RECOVERING THE ELLIOTT INVARIANT FROM THE CUNTZ SEMIGROUP

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ABSTRACT. Let A be a simple, separable C*-algebra of stable rank one. We prove that the Cuntz semigroup of $\mathrm{C}(\mathbb{T}, A)$ is determined by its Murray-von Neumann semigroup of projections and a certain semigroup of lower semicontinuous functions (with values in the Cuntz semigroup of A). This result has two consequences. First, specializing to the case that A is simple, finite, separable and $\mathcal{Z}$-stable, this yields a description of the Cuntz semigroup of $\mathrm{C}(\mathbb{T}, A)$ in terms of the Elliott invariant of A. Second, suitably interpreted, it shows that the Elliott functor and the functor defined by the Cuntz semigroup of the tensor product with the algebra of continuous functions on the circle are naturally equivalent.

INTRODUCTION

The Cuntz semigroup $\mathrm{Cu}(A)$ of a C*-algebra A is intimately related to the classification program of simple, separable, and nuclear algebras. This is a semigroup built out of equivalence classes of positive elements in the stabilization of the algebra A much in an analogous way as the projection semigroup $V(A)$ is, and comes equipped with an order that is not algebraic, except for finite dimensional algebras. One order property – almost unperforation – plays a significant role in classification of such algebras up to isomorphism (see [19]). This property is equivalent to strict comparison, which allows to determine the order in the semigroup by means of traces.

The Elliott conjecture predicts the existence of a K-theoretic functor Ell such that, for unital, simple, separable, nuclear C*-algebras A and B in a certain class, isomorphism between $\mathrm{Ell}(A)$ and $\mathrm{Ell}(B)$ can be lifted to a *-isomorphism of the algebras. The concrete form of the invariant (known as the Elliott invariant) for which this conjecture has had tremendous success is the following:

$$\mathrm{Ell}(A) = ((\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]), \mathrm{K}_1(A), \mathrm{T}(A), r)\,,$$

consisting of (ordered) topological K-Theory, the trace simplex, and the pairing between K-Theory and traces given by evaluating a trace at a projection (see, e.g. [7]) (The category where the said invariant sits will be described later.)

It is possible (and generally agreed) that the largest class for which classification in its original form (i.e. using the Elliott invariant as above) may hold consists of those algebras that absorb the Jiang-Su algebra $\mathcal{Z}$ tensorially. Indeed, $\mathcal{Z}$-stability springs into prominence as a necessary condition for classification to hold (under the assumption of weak unperforation on K_0; see [9]). This property of being $\mathcal{Z}$-stable

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stands out as a regularity property for C*-algebras, together with finite decomposition rank and the condition of strict comparison alluded to above. Among separable, simple, nuclear C*-algebras, a conjecture of Toms and Winter (see [21], and also [23]) asserts that these three conditions are equivalent.

The linkage between the Elliott invariant and the Cuntz semigroup has been explored in a number of papers (see, e.g. [3], [6], [14], [18]). One of the main results in [3] recovers the Cuntz semigroup from the Elliott invariant in a functorial manner, for the class of simple, unital, $\mathcal{Z}$-stable algebras. Tikuisis shows, in [18], that the Elliott invariant is equivalent to the invariant $\mathrm{Cu}(\mathrm{C}(\mathbb{T},\cdot))$, for simple, unital, non-type I ASH algebras with slow dimension growth (which happen to be $\mathcal{Z}$-stable, as follows from results of Toms and Winter ([20], [22])). One of our main results in this paper confirms that this equivalence can be extended to all simple, separable, finite $\mathcal{Z}$-stable algebras. Thus, from a functorial point of view and related to the Elliott conjecture, we prove the following:

Theorem. *Let A be a simple, unital, nuclear, finite C*-algebra that absorbs $\mathcal{Z}$ tensorially. Then:*

(i) *There is a functor which recovers the Elliott invariant* $\mathrm{Ell}(A)$ *from the Cuntz semigroup* $\mathrm{Cu}(\mathrm{C}(\mathbb{T},A))$.
(ii) *Viewing the Elliott invariant as a functor from the category of C*-algebras to the category Cu (where the Cuntz semigroup naturally lives), there is a natural equivalence of functors between* $\mathrm{Ell}(\cdot)$ *and* $\mathrm{Cu}(\mathrm{C}(\mathbb{T},\cdot))$.

Since the Cuntz semigroup is a natural carrier of the ideal structure of the algebra, it is plausible to expect that the object $\mathrm{Cu}(\mathrm{C}(\mathbb{T},\cdot))$ may be helpful in the classification of non-simple algebras.

The natural transformation that yields the equivalence of functors in the theorem above is described in Section 4, and is based on describing the Cuntz semigroup of $\mathrm{C}(\mathbb{T},A)$ for any simple, separable, unital C*-algebra of stable rank one. This is carried out in Sections 2 and 3, and is done in terms of the Murray-von Neumann semigroup of projections of $\mathrm{C}(\mathbb{T},A)$ together with a the subsemigroup of the so-called non-compact lower semicontinuous functions with values in $\mathrm{Cu}(A)$. Some of the methods used are similar to the ones in [2].

1. Notation and Preliminaries

We briefly recall the construction of the Cuntz semigroup and the main technical aspects that we shall be using throughout the paper. As a blanket assumption, A will be a separable C*-algebra.

Given positive elements a, b in A, we say that a is *Cuntz subequivalent to* b, in symbols $a \precsim b$, if there is a sequence (x_n) in A such that $x_n b x_n^* \to a$ in norm. The antisymmetrization $\sim$ of the relation $\precsim$ is referred to as *Cuntz equivalence*.

The *Cuntz semigroup* of A is defined as

$$\mathrm{Cu}(A) = (A\otimes\mathcal{K})_+/\sim\ .$$

Denote the class of a positive element a by $[a]$. Then $\mathrm{Cu}(A)$ is ordered by $[a]\leq[b]$ if $a\precsim b$, and it becomes an abelian semigroup with addition given by $[a]+[b]=[\left(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}\right)]$.

As it was proved in [4], there exists a category of ordered semigroups, termed Cu, with an enriched structure, such that the assignment $A \to \mathrm{Cu}(A)$ defines a sequentially continuous functor. We define this category below.

In an ordered semigroup S, we say that x is *compactly contained* in y if, whenever there is an increasing sequence (z_n) with $y \leq \sup z_n$, there is m such that $x \leq z_m$. This is denoted by $x \ll y$ (see [8]). If $x \ll x$, we say that x is *compact*. An increasing sequence (x_n) is termed *rapidly increasing* provided that $x_n \ll x_{n+1}$ for every n.

Define Cu to be the category whose objects are positively ordered (abelian) semigroups for which: (i) every increasing sequence has a supremum; (ii) every element is a the supremum of a rapidly increasing sequence; and (iii) suprema and $\ll$ are compatible with addition. Maps in Cu will be those semigroup maps that preserve addition, order, suprema, and $\ll$. We shall be using repeatedly the fact that, for any positive element a,

$$[a] = \sup_{n\to\infty} [(a - 1/n)_+] \,,$$

as follows from [16, Proposition 2.4] (see also [12]). We shall be frequently using that, if a and b are positive elements and $\|a-b\| < \epsilon$, then there is a contraction c in A such that $(a-\epsilon)_+ = cbc^*$, so in particular $(a-\epsilon)_+ \precsim b$ (see [13, Lemma 2.2] and also [16, Proposition 2.2]).

For a compact space X and a semigroup S in the category Cu, we shall use $\mathrm{Lsc}(X, S)$ to denote the ordered semigroup of all lower semicontinuous functions from $f\colon X \to S$, with pointwise order and operation. (Here, f is lower semicontinuous if, for any $x \in S$, the set $\{t \in X \mid x \ll f(t)\}$ is open in X.)

