Z})$.*
iii) *There exist $m > 0$ such that ${}_mHH_0(L(E))$ and ${}_mHH_1(L(E))$ are both nonzero.*

Proof. We first reduce to the case where the graph does not have sources. By the proof of [6, Theorem 6.3], there is a finite complete subgraph F of E such that F has no sources, F contains all the non-trivial closed paths of E, $\mathrm{Sink}(F) = \mathrm{Sink}(E)$, and $L(F)$ is a full corner in $L(E)$ with respect to the homogeneous idempotent $\sum_{v \in F^0} p_v$. It follows that $HH_*(L(E))$ and $HH_*(L(F))$ are graded-isomorphic. Therefore we can assume that E has no sources.

The first two assertions are already part of Theorem 4.4. For the last assertion, let α be a primitive closed path in E, and let $m = |\alpha|$. Let σ be the cyclic permutation; then $\{\sigma^i\alpha : i = 0, \ldots, m-1\}$ is a linearly independent set. Hence $N(\alpha) = \sum_{i=0}^{m-1} \sigma^i\alpha$ is a nonzero element of $V_m^\sigma = {}_mHH_1(L(E))$. Since on the other hand N vanishes on the image of $1 - \sigma \colon V_m \to V_m$, it also follows that the class of α in ${}_mHH_0(L(E))$ is nonzero. □

5. Applications

Theorem 5.1. *Let $E_1, \dots, E_n$ and $F_1, \dots, F_m$ be finite quivers. Assume that $n \neq m$ and that each of the E_i and the F_j has at least one non-trivial closed path. Then the algebras $L(E_1) \otimes \cdots \otimes L(E_n)$ and $L(F_1) \otimes \cdots \otimes L(F_m)$ are not Morita equivalent.*

Proof. Immediate from Lemma 2.3 and Corollary 4.5(iii). □

Example 5.2. It follows from Theorem 5.1 that L_2 and $L_2 \otimes_k L_2$ are not Morita equivalent. There is another way of proving this, due to Jason Bell and George Bergman [9]. By Theorem 3.3 of [3], $\mathrm{l.gl.dim}\, L_2 \leq 1$. Using a module-theoretic construction, Bell and Bergman show that $\mathrm{l.gl.dim}(\mathrm{L_2 \otimes_k L_2}) \geq 2$, which forces L_2 and $L_2 \otimes_k L_2$ to be not Morita equivalent. Bergman then asked Warren Dicks whether general results were known about global dimensions of tensor products and was pointed to Proposition 10(2) of [12], which is an immediate consequence of Theorem XI.3.1 of [10], and says that if k is a field and R and S are k-algebras, then $\mathrm{l.gl.dim}\, R + \mathrm{w.gl.dim}\, S \leq \mathrm{l.gl.dim}(R \otimes_k S)$. Consequently, if $\mathrm{l.gl.dim}\, R < \infty$ and $\mathrm{w.gl.dim}\, S > 0$, then $\mathrm{l.gl.dim}\, R < \mathrm{l.gl.dim}(R \otimes_k S)$; in particular, R and $R \otimes_k S$ are then not Morita equivalent. To see that $\mathrm{w.gl.dim}\, L_2 > 0$, write x_1, x_2, x_1^*, x_2^* for the usual generators of L_2 and use normal-form arguments to show that $\{a \in L_2 \mid ax_1 = a + 1\} = \emptyset$ and $\{b \in L_2 \mid x_1 b = b\} = \{0\}$. Hence, in L_2, $x_1 - 1$ does not have a left inverse and is not a left zerodivisor (or see [4]) ; thus, $\mathrm{w.gl.dim}\, L_2 > 0$.

We denote by L_∞ the unital algebra presented by generators $x_1, x_1^*, x_2, x_2^*, \dots$ and relations $x_i^* x_j = \delta_{i,j} 1$.

Proposition 5.3. *Let E be any finite quiver having at least one non-trivial closed path. Then $L_\infty \otimes L(E)$ and $L(E)$ are not Morita equivalent. Similarly $L_\infty \otimes L_\infty$ and L_∞ are not Morita equivalent.*

Proof. Let C_n be the algebra presented by generators $x_1, x_1^*, \dots, x_n, x_n^*$ and relations $x_i^* x_j = \delta_{i,j} 1$, for $1 \leq i, j \leq n$. Then

$$L_\infty = \varinjlim C_n\,, \tag{5.4}$$

and $C_n \cong L(E_n)$, where E_n is the graph having two vertices v, w and $2n$ arrows $e_1, \dots, e_n, f_1, \dots, f_n$, with $s(e_i) = r(e_i) = v = s(f_i)$ and $r(f_i) = w$ for $1 \leq i \leq n$. (The isomorphism $C_n \to L(E_n)$ is obtained by sending x_i to $e_i + f_i$ and x_i^* to $e_i^* + f_i^*$.) It follows from Theorem 4.4 and (5.4) that the formulas in Theorem 4.4 for ${}_mHH_n(L_\infty)$, $m \neq 0$, hold taking as $V_{i,m}$ the vectorspace generated by all the words in $x_1, x_2, \dots$ of length m, and that ${}_0HH_0(L_\infty) = k$ and ${}_0HH_n(L_\infty) = 0$ for $n \geq 1$. As before, Lemma 2.3 gives the result. □

