A SHARP REMEZ INEQUALITY FOR TRIGONOMETRIC POLYNOMIALS

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Abstract. We obtain a sharp Remez inequality for the trigonometric polynomial T_n of degree n on $[0, 2\pi)$:

$$\|T_n\|_{L_\infty([0,2\pi))} \leqslant \Big(1 + 2\tan^2 \frac{n\beta}{4m}\Big) \|T_n\|_{L_\infty\big([0,2\pi)\setminus B\big)},$$

where $\frac{2\pi}{m}$ is the minimal period of T_n and $|B| = \beta < \frac{2\pi m}{n}$ is a measurable subset of $\mathbb{T}$. In particular, this gives the asymptotics of the sharp constant in the Remez inequality:

$$\mathcal{C}(n, \beta) = 1 + \frac{(n\beta)^2}{8} + O(\beta)^4,$$

where

$$\mathcal{C}(n, \beta) := \sup_{|B|=\beta} \sup_{T_n} \frac{\|T_n\|_{L_\infty([0,2\pi))}}{\|T_n\|_{L_\infty\big([0,2\pi)\setminus B\big)}}. \tag{0.1}$$

We also obtain sharp Nikol'skii's inequalities for the Lorentz spaces and Net spaces. Multidimensional variants of Remez and Nikol'skii's inequalities are investigated.

1. Introduction

Consider the space of (complex) trigonometric polynomials of degree at most $n \in \mathbb{N}$, i.e.,

$$\mathfrak{N}_n = \Big\{T_n : \quad T_n(x) = \sum_{|k|\leqslant n} c_k e^{ikx}, \quad c_k \in \mathbb{C}, \quad x \in \mathbb{T}\Big\}. \tag{1.1}$$

An important question in polynomial approximation is the following one. Let $|B| = \mu(B)$ be the linear Lebesgue measure of B. How large can $\|T_n\|_{L_\infty([0,2\pi))}$ be if

$$\Big|\Big\{x \in [0, 2\pi) : |T_n(x)| > 1\Big\}\Big| \leqslant \varepsilon$$

holds for some $0 < \varepsilon \leqslant 1$? The well-known Remez inequality answers this question.

For any Lebesgue measurable set $B \subset \mathbb{T}$ such that $|B| \equiv \beta < \pi/2$ we have

$$\|T_n\|_{L_\infty([0,2\pi))} \leqslant C(n, |B|)\|T_n\|_{L_\infty([0,2\pi)\setminus B)}, \quad T_n \in \mathfrak{N}_n. \tag{1.2}$$

We will write $\mathcal{C}(n, |B|)$ in place of $C(n, |B|)$ while dealing with the sharp constant in this estimate, see (0.1).

Investigation of Remez's inequalities is a well developed topic. For algebraic polynomials sharp inequality was proved by Remez [Re]. In [BE] and [Er1] (1.2) was proved with $C(n, |B|) = \exp(4n|B|)$; the history of the question can be found in e.g. [Er3, Sec. 2], [Ga, Sec. 3], and [LGM]. In [Ga, Th. 3.1], the constant was sharpened as $C(n, |B|) = \exp(2n|B|)$. Also, another improvement was obtained in [Er2, Th. 3.4]: $C(n, |B|) = \exp\big(n(|B| + 1.75|B|^2)\big)$.

In case of $B \equiv [a, b] \subset [-\pi, \pi]$ a sharp Remez inequality is given by (see [Er1])

$$\|T_n\|_{L_\infty([0,2\pi))} \leqslant \frac{1}{2}\Big(\tan^{2n}\Big(\frac{2\pi-\beta}{8}\Big) + \cot^{2n}\Big(\frac{2\pi-\beta}{8}\Big)\Big)\|T_n\|_{L_\infty([0,2\pi)\setminus B)}, \quad |B| = \beta, \quad T_n \in \mathfrak{N}_n \tag{1.3}$$

and equality in (1.3) holds if and only if

$$T_n(x) = C\,\mathfrak{T}_n\Big(\frac{\cos(x - (a + b)/2) - \cos^2((b - a)/4)}{\sin^2((b - a)/4)}\Big), \quad C \in \mathbb{R},$$

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where $\mathfrak{T}_n(x)$ is the Chebyshev polynomial of degree n, i.e.,

$$\mathfrak{T}_n(x) = \frac{1}{2}\Big(\big(x+\sqrt{x^2-1}\big)^n + \big(x-\sqrt{x^2-1}\big)^n\Big)$$

for every $x \in \mathbb{R} \setminus (-1,1)$.

Therefore, the best known bounds of the sharp constant in the Remez inequality are given by

$$\frac{1}{2}\Big(\tan^{2n}\Big(\frac{2\pi-\beta}{8}\Big) + \cot^{2n}\Big(\frac{2\pi-\beta}{8}\Big)\Big) \leqslant \mathcal{C}(n,\beta) \leqslant \exp\big(n\beta\min\{2,(1+1.75\beta)\}\big), \tag{1.4}$$

for $0<\beta\leqslant\pi/2$. Moreover, Erdélyi [Er2] and Ganzburg [Ga] conjectured that

$$\mathcal{C}(n,\beta) = \frac{1}{2}\Big(\tan^{2n}\Big(\frac{2\pi-\beta}{8}\Big) + \cot^{2n}\Big(\frac{2\pi-\beta}{8}\Big)\Big) \tag{1.5}$$

for any $\beta>0$.

In case when the measure $|B|=\beta$ is big, that is, when $\pi/2<\beta<2\pi$, it is known that

$$\mathcal{C}(n,\beta) \leqslant C(n,\beta) = \Big(\frac{17}{2\pi-\beta}\Big)^{2n}$$

(see [Er1, Er2, Ga, Na]). Another upper bound of $\mathcal{C}(n,\beta)$, $\beta>0$, was given by Andrievskii [An].

In this paper we investigate Remez's inequalities for the case of $0<\beta<\frac{2\pi}{n}$. Note that in applications in fact we are usually interested in the case of small β (see, e.g., [MT]). Precisely, we prove

$$\|T_n\|_{L_\infty([0,2\pi))} \leqslant \Big(1+2\tan^2\frac{n\beta}{4m}\Big)\|T_n\|_{L_\infty([0,2\pi)\setminus B)}, \qquad 0<\beta<\frac{2\pi m}{n}, \tag{1.6}$$

where $\frac{2\pi}{m}$ is the minimal period of $T_n \in \mathfrak{N}_n$. Equality in (1.6) holds for $T_n(x)=\cos nx+\frac{1}{2}\Big(1-\cos\frac{\beta}{2}\Big)$. This improves the right-hand side bound of the sharp constant in the Remes inequality:

$$\mathcal{C}(n,\beta) \leqslant 1+2\tan^2\frac{n\beta}{4}, \quad 0<\beta<\frac{\pi}{n}. \tag{1.7}$$

Combining the left-hand side of (1.4) and (1.7) we show (see Corollary 3.5) that

$$\mathcal{C}(n,\beta) = 1+\frac{(n\beta)^2}{8}+O(\beta^4). \tag{1.8}$$

This answers Stechkin-Ulyanov's question ([SU]) on the rate of decrease of $\mathcal{C}(n,\beta)$. Moreover,

$$\mathcal{C}(1,\beta) = 1+2\tan^2\frac{\beta}{4},$$

which proves conjecture (1.5) for $n=1$.

The paper is organized as follows. In section 2, we prove several basic auxiliary results for rearrangements. We recall that the distribution of a measurable function f on $\mathbb{T}$ is defined by $\lambda(\sigma,f)=\mu\{x\in[0,2\pi]:|f(x)|>\sigma\}$. Then $f^*(t)=\inf\{\sigma:\lambda(\sigma,f)\leqslant t\}$ is the decreasing rearrangement of f. We will also consider the decreasing rearrangement of functions on their minimal periods denoting it by $f^\star$.

In section 3, we prove sharp Remez and Remez-Bernstein's inequalities in terms of rearrangements and averages which are defined as follows. Any fixed family M of measurable subsets of $\mathbb{T}$ is called a *net* in $\mathbb{T}$.

For a 2π-periodic locally integrable function f we define the function

$$\overline{f(t,M)} = \sup_{\substack{e\in M\\ |e|\geqslant t}} \frac{1}{|e|}\left|\int_e f(x)dx\right|, \quad t\geqslant 0,$$

which is called *the average function of* f on the net M ([NT]). If $\sup_{e\in M}|e|=\alpha$ and $t>\alpha$, then $\overline{f(t,M)}=0$.

In section 4, we obtain sharp Bernstein and Bernstein-Nikol'skii's inequalities in the Lorentz spaces, in particular Lebesgue spaces, and in the Net spaces. As usual, a measurable function f defined on $\mathbb{T}$ belongs to the Lorentz space $L_{p,q}=L_{p,q}(\mathbb{T})$ (see e.g. [BS]) if

$$\|f\|_{L_{p,q}} = \left(\int_0^{2\pi}\Big(t^{1/p}f^*(t)\Big)^q\frac{dt}{t}\right)^{1/q} < \infty, \quad 0<p,q<\infty,$$

and

$$\|f\|_{L_{p,\infty}} = \sup_{t\in(0,2\pi)} t^{1/p}f^*(t) < \infty, \quad 0<p\leqslant\infty.$$

The scale of net spaces is more general than the scale of Lorentz spaces. To define them, let $0 < p, q \leqslant \infty$, M be a net in $\mathbb{T}$, and a 2π-periodic function f be locally integrable. We say that f belongs to the Net space $N_{p,q}(M) = N_{p,q}(M, \mathbb{T})$ ([NA, NT]) if

$$\|f\|_{N_{p,q}(M)} = \left(\int_0^{2\pi} (t^{1/p}\overline{f(t,M)})^q \frac{dt}{t}\right)^{1/q} < \infty, \quad 0 < p, q < \infty,$$

and

$$\|f\|_{N_{p,\infty}(M)} = \sup_{t\in(0,2\pi)} t^{1/p}\overline{f(t,M)} < \infty, \quad 0 < p \leqslant \infty.$$

The net spaces $N_{p,q}(M)$ are a natural generalization of the Lorentz spaces $L_{p,q}$, since if M is a collection of all compact sets in $\mathbb{T}$, then $N_{p,q}(M) = L_{p,q}(\mathbb{T})$ for $1 < p < \infty, 0 < q \leqslant \infty$.

Finally, multidimensional variants of Remez and Nikol'skii's inequalities are obtained in Section 5. In particular, for trigonometric polynomials

$$T_{\mathbf{n}}(x) = \sum_{|k_1|\leqslant n_1} \cdots \sum_{|k_d|\leqslant n_d} c_{\mathbf{k}} e^{i(\mathbf{k},x)}, \quad c_{\mathbf{k}} \in \mathbb{C}, \quad x \in \mathbb{T}^d, \quad d \geqslant 1,$$

we obtain the following Remez inequality

$$\|T_{\mathbf{n}}\|_{L_\infty(\mathbb{T}^d)} \leqslant \frac{1}{\cos \sqrt[d]{|B| \prod_{j=1}^d \frac{jn_j}{2}}} \|T_{\mathbf{n}}\|_{L_\infty(\mathbb{T}^d\setminus B)}, \quad 0 \leqslant |B| < \frac{\pi^d}{\prod_{j=1}^d jn_j},$$

cf. [DP] and [Kr].