If A is a C*-algebra and X is a one dimensional compact Hausdorff topological space, then there is a natural map:

$$\begin{array}{rccc} \alpha\colon & \mathrm{Cu}(\mathrm{C}(X,A)) & \longrightarrow & \mathrm{Lsc}(X, \mathrm{Cu}(A)) \\ & x & \longmapsto & \hat{x} \end{array}$$

where, if $x = [f]$, then $\hat{x}(t) = [f(t)]$. It is proved in [2, Theorem 5.15] that $\mathrm{Lsc}(X, \mathrm{Cu}(A))$ equipped with the point-wise order and addition is a semigroup in Cu, and that α is a well defined map in Cu which is an order embedding in case A has stable rank one and $K_1(I) = 0$ for all ideals of A. Furthermore, α is surjective provided it is an order embedding (and thus an order isomorphism).

2. THE CUNTZ SEMIGROUP OF $\mathrm{C}([0,1], A)$ FOR A SIMPLE ALGEBRA A

In this section, we prove that if A is a simple C*-algebra with stable rank one, then the Cuntz semigroup of $\mathrm{C}([0,1], A)$ is order-isomorphic to $\mathrm{Lsc}([0,1], \mathrm{Cu}(A))$, thus obtaining the same result as in [2, Theorem 2.1] for a simple algebra, but without requiring that $\mathrm{K}_1(A) = 0$. The key point in the argument is based on the fact that, for certain continuous fields of C*-algebras, unitaries from fibres can be lifted to unitaries in the algebra.

Lemma 2.1. *Let A be a unital continuous field of C*-algebras over $X = [0,1]$ and let $u, v \in \mathrm{U}(A)$. If $u(t_0) \sim_h v(t_0)$ for some $t_0 \in (0,1)$, then there exists $w \in \mathrm{U}(A)$ such that $w(0) = u(0)$ and $w(1) = v(1)$.*

Proof. Since $u(t_0) \sim_h v(t_0)$, we have $(vu^*)(t_0) \sim_h 1_{A(t_0)}$. Therefore, there exists a unitary $\tilde{w} \in \mathrm{U}_0(A)$, such that $\tilde{w}(t_0) = vu^*(t_0)$. Consider a continuous path w_s of unitaries in $\mathrm{U}_0(A)$ such that $w_0 = 1_A$ and $w_{t_0} = \tilde{w}$. Let us define the following element $w \in \prod_{t\in[0,1]} A(t)$ given by

$$w(s) := \begin{cases} w_s(s)u(s) & \text{if } s \leq t_0 \\ v(s) & \text{otherwise.} \end{cases}$$

Clearly $w(0) = u(0)$ and $w(1) = v(1)$. Since A is a continuous field of C*-algebras, to prove $w \in A$ it is enough to find, for each $t \in [0,1]$ and $\epsilon > 0$, a neighborhood V_t of t and an element $z \in A$ such that $\|w(s) - z(s)\| < \epsilon$ for all $s \in V$.

This is obvious if $t \in (t_0, 1]$. If $t \in [0, t_0)$, and $\epsilon > 0$, there exists a neighborhood V_t such that for all $s, s' \in V_t$, $\|w_s - w_{s'}\| < \epsilon$ (since w_s is a continuous path). Hence, considering the element $z = w_t u \in A$, we have, for all $s \in V_t$,

$$\|w(s) - (w_t u)(s)\| = \|w_s(s)u(s) - w_t(s)u(s)\| \leq \|w_s(s) - w_t(s)\| \cdot \|u(s)\| < \epsilon.$$

Now for $t = t_0$, since $\|(w_{t_0}u)(t_0) - v(t_0)\| = 0$ and by the continuity of the norm in A, there exists a neighborhood V_{t_0} such that $\|(w_{t_0}u)(s) - v(s)\| < \epsilon$ for all $s \in V_{t_0}$, and furthermore we can choose V_{t_0} such that $\|w_s - w_{t_0}\| < \epsilon$. Now, with a similar argument as above, we are done taking $z = w_{t_0}u$. □

Given a C^*-algebra A and a hereditary subalgebra $B \subseteq \mathrm{C}(X, A)$, B becomes a continuous field of C*-algebras over X whose fibres B_x can be identified with hereditary subalgebras of A. If A is simple, then for all $x \in X$ such that $B_x \neq 0$, the inclusion $i_x\colon B_x \to A$ induces an isomorphism

$$(i_x)_*\colon \quad \mathrm{K}_1(B_x) \quad \to \quad \mathrm{K}_1(A).$$

If A has stable rank one, then $\mathrm{K}_1(A) = \mathrm{U}(A^\sim)/\mathrm{U}_0(A^\sim)$ and elements can be identified with connected components of unitaries in $\mathrm{U}(A^\sim)$, which we denote by $[v]_A$. Hence, for all x, B_x will also have stable rank one and $(i_x)_*([v]_{B_x}) = [i_x^\sim(v)]_A = [v]_A$ where $i_x^\sim\colon B_x^\sim \to A^\sim$ denotes tha natural extension to the unitizations.

Let $D_B = B + \mathrm{C}(X) \cdot 1_{\mathrm{C}(X,A^\sim)} \subseteq \mathrm{C}(X, A^\sim)$. Then D_B is a unital continuous field of C*-algebras whose fibres $D_B(x) \cong B_x + \mathbb{C} \cdot 1_{\mathrm{C}(X,A^\sim)}(x)$. Assuming A is stable, we have $1_{A^\sim} = 1_{\mathrm{C}(X,A^\sim)}(x) \notin B_x \subseteq A$, hence $D_B(x) \cong B_x^\sim$. Observe furthermore that the following diagram commutes:

$$\begin{array}{ccc} D_B = B + \mathrm{C}(X) \cdot 1_{\mathrm{C}(X,A^\sim)} & \hookrightarrow & \mathrm{C}(X,A) + \mathrm{C}(X) \cdot 1_{\mathrm{C}(X,A^\sim)} \\ \downarrow{\scriptstyle \pi_x} & & \downarrow{\scriptstyle \pi_x} \\ D_B(x) = B_x + \mathbb{C} \cdot 1_{\mathrm{C}(X,A^\sim)}(x) & \xhookrightarrow{i_x^\sim} & A + \mathbb{C} \cdot 1_{\mathrm{C}(X,A^\sim)}(x) = A^\sim. \end{array}$$

Hence we will assume, $D_B(x) = B_x^\sim \subseteq A^\sim = \mathrm{C}(X, A^\sim)(x)$.

Proposition 2.2. *Let A be a simple C^*-algebra with stable rank one and let X be a finite graph. Suppose B is a hereditary subalgebra of $\mathrm{C}(X, A)$ such that $B_x \neq 0$ for all $x \in X$. Let $(i_x)_*$ denote the induced isomorphisms. Let $x_0, \ldots, x_n \in X$ and $u_i \in \mathrm{U}(B_{x_i}^\sim)$ for $i = 0, \ldots, n$. If $(i_{x_k})_*([u_k]) = (i_{x_l})_*([u_l])$ for all k, l, then there exists $u \in \mathrm{U}(D_B)$ such that $u(x_i) = u_i$ for $i = 0, \ldots, n$.*

Proof. Let us view X as a 1-dimensional simplicial complex where its 0-skeleton is $X_0 = \{x_0, \ldots, x_n, \ldots, x_m\}$ (possibly adding vertices to X_0 as new points x_i). To define a unitary $u \in \mathrm{U}(D_B)$, it is enough to define it first in X_0 and then in each of the edges of the 1-skeleton, provided the values in the boundary match the corresponding values in X_0. Since X_0 is a finite set of points, u can be easily defined pointwise (see below) choosing, for all $i = 1, \ldots, n$, $u(x_i) = u_i$. Therefore, in order to define u for the 1-skeleton we can reduce to the case $X = [0,1]$ and $x_0 = 0, x_1 = 1$ with unitaries u_0, u_1 such that $[u_1] = (i_{x_1})_*^{-1}(i_{x_0})_*[u_0]$.

Let us choose, for the remaining $x \in (0,1)$, unitaries $u_x \in \mathrm{U}(B_x^\sim)$ such that

$$[u_x]_{B_x} = (i_x)_*^{-1}(i_{x_0})_*[u_0]_{B_{x_0}},$$

and hence such that $[i_{x_0}(u_0)]_A = [i_x(u_x)]_A$ which means that u_x and u_0 are connected in $\mathrm{U}(A^\sim)$.

