COMPLEX AND REAL HAUSDORFF OPERATORS

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Abstract. Hausdorff operators (Hausdorff summability methods) appeared long ago aiming to solve certain classical problems in analysis. Modern theory of Hausdorff operators started with the work of Siskakis in complex analysis setting and with the work of Georgakis and Liflyand-Móricz in the Fourier transform setting. While Hausdorff operators for power series are still studied mostly in dimension one, the center of attraction of interesting problems for the Hausdorff operators of Fourier integrals lies in the multi-variate setting. One of the most general definitions of the Hausdorff operator reads as

$$(\mathcal{H}f)(x) = (\mathcal{H}_\Phi f)(x) = (\mathcal{H}_{\Phi,A} f)(x) = \int_{\mathbb{R}^n} \Phi(u) f\big(xA(u)\big)\, du,$$

where $A = A(u) = (a_{ij})_{i,j=1}^n = \big(a_{ij}(u)\big)_{i,j=1}^n$ is the $n \times n$ matrix with the entries $a_{ij}(u)$ being measurable functions of u. This matrix may be singular on a set of measure zero at most; $xA(u)$ is the row n-vector obtained by multiplying the row n-vector x by the matrix A.

However, we first give a brief overview of Hausdorff operators in other settings. For Fourier transforms, many details are given in dimension one then. Recent results in which conditions on the couple (Φ, A) are found to provide the boundedness of the operator in the real Hardy space are discussed. There now exist two proofs, one based on the H^1-BMO duality while the other on atomic decomposition. The case of product Hardy spaces is also studied. Many open problems in the subject are formulated.

Contents

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1. Introduction

The main goal of these notes is to give a picture of the status of a modern subject which relates old and classical notions of Hausdorff summability with modern theories of Hardy spaces. To make the picture comprehensive, we are going to briefly overview all the main parts of this topic. The scheme of the relations between different subjects within this area is given below.

$$
\begin{array}{ccc}
Hausdorff\ summability\ of\ number\ series & \Longleftrightarrow & (little)\ moment\ problem \\
\Downarrow & & \\
Hausdorff\ summability\ of\ Fourier\ series & & \\
\Downarrow & & \\
Hausdorff\ summability\ of\ power\ series & \Longleftrightarrow & Hardy\ spaces,\ BMO,\ etc. \\
\Downarrow & & \\
Hausdorff\ summability\ of\ Fourier\ transforms & \Longleftrightarrow & Hardy\ spaces,\ BMO,\ etc. \\
\Downarrow \qquad \Downarrow & & \\
One-dimensional\ case \quad Multidimensional\ case & &
\end{array}
$$

The history of what is called Hausdorff summability methods goes back to 1917 when Hurwitz and Silverman in [33] studied a family of methods (see Section 2 below) within the classical framework of summability of number series when regularity (consistency) and comparison of various methods are the main goal. However, a "genuine" history had started with the paper [29] in 1921, in which Hausdorff not only rediscovered the same summability methods but developed their study by associating them with the famous and important moment problem for a finite interval. Under the popularity of summability theory in those times and later period, Hausdorff methods took their place and proved to be of continuous interest. Let us mention some monographs where enough attention is payed to them, like [28], [56], [47], and [5] - unfortunately never translated, or a survey paper [16]. We will give a brief overview of this topic in Section 2 with the emphasis on the connection to the moment problem. Certain details and some proofs are omitted there but the picture in whole seems to be clear enough.

We would like to mention that many applications of Hausdorff summability were made to Fourier series in one and several dimensions (just a couple of random examples: [31, 43, 53, 24]); we will not touch this topic here.

However, the part we are interested in is related to power series of analytic functions and started with the work of Siskakis [48] on composition operators and Cesàro means in H^p spaces and his nice short proof for H^1 in [49]. Already after appearance of [40], general Hausdorff summability was considered for analytic functions in Hardy spaces in [14] and [15] as well as in some other spaces. We fill unavoidable to give a clear picture of this subject to what Section 3 is devoted.

The next natural step was extension of the results from [48, 49] to the Fourier transform setting on the real line. It was done in [19] and in a slightly different manner in [23]. It was [19] that inspired Móricz and the author to try a more general averaging than Cesàro (Hardy) one. The paper [40] had opened, in a sense, a new period in the study of Hausdorff summability. Besides the mentioned progress it inspired in the analytic functions setting, it also led to a number of new open problems, first of all in several dimensions. This growing interest in these problems was mainly connected not with the type of summability itself - it had already been actually known (see, e.g., [18]), but with involving various spaces in consideration, first of all Hardy spaces. In Section 4 we present the initial proof from [40] and discuss certain related problems.

The natural passage from dimension one to several dimensions was made almost in parallel. The paper [41] was written immediately after [40]. Then other papers followed, [36] is pretty recent. In Section 5, the longest and most detailed, a key point is Theorem 18 from [38] that appeared after acceptance of [36] for publication. In that section we discuss various definitions of Hardy spaces and, correspondingly, various existing and possible proofs of the boundedness of Hausdorff operators on Hardy spaces. In the end of the section we give some results for BMO.

In the last Section 6 we present a collection of open problems. The number of open problems corresponding to the multidimensional case is larger than that for other sections as well as their discussion is more detailed.

Thus, there are two main “personages” in these notes: Hausdorff summability and Hardy space, both in various settings and versions. Their interplay is what makes the whole subject interesting.

It was not our aim to give a complete list of references, we give only those immediately involved in the study. Some others can be found in the papers referred to.

In what follows $a \ll b$ means that $a \le Cb$, with C being an absolute constant in this and any other occurrence. In our study we are not interested in explicit indication of these constants. If a constant depends on certain parameters, they will be indicated as subscripts, like C_p.

2. Hausdorff summability of number series

Hausdorff means, the Cesàro means among them, are known long ago in connection with summability of number series. Let us briefly describe this subject. We follow the nice way it is presented in [56, Ch.III] (see also [16]).