2. Basic lemmas on rearrangements

Let $T_n \in \mathfrak{N}_n$ and $d = d(T_n)$ be the minimal period of T_n. Note that $\frac{2\pi}{n} \leqslant d \leqslant 2\pi$. Let $T_n^\star(t)$ be the decreasing rearrangement of T_n on its minimal period, i.e.,

$$\lambda_{\min}(\sigma, T_n) = \left|\left\{x \in [0, d) : |T_n(x)| > \sigma\right\}\right|$$

and

$$T_n^\star(t) = \inf\left\{\sigma : \lambda_{\min}(\sigma, T_n) \leqslant t\right\}.$$

Remark that this concept describes the order of polynomials more accurately than T_n^*. For example,

$$\big(\cos nx\big)^*(\beta) = \big(\cos x\big)^*(\beta) = \cos\frac{\beta}{4}, \quad 0 \leqslant \beta < 2\pi$$

but

$$\big(\cos nx\big)^\star(\beta) = \cos\frac{n\beta}{4}, \quad 0 \leqslant \beta < \frac{2\pi}{n}. \tag{2.1}$$

Moreover, the following result holds.

Lemma 2.1. *Let $n \in \mathbb{N}$ and $f \in \mathfrak{N}_n$ be such that $|c_n(f)| + |c_{-n}(f)| > 0$. If d is the minimal period of f, then*

- (a) *we have $f^\star(\beta) \leqslant f^*(\beta)$ for $0 < \beta \leqslant d$ and $f^\star(\beta) < f^*(\beta)$ for $d < 2\pi$;*
- (b) *there exists $m \in \mathbb{N}$ such that $d = 2\pi/m$;*
- (c) *we have $n/m \in \mathbb{N}$ and $g(x) = f(x/m)$ is a trigonometric polynomial of degree n/m;*
- (d) *$f^*(\beta) = g^*(\beta)$, $\beta \in [0, 2\pi)$;*
- (e) *$f^\star(\beta) = f^*(m\beta)$ for $0 \leqslant \beta \leqslant d$.*

Proof. Item (a) follows from $\lambda_{\min}(\sigma, f) \leqslant \lambda(\sigma, f)$ and $\lambda_{\min}(\sigma, f) < \lambda(\sigma, f)$ for $d < 2\pi$.

(b). If $d \neq 2\pi/m$, $m \in \mathbb{N}$, then we take an integer r such that $rd \leqslant 2\pi < (r + 1)d$. Suppose that $\alpha = 2\pi - rd$; then $f(x) = f(x + rd)$, $x \in \mathbb{R}$. Using 2π-periodicity, we get $f(x) = f(x + 2\pi) = f(x + \alpha + rd)$, $x \in \mathbb{R}$. Then $f(x) = f(x + \alpha)$, which contradicts $\alpha < d$.

(c) Let $d=2\pi/m$ be the minimal period of f. Putting $g(x)=f(x/m)$, $x\in[0,2\pi)$, we have

$$c_r(f)=\int_0^{2\pi} f(x)e^{-irx}\,dx=\sum_{k=0}^{m-1}\int_{2\pi k/m}^{2\pi(k+1)/m} f(x)e^{-irx}\,dx=\sum_{k=0}^{m-1}\int_0^{2\pi/m} f(x)e^{-ir(x+2\pi k/m)}\,dx$$

$$=\int_0^{2\pi/m} f(x)e^{-irx}dx\left(\sum_{k=0}^{m-1}e^{-2\pi ikr/m)}\right). \tag{2.2}$$

Taking into account $\sum\limits_{k=0}^{m-1}e^{-2\pi ikr/m}=\begin{cases} m, & \text{for } r=ms,\ s\in\mathbb{N},\\ 0, & \text{otherwise}\end{cases}$, we get $c_r(f)=0$ for $r\neq ms$. Since $|c_n(f)|+|c_{-n}(f)|>0$, then $n/m\in\mathbb{N}$. Therefore, using (2.2) we get

$$c_{ms}(f)=m\int_0^{2\pi/m} f(x)e^{-imsx}\,dx=\int_0^{2\pi} f(x/m)e^{-isx}\,dx=c_s(g),$$

and g is the trigonometric polynomial of degree n/m.

(d) Making use of $2\pi/m$-periodicity, we get

$$\lambda(\alpha,f)=\left|\left\{x\in[0,2\pi]:\ \ |f(x)|>\alpha\right\}\right|=m\left|\left\{x\in[0,2\pi/m]:\ \ |f(x)|>\alpha\right\}\right|$$
$$=\left|\left\{x\in[0,2\pi]:\ \ |f(x/m)|>\alpha\right\}\right|=\left|\left\{x\in[0,2\pi]:\ \ |g(x)|>\alpha\right\}\right|=\lambda(\alpha,g).$$

Then $g^*(\beta)=f^*(\beta)$ for any $\beta\in[0,2\pi]$.

(e) We proved that $d=\frac{2\pi}{m}$ and $\lambda_{\min}(\sigma,f)=\frac{1}{m}\lambda(\sigma,f)$. Hence, for $0<\beta<d=\frac{2\pi}{m}$,

$$f^\star(\beta)=\inf\{\sigma:\lambda_{\min}(\sigma,f)\leqslant\beta\}=\inf\{\sigma:\lambda(\sigma,f)\leqslant m\beta\}=f^*(m\beta).$$ □

We denote by $\mathfrak{M}_n$ the set of real trigonometric polynomials, i.e.,

$$\mathfrak{M}_n=\Big\{T_n\in\mathfrak{N}_n:\quad T_n(x)\in\mathbb{R},\quad x\in\mathbb{T}\Big\}. \tag{2.3}$$

Let us prove two auxiliary results on rearrangements. First, we recall the following known lemma. The proof can be found in the papers [Be] by S. Bernstein and [St] by S. Stechkin.

Lemma 2.2. *For $T_n\in\mathfrak{M}_n$ such that $|T_n(x)|\leqslant 1$ and $T_n(0)=1$, we have*

$$T_n(x)\geqslant\cos nx,\qquad -\frac{\pi}{n}\leqslant x\leqslant\frac{\pi}{n}. \tag{2.4}$$

Lemma 2.3. *Let $T_n\in\mathfrak{M}_n$ be such that*

$$\max T_n(x)-\min T_n(x)=2.$$

Then for $0\leqslant\beta<\frac{2\pi}{n}$,

$$T_n^\star(\beta)\geqslant(\cos nx+\alpha)^\star(\beta), \tag{2.5}$$

where $\alpha=\max T_n(x)-1$.

Proof. Denote

$$\Xi:=\max T_n(x)\qquad\text{and}\qquad \xi:=\min T_n(x).$$

Let d be the minimal period of T_n. Then there exist $x_1,x_2\in[0,d)$ such that $T_n(x_1)=\Xi$, $T_n(x_2)=\xi$ and the polynomial

$$\widetilde{T}_n(x)=T_n(x_1-x)-\Xi+1$$

satisfies all conditions of Lemma 2.2. Hence, for $0\leqslant|x|\leqslant\frac{\pi}{n}$, it follows that

$$\widetilde{T}_n(x)\geqslant\cos nx. \tag{2.6}$$

Moreover, $|x_1-x_2|\geqslant\frac{\pi}{n}$ since if $|x_1-x_2|<\frac{\pi}{n}$, then

$$\widetilde{T}_n(x_1-x_2)=T_n(x_2)-\Xi+1=-1<\cos nx,$$

which contradicts (2.6). Inequality (2.6) can be written as follows

$$T_n(x_1-x)\geqslant\cos nx+\alpha, \tag{2.7}$$

where $\alpha=\Xi-1=\xi+1$.

Similarly, Lemma 2.2 gives

$$T_n(x_2 - x) \leqslant \cos n\Big(\frac{\pi}{n} - x\Big) + \alpha = -\cos nx + \alpha, \quad 0 \leqslant |x| \leqslant \frac{\pi}{n} \tag{2.8}$$

(here it is enough to consider $-T_n$).

Assume first that $\alpha \in [-1, 1]$ and $0 \leqslant |x| \leqslant \frac{\arccos(-\alpha)}{n}$. Then using (2.7) we get

$$T_n(x_1 - x) \geqslant \cos nx + \alpha \geqslant 0. \tag{2.9}$$

Also using (2.8) for $\frac{\arccos(-\alpha)}{n} \leqslant x \leqslant \frac{2\pi}{n} - \frac{\arccos(-\alpha)}{n}$, we have

$$T_n\Big(x_2 - \Big(x - \frac{\pi}{n}\Big)\Big) \leqslant \cos nx + \alpha \leqslant 0$$

and therefore

$$\Big|T_n\Big(x_2 - \Big(x - \frac{\pi}{n}\Big)\Big)\Big| \geqslant |\cos nx + \alpha|. \tag{2.10}$$

Further, let

$$A_1 := \Big(x_1 - \frac{\arccos(-\alpha)}{n}, x_1 + \frac{\arccos(-\alpha)}{n}\Big),\ A_2 := \Big(x_2 - \Big(\frac{\pi}{n} - \frac{\arccos(-\alpha)}{n}\Big), x_2 + \Big(\frac{\pi}{n} - \frac{\arccos(-\alpha)}{n}\Big)\Big).$$

Since $|x_1 - x_2| \geqslant \frac{\pi}{n}$, then $A_1 \cap A_2 = \emptyset$. Moreover, $|A_1| + |A_2| = 2\pi/n \leqslant d$ and for $\gamma > 0$

$$\begin{aligned}\Big|\Big\{x \in [0, d) : |T_n(x)| > \gamma\Big\}\Big| &\geqslant \Big|\Big\{x \in A_1 \cup A_2 : |T_n(x)| > \gamma\Big\}\Big| \\ &= \Big|\Big\{x \in A_1 : |T_n(x)| > \gamma\Big\}\Big| + \Big|\Big\{x \in A_2 : |T_n(x)| > \gamma\Big\}\Big|.\end{aligned}$$

Then, making use of (2.9) and (2.10) and translation invariance of the distribution function, we have for $\gamma > 0$

$$\begin{aligned}|\{x \in [0, d) : |T_n(x)| > \gamma\}| &\geqslant \Big|\Big\{x \in \Big(-\frac{\arccos(-\alpha)}{n}, \frac{\arccos(-\alpha)}{n}\Big) : |\cos nx + \alpha| > \gamma\Big\}\Big| \\ &\quad + \Big|\Big\{x \in \Big(\frac{\arccos(-\alpha)}{n}, \frac{2\pi}{n} - \frac{\arccos(-\alpha)}{n}\Big) : |\cos nx + \alpha| > \gamma\Big\}\Big| \\ &= \Big|\Big\{x \in \Big(0, \frac{2\pi}{n}\Big) : |\cos nx + \alpha| > \gamma\Big\}\Big|.\end{aligned}$$

Therefore, for $\alpha \in [-1, 1]$ we get the required inequality (2.5) for $0 \leqslant \beta < \frac{2\pi}{n}$.