For each $x \in X$ we can find an open neighborhood V_x such that $u_x = v_x(x)$ for some $v_x \in D_B$ and $v_x|_{\overline{V_x}}$ is a unitary. Since X is compact, we can find a finite number of such neighborhoods $V_{x_0} := V_0, \ldots, V_r := V_{x_1}$ covering X. Furthermore by restricting the V_x's to be open intervals we can assume that the resulting cover has multiplicity 1 and denote $V_i \cap V_{i+1} = (a_i, b_i)$ for $i = 1, \ldots, r-1$ ($a_i < b_i < a_{i+1}$). For $i = 1, \ldots, r-1$, let us assume $V_i = V_{y_i}$ for some $y_i \in X$, $y_0 = x_0 = 0$ and $y_r = x_1 = 1$.

For each $i = 0, \ldots, r-1$ choose $z_i \in (a_i, b_i)$. Since $D_B \subseteq \mathrm{C}(X, A^\sim)$, both $v_{y_i}|_{\overline{V_i}}$ and $v_{y_{i+1}}|_{\overline{V_{i+1}}}$ are paths of unitaries in $A^\sim$. Hence in $A^\sim$ we have

$$v_{y_i}(z_i) \sim_h v_{y_i}(y_i) = u_{y_i} \sim_h u_0 \sim_h u_{y_{i+1}} = v_{y_{i+1}}(y_{i+1}) \sim_h v_{y_{i+1}}(z_i).$$

This implies $(i_{z_i})_*[v_{y_i}(z_i)] = (i_{z_i})_*[v_{y_{i+1}}(z_i)]$, but since $(i_{z_i})_*$ is an isomorphism, we obtain $v_{y_{i+1}}(z_i) \sim_h v_{y_i}(z_i)$ in $B(z_i)^\sim$. Now, using Lemma 2.1, we can construct a unitary w_i in $D_B(\overline{V_i \cap V_{i+1}})^\sim = D_B([a_i, b_i])^\sim$ such that $w_i(a_i) = v_{y_i}(a_i)$ and $w_i(b_i) = v_{y_{i+1}}(b_i)$.

Therefore, defining $v \in D_B$ as the following element in $\prod_{x \in X} B_x^\sim$

$$v(x) = \begin{cases} v_{y_i}(x) & \text{if } x \in V_i \setminus (V_{i-1} \cup V_{i+1}) \\ w_i(x) & \text{if } x \in V_i \cap V_{i+1} \end{cases}$$

we obtain an element in D_B, which is furthermore a unitary and $v(0) = u_0$, $v(1) = u_1$. □

Remark 2.3. *Observe that, in the particular case of only one point $x_0 \in X$, the Proposition states that the map $\mathrm{U}(D_B) \to \mathrm{U}(B_{x_0}^\sim)$ is surjective, and thus we can lift unitaries from each fibre.*

Let X be a locally compact, Hausdorff space. Suppose A is a continuous field of C*-algebras over X and $a \in A$. We denote by $\mathrm{supp}(a) = \{x \in X \mid a(x) \neq 0\}$. Observe that, since the assignment $x \to \|a(x)\|$ is continuous, $\mathrm{supp}(a)$ is an open subset of X. If $Y \subseteq X$ is a closed subset of X, and $a \in A$, then $a|_Y$ denotes the image of a by the projection $\pi_Y : A \to A(Y)$.

Lemma 2.4. *Let A be a continuous field of C*-algebras over a space X and let $a, b \in A_+$.*

(1) If $X = \sqcup_{i=1}^r X_i$ is a finite disjoint union of open sets, then $a \precsim b$ if and only if $a|_{X_i} \precsim b|_{X_i}$ for $i = 1, \ldots, r$.

(2) *If* $b|_K \precsim a|_K$ *for some* K *such that* $\mathrm{supp}(b) \subseteq K \subseteq \mathrm{supp}(a)$*, then* $a \precsim b$*.*

Proof. (i) Is clear since $A \cong \oplus_{i=1}^r A(X_i)$. Let us prove (ii). Suppose $b|_K \precsim a|_K$ as in the statement. Given $\epsilon > 0$, we can find $d \in A$ such that $\|b(x) - d(x)a(x)d^*(x)\| < \epsilon$ for all $x \in K$, but since A is a continuous field of C*-algebras, this is valid in an open set $K \subseteq U$,

$$||b(x) - dad^*(x)|| < \epsilon \quad \text{for all } x \in U. \tag{1}$$

Now, since $K \cap U^c = \emptyset$ we can consider a continuous function $\lambda\colon X \to [0,1]$ such that $\lambda|_K = 1$ and $\lambda|_{U^c} = 0$. If $x \in \mathrm{supp}(b) \subseteq K \subseteq U$, then

$$\|b(x) - (\lambda d)a(\lambda d)^*(x))\| = \|b(x) - d(x)a(x)d^*(x)\| < \epsilon,$$

by (1), and if $x \notin U$ then $||b(x) - (\lambda d)a(\lambda d)^*(x)|| = 0$. Finally, if $x \in U \setminus \mathrm{supp}(b)$, then $b(x) = 0$, and

$$||b(x) - (\lambda d)a(\lambda d)^*(x)|| = ||\lambda^2 b(x) - (\lambda d)a(\lambda d)^*(x)|| = |\lambda^2(x)| \cdot \|b(x) - dad^*(x)\| < \epsilon,$$

again by (1). Hence, since $||b - (\lambda d)a(\lambda d)^*|| = \sup_{x\in X} ||b(x) - (\lambda d)a(\lambda d)^*(x)|| < \epsilon$, we obtain $b \precsim a$. □

Recall that if X is a locally compact Hausdorff topological space, then the set $\mathcal{O}(X)$ consisting of open sets ordered by inclusion is a continuous lattice. In the case X is second countable, we have that $U \ll V$ whenever there exists a compact set K such that $U \subseteq K \subseteq V$ (the countability condition is needed since our definition of compact containment is only for increasing sequences, and not arbitrary nets). In fact, $\mathcal{O}(X)$ with union as addition is a semigroup in Cu, which can be described as $\mathrm{Lsc}(X, \{0,\infty\})$ through the assignment $f \to \mathrm{supp}(f)$ (since ∞ is a compact element in $\{0,\infty\}$ and thus $\mathrm{supp}(f) = f^{-1}(\{\infty\})$ is an open set, and, by the same argument, the characteristic function (in $\{0,\infty\}$) of any open set, is lower semicontinuous). The following Lemma illustrates the relation of Cuntz order in a continuous field of C*-algebras over X with the ordered structure of $\mathcal{O}(X)$.

Lemma 2.5. *Let A be a (stable) continuous field of C*-algebras over a compact Hausdorff space X and $a, b \in A_+$ such that $[b] \leq [a]$. Then* $\mathrm{supp}(b) \subseteq \mathrm{supp}(a)$ *and, if* $[b] \ll [a]$ *we have* $\mathrm{supp}(b) \ll \mathrm{supp}(a)$*.*

Proof. The first statement is obvious, let us suppose $[b] \ll [a]$ for some $a, b \in A_+$. Since $\mathcal{O}(X)$ is in Cu, let us write $\mathrm{supp}(a) = \cup_{i\geq 0} U_i$ for some $U_i \ll U_{i+1}$ (hence $U_i \subseteq \overline{U_i} \subseteq U_{i+1}$). We can find, by Urysohn's Lemma, continuous functions $\lambda_n\colon X \to [0,1]$ such that $\lambda_n(\overline{U_n}) = 1$ and $\lambda_n(U_{n+1}^c) = 0$. Since X is compact we obtain $\lambda_n a \to a$ and $\lambda_n a \leq \lambda_{n+1} a$, thus $[a] = \sup_n [\lambda_n a]$. Now since $[b] \ll [a]$ we get $[b] \leq [\lambda_N a]$ for some $N > 0$ and therefore $\mathrm{supp}(b) \subseteq \mathrm{supp}(\lambda_N a) \subseteq U_{N+1} \subseteq \overline{U_{N+1}} \subseteq \mathrm{supp}(a)$. Hence $\mathrm{supp}(b) \ll \mathrm{supp}(a)$. □