Let the sequence $s_0, s_1, s_2, ...$ be represented by the infinite matrix S in which it is the first column, while the rest of the entries are zeros. Similarly, the sequence $t_0, t_1, t_2, ...$

is represented by the matrix T. Explicitly,

$$S = \begin{pmatrix} s_1 & 0 & 0 & ... \\ s_2 & 0 & 0 & ... \\ s_3 & 0 & 0 & ... \\ . & . & . & ... \\ . & . & . & ... \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} t_1 & 0 & 0 & ... \\ t_2 & 0 & 0 & ... \\ t_3 & 0 & 0 & ... \\ . & . & . & ... \\ . & . & . & ... \end{pmatrix}.$$

Let M be the infinite diagonal matrix with the sequence $\mu_0, \mu_1, \mu_2, ...$ as its diagonal entries:

$$M = \begin{pmatrix} \mu_1 & 0 & 0 & ... \\ 0 & \mu_2 & 0 & ... \\ 0 & 0 & \mu_3 & ... \\ . & . & . & ... \\ . & . & . & ... \end{pmatrix}.$$

Let, finally,

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & ... \\ 1 & -1 & 0 & 0 & 0 & ... \\ 1 & -2 & 1 & 0 & 0 & ... \\ 1 & -3 & 3 & -1 & 0 & ... \\ . & . & . & . & . & ... \\ . & . & . & . & . & ... \end{pmatrix}$$

be the difference matrix. The latter, with the entries $r_{mn} = (-1)^n \binom{m}{n}$ for $n = 0, 1, 2, ..., m$, and zeros otherwise, is self-reciprocal, that is, $R = R^{-1}$. The reason for the term *difference matrix* becomes clear if one observes that

$$\sum_{n=0}^{\infty} r_{mn} s_n = \sum_{n=0}^{m} (-1)^n \binom{m}{n} s_n = \Delta^m s_0,$$

with $\Delta s_k = s_k - s_{k+1}$ and $\Delta^m = \Delta(\Delta^{m-1})$.

Now, given a matrix $A = (a_{mn})$, $m, n = 0, 1, 2, ...,$

$$T = AS \tag{1}$$

transforms the matrix S into T or, if we consider only the first columns of S and T, the sequence $\{s_n\}$ into the sequence $\{t_n\}$. Then the sequence $\{s_n\}$ is summable by the matrix A to the sum s if the sequence $\{t_n\}$ is defined by (1) and if

$$\lim_{m\to\infty} t_m = s. \tag{2}$$

More explicitly, this means that the series

$$t_m = \sum_{n=0}^{\infty} a_{mn} s_n, \tag{3}$$

where $m = 0, 1, 2, ...$, and the numbers a_{mn} are the entries of the matrix A, all converge and (2) holds.

One of the basic examples is that when

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & ... \\ 1/2 & 1/2 & 0 & 0 & ... \\ 1/3 & 1/3 & 1/3 & 0 & ... \\ . & . & . & . & ... \\ . & . & . & . & ... \end{pmatrix},$$

then

$$t_m=\frac{s_0+s_1+\cdots+s_m}{m+1},$$

which leads to the Cesàro summability.

A method of summability is called *regular* (the term *consistent* is used sometimes) if every convergent sequence is summable by it to the actual limit of the sequence.

We are now in a position to define Hausdorff summability.

Definition 1. The matrix A is a *Hausdorff matrix* corresponding to the sequence $\{\mu_n\}$, $n=0,1,2,...$, if $A=RMR^{-1}$.

It is easily seen that multiplication of Hausdorff matrices is commutative. Indeed, for two Hausdorff matrices A_1 and A_2, with corresponding diagonal matrices M_1 and M_2, respectively, we have

$$\begin{aligned} A_1A_2 &= RM_1R^{-1}\,RM_2R^{-1}=RM_1\,M_2R^{-1}=RM_2\,M_1R^{-1} \\ &= RM_2R^{-1}\,RM_1R^{-1}=A_2A_1. \end{aligned}$$

We now wish to determine what sequences $\{\mu_n\}$ lead to *regular* Hausdorff matrices. The celebrated Toeplitz theorem is a natural tools to test this.

Theorem 2. *Summability by the matrix A is regular if and only if a constant K exists such that*

$$\sum_{n=0}^{\infty}|a_{mn}|<K, \qquad m=0,1,2,...; \tag{4}$$

$$\lim_{m\to\infty}a_{mn}=0, \qquad n=0,1,2,...; \tag{5}$$

$$\lim_{m\to\infty}\sum_{n=0}^{\infty}a_{mn}=1. \tag{6}$$

Before applying this theorem to the Hausdorff summability method we first obtain the Hausdorff matrix in terms of the given sequence $\{\mu_n\}$. By definition,

$$T=RMR^{-1}S. \tag{7}$$

We wish to determine the elements a_{mn} so that (7) will be of the form (3). Equation (7) means

$$t_m = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\mu_j\sum_{n=0}^{j}(-1)^n\binom{j}{n}s_n$$

$$= \sum_{n=0}^{m} s_n \sum_{j=n}^{m} (-1)^{j+n} \binom{m}{j} \binom{j}{n} \mu_j.$$

Employing the identity

$$\binom{m}{j}\binom{j}{n} = \binom{m}{n}\binom{m-n}{j-n}$$

for $n \le j \le m$, we obtain

$$\begin{aligned} t_m &= \sum_{n=0}^{m} \binom{m}{n} s_n \sum_{j=0}^{m-n} (-1)^j \binom{m-n}{j} \mu_{j+n} \\ &= \sum_{n=0}^{m} \binom{m}{n} \Delta^{m-n}\mu_n \, s_n. \end{aligned}$$

We have thus proved that

$$a_{mn} = \binom{m}{n} \Delta^{m-n}\mu_n$$

for $n = 0, 1, ..., m$, and zero otherwise. This is the *necessary and sufficient* condition for the matrix A to correspond to the sequence $\{\mu_n\}$.