Theorem 5.5. *Let $E_1, \dots, E_n$ and $F_1, \dots, F_m, \dots$ be a finite and an infinite sequence of quivers. Assume that the number of indices i such that F_i has at least* *one non-trivial closed path is infinite. Then the algebras* $L(E_1)\otimes\cdots\otimes L(E_n)$ *and* $\bigotimes_{i=1}^{\infty} L(F_i)$ *are not Morita equivalent.*

Proof. Immediate from Lemma 2.3 and Corollary 4.5(iii). □

Example 5.6. Let $L^{(\infty)} = \bigotimes_{i=1}^{\infty} L_2$, and let E be any quiver having at least one non-trivial closed path. Then $L^{(\infty)} \otimes L(E)$ and $L(E)$ are not Morita equivalent.

It would be interesting to know the answer to the following question:

Question 5.7. Is there a unital homomorphism $\phi\colon L_2 \otimes L_2 \to L_2$?

Observe that, to build a unital homomorphism $\phi\colon L_2 \otimes L_2 \to L_2$, it is enough to exhibit a *non-zero* homomorphism $\psi\colon L_2 \otimes L_2 \to L_2$, because $eL_2e \cong L_2$ for every non-zero idempotent e in L_2.

6. K-theory

To conclude the paper we note that algebraic K-theory cannot distinguish between L_2 and $L_2\otimes L_2$ or between L_∞ and $L_\infty \otimes L_\infty$. For this we need a lemma, which might be of independent interest. Recall that a unital ring R is said to be *regular supercoherent* in case all the polynomial rings $R[t_1,\dots,t_n]$ are regular coherent in the sense of [13].

Lemma 6.1. *Let E be a finite graph. Then $L(E)$ is regular supercoherent.*

Proof. Let $P(E)$ be the usual path algebra of E. It was observed in the proof of [4, Lemma 7.4] that the algebra $P(E)[t]$ is regular coherent. The same proof gives that all the polynomial algebras $P(E)[t_1,\dots,t_n]$ are regular coherent. This shows that $P(E)$ is regular supercoherent. By [4, Proposition 4.1], the universal localization $P(E)\to L(E)=\Sigma^{-1}P(E)$ is flat on the left. It follows that $L(E)$ is left regular supercoherent (see [6, page 23]). Since $L(E)\otimes k[t_1,\dots,t_n]$ admits an involution, it follows that $L(E)$ is regular supercoherent. □

Proposition 6.2. *Let R be regular supercoherent. Then the algebraic K-theories of L_2 and of $L_2\otimes R$ are both trivial.*

Proof. Let E be the quiver with one vertex and two arrows. Then $L_2\cong L(E)$, and we have

$$L_2\otimes R = L_R(E).$$

Applying [6, Theorem 7.6] we obtain that $K_*(L_R(E)) = K_*(L(E)) = 0$. The result follows. □

We finally obtain a K-absorbing result for Leavitt path algebras of finite graphs, indeed for any regular supercoherent algebra.

Proposition 6.3. *Let R be a regular supercoherent algebra. Then the natural inclusion $R\to R\otimes L_\infty$ induces an isomorphism $K_i(R)\to K_i(R\otimes L_\infty)$ for all $i\in\mathbb{Z}$.*

Proof. Adopting the notation used in the proof of Proposition 5.3, we see that it is enough to show that the natural map $R \to R \otimes L(E_n)$ induces isomorphisms $K_i(R) \to K_i(R \otimes L(E_n))$ for all $i \in \mathbb{Z}$ and all $n \geq 1$. Since R is regular supercoherent the K-theory of $R \otimes L(E_n) \cong L_R(E_n)$ can be computed by using [6, Theorem 7.6]. By the explicit form of the quiver E_n, we thus obtain that

$$K_i(R \otimes L(E_n)) \cong (K_i(R) \oplus K_i(R))/(-n, 1-n)K_i(R).$$

The natural map $R \to L_R(E_n)$ factors as

$$R \to Rv \oplus Rw \to L_R(E_n)\,.$$

The first map induces the diagonal homomorphism $K_i(R) \to K_i(R) \oplus K_i(R)$ sending x to (x, x). The second map induces the natural surjection

$$K_i(R) \oplus K_i(R) \to (K_i(R) \oplus K_i(R))/(-n, 1-n)K_i(R).$$

Therefore the natural homomorphism $R \to L_R(E_n)$ induces an isomorphism

$$K_i(R) \xrightarrow{\sim} K_i(L_R(E_n)).$$

This concludes the proof. □

Corollary 6.4. *The natural maps $k \to L_\infty \to L_\infty \otimes L_\infty$ induce K-theory isomorphisms $K_*(k) = K_*(L_\infty) = K_*(L_\infty \otimes L_\infty)$.*

Proof. A first application of Proposition 6.3 gives $K_*(k) = K_*(L_\infty)$. A second application shows that for E_n as in the proof above, the inclusion $L(E_n) \to L(E_n) \otimes L_\infty$ induces a K-theory isomorphism; passing to the limit, we obtain the corollary. □

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