Theorem 2.6. *Let A be a C*-algebra which is separable, simple and has stable rank one. Then, the map* $\alpha\colon \mathrm{Cu}(\mathrm{C}([0,1], A)) \to \mathrm{Lsc}([0,1], \mathrm{Cu}(A))$ *is an order isomorphism.*

Proof. We may assume A is stable. Suppose $f, g \in \mathrm{C}([0,1], A)$ are such that $f(t) \precsim g(t)$. It is enough to prove that $(f-\delta)_+ \precsim g$ for all $\delta > 0$, so let us first assume that $[f] \ll [g]$. Hence, by Lemma 2.5 we have $\mathrm{supp}(f) \ll \mathrm{supp}(g)$, and thus there exists a compact set K such that $\mathrm{supp}(f) \subseteq K \subseteq \mathrm{supp}(g)$. Since finite unions of open intervals form a dense subset of $\mathcal{O}([0,1])$, and K is compact, we may further assume that K is a finite union of closed intervals. Now by virtue of Lemma 2.4 (i) and (ii), we may finally assume that $\mathrm{supp}(g) = [0,1]$.

Now the proof follows the lines of [2, Theorem 2.1]. In there, $\mathrm{K}_1(A) = 0$ was assumed in order to lift unitaries from $\mathrm{Her}(g(t))^\sim$ to $D_{\mathrm{Her}(g)} = \mathrm{Her}(g) + \mathrm{C}(X) \cdot 1_{M(\mathrm{C}(X,A))}$. From our argument in the previous paragraph we can reduce to the case where $\mathrm{Her}(g(t))^\sim \neq 0$ for all t and then use Proposition 2.2 with $B := \mathrm{Her}(g)$ (see also Remark 2.3). ☐

3. THE INVARIANT $\mathrm{Cu}_{\mathbb{T}}(A)$

In this section we give a complete description of the Cuntz semigroup of $\mathrm{C}(\mathbb{T}, A)$, for a simple, separable C*-algebra A that has stable rank one. We also show that, in the simple, $\mathcal{Z}$-stable, finite case, the information it contains is equivalent to that of the Elliott invariant (see next section and also [18]).

We start with the following:

Lemma 3.1. *Let A be a simple C*-algebra with stable rank one, and let $y \in A$ be a contraction. Let $\epsilon > 0$ be such that $\epsilon \in \sigma(yy^*)$, and let B be a hereditary subalgebra of A with $yy^* \in B$. If u, v are unitaries in $B^\sim$, then there is $u_0 \in U(B^\sim)$ with $[u_0] = [v]$ in $\mathrm{K}_1(B)$, and*

$$\|uy - u_0 y\| < 5\sqrt{\epsilon}\,.$$

Proof. We know that $\epsilon \in \sigma(yy^*)$, so there exists $0 < c$ with $\|c\| < 2\epsilon$ and such that

$$(yy^* - 2\epsilon)_+ \perp c\,, \text{ and } (yy^* - 2\epsilon)_+ + c \leq yy^*\,.$$

Note that $c \in B$. Write $d = (yy^* - 2\epsilon)_+$. As $c \neq 0$, inclusion induces an isomorphism $\mathrm{K}_1(\overline{cAc}) \cong \mathrm{K}_1(B)$, so there is $w \in U(\overline{cAc}^\sim)$ of the form $1 + a$, where $a \in \overline{cAc}$ such that $[w] \mapsto [v] - [u]$.

Put $u_0 = uw$, a unitary in $B^\sim$. Note that, in $\mathrm{K}_1(B)$, we have $[u_0] = [u] + [w] = [u] + [v] - [u] = [v]$.

Next, choose $czc \in cAc$ such that $\|a - czc\| < \epsilon/2$. Then, $\|w - (1 + czc)\| = \|a - czc\| < \epsilon/2$, so $\|czc\| < \epsilon/2 + 2$. Compute that

$$u(1 + czc)(d + c) = u(d + c + czc^2) = ud + uc + uczc^2\,,$$

whence

$$\|u(1 + czc)(d + c) - u(d + c)\| = \|uczc^2\| < (2 + \epsilon/2)2\epsilon = 4\epsilon + \epsilon^2\,.$$

Therefore

$$\begin{aligned}
\|uyy^* - u_0yy^*\| \leq& \|uyy^* - u(d + c)\| + \|u(d + c) - u_0(d + c)\| + \|u_0(d + c) - u_0yy^*\| \\
<& 4\epsilon + \|u(d + c) - u_0(d + c)\| \\
\leq& 4\epsilon + \|u(d + c) - u(1 + czc)(d + c)\| + \|u(1 + czc)(d + c) - uw(d + c)\| \\
<& 8\epsilon + \epsilon^2 + \|u(1 + czc - w)(d + c)\| < 8\epsilon + \epsilon^2 + \epsilon/2\,.
\end{aligned}$$

Thus

$$\|uy - u_0y\|^2 = \|(u - u_0)yy^*(u - u_0)^*\| < 2(8\epsilon + \epsilon^2 + \epsilon/2) < 19\epsilon\,,$$

so that $\|uy - u_0y\| < \sqrt{19\epsilon} < 5\sqrt{\epsilon}$. □

Proposition 3.2. *Let A be a simple, separable C^*-algebra of stable rank one. Let f and g be elements in $\mathrm{C}(\mathbb{T}, A)$ such that f is not equivalent to a projection, and g is never zero. If $f(t) \precsim g(t)$ for all $t \in \mathbb{T}$, then $f \precsim g$.*

Proof. Since f is not equivalent to a projection, zero is an isolated point of $\sigma(f)$ and this implies, as $\sigma(f) = \overline{\cup_{t\in\mathbb{T}}\sigma(f(t))}$, that for every n, there is $t_n \in \mathbb{T}$ and $\lambda_n \in \sigma(f(t_n))$ with $0 < \lambda_n < 1/2^n$. By compactness, and passing to a subsequence if necessary, we may assume that (t_n) converges to a point t_0. We shall assume that the sequence (t_n) is not eventually constant, since otherwise (that is, $f(t_0)$ itself is not equivalent to a projection), the argument is similar, and easier.

Let $\varphi\colon [0,1] \to \mathbb{T}$ be the map that $\varphi(0) = \varphi(1) = t_0$. Since $f(t) \precsim g(t)$ for all t, this also holds when composing with φ, so $(f \circ \varphi)(s) \precsim (g \circ \varphi)(s)$ for all $s \in [0,1]$.

Let $0 < \epsilon < 1$. There exists $d \in A$ such that $\|(f\circ\varphi)(0) - d^*(g\circ\varphi)(0)d\| < \epsilon$. There is then a neighbourhood U of 0 and 1 such that, with $h(s) = d$, we have

$$\|(f\circ\varphi - h^*(g\circ\varphi)h)|_U\| < \epsilon\,.$$

Write $U = [0, s_0) \cup (s_0', 1]$, with $s_0 < s_0'$. Now, there exists $\epsilon' < \epsilon^2$, $s_1 \in U$ and $\lambda_{\varphi(s_1)} \in \sigma(f(\varphi(s_1)))$ such that $\epsilon' < \lambda_{\varphi(s_1)} < \epsilon^2$, and we may assume (without loss of generality) that $0 < s_1 < s_0$. Choose also $s_0' < s_2 < 1$.