With this and Theorem 2 in hand, we are in a position to obtain a criterion for regularity of the Hausdorff method.

Theorem 3. *The Hausdorff summability method corresponding to the sequence* $\{\mu_n\}$ *is regular if and only if*

$$\mu_n = \int_0^1 t^n d\mu(t), \qquad n = 0, 1, ..., \tag{8}$$

where $\mu(t)$ *is of bounded variation in* $(0, 1)$, $\mu(0) = \mu(0+) = 0$, *and* $\mu(1) = 1$.

To prove this criterion, we have only to apply Theorem 2 to the matrix A. To show that the main condition (4) reduces to the boundedness of variation needs certain efforts. We omit this by referring the reader to the mentioned bibliography.

Condition (5) and (6) of Theorem 2 become

$$\lim_{m\to\infty} \binom{m}{n} \int_0^1 t^n (1-t)^{m-n} d\mu(t) = 0, \quad n = 0, 1, 2, ...,$$

and

$$\lim_{m\to\infty} \int_0^1 d\mu(t) = 1,$$

from which the rest of the conditions is easily derived.

It was in fact the study of the summability of divergent series which led Hausdorff to investigation of the (little) moment problem: given a sequence $\{\mu_n\}$ under what conditions it is possible to determine a function $\mu(t)$ of bounded variation in $(0, 1)$ such that (8) holds. Such $\{\mu_n\}$ is called a *moment sequence*. The solution is as follows: the *necessary and sufficient* condition that $\{\mu_n\}$ should be a moment sequence is that a constant K exists such that for $a_{mn} = \binom{m}{n}\Delta^{m-n}\mu_n$

$$\sum_{n=0}^{m} |a_{mn}| < K, \qquad m = 0, 1, 2, ...$$

It might be of interest to consider various multidimensional generalizations of this subject, with respect to various types of ordering the elements of the series, or in other words with respect to various types of summation. Certain partial results do exist see, e.g., [1]) but to our best knowledge the topic is not generalized in full.

3. Hausdorff summability of power series

It was the analytic functions setting where connections of Hardy spaces as well as certain related ones to Hausdorff summability historically came into play. We follow the way the subject and relevant results are given in [48, 49, 14, 15]. We first give all needed prerequisites.

3.1. **Hardy spaces.**

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$. For $1 \le p < \infty$ the Hardy space H^p is the space of analytic functions $f : D \to \mathbb{C}$ such that

$$||f||_{H^p} = \sup_{r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

With this norm H^p is a Banach space (and Hilbert for $p = 2$). If $1 \le p \le q < \infty$ then $H^1 \supset H^p \supset H^q$. Functions $f \in H^p$ possess boundary values (non-tangential limits) $f(e^{i\theta})$ which are p-integrable on ∂D. Identifying f with its boundary function provides an isometric embedding of H^p into $L^p(\partial D)$, the norm in the latter will be denoted by $|| \cdot ||_p$. If $f \in H^p$ then for $z \in D$

$$|f(z)| \le 2^{1/p} \frac{||f||_p}{(1 - |z|)^{1/p}},$$

see [11, p.36].

For each function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^1$, Hardy's inequality (see, e.g., [11, p.48]) holds true

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \le \pi ||f||_{H^1}. \tag{9}$$

Every analytic function $a(z) : D \to D$, that is, mapping the unit disk into itself, induces a bounded composition operator

$$W_a f(z) = f(a(z))$$

on the Hardy space H^p; see [11, p.29]. In addition, if $b(z)$ is a bounded analytic function on D then the weighted composition operator

$$W_{a,b} f(z) = b(z) f(a(z))$$

is bounded on H^p as well.

3.2. **Cesàro means for power series.**

Cesàro means for power series from the Hardy space H^1 in the unit disk were considered by Siskakis. Such Cesàro means are constructed by replacing the coefficients a_k in the expansion

$$f(z)=\sum_{k=0}^{\infty} a_k z^k,$$

with $f\in H^1$, by their Hardy transform

$$\frac{1}{k+1}\sum_{p=0}^{k} a_p,$$

which results in

$$Cf(z)=\sum_{k=0}^{\infty}\left(\frac{1}{k+1}\sum_{p=0}^{k} a_p\right) z^k. \tag{10}$$

The following elegant proof of the boundedness of the corresponding operator on H^1 is given in [49]. It is worth considering it not only because of its brevity and elegance, but since generalizing this proof to several dimensions might give certain hints on the ways and directions of such generalizations.

Let us proceed to this proof. By computation we obtain

$$Cf(z)=\int_0^1 f(tz)\frac{1}{1-tz}\,dt.$$

Indeed,

$$\begin{aligned}\int_0^1 f(tz)\frac{1}{1-tz}\,dt &= \sum_{p=0}^{\infty} a_p z^p\int_0^1 \frac{t^p}{1-tz}\,dt\\ &= \sum_{p=0}^{\infty} a_p \sum_{k=0}^{\infty} z^{k+p}\int_0^1 t^{k+p}dt\\ &= \sum_{p=0}^{\infty} a_p \sum_{k=p}^{\infty} z^{k}\int_0^1 t^{k}dt\\ &= \sum_{p=0}^{\infty} a_p \sum_{k=p}^{\infty} \frac{z^k}{k+1}=\sum_{k=0}^{\infty}\left(\frac{1}{k+1}\sum_{p=0}^{k} a_p\right) z^k.\end{aligned}$$

For the series convergent absolutely and uniformly we change the order of summation and integration freely.