Let now $\alpha > 1$ and therefore $\xi > 0$. Applying (2.5) to $T_n - \xi$, it follows that

$$(T_n - \xi)^\star(\beta) \geqslant (\cos nx + 1)^\star(\beta), \quad 0 \leqslant \beta < \frac{2\pi}{n}.$$

Taking into account that

$$T_n^\star(\beta) = (T_n - \xi)^\star(\beta) + \xi$$

and

$$(\cos nx + \alpha)^\star(\beta) = (\cos nx + \xi + 1)^\star(\beta) = (\cos nx + 1)^\star(\beta) + \xi,$$

we get (2.5) for $\alpha > 1$. For $\alpha < -1$ it is enough to consider $-T_n$. □

Lemma 2.4. *Let $n \in \mathbb{N}$ and $0 \leqslant \beta < \frac{2\pi}{n}$. Then*

(a)

$$\sup_{0 \leqslant \alpha \leqslant 1} \frac{1 + \alpha}{(\cos nx + \alpha)^\star(\beta)} \leqslant \frac{1 + \sin^2 \frac{n\beta}{4}}{1 - \sin^2 \frac{n\beta}{4}};$$

(b)

$$\inf_{0 \leqslant \alpha < \infty} (\cos nx + \alpha)^\star(\beta) \geqslant \cos^2 \frac{n\beta}{4};$$

(c)

$$\Big(\cos nx + \frac{1 - \cos\frac{n\beta}{2}}{2}\Big)^\star(\beta) = \cos^2 \frac{n\beta}{4}.$$

Proof. Let $\cos\frac{n\beta}{2}+\alpha \geqslant 1-\alpha$, i.e., $\alpha \geqslant \frac{(1-\cos\frac{n\beta}{2})}{2}$ (see Figure 1 showing the graph of the function $f(x) = |\cos nx + \alpha|$).

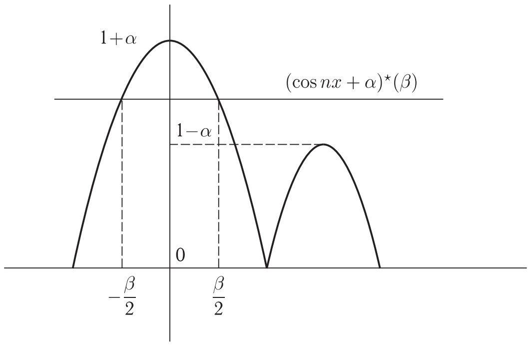

FIGURE 1.

Then

$$(\cos nx+\alpha)^{\star}(\beta)=\cos\frac{n\beta}{2}+\alpha. \tag{2.11}$$

Note that the function

$$\varphi(\alpha)=\frac{1+\alpha}{\cos\frac{n\beta}{2}+\alpha}$$

is monotone with respect to α and therefore

$$\sup_{0\leqslant\alpha\leqslant1}\frac{1+\alpha}{(\cos nx+\alpha)^{\star}(\beta)}=\sup_{0\leqslant\alpha\leqslant(1-\cos\frac{n\beta}{2})/2}\frac{1+\alpha}{(\cos nx+\alpha)^{\star}(\beta)}. \tag{2.12}$$

Next let $0\leqslant\alpha\leqslant\frac{1-\cos\frac{n\beta}{2}}{2}$, then

$$(\cos nx+\alpha)^{\star}(\beta)=\frac{\cos\frac{nu}{2}+\cos\frac{nv}{2}}{2},$$

where $0\leqslant u\leqslant v\leqslant\beta$ satisfy the following equations

$$\begin{cases} u+v=\beta \\ \cos\frac{nu}{2}-\alpha=\cos\frac{nv}{2}+\alpha \end{cases} \tag{2.13}$$

(see Figure 2).

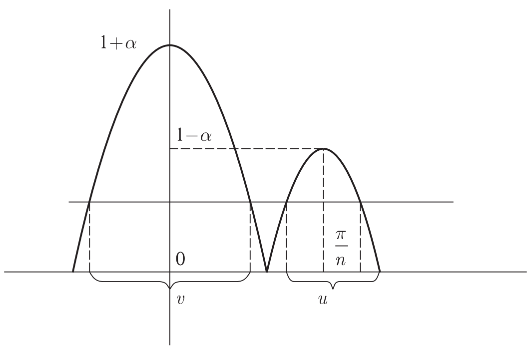

FIGURE 2.

Hence,

$$\frac{1+\alpha}{(\cos nx+\alpha)^{\star}(\beta)}=\frac{2+\cos\frac{nu}{2}-\cos\frac{n(\beta-u)}{2}}{\cos\frac{nu}{2}+\cos\frac{n(\beta-u)}{2}}.$$

Let us find the maximum of the right-hand side for $u \in [0, \beta/2]$. Take

$$f(z) = \frac{2 + \cos z - \cos(a - z)}{\cos z + \cos(a - z)}, \quad z \in [0, a/2),$$

where $0 \leqslant a < \pi$. Then

$$f'(z) = \frac{-2\sin a - 2\sin(a - z) + 2\sin z}{(\cos z + \cos(a - z))^2} < 0, \quad z \in [0, a/2).$$

Therefore, $f(z)$ is non-increasing on $[0, \frac{a}{2})$ and has a maximum at $z = 0$. Then, by (2.12),

$$\begin{aligned}\sup_{0 \leqslant \alpha \leqslant 1} \frac{1 + \alpha}{(\cos nx + \alpha)^\star(\beta)} &= \max_{0 \leqslant u \leqslant \frac{\beta}{2}} \frac{2 + \cos\frac{nu}{2} - \cos\frac{n(\beta - u)}{2}}{\cos\frac{nu}{2} + \cos\frac{n(\beta - u)}{2}} \\ &= \frac{3 - \cos\frac{n\beta}{2}}{1 + \cos\frac{n\beta}{2}} = \frac{1 + \sin^2\frac{n\beta}{4}}{1 - \sin^2\frac{n\beta}{4}},\end{aligned}$$

i.e., (a) is verified.

To prove (b), let $\alpha > 1$ and then

$$(\cos nx + \alpha)^\star = (\cos nx + 1)^\star + \alpha - 1 > (\cos nx + 1)^\star.$$

Therefore, in (b), infimum is attained for $0 \leqslant \alpha \leqslant 1$. Further we remark that for $\alpha > \frac{1 - \cos\frac{n\beta}{2}}{2}$ we have

$$(\cos nx + \alpha)^\star(\beta) = \cos\frac{n\beta}{2} + \alpha,$$

which is the non-decreasing function with respect to α. Then

$$\inf_{0 \leqslant \alpha < \infty} (\cos nx + \alpha)^\star(\beta) \geqslant \inf_{0 \leqslant \alpha \leqslant (1 - \cos\frac{n\beta}{2})/2} (\cos nx + \alpha)^\star(\beta).$$

Next, for $0 \leqslant \alpha \leqslant (1 - \cos\frac{n\beta}{2})/2$, we have

$$(\cos nx + \alpha)^\star(\beta) = \frac{\cos\frac{nu}{2} + \cos\frac{nv}{2}}{2}, \quad 0 \leqslant u \leqslant v \leqslant \beta,$$

where u, v, and α are related by (2.13). Taking into account that

$$\min_{0 \leqslant u \leqslant \frac{\beta}{2}} \frac{\cos\frac{n(\beta - u)}{2} + \cos\frac{nu}{2}}{2} = \cos^2\frac{n\beta}{4},$$

we get (b).

To prove (c), we just use (2.11):

$$(\cos nx + \alpha)^\star(\beta) = \cos\frac{n\beta}{2} + \frac{(1 - \cos\frac{n\beta}{2})}{2} = \cos^2\frac{n\beta}{4}. \tag{2.14}$$

□

3. Sharp Remez inequalities

First, we note that the classical Remez inequality can be written as an inequality in terms of rearrangements (see also [DP]).

Remark 3.1. Let Ω be a measurable subset of $\mathbb{R}$, $f \in L_\infty(\Omega)$ and a right continuous function $C(\cdot)$ be nondecreasing. Then the following statements are equivalent:

$$f^*(0) \leqslant C(\beta) f^*(\beta), \quad \beta > 0; \tag{3.1}$$

$$\|f\|_{L_\infty(\Omega)} \leqslant C(|B|)\|f\|_{L_\infty(\Omega\setminus B)} \quad \text{for any measurable set} \quad B \subset \Omega; \tag{3.2}$$

$$\text{if} \quad \Big|\{x \in \Omega : |f(x)| > 1\}\Big| < \beta, \quad \text{then} \quad \|f\|_{L_\infty(\Omega)} \leqslant C(\beta). \tag{3.3}$$

Proof. Let us verify that (3.1) ⇒ (3.2) ⇒ (3.3) ⇒ (3.1). First assume that (3.1) holds. Let B be a subset of Ω such that $|B| = \beta$. Note that for any $\varepsilon > 0$

$$A = \left|\left\{x \in \Omega \ : \ |f(x)| \geqslant f^*(\beta+\varepsilon)\right\}\right| \geqslant (\beta+\varepsilon).$$

Then

$$\left|\left\{x \in \Omega \ : \ |f(x)| \geqslant f^*(\beta+\varepsilon)\right\} \setminus B\right| > 0$$

and

$$\|f\|_{L_\infty(\Omega\setminus B)} \geqslant \|f\|_{L_\infty(A\setminus B)} \geqslant f^*(\beta+\varepsilon).$$

Using (3.1), we get

$$\|f\|_{L_\infty(\Omega)} = f^*(0) \leqslant C(\beta+\varepsilon) f^*(\beta+\varepsilon) \leqslant C(\beta+\varepsilon)\|f\|_{L_\infty(\Omega\setminus B)},$$

and therefore (3.2) holds.

Let now (3.2) hold. Denoting $B := \{x \in \Omega : |f(x)| > 1\}$ and assuming $|B| < \beta$, we have $|f(x)| \leqslant 1$ for any $x \in \Omega \setminus B$. Then (3.2) gives

$$\|f\|_{L_\infty(\Omega)} \leqslant C\big(|B|\big)\|f\|_{L_\infty(\Omega\setminus B)} \leqslant C\big(|B|\big) \leqslant C\big(\beta\big).$$

Assuming (3.3), consider $\beta > 0$ and $\psi(x) = \frac{f(x)}{f^*(\beta)}$, we get

$$\Big|\{x \in \Omega : |\psi(x)| > 1\}\Big| = \Big|\{x \in \Omega : |f(x)| > f^*(\beta)\}\Big| \leqslant \beta < \beta+\varepsilon.$$

By (3.3) we have $\|\psi\|_{L_\infty(\Omega)} \leqslant C(\beta+\varepsilon)$ for any $\varepsilon > 0$. Since $C(\cdot)$ is right continuous, $\|\psi\|_{L_\infty(\Omega)} \leqslant C(\beta)$, i.e., (3.1) follows. □

Thus, to investigate the Remez inequality, one can use its rearrangement form

$$T_n^*(0) \leqslant C(n,\beta)\, T_n^*(\beta), \quad \beta > 0. \tag{3.4}$$

Note that dealing with polynomials with the minimal periods less than 2π allows us to decrease the constant $C(n,\beta)$. Indeed, if $d = 2\pi/m$ is a period of T_n, then by Lemma 2.1, $T_n(x/m)$ is a polynomial of degree n/m and

$$\overline{T}_{\frac{n}{m}}(x) := T_n(x/m), \quad \overline{T}^*_{\frac{n}{m}}(\beta) = T_n^*(\beta), \quad \beta > 0.$$

Then inequality (3.4) is true with the constant $C(\frac{n}{m},\beta)$.

To overcome this, we are going to study Remez's inequality with the $\star$-rearrangement, i.e.,

$$T_n^\star(0) \leqslant \ C(n,\beta)\, T_n^\star(\beta), \quad \beta > 0. \tag{3.5}$$

Let us now recall that $\mathfrak{N}_n$ and $\mathfrak{M}_n$ are the sets of complex and real polynomials of degree at most n, respectively. The next remark shows that to study the Remez inequality in the form (3.4) or in the form (3.5), it suffices to deal with real polynomials.