By Theorem 2.6, there exists $c \in \mathrm{C}([0,1], A)$ such that $\|f\circ\varphi - c^*(g\circ\varphi)c\| < \epsilon'/2$. By [13, Lemma 2.2], there is a contraction $e \in \mathrm{C}([0,1], A)$ such that, with $y_1 = (g\circ\varphi)^{1/2}ce$, we have $((f\circ\varphi) - \epsilon'/2)_+ = y_1^*y_1$. If we let $y_2 = (g\circ\varphi)^{1/2}h$, we have

$$\|f\circ\varphi - y_1^*y_1\| \le \epsilon'/2 < \epsilon'\,,\ \|f\circ\varphi - y_2^*y_2\| < \epsilon \text{ and } y_iy_i^* \in \mathrm{Her}(g\circ\varphi) \text{ for } i = 1,2\,.$$

By evaluating at the s_i, for $i = 1, 2$, we get

$$\|(f\circ\varphi)(s_i) - y_1^*y_1(s_i)\| < \epsilon' \text{ and } \|(f\circ\varphi)(s_i) - y_2^*y_2(s_i)\| < \epsilon\,,$$

so we may apply [2, Lemma 1.4] to find unitaries

$$u_1' \in \mathrm{Her}((g\circ\varphi)(s_1))^{\sim} \text{ and } u_2 \in \mathrm{Her}((g\circ\varphi)(s_2))^{\sim}$$

such that

$$\|u_1'y_1(s_1) - y_2(s_1)\| < 9\epsilon \text{ and } \|u_2y_1(s_2) - y_2(s_2)\| < 9\epsilon\,.$$

Let u_1'' be a unitary such that

$$[u_1''] = (i_{s_1})_*^{-1} \circ (i_{s_2})_*([u_2])\,.$$

Since $\lambda_{\varphi(s_1)} \in \sigma((f\circ\varphi)(s_1))$, we have that $0 < \lambda_{\varphi(s_1)} - \epsilon'/2 \in \sigma(((f\circ\varphi)(s_1) - \epsilon'/2)_+) = \sigma(y_1^*y_1(s_1))$, so $\lambda_{\varphi(s_1)} - \epsilon'/2 \in \sigma(y_1y_1^*(s_1))$. By Lemma 3.1, there is a unitary $u_1 \in \mathrm{Her}((g\circ\varphi)(s_1))^{\sim}$ such that

$$[u_1] = [u_1''] \text{ in } \mathrm{K}_1(\mathrm{Her}((g\circ\varphi)(s_1)))$$

and

$$\|u_1'y_1(s_1) - u_1y_1(s_1)\| < 5\sqrt{\lambda_{\varphi(s_1)} - \epsilon'/2} < 5\epsilon\,.$$

Thus

$$\|u_1y_1(s_1) - y_2(s_1)\| \le \|u_1y_1(s_1) - u_1'y_1(s_1)\| + \|u_1'y_1(s_1) - y_2(s_1)\| < 5\epsilon + 9\epsilon = 14\epsilon\,.$$

By Proposition 2.2 there is a unitary $w \in D_{\mathrm{Her}(g)}$ such that $w(\varphi(s_1)) = u_1$ and $w(\varphi(s_2)) = u_2$.

Put $y_1' = (w \circ \varphi)y_1$, and notice that $\|f \circ \varphi - (y_1')^*(y_1')\| = \|f \circ \varphi - y_1^* y_1\| < \epsilon' < \epsilon$, and also that

$$\|y_1'(s_1) - y_2(s_1)\| = \|(w(\varphi(s_1))y_1(s_1) - y_2(s_1)\| = \|u_1 y_1(s_1) - y_2(s_1)\| < 14\epsilon$$

and

$$\|y_1'(s_2) - y_2(s_2)\| = \|w(\varphi(s_2))y_1(s_2) - y_2(s_2)\| = \|u_2 y_1(s_2) - y_2(s_2)\| < 9\epsilon\,.$$

Therefore, there exists a neighbourhood $W \subset U$ of s_1 and s_2, that neither contains 0 nor 1, with

$$\|y_1'(s) - y_2(s)\| < 14\epsilon \text{ for all } s \in W\,.$$

Let $V = [0, s_1) \cup (s_2, 1]$, and let μ_1, μ_2 be a partition of unity associated to the covering $V \cup W$, $V^c \cup W$, and consider the element

$$z = \mu_1 y_1' + \mu_2 y_2\,.$$

Note that $z(0) = y_2(0) = y_2(1) = z(1)$, so $z \in \mathrm{C}(\mathbb{T}, A)$. Also $zz^* \in \mathrm{Her}(g)$.

We need to estimate $\|f - z^*z\|$. It is enough to consider $(f - z^*z)|_W$. Since $(y_1' - z)|_W = (y_1' - \mu_1 y_1' - \mu 2 y_2)|_W = \mu_2(y_1' - y_2)|_W$, we see that $\|(y_1' - z)|_W\| \leq \|(y_1' - y_2)|_W\| < 14\epsilon$. Therefore, a standard argument shows that

$$\|((y_1')^* y_1' - z^*z)|_W\| < 28\epsilon\sqrt{1+\epsilon} < 42\epsilon\,,$$

whence

$$\|(f - z^*z)|_W\| < \epsilon + 42\epsilon = 43\epsilon\,.$$

This implies that $(f - 43\epsilon)_+ \precsim g$, and since $\epsilon > 0$ is arbitrary, it follows that $f \precsim g$ in $\mathrm{C}(\mathbb{T}, A)$, as desired. □

Remark 3.3. In view of the previous result, the reader may wonder whether if an element $f \in \mathrm{C}(X, A)$ is not equivalent to a projection, then there is some point $x \in X$ such that $f(x)$ is itself not equivalent to a projection. We remark this is not true, as is seen by taking, e.g. $X = [0, 1]$, p any non-zero projection in A, $\lambda(t) = (1/2 - t)_+$, and $f = \lambda p$. Then, clearly f is equivalent to a projection pointwise, but not globally.

Proposition 3.4. *Let A be a simple, separable C^*-algebra of stable rank one. Let f and g be elements in $\mathrm{C}(\mathbb{T}, A)$ such that f is not equivalent to a projection. If $f(t) \precsim g(t)$ for all $t \in \mathbb{T}$, then $f \precsim g$.*

Proof. If g is never zero, then the result follows from Proposition 3.2. We may therefore assume that, without loss of generality, $g(1) = 0$ (and then also $f(1) = 0$).

Let $\varphi\colon [0, 1] \to \mathbb{T}$ be the map such that $\varphi(0) = \varphi(1) = 1$. Since $f \circ \varphi(s) \precsim g \circ \varphi(s)$ for every $s \in [0, 1]$, it follows from Theorem 2.6 that $(f \circ \varphi) \precsim (g \circ \varphi)$. Let $\epsilon > 0$. Find $c \in \mathrm{C}([0, 1], A)$ such that

$$\|f \circ \varphi - c(g \circ \varphi)c^*\| < \epsilon/2\,.$$

Since $f(1) = g(1) = 0$, there is a neighbourhood U of 0 and 1 such that $\|(f \circ \varphi)|_U\| < \epsilon$ and $\|(g \circ \varphi)|_U\| < \epsilon/(2\|c\|^2)$. Let $\lambda\colon [0, 1] \to \mathbb{C}$ be a continuous function such that

$0 \leq \lambda \leq 1$, $\lambda|_{U^c} = 1$, and $\lambda(0) = \lambda(1) = 0$, and let $d = \lambda^{1/2}c$, which defines an element in $\mathrm{C}(\mathbb{T}, A)$. Then $(f \circ \varphi - \lambda c(g \circ \varphi)c^*)|_{U^c} = (f \circ \varphi - c(g \circ \varphi)c^*)|_{U^c}$, and

$$\|(f \circ \varphi - \lambda c(g \circ \varphi)c^*)|_U\| \leq \|(f \circ \varphi - c(g \circ \varphi)c^*)|_U\| + \|(1-\lambda)|_U\|\|g\|\|c\|^2 < \epsilon/2 + \epsilon/2 = \epsilon\,,$$

whence $\|f - dgd^*\| < \epsilon$ so $(f-\epsilon)_+ \precsim g$. Since $\epsilon > 0$ is arbitrary, this implies that $f \precsim g$, as was to be shown. □

We are now ready to describe the Cuntz semigroup of $\mathrm{C}(\mathbb{T}, A)$, whenever A is simple and has stable rank one. As A is, in particular, stably finite, this is also the case for $\mathrm{C}(\mathbb{T}, A)$. Thus, upon identification of $\mathrm{V}(\mathrm{C}(\mathbb{T}, A))$ with its image in $\mathrm{Cu}_{\mathbb{T}}(A)$, we have

$$\mathrm{Cu}_{\mathbb{T}}(A) = V(\mathrm{C}(\mathbb{T}, A)) \sqcup \mathrm{Cu}_{\mathbb{T}}(A)_{\mathrm{nc}}\,,$$

where $\mathrm{Cu}_{\mathbb{T}}(A)_{\mathrm{nc}}$ stands for the subsemigroup of non-compact elements.