For $0<r<1$ and $f\in H^1$, we denote by

$$M_q(f;r)=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(rte^{i\theta})|^q\,d\theta\right)^{1/q}$$

the integral means on $|z|=r$ of an analytic f. We thus have

$$\begin{aligned} M_1(Cf;r) &= \frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^1 f(rte^{i\theta})\frac{1}{1-rte^{i\theta}}\,dt\right|d\theta \\ &\le \int_0^1 \frac{1}{2\pi}\int_0^{2\pi}|f(rte^{i\theta})|\,\frac{1}{|1-rte^{i\theta}|}\,d\theta\,dt. \end{aligned}$$

Applying Hölder's inequality and denoting by

$$P(r,\theta)=\frac{1-r^2}{1-2r\cos\theta+r^2}=\mathrm{Re}\frac{1+re^{i\theta}}{1-re^{i\theta}}$$

the Poisson kernel, we obtain

$$\begin{aligned} M_1(Cf;r) &\le \int_0^1\left(\frac{1}{2\pi}\int_0^{2\pi}|f(rte^{i\theta})|^2\,d\theta\right)^{1/2}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{1-rte^{i\theta}}\right|^2 d\theta\right)^{1/2} dt \\ &= \int_0^1 M_2(f;rt)\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-r^2t^2}\,P(rt,\theta)d\theta\right)^{1/2} dt. \end{aligned}$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}P(r,\theta)\,d\theta=1,$$

the last integral is dominated by

$$\int_0^1 M_2(f;rt)(1-r^2t^2)^{-1/2}dt.$$

Since the absolute value of an analytic function in any positive power is a subharmonic function, $M_q(f;s)$ increases in s (see, e.g., [30, Th.2.12]), and the last integral is dominated by

$$\int_0^1 M_2(f;t)(1-t)^{-1/2}dt.$$

We now need the following Hardy-Littlewood result (see [26]).

Lemma 1. *If $f\in H^1$ and $q>1$, then*

$$\int_0^1 M_q(f;s)(1-s)^{-1/q}ds\le C_q||f||_{H^1},$$

where the constant C_q depends only on q.

Applying this lemma with $q=2$ to the last integral yields

$$\int_0^1 M_2(f;t)(1-t)^{-1/2}dt\le C||f||_{H^1},$$

which completes the proof.

It is worth mentioning that this proof from [49] is not applicable to H^p for any p except $p=1$. More general approach is used in [48] but, again, we think that it is worth considering this nice partial case separately.

3.3. **Hausdorff means for power series.**

Later, already after appearance of the paper [40], general Hausdorff matrices were considered in [14] (for a continuation, see [15]) as follows.

Let, as in the previous section, Δ be the forward difference operator defined on scalar sequences $\mu = (\mu_n)_{n=0}^{\infty}$ by $\Delta\mu_n = \mu_n - \mu_{n+1}$ and $\Delta^k\mu_n = \Delta(\Delta^{k-1}\mu_n)$ for $k = 1, 2, ...$ with $\Delta^0\mu_n = \mu_n$.

Setting

$$c_{n,k} = \binom{n}{k}\Delta^{n-k}\mu_k, \qquad k \le n,$$

we define the *Hausdorff matrix* $\mathcal{H} = \mathcal{H}_\mu$ with generating sequence μ to be the lower triangular matrix with the entries

$$\mathcal{H}_\mu(i,j) = \begin{cases} 0, & i < j \\ c_{i,j}, & i \ge j \end{cases}.$$

It induces two operators on spaces of power series which are formally given by

$$\mathcal{H}_\mu f(z) = \mathcal{H}_\mu\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} c_{n,k}a_k\right)z^n,$$

which is obtained by letting the matrix $\mathcal{H}_\mu$ multiply the Taylor coefficients of f, and

$$\mathcal{A}_\mu f(z) = \mathcal{A}_\mu\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} c_{n,k}a_n\right)z^k,$$

which is obtained by letting the transposed matrix $\mathcal{A}_\mu = \mathcal{H}_\mu^*$ to act on the Taylor coefficients of f. Such a matrix $\mathcal{A}_\mu$ is called a *quasi-Hausdorff matrix.* The convergence of the power series $\mathcal{A}_\mu f$ is more delicate than that of $\mathcal{H}_\mu$. However, it is clear that if f is a polynomial then $\mathcal{A}_\mu f$ is also a polynomial. If the space considered contains the polynomials, we may ask whether $\mathcal{A}_\mu$ extends to a bounded operator on the corresponding space.

An important special case of such matrices occurs when μ_n is the moment sequence of a finite (positive) Borel measure μ on $(0, 1]$:

$$\mu_n = \int_0^1 t^n d\mu(t), \qquad n = 0, 1, ...$$

In this case for $k \le n$

$$\begin{aligned} c_{n,k} &= \binom{n}{k}\Delta^{n-k}\int_0^1 t^k\, d\mu(t) \\ &= \binom{n}{k}\int_0^1 [t^k - \binom{n-k}{1}t^{k+1} + ... + t^n]\, d\mu(t) \\ &= \binom{n}{k}\int_0^1 t^k(1-t)^{n-k}d\mu(t). \end{aligned}$$

All this becomes more transparent if one recalls the structure and properties of Hausdorff means from the previous section.

It follows from the work of Hardy [27] that if the measure μ satisfies

$$\int_0^1 t^{-1/p} d\mu(t) < \infty,$$

then $\mathcal{H}_\mu$ determines a bounded linear operator

$$\mathcal{H}_\mu : \{a_n\} \to \{A_n\}, \qquad A_n = \sum_{k=0}^{n} c_{n,k} a_k, \quad n = 0, 1, ...,$$

on the sequence space l^p, $1 < p < \infty$, whose norm is exactly the last integral.