Remark 3.2. Let $0 \leqslant \beta < 2\pi$. Then

$$\sup_{T_n\in\mathfrak{N}_n} \frac{T_n^*(0)}{T_n^*(\beta)} = \sup_{T_n\in\mathfrak{M}_n} \frac{T_n^*(0)}{T_n^*(\beta)}, \tag{3.6}$$

and

$$\sup_{T_n\in\mathfrak{N}_n} \frac{T_n^\star(0)}{T_n^\star(\beta)} = \sup_{T_n\in\mathfrak{M}_n} \frac{T_n^\star(0)}{T_n^\star(\beta)}. \tag{3.7}$$

Proof. It is clear that

$$\sup_{T_n\in\mathfrak{N}_n} \frac{T_n^*(0)}{T_n^*(\beta)} \geqslant \sup_{T_n\in\mathfrak{M}_n} \frac{T_n^*(0)}{T_n^*(\beta)}.$$

Let now $T_n \in \mathfrak{N}_n$ and $|T_n(x_0)| = \max_{x\in\mathbb{T}} |T_n(x)|$. Suppose $\alpha = \frac{T_n(x_0)}{|T_n(x_0)|}$, then $Re\,(\bar{\alpha}T_n) \in \mathfrak{M}_n$ and

$$\max_{x\in\mathbb{T}} |Re\,(\bar{\alpha}T_n)\,(x)| = |T_n(x_0)| = T_n^*(0).$$

Further, since $|Re\,(\bar{\alpha}T_n(x))\,| \leqslant |T_n(x)|$, we get

$$\frac{T_n^*(0)}{T_n^*(\beta)} = \frac{\left(Re\,(\bar{\alpha}T_n)\right)^*(0)}{T_n^*(\beta)} \leqslant \frac{\left(Re\,(\bar{\alpha}T_n)\right)^*(0)}{\left(Re\,(\bar{\alpha}T_n)\right)^*(\beta)} \leqslant \sup_{T_n\in\mathfrak{M}_n} \frac{T_n^*(0)}{T_n^*(\beta)}$$

and (3.6) follows. Similarly, we can show (3.7). □

3.1. **Sharp Remez inequalities in terms of rearrangements.** The main result of this section is the following sharp Remez type inequalities.

Theorem 3.3. *Let $n, k \in \mathbb{N}$ and $0 \leqslant \beta < \frac{2\pi}{n}$. Then*

$$\sup_{T_n\in\mathfrak{M}_n} \frac{T_n^\star(0)}{T_n^\star(\beta)} = 1 + 2\tan^2\frac{n\beta}{4} \tag{3.8}$$

and

$$\sup_{T_n\in\mathfrak{M}_n} \frac{\|n^{-k}T_n^{(k)}\|_{L_\infty}}{T_n^\star(\beta)} = 1 + \tan^2\frac{n\beta}{4}. \tag{3.9}$$

Proof. To prove (3.8), we define two subsets of $\mathfrak{M}_n$:

$$\mathfrak{M}_n^0 = \left\{T_n \in \mathfrak{M}_n : \quad \max_x T_n(x) - \min_x T_n(x) = 2\right\} \tag{3.10}$$

and

$$\mathfrak{M}_n^1 = \left\{T_n \in \mathfrak{M}_n^0 : \;\; 1 \leqslant \max_x T_n(x) \leqslant 2\right\}. \tag{3.11}$$

Note that for any $T_n \neq \mathit{const}$ there exists $\gamma > 0$ such that $\gamma T_n \in \mathfrak{M}_n^0$. Then it is clear that

$$\sup_{T_n\in\mathfrak{M}_n} \frac{T_n^\star(0)}{T_n^\star(\beta)} = \sup_{T_n\in\mathfrak{M}_n^0} \frac{T_n^\star(0)}{T_n^\star(\beta)}.$$

Let us now show that

$$\sup_{T_n\in\mathfrak{M}_n^0} \frac{T_n^\star(0)}{T_n^\star(\beta)} = \sup_{T_n\in\mathfrak{M}_n^1} \frac{T_n^\star(0)}{T_n^\star(\beta)}.$$

Suppose that $T_n \in \mathfrak{M}_n^0$ but $T_n \notin \mathfrak{M}_n^1$ and $-T_n \notin \mathfrak{M}_n^1$. Denoting $\Xi := \max T_n(x)$, note that $\Xi > 2$. Indeed, assume $0 \leqslant \Xi < 1$, then it is clear $-T_n \in \mathfrak{M}_n^1$, which contradicts the choice of T_n. The case $\Xi < 0$ can be reduced to the case $\Xi > 0$ by taking $-T_n$.

Therefore, $\Xi > 2$ and $\xi := \min T_n(x) > 0$. Then for $P_n(x) := T_n(x) - \xi \geqslant 0$ we have

$$P_n^\star(\beta) = T_n^\star(\beta) - \xi, \quad \beta \geqslant 0.$$

On the other hand, $P_n \in \mathfrak{M}_n^1$,

$$\frac{T_n^\star(0)}{T_n^\star(\beta)} = \frac{\left(T_n^\star(0) - \xi\right)^\star + \xi}{\left(T_n(\beta) - \xi\right)^\star + \xi} = \frac{P_n^\star(0) + \xi}{P_n^\star(\beta) + \xi} \leqslant \frac{P_n^\star(0)}{P_n^\star(\beta)},$$

because of $P_n^\star(0) \geqslant P_n^\star(\beta)$ and $\xi > 0$. Hence, we get

$$\sup_{T_n\in\mathfrak{M}_n} \frac{T_n^\star(0)}{T_n^\star(\beta)} = \sup_{T_n\in\mathfrak{M}_n^1} \frac{T_n^\star(0)}{T_n^\star(\beta)}.$$

Therefore, it suffices to verify (3.8) on the class $\mathfrak{M}_n^1$.

Let further $T_n \in \mathfrak{M}_n^1$, then by Lemma 2.3,

$$T_n^\star(\beta) \geqslant \left(\cos nx + \alpha\right)^\star(\beta),$$

where $0 \leqslant \alpha = \Xi - 1 \leqslant 1$. Then

$$\frac{T^\star(0)}{T^\star(\beta)} \leqslant \frac{\Xi}{(\cos nx + \alpha)^\star(\beta)} = \frac{1+\alpha}{(\cos nx + \alpha)^\star(\beta)}$$

and therefore,

$$\sup_{T_n\in\mathfrak{M}_n^1} \frac{T^\star(0)}{T^\star(\beta)} \leqslant \sup_{0\leqslant\alpha\leqslant 1} \frac{1+\alpha}{(\cos nx + \alpha)^\star(\beta)}.$$

Making use of Lemma 2.4(a), we get

$$\sup_{T_n\in\mathfrak{M}_n}\frac{T_n^\star(0)}{T_n^\star(\beta)}=\sup_{T_n\in\mathfrak{M}_n^1}\frac{T^\star(0)}{T^\star(\beta)}\leqslant\frac{1+\sin^2\frac{\beta n}{4}}{1-\sin^2\frac{\beta n}{4}}=1+2\tan^2\frac{\beta n}{4},$$

where $0\leqslant\beta<\frac{2\pi}{n}$. The constant in (3.8) is sharp, take

$$T_n(x)=\cos nx+\frac{1-\cos\frac{n\beta}{2}}{2} \tag{3.12}$$

and use Lemma 2.4(c).

Next, let us show the accuracy of (3.9). Remark that

$$\sup_{T_n\in\mathfrak{M}_n}\frac{\|n^{-k}T_n^{(k)}\|_{L_\infty}}{T_n^\star(\beta)}=\sup_{T_n\in\mathfrak{M}_n^1}\frac{\|n^{-k}T_n^{(k)}\|_{L_\infty}}{T_n^\star(\beta)}.$$

Let $T_n\in\mathfrak{M}_n^1$, then $1\leqslant\Xi\leqslant2$, $\max_x|T_n(x)-(\Xi-1)|=1$. Using Bernstein's inequality, we have

$$\sup_{T_n\in\mathfrak{M}_n^1}\frac{\|n^{-k}T_n^{(k)}\|_{L_\infty}}{T_n^\star(\beta)}=\sup_{T_n\in\mathfrak{M}_n^1}\frac{\|n^{-k}\big(T_n-(\Xi-1)\big)^{(k)}\|_{L_\infty}}{T_n^\star(\beta)}\leqslant\sup_{T_n\in\mathfrak{M}_n^1}\frac{1}{T_n^\star(\beta)}.$$

By Lemma 2.3,

$$\sup_{T_n\in\mathfrak{M}_n^1}\frac{\|n^{-k}T_n^{(k)}\|_{L_\infty}}{T_n^\star(\beta)}\leqslant\sup_{T_n\in\mathfrak{M}_n^1}\frac{1}{\Big(\cos nx+\Xi-1\Big)^\star(\beta)}=\frac{1}{\inf_{0\leqslant c<\infty}\Big(\cos nx+c\Big)^\star(\beta)}.$$

Also, taking into account Lemma 2.4 (b), we get

$$\sup_{T_n\in\mathfrak{M}_n}\frac{\|n^{-k}T_n^{(k)}\|_{L_\infty}}{T_n^\star(\beta)}=\sup_{T_n\in\mathfrak{M}_n^1}\frac{\|n^{-k}T_n^{(k)}\|_{L_\infty}}{T_n^\star(\beta)}\leqslant\frac{1}{\cos^2\frac{n\beta}{4}}.$$

Sharpness in (3.9) can be verified by considering the function (3.12); see Lemma 2.4(c). ☐

Note that first applying Bernstein's inequality and then Remez's inequality (3.8) does not give the sharp constant in the Remez-Bernstein inequality (3.9).

Corollary 3.4. *Let $n,k\in\mathbb{N}$, $T_n\in\mathfrak{M}_n$, $2\pi/m$ be the minimal period of T_n, and $0\leqslant\beta<\frac{2\pi m}{n}$. Then*

$$T_n^*(0)\leqslant\Big(1+2\tan^2\frac{n\beta}{4m}\Big)\,T_n^*(\beta) \tag{3.13}$$

and

$$\|n^{-k}T_n^{(k)}\|_{L_\infty}\leqslant\Big(1+\tan^2\frac{n\beta}{4m}\Big)\,T_n^*(\beta). \tag{3.14}$$

Inequalities (3.13) and (3.14) are sharp; they become equalities for $T_n(x)=\cos nx+\frac12\Big(1-\cos\frac{\beta}{2}\Big)$.

Proof. Let $0\leqslant\beta<\frac{2\pi m}{n}$ and $k\in\mathbb{N}$. Then Theorem 3.3, where (3.8) and (3.9) are applied at the point β/m, and Lemma 2.1(e) give

$$T_n^*(0)=T_n^\star(0)\leqslant\Big(1+2\tan^2\frac{n\beta}{4m}\Big)\,T_n^\star(\beta/m)=\Big(1+2\tan^2\frac{n\beta}{4m}\Big)\,T_n^*(\beta)$$

and

$$\|n^{-k}T_n^{(k)}\|_{L_\infty}\leqslant\Big(1+\tan^2\frac{n\beta}{4m}\Big)\,T_n^\star(\beta/m)=\Big(1+\tan^2\frac{n\beta}{4m}\Big)\,T_n^*(\beta).$$ ☐

The next result provides the asymptotics of the sharp constant $\mathcal{C}(n,\beta)$ in the Remez inequality.