Observe that $\mathrm{Cu}_{\mathbb{T}}(A) \to \mathrm{Lsc}(\mathbb{T}, \mathrm{Cu}(A))$ sends compact elements to compact elements. Using the arguments in [2, Corollary 3.8], those are the functions that take a constant value in $\mathrm{V}(A)$.

If X is a compact Hausdorff, connected space, and S is a semigroup in the category Cu, let us denote by $\mathrm{Lsc}_{\mathrm{nc}}(X, S)$ the set of non-compact elements in $\mathrm{Lsc}(X, S)$.

Remark 3.5. Observe that, if X is a connected, compact Hausdorff space, A is a (stable) C*-algebra and $f \sim p$, for $f \in \mathrm{C}(X, A)_+$ and p a projection in $\mathrm{C}(X, A)$, then f is pointwise equivalent to a projection $q \in A$. This is easy to verify by a direct argument, but can also be obtained as a consequence of the fact that, for a semigroup S in Cu, the compact elements in $\mathrm{Lsc}(X, S)$ are precisely the constant, compact-valued functions (see, e.g. the arguments in [2, Corollary 3.8]). In particular, such an f is either identically zero or always non-zero.

Lemma 3.6. *Let X be compact Hausdorff and connected, and let S be a semigroup in Cu with cancellation of compact elements and such that the set of non-compact elements is closed under addition. Then $\mathrm{Lsc}_{\mathrm{nc}}(X, S)$ is a subsemigroup of $\mathrm{Lsc}(X, S)$.*

Proof. Let $f, g \in \mathrm{Lsc}_{\mathrm{nc}}(X, S)$ and assume that $f + g$ is compact. The arguments in [2, Corollary 3.8] show that there is a compact element $c \in S$ such that $(f + g)(t) = c$ for every $t \in X$. By our assumptions on S, it follows that $f(t)$ and $g(t)$ are compact for every $t \in X$.

Using that $f(t) \ll f(t)$ and $g(t) \ll g(t)$, and that f and g are lower semicontinuous, find a neighbourhood U_t of t such that $f(t) \ll f(s)$ and $g(t) \ll g(s)$ for every $s \in U_t$ (see, e.g. [2, Lemma 5.1]). It then follows that

$$f(t) + g(s) \ll f(s) + g(s) = c = f(t) + g(t)\,.$$

By cancellation of compact elements, $g(s) \leq g(t) \leq g(s)$ in U_t, so that g is constant in a neighbourhood of t. Since X is connected, it follows that g is constant. Likewise, f is constant. □

When S as above comes as a Cuntz semigroup of a C*-algebra, then it satisfies the additional axiom of having an "almost algebraic order" (see [17, Lemma 7.1 (i)], and also [15]): if $x \leq y$ and $x' \ll x$, then there is $z \in S$ such that $x' + z \leq y \leq x + z$. One can then prove that, if such an S has moreover cancellation of compact elements, then the set S_{nc} of non-compact elements is a subsemigroup of S. Indeed, if $x+y$ is compact, choose $x' \ll x'' \ll x$ such that $x'+y = x''+y = x+y$. By the almost algebraic order axiom, there is $z \in S$ with $x'+z \leq x \leq x''+z$. Adding y to this inequality yields $(x+y)+z \leq x+y$, and since $x+y$ is compact, it follows that $z=0$, and this implies that $x \leq x'' \ll x$.

For a simple, separable C*-algebra with stable rank one, consider the semigroup

$$\mathrm{V}(\mathrm{C}(\mathbb{T},A)) \sqcup \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T},\mathrm{Cu}(A))\,,$$

equipped with addition that extends both the natural operations in both components, and with

$$x+f = \hat{x}+f\,,\ \text{whenever } x \in \mathrm{V}(\mathrm{C}(\mathbb{T},A)) \text{ and } f \in \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T},\mathrm{Cu}(A))\,.$$

We can order this semigroup by taking the algebraic ordering in $\mathrm{V}(\mathrm{C}(\mathbb{T},A))$, the pointwise ordering on $\mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T},\mathrm{Cu}(A))$, and we order mixed terms as follows:

(i) $f \leq x$ if $f(t) \leq \hat{x}(t)$ for every $t \in \mathbb{T}$.
(ii) $x \leq f$ if there is $g \in \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T},\mathrm{Cu}(A))$ such that $\hat{x}+g = f$.

That this ordering is transitive is not entirely trivial, but it follows from the arguments in Theorem 3.7 below.

We may now define an ordered map in the category of semigroups:

$$\begin{array}{rccl} \alpha\colon & \mathrm{Cu}_{\mathbb{T}}(A) & \longrightarrow & \mathrm{V}(C(\mathbb{T},A)) \sqcup \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T},\mathrm{Cu}(A)) \\ & x & \longmapsto & \begin{cases} x & \text{if } x \in \mathrm{V}(\mathrm{C}(\mathbb{T},A)) \\ \hat{x} & \text{otherwise.} \end{cases} \end{array}$$

Theorem 3.7. *If A is a simple C*-algebra with stable rank one, then there is an order-isomorphism*

$$\mathrm{Cu}_{\mathbb{T}}(A) \cong \mathrm{V}(\mathrm{C}(\mathbb{T},A)) \sqcup \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T},\mathrm{Cu}(A))\,.$$

Proof. We will show that the map α just defined is a surjective order-embedding.

First note that $\mathrm{C}(\mathbb{T},A)$ is the following pullback

$$\begin{array}{ccc} \mathrm{C}(\mathbb{T},A) & \xrightarrow{\mathrm{ev}_1} & A \\ \downarrow & & \downarrow \\ \mathrm{C}([0,1],A) & \xrightarrow{\mathrm{ev}_{0,1}} & A \oplus A\,. \end{array}$$

Since, by Theorem 2.6, the natural map $\mathrm{Cu}(\mathrm{C}([0,1],A) \to \mathrm{Lsc}([0,1],\mathrm{Cu}(A))$ is an order-embedding, we may use [2, Theorem 3.3] to conclude that the pullback map

$$\mathrm{Cu}_{\mathbb{T}}(A) \to \mathrm{Cu}(\mathrm{C}([0,1],A)) \oplus_{\mathrm{Cu}(A\oplus A)} \mathrm{Cu}(A)$$

is a surjective map in the category Cu. Upon identifying $\mathrm{Cu}(\mathrm{C}([0,1],A)) \oplus_{\mathrm{Cu}(A\oplus A)} \mathrm{Cu}(A)$ with $\mathrm{Lsc}(\mathbb{T},\mathrm{Cu}(A))$, we obtain that the map

$$\mathrm{Cu}_{\mathbb{T}}(A) \to \mathrm{Lsc}(\mathbb{T},\mathrm{Cu}(A)),\ \text{given by } x \mapsto \hat{x}\,,$$

is also surjective. This implies in particular that the map α is surjective.

To prove that α is an order-embedding, let $x,y \in \mathrm{Cu}_{\mathbb{T}}(A)$ and assume that $\alpha(x) \leq \alpha(y)$. There is nothing to prove if $x,y \in \mathrm{V}(\mathrm{C}(\mathbb{T},A))$.

If $x \notin \mathrm{V}(\mathrm{C}(\mathbb{T},A))$, then write $x=[f]$, $y=[g]$, and our assumption just means that $f(t) \precsim g(t)$ for every $t \in \mathbb{T}$. We may then apply Proposition 3.4 to conclude that $f \precsim g$.