Various choices of the measure μ give rise to well known classical matrices. For example, when μ is the Lebesgue measure one has the Cesàro matrix. Indeed, since

$$\sum_{k=0}^{n} \binom{n}{k} \int_0^1 t^k (1-t)^{n-k} dt = \int_0^1 dt = 1,$$

it suffices to prove that all $c_{n,k}$ are equal to each other in this case. Integrating by parts, we obtain

$$\binom{n}{k+1} \int_0^1 t^{k+1} (1-t)^{n-k-1} dt = \binom{n}{k+1} \frac{k+1}{n-k} \int_0^1 t^k (1-t)^{n-k} dt.$$

Since

$$\binom{n}{k+1} \frac{k+1}{n-k} = \binom{n}{k},$$

we have $c_{n,k+1} = c_{n,k}$, which completes the proof.

3.4. **Hausdorff matrices and composition operators.**

The study of Hausdorff means for analytic functions is based on relating them with certain families of composition operators. The latter under certain conditions, as we have seen above, bring us into the Hardy space.

For $t \in (0, 1]$ and $z \in D$ the two families of mappings of the disk into itself

$$\phi_t(z) = \frac{tz}{(t-1)z + 1}$$

and

$$\psi_t(z) = tz + 1 - t,$$

and the family of bounded functions on D

$$w_t(z) = \frac{1}{(t-1)z + 1}$$

will be used to construct composition operators associated with Hausdorff matrices.

We define then

$$S_\mu f(z) = \int_0^1 w_t(z) f(\phi_t(z))\, d\mu(t).$$

The integral is finite. Indeed,

$$|(t-1)z + 1| \geq 1 - |(t-1)z| \geq 1 - |z|,$$

and hence $|w_t(z)| \le \frac{1}{1-|z|}$. On the other hand, the last inequality can be finished in a different way

$$1 - |(t-1)z| \ge 1 - (1-t) = t,$$

and $|\phi_t(z)| \le |z|$. By this

$$|w_t(z) f(\phi_t(z))| \le \frac{1}{1-|z|} \sup_{|\zeta| \le |z|} |f(\zeta)|.$$

There is also a need in the integral

$$T_\mu f(z) = \int_0^1 f(\psi_t(z))\, d\mu$$

for those analytic functions f and points z for which it is defined.

Lemma 2. *Let μ be a finite positive Borel measure on $(0,1]$ and let f be analytic in D. Then the power series $\mathcal{H}_\mu f(z)$ absolutely converges in D and $\mathcal{H}_\mu f(z) = S_\mu f(z)$ for every $z \in D$.*

Proof. The absolute convergence of $\mathcal{H}_\mu f(z)$ follows from

$$\begin{aligned}
&\sum_{n=0}^\infty \left| \sum_{k=0}^n c_{n,k} a_k \right| |z|^n \\
\le\ & \int_0^1 \sum_{k=0}^\infty |a_k| \left[\sum_{n=k}^\infty \binom{n}{k} (1-t)^{n-k} |z|^{n-k} \right] t^k |z|^k d\mu(t) \\
=\ & \int_0^1 \frac{1}{1-(1-t)|z|} \sum_{k=0}^\infty |a_k| \left(\frac{t|z|}{1-(1-t)|z|} \right)^k d\mu(t) \\
\le\ & \frac{1}{1-|z|} \mu(0,1] \sum_{k=0}^\infty |a_k||z|^k < +\infty.
\end{aligned}$$

Similar calculations but with no absolute values give $\mathcal{H}_\mu f(z) = S_\mu f(z)$ and thus complete the proof. □

The following trivial lemma is a counterpart of Lemma 2 for $\mathcal{A}_\mu f$. The reason the two lemmas are different is that $\sum\limits_{n=k}^\infty c_{n,k} a_n$ may not converge.

Lemma 3. *Let μ be a finite positive Borel measure on $(0,1]$. Then for each polynomial f the function $\mathcal{A}_\mu f(z)$ is also a polynomial and $\mathcal{A}_\mu f(z) = T_\mu f(z)$ for every $z \in D$.*

3.5. **Hausdorff means on the Hardy space.**

We are now in a position to formulate and prove a criterion for boundedness of the Hausdorff means on H^1.

Theorem 4. *Let μ be a finite positive Borel measure on $(0,1]$. Then $\mathcal{H}_\mu : H^1 \to H^1$ is a bounded operator if and only if*

$$L_1 = \int_0^1 \left(1 + \ln \frac{1}{t}\right) d\mu(t) < +\infty. \tag{11}$$

In this case $||\mathcal{H}_\mu||_{H^1} \asymp L_1$.

Proof. We first prove the sufficiency. Since $\mathcal{H}_\mu = S_\mu$, we have

$$||S_\mu||_{H^1} \leq \int_0^1 \frac{1}{2\pi} \int_0^{2\pi} |w_t(e^{i\theta})||f(\phi_t(e^{i\theta}| \, d\theta \, d\mu(t).$$

Fixing $t \in (0,1]$, we work with the inner integral

$$A(t) = \int_{-\pi}^{\pi} \frac{1}{|1-(1-t)e^{i\theta}|} \left| f\left(\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}\right)\right| \frac{d\theta}{2\pi}.$$

We define e^{is} to be the radial projection of $\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}$ on the boundary ∂D of the unit disk. This means that

$$e^{is} = \frac{te^{i\theta}}{1-(1-t)e^{i\theta}} \frac{|1-(1-t)e^{i\theta}|}{t}.$$

By calculation or geometrically, one can see that for $0 < t \leq 1$

$$\left|\frac{ds}{d\theta}\right| \geq C \, \frac{1}{t}. \tag{12}$$

If $Mf(e^{is}) = \sup\limits_{0<r\leq 1} |f(re^{is})|$ is the radial maximal function, then taking $1/2 \leq t \leq 1$, we derive from (12)

$$A(t) \ll |Mf(e^{is}| \, ds \ll ||f||_{H^1}.$$

Let now $0 < t < 1/2$. Splitting

$$A(t) = \int_{0<|\theta|\leq t} + \int_{t<|\theta|\leq\pi} = A_1(t) + A_2(t),$$

we obtain by (12)

$$A_1(t) \leq \frac{1}{t} \int_{0<|\theta|\leq t} |Mf(e^{is})| \left|\frac{d\theta}{ds}\right| \frac{ds}{2\pi} \ll ||f||_{H^1}.$$

In $A_2(t)$ we have

$$\left|\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}\right| \leq C < 1,$$

which yields

$$\left| f\left(\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}\right)\right| \ll ||f||_{H^1}.$$

Hence,

$$A_2(t) \ll \int_{t<|\theta|\le\pi} |\theta|^{-1} d\theta\, ||f||_{H^1} \ll \ln\frac{1}{t}\, ||f||_{H^1},$$

and sufficiency is proved.