Corollary 3.5. *We have*

$$\mathcal{C}(n,\beta)=1+\frac{(n\beta)^2}{8}+O(\beta^4) \tag{3.15}$$

and

$$\mathcal{C}(1,\beta)=1+2\tan^2\frac{\beta}{4}. \tag{3.16}$$

Proof. It follows from (1.4) and (1.7) that

$$A_n(\beta) := \frac{1}{2}\Big(\tan^{2n}\Big(\frac{2\pi-\beta}{8}\Big) + \cot^{2n}\Big(\frac{2\pi-\beta}{8}\Big)\Big) \leqslant \mathcal{C}(n,\beta) \leqslant 1 + 2\tan^2\frac{n\beta}{4}, \qquad 0<\beta<\frac{2\pi}{n}.$$

Let $x=\beta/8$. Then

$$\begin{aligned} A_n(x)-1 &= \frac{1}{2}\Big[\big(\cot^{2n}(\pi/4-x)-1\big) + \big(\tan^{2n}(\pi/4-x)-1\big) \Big] \\ &= \frac{1}{2}\left[\left(\frac{\cos(\pi/2-2x)}{\sin^2(\pi/4-x)}\sum_{k=0}^{n-1}\cot^{2k}(\pi/4-x)\right) - \left(\frac{\cos(\pi/2-2x)}{\cos^2(\pi/4-x)}\sum_{k=0}^{n-1}\tan^{2k}(\pi/4-x)\right)\right] \\ &= \frac{\sin 2x}{2}\left[\frac{1}{\sin^2(\pi/4-x)}\sum_{k=0}^{n-1}\big(\cot^{2k}(\pi/4-x)-1\big) - \frac{1}{\cos^2(\pi/4-x)}\sum_{k=0}^{n-1}\big(\tan^{2k}(\pi/4-x)-1\big)\right. \\ &\quad \left. + \frac{n}{\sin^2(\pi/4-x)} - \frac{n}{\cos^2(\pi/4-x)}\right] \\ &= \frac{\sin 2x}{2}\left[\frac{1}{\sin^2(\pi/4-x)}\sum_{k=0}^{n-1}\frac{\cos(\pi/2-2x)}{\sin^2(\pi/4-x)}\sum_{m=0}^{k-1}\cot^{2m}(\pi/4-x)\right. \\ &\quad \left. + \frac{1}{\cos^2(\pi/4-x)}\sum_{k=0}^{n-1}\frac{\cos(\pi/2-2x)}{\cos^2(\pi/4-x)}\sum_{m=0}^{k-1}\tan^{2m}(\pi/4-x) + \frac{n\cos(\pi/2-2x)}{\sin^2(\pi/4-x)\cos^2(\pi/4-x)}\right] \\ &= \frac{\sin^2 2x}{2}\left[\frac{1}{\sin^2(\pi/4-x)}\sum_{k=0}^{n-1}\frac{1}{\sin^2(\pi/4-x)}\sum_{m=0}^{k-1}\cot^{2m}(\pi/4-x)\right. \\ &\quad \left. + \frac{1}{\cos^2(\pi/4-x)}\sum_{k=0}^{n-1}\frac{1}{\cos^2(\pi/4-x)}\sum_{m=0}^{k-1}\tan^{2m}(\pi/4-x) + \frac{n}{\sin^2(\pi/4-x)\cos^2(\pi/4-x)}\right]. \end{aligned}$$

Since $A_n(x)$ is even, we get

$$A_n(x)-1 = \frac{\sin^2 2x}{2(\frac{\sqrt{2}}{2})^4}\left(2\sum_{k=1}^{n-1}k \ + \ n + O(x^2)\right) = 2\sin^2\frac{\beta}{4}\Big(n^2+O(\beta^2)\Big) = \frac{(n\beta)^2}{8} + O(\beta^4)$$

or

$$A_n(\beta) = 1 + \frac{(n\beta)^2}{8} + O(\beta^4).$$

On the other hand,

$$1+2\tan^2\frac{n\beta}{4} = 1 + \frac{(n\beta)^2}{8} + O(\beta^4),$$

which implies (3.15).

Further, (3.16) follows from ($n=1$ and $x=\beta/8$)

$$\begin{aligned} A_n(\beta)-1 &= \frac{1}{2}\Big[\big(\cot^2(\pi/4-x)-1\big) + \big(\tan^2(\pi/4-x)-1\big) \Big] \\ &= \frac{\sin^2 2x}{2}\left[\frac{1}{\sin^2(\pi/4-x)\cos^2(\pi/4-x)}\right] = 2\tan^2 2x = 2\tan^2\frac{\beta}{4}. \end{aligned}$$ □

Remark 3.6. Corollary 3.4 yields the following Remez inequality with the constant written in the exponential form. If $d=2\pi/m$ is the minimal period of $T_n\in\mathfrak{M}_n$, then

$$\|T_n\|_{L_\infty[0,2\pi)} \leqslant \exp\Big(\frac{(n\beta)^2}{2m^2}\Big)\|T_n\|_{L_\infty([0,2\pi)\setminus B)}, \qquad |B|=\beta<\frac{m\pi}{n};$$

compare with the known bound $\exp\Big(n\beta\min\{2,(1+1.75\beta)\}\Big)$, cf. (1.4). This follows from (3.13) and the inequality $1+2\tan^2 x/4 \leqslant \exp\big(\frac{x^2}{2}\big)$ for $0<x<\pi$.

Next we provide the following result on the Remez inequality in L_p (see also [Er2] and [EMN]).

Corollary 3.7. *Let $p \in (0,\infty)$ and $2\pi/m$ be a minimal period of $T_n \in \mathfrak{M}_n$. Then*
$$\int_0^{2\pi} \Big|T_n(t)\Big|^p dt \leqslant \left[1 + \left(1 + 2\tan^2 \frac{n|B|}{2m}\right)^p\right] \int_{\mathbb{T}\setminus B} \Big|T_n(t)\Big|^p dt,$$
where $|B| < \frac{\pi m}{n}$.

Proof. Using Corollary 3.4, by (3.13), we get
$$\begin{aligned}\int_B \Big|T_n(x)\Big|^p dx &\leqslant \int_0^{|B|} (T_n^*(t))^p\, dt \\ &\leqslant |B| \left(1 + 2\tan^2 \frac{n|B|}{2m}\right)^p (T_n^*(2|B|))^p \\ &\leqslant \left(1 + 2\tan^2 \frac{n|B|}{2m}\right)^p \int_{|B|}^{2|B|} (T_n^*(t))^p\, dt \\ &\leqslant \left(1 + 2\tan^2 \frac{n|B|}{2m}\right)^p \int_{|B|}^{2\pi} (T_n^*(t))^p\, dt \\ &\leqslant \left(1 + 2\tan^2 \frac{n|B|}{2m}\right)^p \int_{\mathbb{T}\setminus B} |T_n(x)|^p\, dx.\end{aligned}$$
□

3.2. **Sharp Remez inequalities in terms of averages.** Remez's inequality allows us to control how fast polynomials of degree n change on the period. Another inequality that characterizes such properties of polynomials is the following one:
$$\|T_n\|_{L_\infty} \leqslant C(n,\beta)\, \overline{T_n(\beta, M)}. \tag{3.17}$$
Let $[a,b]$ be an interval of $\mathbb{T}$ of the measure $\leqslant d$. We say that Q is a *harmonic interval* if
$$Q = \bigcup_{i=0}^{r} \Big([a,b] + id\Big), \qquad r \in \mathbb{N}$$
(see e.g. [Nu]). By $M_{h.int.}$ we denote the collection of all harmonic intervals in $\mathbb{T}$. Let also $M_{int.}$ be the collection of all intervals in $\mathbb{T}$.

Theorem 3.8. *Let M be a net that contains $M_{h.int.}$. Let $T_n \in \mathfrak{M}_n$ and $\frac{2\pi}{m}$ be the minimal period of T_n. Then*
$$\|T_n\|_{L_\infty} \leqslant \frac{n\beta}{2m \sin \frac{n\beta}{2m}}\, \overline{T_n(\beta, M)}, \quad 0 < \beta < \frac{\pi m}{n}, \tag{3.18}$$
and
$$\|T_n\|_{L_\infty} \leqslant \frac{n\beta}{2 \sin \frac{n\beta}{2}}\, \overline{T_n(\beta, M_{int.})}, \quad 0 < \beta < \frac{\pi}{n}. \tag{3.19}$$
Inequalities (3.18) and (3.19) are sharp; they become equalities for $T_n(x) = \cos nx$.

Remark 3.9. An analogue of (3.19) for a net M that contains $M_{int.}$ also holds, i.e.,
$$\|T_n\|_{L_\infty} \leqslant \frac{n\beta}{2 \sin \frac{n\beta}{2}}\, \overline{T_n(\beta, M)}, \quad 0 < \beta < \frac{\pi}{n}.$$
However, unlike (3.18) this inequality is not sharp.

Proof of Theorem 3.8. Since $\frac{\|T_n\|_{L_\infty}}{T_n(\beta,M)}$ is invariant under multiplication by $\gamma \in \mathbb{R}\setminus\{0\}$, we get
$$\sup_{T_n \in \mathfrak{M}_n} \frac{\|T_n\|_{L_\infty}}{T_n(\beta, M)} = \sup_{T_n \in \mathfrak{M}_n^0} \frac{\|T_n\|_{L_\infty}}{T_n(\beta, M)}.$$
Let then $T_n \in \mathfrak{M}_n^0$. Without loss of generality we assume that $\Xi = \max_{x\in[-\pi,\pi)} T_n(x) = T_n(0) \geqslant 1$. Then Lemma 2.2 implies
$$T_n(x) \geqslant \cos nx + (\Xi - 1), \quad x \in \big[-\frac{\pi}{2n}, \frac{\pi}{2n}\big].$$

This and $\frac{2\pi}{m}$-periodicity of T_n give for $\beta \in (0, \pi m/n)$

$$\frac{1}{\beta}\sum_{i=0}^{m-1}\int_{-\frac{\beta}{2m}+\frac{2\pi i}{m}}^{\frac{\beta}{2m}+\frac{2\pi i}{m}} T_n(x)dx = \frac{m}{\beta}\int_{-\frac{\beta}{2m}}^{\frac{\beta}{2m}} T_n(x)dx$$
$$\geqslant \frac{m}{\beta}\int_{-\frac{\beta}{2m}}^{\frac{\beta}{2m}} (\cos nx + \Xi - 1)\, dx = \frac{2m\sin\frac{n\beta}{2m}}{n\beta} + \Xi - 1 \geqslant 0.$$

Hence,

$$\sup_{\substack{e\in M_{h.int.}\\ |e|=\beta}} \frac{1}{|e|}\left|\int_e T_n(y)dy\right| \geqslant \frac{1}{\beta}\sum_{i=0}^{m-1}\int_{-\frac{\beta}{2m}+\frac{2\pi i}{m}}^{\frac{\beta}{2m}+\frac{2\pi i}{m}} T_n(x)dx \geqslant \frac{2m\sin\frac{n\beta}{2m}}{n\beta} + \Xi - 1$$

and

$$\sup_{T_n\in\mathfrak{M}_n}\frac{\|T_n\|_{L_\infty}}{T_n(\beta,M)} \leqslant \sup_{T_n\in\mathfrak{M}_n^0}\frac{\|T_n\|_{L_\infty}}{T_n(\beta,M_{h.int.})} \leqslant \sup_{T_n\in\mathfrak{M}_n^0}\frac{\|T_n\|_{L_\infty}}{\sup_{\substack{e\in M_{h.int.}\\ |e|=\beta}}\frac{1}{|e|}\left|\int_e T_n(y)dy\right|}$$
$$\leqslant \sup_{1\leqslant\Xi<\infty}\frac{\Xi}{\frac{2m\sin\frac{n\beta}{2m}}{n\beta}+\Xi-1} = \frac{n\beta}{2m\sin\frac{n\beta}{2m}},$$

the required inequality.