Finally, assume that $x \in \mathrm{V}(\mathrm{C}(\mathbb{T},A))$ and $y \notin \mathrm{V}(\mathrm{C}(\mathbb{T},A))$. Then $\alpha(x) \leq \alpha(y)$ means, by definition, that there is $g \in \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T},\mathrm{Cu}(A))$ with $\hat{x}+g=\hat{y}$. Let $z \in \mathrm{Cu}_{\mathbb{T}}(A)$ be such that $\hat{z}=g$. Then

$$(x+z)\hat{} = \hat{x}+g=\hat{y}\,.$$

Note that $x+z \notin \mathrm{V}(\mathrm{C}(\mathbb{T},A))$, as otherwise $(x+z)\hat{}$ would be a compact element in $\mathrm{Lsc}(\mathbb{T},\mathrm{Cu}(A))$. By Lemma 3.6 (or rather, its proof – see also Remark 3.5), $g=\hat{z}$ would be constant (and compact), a contradiction.

The argument in the previous paragraph then shows that $x+z=y$, as wanted. □

Theorem 3.8. *Let A be a separable, finite $\mathcal{Z}$-stable C*-algebra. Then, there is an order-isomorphism*

$$\mathrm{Cu}_{\mathbb{T}}(A) \cong (\{0\} \sqcup (\mathrm{V}(A)^* \times \mathrm{K}_1(A))) \sqcup \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T},\mathrm{Cu}(A))\,,$$

where $\mathrm{V}(A)^* = \mathrm{V}(A) \setminus \{0\}$.

Proof. By Theorem 3.7, we only need to show that $\mathrm{V}(\mathrm{C}(\mathbb{T},A)) \cong \{0\} \sqcup (\mathrm{V}(A)^* \times \mathrm{K}_1(A))$. This follows once we notice that $\mathrm{C}(\mathbb{T},A))$ has cancellation of projections (see, e.g. [18]). Since A is $\mathcal{Z}$-stable, then $\mathrm{C}(\mathbb{T},A)$ is also $\mathcal{Z}$-stable, whence $\mathrm{C}(\mathbb{T},A)$ has cancellation of full projections by [11, Theorem 1]. We have already observed (see Remark 3.5) that every projection in (matrices over) $\mathrm{C}(\mathbb{T},A)$ is either identically zero or always non-zero, and in that case it is a full projection as A is simple, by an application of [5, Lemma 10.4.2]. □

Remark 3.9. In light of these results, one might expect that the same description of the Cuntz semigroup will hold for more general spaces (of dimension at most 1). However, the following example provided by N. C. Phillips shows that this is not the case.

Let A be a simple C*-algebra with stable rank one, $\mathrm{K}_1(A) \neq 0$ and such that $\mathrm{V}(\mathrm{C}(\mathbb{T},A)) \cong \{0\} \sqcup \mathrm{V}(A)^* \times \mathrm{K}_1(A)$ (for example, A could be $\mathcal{Z}$-stable as above). Let $X = \mathbb{T} \cup [1,2]$, and take f', $g' \in \mathrm{C}(\mathbb{T},A)$ be elements such that $f'(t) \sim g'(t)$ for all $t \in \mathbb{T}$, yet f' and g' are not comparable. For example, we could take a non-zero element $[p] \in \mathrm{V}(A)^*$, a non-trivial class $[u] \in \mathrm{K}_1(A)$, and f' corresponding to $([p],[1])$ and g' corresponding to $([p],[u])$. Define $f,g \in \mathrm{C}(X,A)$ as f',g' over $\mathbb{T}$, and $f(t)=(2-t)f(1)$, $g(t)=g(1)$ for $t \in [1,2]$. Then clearly $f(t) \precsim g(t)$ for all $t \in X$, but $f \not\precsim g$.

4. A CATEGORICAL APPROACH

As already shown in [18], the Elliott invariant and the invariant defined by $\mathrm{Cu}_{\mathbb{T}}(-)$ are equivalent in a functorial way, for simple, unital non-type I ASH algebras with slow dimension growth. Because of Theorem 3.8, this is actually true in the more general setting of separable $\mathcal{Z}$-stable, simple C*-algebras with stable rank one. Our aim in this section is to develop a (somewhat) abstract approach that makes the functorial equivalence explicit, thus also proving the Theorem announced in the Introduction.

Let S be a semigroup in Cu. Assume that the subset S_{nc} of non-compact elements is an absorbing subsemigroup, in the sense that $S_{nc} + S \subseteq S_{\mathrm{nc}}$. Denote by S_{c} the subsemigroup of compact elements and $S_{\mathrm{c}}^* = S_{\mathrm{c}} \setminus \{0\}$. Let G be an abelian group and consider the semigroup

$$S_G = (\{0\} \sqcup (G \times S_{\mathrm{c}}^*)) \sqcup S_{\mathrm{nc}}\,,$$

with natural operations in both components, and $(g, x) + y = x + y$ whenever $x \in S_{\mathrm{c}}^*$, $y \in S_{\mathrm{nc}}$, and $g \in G$. This semigroup can be ordered by

(i) For $x, y \in S_{\mathrm{c}}^*$, and $g, h \in G$, $(g, x) \leq (h, y)$ if and only if $x = y$ and $g = h$, or else $x < y$.
(ii) For $x \in S_{\mathrm{c}}^*$, $y \in S_{\mathrm{nc}}$, $g \in G$, (g, x) is comparable with y if x is comparable with y.

The proof of the following lemma is rather straightforward, hence we omit the details.

Lemma 4.1. *Let S be an object of Cu such that S_{nc} is an absorbing subsemigroup. If G is an abelian group, then S_G is also an object of Cu.*

As in [14], let us write $\mathcal{I}$ to denote the category whose objects are 4-tuples

$$I = ((G_0, G_0^+, u), G_1, X, r)\,,$$

where (G_0, G_0^+, u) is a (countable) simple partially ordered abelian group with order-unit u, G_1 is a (countable) abelian group, X is a (metrizable) Choquet simplex, and $r\colon X \to \mathrm{S}(G_0, u)$ is an affine map, where $\mathrm{S}(G_0, u)$ denotes the state space of (G_0, u).

Maps between objects $((G_0, G_0^+, u), G_1, X, r)$ and $((H_0, H_0^+, v), H_1, Y, s)$ of $\mathcal{I}$ are described as 3-tuples $(\theta_0, \theta_1, \gamma)$, where θ_0 is a morphism of ordered groups with order unit, θ_1 is a morphism of abelian groups, and $\gamma\colon Y \to X$ is an affine and continuous map such that $r \circ \gamma = \theta_0^* \circ s$, where $\theta_0^*\colon \mathrm{S}(H_0, v) \to \mathrm{S}(G_0, u)$ is the naturally induced map.

Let $\mathcal{C}_s$ denote the class of simple, unital, separable and nuclear C*-algebras. Then, the Elliott invariant defines a functor

$$\mathrm{Ell}\colon \mathcal{C}_s \to \mathcal{I}$$

by

$$\mathrm{Ell}(A) = ((\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]), \mathrm{K}_1(A), \mathrm{T}(A), r)\,,$$

where $\mathrm{T}(A)$ is the trace simplex and r is the pairing between K-Theory and traces.

Let us define a functor

$$F\colon \mathcal{I} \to \mathrm{Cu}$$

as follows. If $I = ((G_0, G_0^+, u), G_1, X, r)$ is an object of $\mathcal{I}$, set

$$F(I) = (\{0\} \sqcup (G_1 \times G_0^{++})) \sqcup \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T}, G_0^+ \sqcup \mathrm{LAff}(X)^{++})\,,$$

where $G_0^{++} = G_0^+ \setminus \{0\}$.

Since $G_0^+ \sqcup \mathrm{LAff}(X)^{++}$ is an object of Cu (see, e.g. [1, Lemma 6.3]), it follows from Lemma 4.1 above that $F(I)$ is also an object of Cu. (The addition on $G_0^+ \sqcup \mathrm{LAff}(X)^{++}$ is given by $(g + f)(x) = r(x)(g) + f(x)$, where $g \in G$, $f \in \mathrm{LAff}(X)$ and $x \in X$.)