We now prove the necessity. Let $\mathcal{H}_\mu$ be bounded on H^1. Observing that

$$\mathcal{H}_\mu(1)(z) = \sum_{n=0}^{\infty} \int_0^1 (1-t)^n \, d\mu(t)\, z^n$$

and using Hardy's inequality (9), we get

$$\int_0^1 \Big(1+\ln\frac{1}{t}\Big)\, d\mu(t) \quad\ll\quad \int_0^1 \frac{1}{1-t}\ln\frac{1}{t}\, d\mu(t).$$

Indeed, we obviously have

$$\int_0^{1/2} \ln\frac{1}{t}\, d\mu(t) \quad\ll\quad \int_0^{1/2} \frac{1}{1-t}\ln\frac{1}{t}\, d\mu(t).$$

Further,

$$\begin{aligned} &\int_{1/2}^1 \Big(1+\ln\frac{1}{t}\Big)\, d\mu(t) \\ \ll\quad &\int_{1/2}^1 \ln\frac{1}{t}\, d\mu(t) + \int_{1/2}^1 \frac{t}{1-t}\ln\Big(1+\frac{1-t}{t}\Big)\, d\mu(t) \\ =\quad &\int_{1/2}^1 \frac{1}{1-t}\ln\frac{1}{t}\, d\mu(t). \end{aligned}$$

Thus

$$\begin{aligned} \int_0^1 \Big(1+\ln\frac{1}{t}\Big)\, d\mu(t) \quad&\ll\quad \int_0^1 \frac{1}{1-t}\ln\frac{1}{t}\, d\mu(t) \\ &\ll\quad \sum_{n=0}^{\infty}\frac{1}{n+1}\int_0^1 (1-t)^n\, d\mu(t) \\ &\ll\quad ||\mathcal{H}_\mu(1)||_{H^1} \ll ||\mathcal{H}_\mu||_{H^1}. \end{aligned}$$

The proof is complete. □

In like manner one can prove (see [14] and [15])

Theorem 5. *Let μ be a finite positive Borel measure on $(0,1]$. Then $\mathcal{A}_\mu : H^1 \to H^1$ defines a bounded operator if and only if*

$$||\mathcal{A}_\mu||_{H^1} = \int_0^1 t^{-1}\, d\mu(t) < +\infty.$$

In this case $\mathcal{A}_\mu f = T_\mu f$ for every $f \in H^1$.

We mention that in the cited papers conditions were given and proved not only for H^1 but for all H^p, $1 \le p < \infty$, with corresponding dependance on p (in the case of H^∞ conditions are also found and look different from those for $p < \infty$).

Theorem 6. *Let μ be a finite positive Borel measure on $(0,1]$. If $1 < p \le \infty$, then $\mathcal{H}_\mu : H^p \to H^p$ is a bounded operator if and only if*

$$||\mathcal{H}_\mu||_{H^p \to H^p} = \int_0^1 t^{1/p-1} d\mu(t) < \infty. \tag{13}$$

If $1/p + 1/p' = 1$, then, under the above conditions, $\mathcal{H}_\mu : H^p \to H^p$ and $\mathcal{A}_\mu : H^{p'} \to H^{p'}$ are adjoint.

Theorem 7. *Let μ be a finite positive Borel measure on $(0,1]$ and $1 \le p < \infty$. Then $\mathcal{A}_\mu : H^p \to H^p$ defines a bounded operator if and only if*

$$||\mathcal{A}_\mu||_{H^p \to H^p} = \int_0^1 t^{-1/p}\, d\mu(t) < +\infty. \tag{14}$$

Furthermore, $\mathcal{A}_\mu$ is bounded on H^∞ if and only if

$$\lim_{n\to\infty} \ln n \int_0^1 (1-t)^n d\mu(t) = 0. \tag{15}$$

In this case

$$||\mathcal{A}_\mu||_{H^\infty \to H^\infty} = \mu(0,1].$$

We have by intention restricted ourselves to the case of H^1 here and in the sequel. This is partially due to our taste, partially for brevity (relative, of course), and partially because in most situations the proofs for H^p go along the same lines. Pretty often H^1 results need additional tools and efforts and thus are of more interest and challenge. By this, the only other space that appears in our study is BMO, since it is intimately related to H^1 and certain results for it follow from those for H^1 or are obtained as by-product just "for free".

4. Hausdorff operators on the real line

We have now arrived at consideration of Hausdorff operators in our basic, Fourier transform setting. We first study a bundle of problems in dimension one, on the real axis. We will further go on to those in several dimensions.