Consider $T_n(x) = \cos nx$. For this function $m = n$, $\max|T_n(x)| = 1$ and for $0 < \beta < 2\pi$,

$$\sup_{\substack{e\in M\\ |e|\geqslant\beta}}\frac{1}{|e|}\left|\int_e T_n(y)dy\right| = \sup_{\substack{e\in M_{h.int.}\\ |e|\geqslant\beta}}\frac{1}{|e|}\left|\int_e T_n(y)dy\right| = \frac{n}{\beta}\int_{-\frac{\beta}{2n}}^{\frac{\beta}{2n}}\cos nx dx = \frac{2\sin\frac{\beta}{2}}{\beta},$$

i.e., inequality (3.18) is sharp.

The proof of inequality (3.19) is similar to the proof above using

$$\sup_{T_n\in\mathfrak{M}_n^0}\frac{\|T_n\|_{L_\infty}}{T_n(\beta,M_{int.})} \leqslant \sup_{T_n\in\mathfrak{M}_n^0}\frac{\|T_n\|_{L_\infty}}{\sup_{\substack{e\in M_{int.}\\ |e|=\beta}}\frac{1}{|e|}\left|\int_e T_n(y)dy\right|} \leqslant \frac{n\beta}{2\sin\frac{n\beta}{2}}. \tag{3.20}$$ □

The next theorem plays an important role in approximation theory (see Stechkin [St] and Nikol'skii [Ni] in case $\beta = \pi/n$). In particular, it generalizes Bernstein's inequality $\|T_n'\|_\infty \leqslant n\|T_n\|_\infty$.

Corollary 3.10. *Let $n \in \mathbb{N}$. We have*

$$\sup_{T_n\in\mathfrak{M}_n}\frac{\|T_n'\|_{L_\infty}}{\|T_n(x+\beta)-T_n(x)\|_{L_\infty}} = \left(\frac{n}{2\sin\frac{\beta n}{2}}\right), \quad 0 < \beta < \frac{2\pi}{n}.$$

Proof. Without loss of generality, assume $\max_{x\in[-\pi,\pi)} T_n'(x) = T_n'(0) \geqslant 1$. The estimate from above follows from (3.20):

$$\frac{\|T_n'\|_{L_\infty}}{\max\limits_{x\in[0,2\pi)}|T_n(x+\beta)-T_n(x)|} = \frac{\|T_n'\|_{L_\infty}}{\beta\max\limits_{x\in[0,2\pi)}\frac{1}{\beta}\left|\int_x^{x+\beta}T_n'(y)dy\right|} \leqslant \frac{n}{2\sin\frac{\beta n}{2}}.$$

To prove the estimate from below, we consider $T_n(x) = \sin nx$. □

4. Sharp Nikol'skii and Bernstein-Nikol'skii inequalities in the Lorentz and Net spaces

We start this section with Nikol'skii's inequalities in the Lorentz spaces $L_{p,q} \equiv L_{p,q}(\mathbb{T})$.

Theorem 4.1. *Let $0 < p < \infty$, $0 < q \leqslant \infty$, and $n \in \mathbb{N}$. Let also $T_n \in \mathfrak{M}_n$ and $\frac{2\pi}{m}$ be a minimal period of T_n. We have*

$$\|T_n\|_{L_\infty} \leqslant \left(\frac{n}{m}\right)^{1/p}\sup_{0\leqslant\alpha\leqslant1}\frac{1+\alpha}{\|\cos(\cdot)+\alpha\|_{L_{p,q}}}\|T_n\|_{L_{p,q}}; \tag{4.1}$$

$$\|T_n^{(k)}\|_{L_\infty} \leqslant \left(\frac{n}{m}\right)^{1/p}\frac{n^k}{\inf_{\alpha\in[0,1]}\|\cos n(\cdot)+\alpha\|_{L_{p,q}}}\|T_n\|_{L_{p,q}}. \tag{4.2}$$

Inequalities (4.1) and (4.2) are sharp; they become equalities, respectively, for $T_{n,i}(x) = \cos nx + \alpha_i, \quad i = 1, 2,$ *where* α_1 *is the point where the maximum of*

$$\frac{1+\alpha}{\|\cos(\cdot)+\alpha\|_{L_{p,q}}}, \quad \alpha \in [0,1]$$

is attained and α_2 *is the point where the minimum of*

$$\|\cos(\cdot)+\alpha\|_{L_{p,q}}, \quad \alpha \in [0,1]$$

is attained.

Remark 4.2. Note that in the papers [BKP1] and [BKP2] it is proved that

$$\sup_{n\in\mathbb{N}} \sup_{T_n\in\mathfrak{M}_n} \frac{\|T_n\|_{L_r}}{n^{1/p-1/r}\|T_n\|_{L_p}} = \sup_{0\leqslant\alpha\leqslant1} \frac{\|\cos(\cdot)+\alpha\|_{L_r}}{\|\cos(\cdot)+\alpha\|_{L_p}}, \quad 0 < p < r \leqslant \infty \tag{4.3}$$

and

$$\sup_{n\in\mathbb{N}} \sup_{T_n\in\mathfrak{M}_n} \frac{\|T_n^{(k)}\|_{L_\infty}}{n^{k+1/p}\|T_n\|_{L_p}} = \left(\|\cos(\cdot)\|_{L_p}\right)^{-1}, \quad 1 \leqslant p < \infty, \ k \in \mathbb{N}. \tag{4.4}$$

Therefore, inequalities (4.1) and (4.2) supplement (4.3) and (4.4) for the case of the Lorentz spaces (take $p = q$). At the same time, considering minimal period of polynomials allows us to prove sharp Nikol'skii and Bernstein-Nikol'skii inequalities (4.1) and (4.2). The sharp constant in (4.1) was also discussed in [DP, Rem. 6.3].

We will need the following technical result.

Lemma 4.3. *Suppose* $0 < p < \infty$ *and* $0 < q \leqslant \infty$*; then*

$$\sup_{0\leqslant\alpha<\infty} \frac{1+\alpha}{\|\cos(\cdot)+\alpha\|_{L_{p,q}}} = \sup_{0\leqslant\alpha\leqslant1} \frac{1+\alpha}{\|\cos(\cdot)+\alpha\|_{L_{p,q}}} \tag{4.5}$$

and

$$\inf_{0\leqslant\alpha<\infty} \|\cos n(\cdot)+\alpha\|_{L_{p,q}} = \inf_{0\leqslant\alpha\leqslant1} \|\cos n(\cdot)+\alpha\|_{L_{p,q}}. \tag{4.6}$$

Proof. Let $\alpha \geqslant 1$. Let us find the extremal point of the function

$$g(\alpha) = \frac{(1+\alpha)^q}{\|\cos(\cdot)+\alpha\|_{L_{p,q}}^q}.$$

Since

$$(\cos(\cdot)+\alpha)^*(t) = (\cos(\cdot)+1)^*(t) + \alpha - 1 = \cos\frac{t}{2} + \alpha,$$

we get

$$g'(\alpha) = \left(q(1+\alpha)^{q-1} \int_0^{2\pi} t^{\frac{q}{p}-1} \left(\cos\frac{t}{2}+\alpha\right)^{q-1} \left(\cos\frac{t}{2}-1\right) dt\right) \|\cos(\cdot)+\alpha\|_{L_{p,q}}^{-2q} < 0.$$

Hence, $g(\alpha) \leqslant g(1)$, and (4.5) follows. Inequality (4.6) can be shown similarly. □

Proof of Theorem 4.1. Since $\frac{\|T_n\|_{L_\infty}}{\|T_n\|_{L_{p,q}}}$ is invariant under multiplication by $\gamma \in \mathbb{R}\setminus\{0\}$, we have

$$\sup_{T_n\in\mathfrak{M}_n} \frac{m^{1/p}\|T_n\|_{L_\infty}}{n^{1/p}\|T_n\|_{L_{p,q}}} = \sup_{T_n\in\mathfrak{M}_n^0} \frac{m^{1/p}\|T_n\|_{L_\infty}}{n^{1/p}\|T_n\|_{L_{p,q}}},$$

where $\mathfrak{M}_n^0$ is defined by (3.10).

For $T_n \in \mathfrak{M}_n^0$ we can assume that $\Xi := \max T_n(x) \geqslant 1$. Let $\frac{2\pi}{m}$ be the minimal period of T_n. By Lemma 2.3,

$$T_n^\star(\beta) \geqslant (\cos nx + \Xi - 1)^\star(\beta) \quad 0 < \beta < 2\pi/n,$$

and hence,

$$\begin{aligned}\|T_n\|_{L_{p,q}} &= m^{1/p}\left(\int_0^{2\pi/m} t^{\frac{q}{p}}(T^*(tm))^q\frac{dt}{t}\right)^{1/q}\\ &= m^{1/p}\left(\int_0^{2\pi/m} t^{\frac{q}{p}}(T^\star(t))^q\frac{dt}{t}\right)^{1/q}\\ &\geqslant m^{1/p}\left(\int_0^{2\pi/n} t^{\frac{q}{p}}(T^\star(t))^q\frac{dt}{t}\right)^{1/q}\\ &\geqslant m^{1/p}\left(\int_0^{2\pi/n} t^{\frac{q}{p}}\Big((\cos nx+\Xi-1)^\star(t)\Big)^q\frac{dt}{t}\right)^{1/q}.\end{aligned}$$

Using Lemma 2.1(e), we have

$$\|T_n\|_{L_{p,q}} \geqslant m^{1/p}n^{-1/p}\left(\int_0^{2\pi} t^{\frac{q}{p}}\Big((\cos x+\Xi-1)^*(t)\Big)^q\frac{dt}{t}\right)^{1/q}.$$

Then

$$\begin{aligned}\sup_{\substack{T_n\in\mathfrak{M}_n^0\\ \Xi\geqslant 1}}\frac{m^{1/p}\|T_n\|_{L_\infty}}{n^{1/p}\|T_n\|_{L_{p,q}}} &\leqslant \sup_{\Xi\geqslant 1}\frac{\Xi}{\|\cos(\cdot)+\Xi-1\|_{L_{p,q}}}\\ &= \sup_{0\leqslant\alpha<\infty}\frac{1+\alpha}{\|\cos(\cdot)+\alpha\|_{L_{p,q}}}.\end{aligned}\tag{4.7}$$

By Lemma 4.3, we obtain

$$\sup_{T_n\in\mathfrak{M}_n}\frac{m^{1/p}\|T_n\|_{L_\infty}}{n^{1/p}\|T_n\|_{L_{p,q}}}\leqslant \sup_{0\leqslant\alpha<\infty}\frac{1+\alpha}{\|\cos(\cdot)+\alpha\|_{L_{p,q}}}=\sup_{0\leqslant\alpha\leqslant 1}\frac{1+\alpha}{\|\cos(\cdot)+\alpha\|_{L_{p,q}}}.$$

To show the sharpness of (4.1), take $T_n(x)=\cos nx+\alpha_1$.