That F is a functor follows almost by definition. The only non-trivial detail that needs to be checked is that, if

$$(\theta_0, \theta_1, \gamma)\colon ((G_0, G_0^+, u), G_1, X, r) \to ((H_0, H_0^+, v), H_1, Y, s)$$

is a morphism in $\mathcal{I}$ and $f\colon \mathbb{T} \to G_0^+ \sqcup \mathrm{LAff}(X)^{++}$ is non-compact, then $(\theta_0 \sqcup \gamma^*) \circ f\colon \mathbb{T} \to H_0^+ \sqcup \mathrm{LAff}(Y)^{++}$ is also non-compact. Here

$$\theta_0 \sqcup \gamma^*\colon G_0^+ \sqcup \mathrm{LAff}(X)^{++} \to H_0^+ \sqcup \mathrm{LAff}(Y)^{++}$$

is defined as θ_0 on G_0^+ and γ^* on $\mathrm{LAff}(X)^{++}$.

If $(\theta_0 \sqcup \gamma^*) \circ f$ is compact, then there is $h \in H_0^+$ such that $\theta_0(f(\mathbb{T})) = \{h\}$ and $f(\mathbb{T}) \subseteq G_0^+$. As f is non-compact and lower semicontinuous, there are $s, t \in \mathbb{T}$ with $f(t) < f(s)$, whence $f(s) - f(t) \in G_0^{++}$ is an order-unit. Thus, there exists $n \in \mathbb{N}$ with $f(s) \leq n(f(s) - f(t))$. After applying θ_0, we obtain that $h \leq 0$, so that $h = 0$. But this is not possible since, as f is not constant, it takes some non-zero value a, which will be an order-unit with $\theta_0(a) = 0$, contradicting that $\theta_0(u) = v$.

Let us show that $F\colon \mathcal{I} \to F(\mathcal{I})$ is a full, faithful and dense functor, so it yields an equivalence of categories. Therefore, by standard category theory, there exists a functor $G\colon F(\mathcal{I}) \to \mathcal{I}$ such that $F \circ G$ and $G \circ F$ are naturally equivalent to the (respective) identities.

We only need to prove that F is a faithful functor. If

$$(\theta_0, \theta_1, \gamma)\colon ((G_0, G_0^+, u), G_1, X, r) \to ((H_0, H_0^+, v), H_1, Y, s)$$

is a morphism in $\mathcal{I}$, we shall write $F((\theta_0, \theta_1, \gamma)) = (\theta_1 \times \theta_0) \sqcup (\theta_0 \sqcup (\gamma^*)_*)$, where

$$(\theta_0 \sqcup (\gamma^*))_*(f) = (\theta_0 \sqcup (\gamma^*)) \circ f\,,$$

for $f \in \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T}, G_0^+ \sqcup \mathrm{LAff}(X)^{++})$. If now $F((\theta_0, \theta_1, \gamma)) = F((\theta_0', \theta_1', \gamma'))$, we readily see that $\theta_0 \times \theta_1 = \theta_0' \times \theta_1'$, whence $\theta_i = \theta_i'$. It also follows that $\gamma^*(h) = h \circ \gamma = h \circ \gamma' = \gamma'^*(h)$, for every affine continuous function h on X. Since X is homeomorphic to the state space on $\mathrm{Aff}(X)$ (normalized at the constant function 1) via the natural evaluation map $\psi\colon X \to \mathrm{S}(\mathrm{Aff}(X), 1)$ (e.g. [10, Theorem 7.1]), the compositions

$$Y \xrightarrow{\cong} \mathrm{S}(\mathrm{Aff}(Y), 1) \underset{\gamma'^*}{\overset{\gamma^*}{\rightrightarrows}} \mathrm{S}(\mathrm{Aff}(X), 1) \xrightarrow{\cong} X$$

yield that $\gamma = \gamma'$.

Assembling our observations above (together with Theorem 3.8 and [3, Corollary 5.7]), we get the following:

Theorem 4.2. (Cf. [18]) *Upon restriction to the class of unital, simple, separable and finite $\mathcal{Z}$-stable algebras, there are natural equivalences of functors*

$$F \circ \mathrm{Ell} \simeq \mathrm{Cu}_{\mathbb{T}} \;\; \textit{and} \;\; \mathrm{Ell} \simeq G \circ \mathrm{Cu}_{\mathbb{T}}\ .$$

Therefore, for these algebras, Ell *is a classifying functor if, and only if, so is* $\mathrm{Cu}_{\mathbb{T}}$.

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REFERENCES

[1] R. Antoine, J. Bosa, and F. Perera. Completions of monoids with applications to the Cuntz semigroup. *Int. J. Math.*, 22(5):1–25, 2011.

[2] R. Antoine, F. Perera, and L. Santiago. Pullbacks, $\mathrm{C}(X)$-algebras, and their Cuntz semigroup. *J. Funct. Anal.*, 260(10):2844–2880, 2011.

[3] N. P. Brown, F. Perera, and A. S. Toms. The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^*-algebras. *J. Reine Angew. Math.*, 621:191–211, 2008.

[4] K. T. Coward, G. A. Elliott, and C. Ivanescu. The Cuntz semigroup as an invariant for C^*-algebras. *J. Reine Angew. Math.*, 623:161–193, 2008.

[5] J. Dixmier, *C^*-algebras*, volume 15 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[6] G. A. Elliott, L. Robert, and L. Santiago. The cone of lower semicontinuous traces on a C^*-algebra. *Amer. J. Math.*, 133(4):969–1005, 2011.

[7] G. A. Elliott and A. S. Toms. Regularity properties in the classification program for separable amenable C^*-algebras. *Bull. Amer. Math. Soc. (N.S.)*, 45(2):229–245, 2008.

[8] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous lattices and domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2003.

[9] G. Gong, X. Jiang, and H. Su. Obstructions to $\mathcal{Z}$-stability for unital simple C^*-algebras. *Canad. Math. Bull.*, 43(4):418–426, 2000.

[10] K. R. Goodearl. *Partially ordered abelian groups with interpolation*, volume 20 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1986.

[11] X. Jiang. Non-stable K-theory for Z-stable C*-algebras. *arXiv:math/9707228v1 [math.OA]*, pages 1–18, 1997.

[12] E. Kirchberg and M. Rørdam. Non-simple purely infinite C^*-algebras. *Amer. J. Math.*, 122(3):637–666, 2000.

[13] E. Kirchberg and M. Rørdam. Infinite non-simple C^*-algebras: absorbing the Cuntz algebras $\mathcal{O}_\infty$. *Adv. Math.*, 167(2):195–264, 2002.

[14] F. Perera and A. S. Toms. Recasting the Elliott conjecture. *Math. Ann.*, 338(3):669–702, 2007.

[15] L. Robert. The cone of functionals on the Cuntz semigroup. *arXiv:1102.1451v2 [math.OA]*, pages 1–22, 2011.

[16] M. Rørdam. On the structure of simple C^*-algebras tensored with a UHF-algebra. II. *J. Funct. Anal.*, 107(2):255–269, 1992.

[17] M. Rørdam and W. Winter. The Jiang-Su algebra revisited. *J. Reine Angew. Math.*, 642:129–155, 2010.

[18] A. Tikuisis. The Cuntz semigroup of continuous functions into certain simple C^*-algebras. *Int. J. Math.*, 22(7):1–37, 2011.

[19] A. S. Toms. On the classification problem for nuclear C^*-algebras. *Ann. of Math. (2)*, 167(3):1029–1044, 2008.

[20] A. S. Toms. K-theoretic rigidity and slow dimension growth. *Invent. Math.*, 183(2):225–244, 2011.
[21] A. S. Toms. Characterizing classifiable AH algebras. *arXiv:1102.0932v1 [math.OA]*, pages 1–3, 2011.
[22] W. Winter. Nuclear dimension and $\mathcal{Z}$-stability of pure C*-algebras. *to appear in Invent. Math.; arXiv:1006.2731v2 [math.OA]*, pages 1–65, 2011.
[23] W. Winter and J. Zacharias. The nuclear dimension of C^*-algebras. *Adv. Math.*, 224(2):461–498, 2010.

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