4.1. **Preliminaries.**

We recall that the Fourier transform $\hat{f}$ of a (complex-valued) function f in $L^1(\mathbb{R})$ is defined by

$$\hat{f}(t) := \int_{\mathbb{R}} f(x) e^{-itx} dx, \quad t \in \mathbb{R},$$

while its Hilbert transform $\tilde{f}$ is defined by

$$\begin{aligned} \tilde{f}(x) &:= \frac{1}{\pi} (\text{P.V.}) \int_{\mathbb{R}} f(x-u) \frac{du}{u} \\ &= \frac{1}{\pi} \lim_{\delta \downarrow 0} \int_\delta^\infty \{f(x-u) - f(x+u)\} \frac{du}{u}, \quad x \in \mathbb{R}. \end{aligned}$$

As is well known, this limit exists for almost all x in $\mathbb{R}$, and the real Hardy space $H^1(\mathbb{R})$ is defined to be

$$H^1(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : \tilde{f} \in L^1(\mathbb{R})\},$$

endowed with the norm

$$\|f\|_{H^1} := \|f\|_{L^1} + \|\tilde{f}\|_{L^1}, \quad \text{where} \quad \|f\|_{L^1} := \int_{\mathbb{R}} |f(x)|\, dx.$$

This space is a Banach algebra under point-wise addition and scalar multiplication, and convolution.

The familiar identity

$$(\tilde{f})^\wedge(t) = (-i\,\mathrm{sign}t)\hat{f}(t), \quad t \in \mathbb{R}, \tag{16}$$

is valid for all f in $H^1(\mathbb{R})$ and plays an important role in the sequel.

For example, it implies immediately that if $f \in H^1(\mathbb{R})$, then $\hat{f}(0) = 0$ and, by uniqueness of Fourier transform, almost everywhere

$$(\tilde{f})^\sim(t) = -f(t). \tag{17}$$

In particular, if $f \in H^1(\mathbb{R})$, then $\tilde{f} \in H^1(\mathbb{R})$ and

$$||\tilde{f}||_{H^1} = ||f||_{H^1}.$$

A question here is how $(\tilde{f})^\wedge$ is defined. As often happens, the distributional approach is the most general and natural. If we introduce an appropriate principal value distribution, then the Fourier transform $(\tilde{f})^\wedge$ can be defined as a tempered distribution in such a way that (16) holds true.

In a previous section we have considered Hardy spaces of analytic functions in the unit disk. Let us briefly discuss how these two cases related to each other. First, let us consider Hardy spaces in the upper half-plane instead of those in the unit disk. Both settings are related via a conformal mapping. The Hardy space $H^1(\mathbb{C}_+)$ of analytic functions $F(z)$ in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}z > 0\}$ is defined by the condition

$$||F||_{H^1} = \sup_{y>0} \int_{\mathbb{R}} |F(x+iy|\, dx < \infty.$$

This space is complete with respect to the indicated norm. It is known that each such F has a finite limit

$$\lim_{y\to 0+} F(x+iy) = f(x) + ig(x)$$

almost everywhere on the real axis; in addition, the real-valued functions f and g belong to the space $L^1(\mathbb{R})$. Moreover, $g(x) = \tilde{f}(x)$. On the other side, it is known that if f is a real-valued function in $L^1(\mathbb{R})$ such that its Hilbert transform $\tilde{f}$ also belongs to $L^1(\mathbb{R})$, then $f(x) + i\tilde{f}(x)$ coincides with the limit values as $y \to 0+$ of some function $F(z) = F(x+iy) \in H^1(\mathbb{C}_+)$ almost everywhere in $\mathbb{R}$.

4.2. **Definition and basic properties.**

The Hausdorff operator $\mathcal{H}$ generated by a function φ in $L^1(\mathbb{R})$ as introduced in [40], can be defined via the Fourier transform as follows:

$$(\mathcal{H}f)^\wedge(t) = (\mathcal{H}_\varphi f)^\wedge(t) := \int_{\mathbb{R}} \hat{f}(tx)\varphi(x)\,dx, \quad t \in \mathbb{R}, \tag{18}$$

where f is also in $L^1(\mathbb{R})$. The existence of such a function $\mathcal{H}f$ in $L^1(\mathbb{R})$ will be clear from the proof of Theorem 8 below. In fact, one can find a close definition already in [28, Ch.XI, 11.18], along with its summability properties. Later on the Hausdorff mean (of a Fourier-Stieltjes transform) was studied on $L^1(\mathbb{R})$ in [18].

We note that if $\varphi(x) := \chi_{(0,1)}(x)$, the indicator function of the unit interval $(0, 1)$, then (18) is of the following form:

$$(\mathcal{H}f)^\wedge(t) := \int_0^1 \hat{f}(tx)dx = \frac{1}{t}\int_0^t \hat{f}(u)du, \quad t \neq 0.$$

In this case, $\mathcal{H}$ is the well-known Cesàro operator; its properties were studied in [19].

The objective here is to determine the Fourier analytic properties of $\mathcal{H}$ on the Hardy space. One of the points is as follows. Since, generally speaking, the inverse formula

$$f(x) = (2\pi)^{-1}\int_{\mathbb{R}} \widehat{f}(t)e^{ixt}dt$$

does not take place for $f \in L^1(\mathbb{R})$ as well as for $f \in H^1(\mathbb{R})$, expected is that

$$\int_{\mathbb{R}} (\mathcal{H}f)\widehat{\ }(y)e^{ixy}dy \tag{19}$$

behaves better and characterizes f properly, in a sense.

But first we establish two its properties without of which further study is meaningless.

Theorem 8. *If $\varphi \in L^1(\mathbb{R})$, then the Hausdorff operator $\mathcal{H} = \mathcal{H}_\varphi : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is bounded and*

$$||\mathcal{H}_\varphi|| = \sup_{||f||_{L^1(\mathbb{R})}\le 1} ||\mathcal{H}_\varphi||_{L^1(\mathbb{R})} \le ||\varphi||_{L^1(\mathbb{R})}. \tag{20}$$

We will obtain the proof of this result as a by-product of the following theorem, in which additional facts are contained. But before this we need an auxiliary result in which equivalent representations for $(\mathcal{H}f)^\wedge$ are given.