Let us now show the accuracy of (4.2). Estimating as above and using Bernstein's inequality, we get

$$\begin{aligned}\sup_{T_n\in\mathfrak{M}_n}\frac{m^{1/p}\|T_n^{(k)}\|_{L_\infty}}{n^{k+1/p}\|T_n\|_{L_{p,q}}} &= \sup_{T_n\in\mathfrak{M}_n^0}\frac{m^{1/p}\|T_n^{(k)}\|_{L_\infty}}{n^{k+1/p}\|T_n\|_{L_{p,q}}}\\ &\leqslant \sup_{T_n\in\mathfrak{M}_n^0}\frac{\|n^{-k}\big(T_n-(\Xi-1)\big)^{(k)}\|_{L_\infty}}{n^{1/p}\left(\int_0^{2\pi/n}\Big(t^{\frac{1}{p}}T_n^\star(t)\Big)^q\frac{dt}{t}\right)^{1/q}}\\ &\leqslant \sup_{T_n\in\mathfrak{M}_n^0}\frac{1}{n^{1/p}\left(\int_0^{2\pi/n}\Big(t^{\frac{1}{p}}T_n^\star(t)\Big)^q\frac{dt}{t}\right)^{1/q}}\\ &\leqslant \sup_{0\leqslant\alpha<\infty}\frac{1}{\left(\int_0^{2\pi}\Big(t^{\frac{1}{p}}(\cos(\cdot)+\alpha)^*(t)\Big)^q\frac{dt}{t}\right)^{1/q}}=\sup_{0\leqslant\alpha\leqslant 1}\frac{1}{\|\cos(\cdot)+\alpha\|_{L_{p,q}}}.\end{aligned}$$

The polynomial $T_n(x)=\cos nx+\alpha_2$ provides the sharpness of (4.2). □

The following remark gives the (L_p, L_{p_1})-Nikol'skii inequality.

Remark 4.4. Let $0<p<p_1<\infty$ and $0<q_1,q\leqslant\infty$ such that $q/p=q_1/p_1$. Let $2\pi/m$ be a minimal period of $T_n\in\mathfrak{M}_n$. Then

$$\|T_n\|_{L_{p_1,q_1}}\leqslant\left(\frac{n}{m}\right)^{1/p-1/p_1}\left(\sup_{0\leqslant\alpha\leqslant 1}\frac{1+\alpha}{\|\cos(\cdot)+\alpha\|_{L_{p,q}}}\right)^{1-p/p_1}\|T_n\|_{L_{p,q}}.$$

In particular,

$$\|T_n\|_{L_{p_1}}\leqslant\left(\frac{n}{m}\right)^{1/p-1/p_1}\left(\sup_{0\leqslant\alpha\leqslant 1}\frac{1+\alpha}{\|\cos(\cdot)+\alpha\|_{L_p}}\right)^{1-p/p_1}\|T_n\|_{L_p}.$$

To finish this section, we give Nikol'skii's inequality for the net spaces.

Theorem 4.5. *Let $0 < p < \infty$, $0 < q \leqslant \infty$, and a net M contain $M_{h.i.}$, the set of all harmonic intervals. Let $T_n \in \mathfrak{M}_n$ and $\frac{2\pi}{m}$ be the minimal period of T_n. We have*

$$\|T_n\|_{L_\infty} \leqslant \left(\frac{n}{m}\right)^{1/p} \left(\left(\int_0^{2\pi} \left(2t^{\frac{1}{p}-1} \sin \frac{t}{2} \right) \right)^q \frac{dt}{t} \right)^{-1/q} \|T_n\|_{N_{p,q}(M)}, \quad q < \infty; \tag{4.8}$$

$$\|T_n\|_{L_\infty} \leqslant \left(\frac{n}{m}\right)^{1/p} \left(\sup_{0<\alpha<\pi} (2\alpha)^{\frac{1}{p}} \frac{\sin \alpha}{\alpha} \right)^{-1} \|T_n\|_{N_{p,\infty}(M)}. \tag{4.9}$$

Inequalities (4.8) and (4.9) are sharp; they become equalities for $T_n(x) = \cos nx$.

Remark 4.6. An advantage of considering the net spaces is, in particular, that the constants in inequalities (4.8) and (4.9) do not have supremum form as in inequalities (4.1) and (4.2). On the other hand, if M is the collection of all compact sets, then $N_{p,q}(M) = L_{p,q}$, $1 < p \leqslant \infty$; see [NT]. Moreover, $\|T_n\|_{N_{p,q}(M)} \leqslant \|T_n^{**}\|_{L_{p,q}}$ for any p and q, where $T_n^{**}(x) = \frac{1}{x}\int_0^x T_n^*(t)\,dt$.

Proof. Let $q < \infty$. Using (3.18), we get

$$\begin{aligned}
\sup_{T_n \in \mathfrak{M}_n} \frac{m^{1/p}\|T_n\|_{L_\infty}}{n^{1/p}\|T_n\|_{N_{p,q}(M)}} &\leqslant \sup_{T_n \in \mathfrak{M}_n} \frac{m^{1/p}\|T_n\|_{L_\infty}}{n^{1/p}\left(\int_0^{2\pi m/n} \left(t^{1/p}\overline{T_n(t,M)}\right)^q \frac{dt}{t}\right)^{1/q}} \\
&\leqslant \sup_{T_n \in \mathfrak{M}_n} \frac{m^{1/p}}{n^{1/p}\left(\int_0^{2\pi m/n} \left(t^{1/p}\frac{2m}{nt}\sin\frac{nt}{2m}\right)^q \frac{dt}{t}\right)^{1/q}} \\
&= \left(\left(\int_0^{2\pi} \left(2t^{1/p} \frac{\sin\frac{t}{2}}{t} \right) \right)^q \frac{dt}{t} \right)^{-1/q}.
\end{aligned}$$

If $q = \infty$, similarly, using again (3.18), we have

$$\begin{aligned}
\sup_{T_n \in \mathfrak{M}_n} \frac{m^{1/p}\|T_n\|_{L_\infty}}{n^{1/p}\|T_n\|_{N_{p,\infty}(M)}} &\leqslant \sup_{T_n \in \mathfrak{M}_n} \frac{\|T_n\|_{L_\infty}}{\sup\limits_{0<\beta<2\pi m/n} \left(\left(\frac{n\beta}{m}\right)^{\frac{1}{p}} \overline{T_n(\beta,M)}\right)} \\
&\leqslant \sup_{T_n \in \mathfrak{M}_n} \frac{1}{\sup\limits_{0<\beta<2\pi m/n} \left(\left(\frac{n\beta}{m}\right)^{\frac{1}{p}} \frac{2m}{n\beta}\sin\frac{n\beta}{2m}\right)} = \left(\sup_{0<\alpha<\pi} (2\alpha)^{\frac{1}{p}} \frac{\sin\alpha}{\alpha} \right)^{-1}.
\end{aligned}$$

Considering $T_n(x) = \cos nx$ gives sharpness of inequalities (4.8) and (4.9). □

5. Remez inequalities in the multidimensional case

Let $d \in \mathbb{N}$. The following multidimensional Remez inequality for trigonometric polynomials

$$T_n(x) = \sum_{|\mathbf{k}| \leqslant n} c_{\mathbf{k}} e^{i(\mathbf{k},x)}, \quad \mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{Z}^d, \quad c_{\mathbf{k}} \in \mathbb{C}, \quad x \in \mathbb{T}^d,$$

where $|\mathbf{k}| = \|\mathbf{k}\|_{\ell_\infty} = \max\{|k_1|, \cdots, |k_d|\}$, is well known: *For any Lebesgue measurable set $B \subset \mathbb{T}^d$ such that $|B| < \left(\pi/2\right)^d$ we have*

$$\|T_n\|_{L_\infty(\mathbb{T}^d)} \leqslant \exp\left(4dn|B|^{1/d}\right)\|T_n\|_{L_\infty(\mathbb{T}^d \setminus B)}. \tag{5.1}$$

For $d \geqslant 1$ the proof is given in [DP, Kr]; the proof is by induction.

We consider more general trigonometric polynomials. Let $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d$ and

$$\mathfrak{M}_{\mathbf{n}}(d) = \mathfrak{M}_{n_1 \cdots n_d}(d) = \Big\{ T_{\mathbf{n}} \,:\, \mathbb{R} \ni T_{\mathbf{n}}(x) = \sum_{|k_1| \leqslant n_1} \cdots \sum_{|k_d| \leqslant n_d} c_{\mathbf{k}} e^{i(\mathbf{k},x)}, \quad \mathbf{k} \in \mathbb{Z}^d, \quad c_{\mathbf{k}} \in \mathbb{C}, \quad x \in \mathbb{T}^d \Big\}.$$

Note that Remark 3.1 holds for multivariate polynomials from $\mathfrak{M}_{\mathbf{n}}(d)$, that is, Remez's inequality can be studied in the form $T^*_{\mathbf{n}}(0) \leqslant C(\beta) T^*_{\mathbf{n}}(\beta)$, $\beta > 0$, where f^* is the decreasing rearrangement of f on $\mathbb{T}^d$, i.e., $f^*(t) = \inf \Big\{\sigma : \mu\big\{x \in \mathbb{T}^d : |f(x)| > \sigma\big\} \leqslant t\Big\}$ and μ is the d-dimensional Lebesgue measure.

Below we prove analogues of the one-dimensional Remez inequality. For the sake of simplicity, we consider only the case when $d = 2$.

Theorem 5.1. *Let $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$ and $T_{\mathbf{n}}(x_1, x_2) \in \mathfrak{M}_{\mathbf{n}}(2)$. Then*

$$T^*_{\mathbf{n}}(0) \leqslant \frac{1}{\cos\sqrt{\frac{n_1 n_2 \beta}{2}}}\, T^*_{\mathbf{n}}(\beta), \qquad \text{where} \qquad 0 \leqslant \beta < \frac{\pi^2}{2 n_1 n_2}. \tag{5.2}$$

Proof. Let

$$T_{\mathbf{n}}(x_1, x_2) = \sum_{k_1=-n_1}^{n_1} \sum_{k_2=-n_2}^{n_2} c_{k_1,k_2} \exp\Big(i k_1 x_1 + i k_2 x_2\Big).$$

Without loss of generality we assume that

$$\max T_{\mathbf{n}}(x_1, x_2) - \min T_{\mathbf{n}}(x_1, x_2) = 2$$

and

$$T_{\mathbf{n}}(0,0) = \max T_{\mathbf{n}}(x_1, x_2) := \Xi \geqslant 1.$$

Let $\tau = \frac{p}{q} \in \mathbb{Q}$, $p \in \mathbb{Z}$, and $q \in \mathbb{N}$. Consider the polynomial $T_{\mathbf{n}}(x_1, x_2)$ on the line $x_1 = x, x_2 = \tau x$, i.e.,

$$T_{\mathbf{n}}(x, \tau x) = \sum_{k_1=-n_1}^{n_1} \sum_{k_2=-n_2}^{n_2} c_{k_1,k_2} \exp\Big(i(k_1 q + k_2 p)\frac{x}{q}\Big) =: \widetilde{T}\big(\frac{x}{q}\big),$$

where $\widetilde{T}(y)$ is the trigonometric polynomial of degree at most $n_1 q + n_2|p|$ with respect to y. Using Lemma 2.2,

$$\widetilde{T}(y) \geqslant \cos(n_1 q + n_2|p|)y + \Xi - 1, \qquad -\frac{\pi}{n_1 q + n_2|p|} < y < \frac{\pi}{n_1 q + n_2|p|}.$$

Let $z = (n_1 q + n_2|p|)y$, i.e., $y = \frac{z}{(n_1 q + n_2|p|)}$,

$$\cos z + \Xi - 1 \leqslant \widetilde{T}\Big(\frac{z}{n_1 q + n_2|p|}\Big) = T_{\mathbf{n}}\Big(\frac{qz}{n_1 q + n_2|p|}, \frac{pz}{n_1 q + n_2|p|}\Big), \tag{5.3}$$

where $-\pi < z < \pi$.

Denote

$$Q_z = Q_z(\mathbf{n}) = \Big\{(x_1, x_2) : |x_1 n_1 + x_2 n_2| \leqslant z, |x_1 n_1 - x_2 n_2| \leqslant z\Big\}, \qquad z > 0.$$

Then inequality (5.3) implies

$$T_{\mathbf{n}}(x_1, x_2) \geqslant \cos z + \Xi - 1 \geqslant 0, \qquad 0 \leqslant z \leqslant \frac{\pi}{2}$$

for any $(x_1, x_2) \in \Big\{(x_1, x_2) : x_2 = \tau x_1, (x_1, x_2) \in \partial Q_z\Big\}$ and $\tau \in \mathbb{Q}$. Here ∂Q_z is the boundary of Q_z.

Since $T_{\mathbf{n}}$ is continuous,

$$T_{\mathbf{n}}(x_1, x_2) \geqslant \cos z + \Xi - 1 \geqslant 0, \quad 0 \leqslant z \leqslant \frac{\pi}{2}$$

holds for any points of Q_z. Indeed, if now $(x_1^0, x_2^0) \in Q_z$, there exists $0 \leqslant z_0 \leqslant z$ such that $(x_1^0, x_2^0) \in \partial Q_{z_0}$. Therefore, on the boundary ∂Q_{z_0}, we get

$$T_{\mathbf{n}}(x_1^0, x_2^0) \geqslant \cos z_0 + \Xi - 1 \geqslant \cos z + \Xi - 1.$$

Hence,

$$T_{\mathbf{n}}(x_1, x_2) \geqslant \cos z + \Xi - 1 \geqslant 0, \quad 0 \leqslant z \leqslant \frac{\pi}{2}$$

for any $(x_1, x_2) \in Q_z$ and then

$$T^*_{\mathbf{n}}\Big(\mu(Q_z)\Big) \geqslant \cos z + \Xi - 1 \geqslant 0.$$

Finally,

$$\sup_{T_{\mathbf{n}} \in \mathfrak{M}_{\mathbf{n}}(2)} \frac{T^*_{\mathbf{n}}(0)}{T^*_{\mathbf{n}}(\mu(Q_z))} \leqslant \sup_{\Xi \geqslant 1} \frac{\Xi}{\cos z + \Xi - 1} = \frac{1}{\cos z}$$

and since $\mu(Q_z) = \frac{2z^2}{n_1 n_2} = \beta$, we arrive at (5.2). $\square$

In the general case ($d \geqslant 2$) the proof is similar.

Remark 5.2. Let $d \geqslant 2$, $\mathbf{n} \in \mathbb{N}^d$, and $T_{\mathbf{n}} \in \mathfrak{M}_{\mathbf{n}}(d)$. Then

$$T^*_{\mathbf{n}}(0) \leqslant \frac{1}{\cos \sqrt[d]{\beta \prod_{j=1}^d \frac{jn_j}{2}}} T^*_{\mathbf{n}}(\beta), \tag{5.4}$$

where $0 \leqslant \beta < \frac{\pi^d}{\prod_{j=1}^d jn_j}$.

In particular, we have the following generalization of (5.1) for $T_{\mathbf{n}} \in \mathfrak{M}_{\mathbf{n}}(d)$:

$$\|T_{\mathbf{n}}\|_{L_\infty(\mathbb{T}^d)} \leqslant \exp\left(0,5d(n_1 \dots n_d |B|)^{1/d}\right) \|T_{\mathbf{n}}\|_{L_\infty(\mathbb{T}^d \setminus B)}, \quad \mu(B) = |B| < \prod_{j=1}^d \frac{2}{jn_j}. \tag{5.5}$$

Indeed, since $\cos t \exp(t) \geqslant 1$ for $0 < t \leqslant 1$, then for $d \geqslant 2$

$$\frac{1}{\cos \sqrt[d]{|B| \prod_{j=1}^d \frac{jn_j}{2}}} \leqslant \exp\left(\sqrt[d]{|B| \prod_{j=1}^d \frac{jn_j}{2}}\right) \leqslant \exp\left(0.5\, d \big(n_1 \dots n_d |B|\big)^{1/d}\right), \quad |B| < \prod_{j=1}^d \frac{2}{jn_j}.$$

Let us now present the multidimensional Nikol'skii inequality in the Lorentz spaces (cf. [DP]).

Corollary 5.3. *Let $0 < p < \infty$, $0 < q \leqslant \infty$, $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{N}^d$, $d \geqslant 2$ and $T_{\mathbf{n}} \in \mathfrak{M}_{\mathbf{n}}(d)$. Then*

$$\sup_{T_{\mathbf{n}} \in \mathfrak{M}_{\mathbf{n}}} \frac{\|T_{\mathbf{n}}\|_{L_\infty(\mathbb{T}^d)}}{\left(\prod_{j=1}^d n_j\right)^{1/p} \|T_{\mathbf{n}}\|_{L_{p,q}(\mathbb{T}^d)}} \leqslant \left(\frac{d!}{2^d}\right)^{1/p} \left(d \int_0^{\frac{\pi}{2}} \left(t^{\frac{d}{p}} \cos t\right)^q \frac{dt}{t}\right)^{-\frac{1}{q}}. \tag{5.6}$$

Proof. Making use of Theorem 5.1 and Remark 5.2 for $T_{\mathbf{n}} \in \mathfrak{M}_{\mathbf{n}}$, we get

$$\begin{aligned}
\|T_{\mathbf{n}}\|_{L_{p,q}(\mathbb{T}^d)} &= \left(\int_0^{(2\pi)^d} \left(t^{\frac{1}{p}} T^*_{\mathbf{n}}(t)\right)^q \frac{dt}{t}\right) \\
&\geqslant \left(\int_0^{\frac{\pi^d}{\prod_{j=1}^d jn_j}} \left(t^{\frac{1}{p}} T^*_{\mathbf{n}}(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \geqslant \left(\int_0^{\frac{\pi^d}{\prod_{j=1}^d jn_j}} \left(t^{\frac{1}{p}} \cos \sqrt[d]{t \prod_{j=1}^d \frac{jn_j}{2}}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} T^*_{\mathbf{n}}(0) \\
&= \left(\prod_{j=1}^d \frac{jn_j}{2}\right)^{-1/p} \left(d \int_0^{\frac{\pi}{2}} \left(t^{\frac{d}{p}} \cos t\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} T^*_{\mathbf{n}}(0),
\end{aligned}$$

and we arrive at (5.6). $\square$

In some cases the integral on the right-hand side of (5.6) can be simply calculated. For example, for $d = 2$, we have

$$\|T_{\mathbf{n}}\|_\infty \leqslant \frac{n_1 n_2}{4\left(\frac{\pi}{2} - 1\right)} \|T_{\mathbf{n}}\|_{L_1},$$

$$\|T_{\mathbf{n}}\|_\infty \leqslant \frac{2\sqrt{n_1 n_2}}{\sqrt{\pi^2 - 4}} \|T_{\mathbf{n}}\|_{L_2},$$

$$\|T_{\mathbf{n}}\|_\infty \leqslant \frac{\sqrt{n_1 n_2}}{2^{3/2}} \|T_{\mathbf{n}}\|_{L_{2,1}}.$$

Similar to Remark 4.4 we can prove the following (L_p, L_{p_1})-Nikol'skii inequality for $T_{\mathbf{n}} \in \mathfrak{M}_{\mathbf{n}}(d), d \geqslant 2$,

$$\|T_{\mathbf{n}}\|_{L_{p_1}(\mathbb{T}^d)} \leqslant C^{1-p/p_1} \left(\prod_{j=1}^d n_j\right)^{\frac{1}{p} - \frac{1}{p_1}} \|T_{\mathbf{n}}\|_{L_p(\mathbb{T}^d)}, \quad 0 < p < p_1 < \infty,$$

where C is the constant in the right-hand side of (5.6) with $p = q$.

Finally, we note that Remez type inequalities hold not only for trigonometric polynomials but also for wider classes of functions. Denote by $\mathcal{E}_{\nu,\mu}$, $\nu, \mu > 0$ the collection of functions $f \in L_\infty(\mathbb{R}^2)$ such that $\frac{\partial^2 f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \in C(\mathbb{R}^2)$, $\alpha_1 + \alpha_2 = 2$, $\alpha_i \geqslant 0$ and

$$\left\| \frac{\partial^2 f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right\|_{L_\infty(\mathbb{R}^2)} \leqslant \nu^{\alpha_1} \mu^{\alpha_2} \left\| f \right\|_{L_\infty(\mathbb{R}^2)}, \qquad \alpha_1 + \alpha_2 = 2,\ \alpha_i \geqslant 0.$$

This class in particular contains all functions of exponential type (ν, μ) with regards to x_1 and x_2, respectively.

Theorem 5.4. *Let $\nu, \mu > 0$ and $0 \leqslant \beta < \frac{4}{\nu\mu}$. For any $f \in \mathcal{E}_{\nu,\mu}$ we have*

$$\sup_{f \in \mathcal{E}_{\nu,\mu}} \frac{f^*(0)}{f^*_{\nu,\mu}(\beta)} \leqslant \left(1 - \frac{(\nu\mu\beta)}{4} \right)^{-1}. \tag{5.7}$$

Proof. Without loss of generality we assume that

$$\max_{(x,y) \in \mathbb{R}^2} |f(x,y)| = f(0,0) = 1.$$

Then since $(0,0)$ is an extreme point, taking into account the definition of $\mathcal{E}_{\nu,\mu}$ and Taylor's formula, there exists $(c_1, c_2) \in (-|x|, |x|) \times (-|y|, |y|)$ such that

$$f(x,y) = f(0,0) + \frac{1}{2} \left(\frac{\partial^2 f}{\partial x^2}(c_1,c_2) x^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(c_1,c_2) xy + \frac{\partial^2 f}{\partial y}(c_1,c_2) y^2 \right) \geqslant 1 - \frac{1}{2}(\nu|x| + \mu|y|)^2.$$

Suppose $z = |x|\nu + |y|\mu$, then using the same argument as in the proof of Theorem 5.1,

$$f(x,y) \geqslant 1 - \frac{z^2}{2} \geqslant 0, \quad 0 < z < \sqrt{2}$$

for any point (x,y) belonging to $Q_z = \Big\{ (x,y) : |x|\nu + |y|\mu \leqslant z \Big\}$. Finally, for $0 \leqslant \beta < \frac{4}{\nu\mu}$ we have

$$f^*(\beta) \geqslant 1 - \frac{\nu\mu\beta}{4} > 0$$

and therefore (5.7) follows. □

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