Lemma 4. *If f and φ both belong to $L^1(\mathbb{R})$, and $\mathcal{H}f$ is defined in (18), then*

$$(\mathcal{H}f)^\wedge(t) = \frac{1}{|t|}\int_{\mathbb{R}} \hat{f}(u)\varphi(u/t)\,du, \quad t \neq 0, \tag{21}$$

and

$$(\mathcal{H}f)^\wedge(t) = \int_{\mathbb{R}} f(u)\hat{\varphi}(tu)\,du, \quad t \in \mathbb{R}. \tag{22}$$

The proof is routine: integrate by substitution and make use of Fubini's theorem.

Theorem 9. *The function $\mathcal{H}f$ defined, for $x\in\mathbb{R}$, by*

$$\mathcal{H}f(x)=\int_{\mathbb{R}}\frac{f(t)}{|t|}\varphi\left(\frac{x}{t}\right)\,dt \tag{23}$$

is in $L^1(\mathbb{R})$ and satisfies (18).

Proof. By Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}}|\mathcal{H}f(x)|\,dx &\le \int_{\mathbb{R}}\int_{\mathbb{R}}\left|\frac{f(t)}{t}\right|\left|\varphi\left(\frac{x}{t}\right)\right|\,dt\,dx\\ &= \int_0^\infty\frac{|f(t)|}{t}\int_{\mathbb{R}}\left|\varphi\left(\frac{x}{t}\right)\right|\,dx\,dt\\ &- \int_{-\infty}^0\frac{|f(t)|}{t}\int_{\mathbb{R}}\left|\varphi\left(\frac{x}{t}\right)\right|\,dx\,dt\\ &= \int_{\mathbb{R}}|f(t)|\,dt\int_{\mathbb{R}}|\varphi(u)|\,du,\end{aligned}$$

which proves (20) as well.

By (23) and Fubini's theorem,

$$\begin{aligned}(\mathcal{H}f)^\wedge(t) &= \int_{\mathbb{R}}e^{-ixt}\int_{\mathbb{R}}\frac{f(u)}{|u|}\varphi\left(\frac{x}{u}\right)\,du\,dx\\ &= \int_0^\infty\frac{f(u)}{u}\int_{\mathbb{R}}\varphi(x/u)e^{-ixt}dx\,du\\ &- \int_{-\infty}^0\frac{f(u)}{u}\int_{\mathbb{R}}\varphi(x/u)e^{-ixt}dx\,du\\ &= \int_0^\infty f(u)\int_{\mathbb{R}}\varphi(x)e^{-ixut}dx\,du,\end{aligned}$$

which is (18) due to (22). The proof is complete. □

In fact, (23) is a direct definition of Hausdorff operator, the argument around (19) is the (only) reason to define it via the Fourier transform.

4.3. Boundedness of the Hausdorff operator on the Hardy space.

Let us now proceed to the boundedness of the Hausdorff operator on the Hardy space.

Theorem 10. *If $\varphi\in L^1(\mathbb{R})$, then the Hausdorff operator $\mathcal{H}=\mathcal{H}_\varphi : H^1(\mathbb{R})\to H^1(\mathbb{R})$ is bounded.*

The proof we present (see [40]) is not unique and probably not the best possible. But it leads to another interesting problem and is of interest by itself. We need two auxiliary well known results.

Lemma 5. *If $f\in L^1(\mathbb{R})$ is such that*

$$\hat{f}(t)=0\quad\text{for}\quad t<0, \tag{24}$$

then $f\in H^1(\mathbb{R})$.

Proof. By (16) and (24), we see that

$$(\tilde{f})^\wedge(t) = (-i\,\mathrm{sign}t)\hat{f}(t) = -i\hat{f}(t).$$

By uniqueness,

$$\tilde{f}(x) = -i\hat{f}(x)$$

for almost all $x \in \mathbb{R}$. Clearly, $\tilde{f} \in L^1(\mathbb{R})$, and consequently $f \in H^1(\mathbb{R})$, which completes the proof. □

The symmetric counterpart of this lemma says that if $f \in L^1(\mathbb{R})$ is such that $\hat{f}(t) = 0$ for $t > 0$, then $f \in H^1(\mathbb{R})$.

Lemma 6. *If $f \in H^1(\mathbb{R})$, then there exist two functions f_1 and f_2, both in $H^1(\mathbb{R})$ such that $f = f_1 + f_2$, and $\hat{f}_1(t) = 0$ for $t < 0$, while $\hat{f}_2(t) = 0$ for $t > 0$.*

We are now in a position to prove Theorem 10.

Proof. Suppose for a moment that $\varphi \in L^1(\mathbb{R})$ is such that $\varphi(x) = 0$ for $x < 0$; then $\varphi(u/t) = 0$ for $u > 0$ and $t < 0$. Given $f \in H^1(\mathbb{R})$, we consider the decomposition $f = f_1 + f_2$ provided by Lemma 6. The Hausdorff operator is clearly linear and therefore

$$\mathcal{H}f = \mathcal{H}f_1 + \mathcal{H}f_2. \tag{25}$$

By Lemma 6, $\hat{f}_1(u) = 0$ for $u < 0$; and therefore for $t < 0$ we have

$$(\mathcal{H}f_1)^\wedge(t) = \frac{1}{|t|}\int_0^\infty \hat{f}_1(u)\varphi(u/t)\,du = 0.$$

In a similar manner, for $t > 0$

$$(\mathcal{H}f_2)^\wedge(t) = \frac{1}{|t|}\int_{-\infty}^0 \hat{f}_2(u)\varphi(u/t)\,du = 0.$$

By Theorem 8 and Lemma 5, both $\mathcal{H}f_1$ and $\mathcal{H}f_2$ belong to $H^1(\mathbb{R})$. From (25) it follows that $\mathcal{H}f \in H^1(\mathbb{R})$.

It is plain that the above argument works in the other particular case where φ vanishes on the other half-axis.

In the general case, we decompose $\varphi \in L^1(\mathbb{R})$ in the trivial way $\varphi = \varphi_1 + \varphi_2$, with $\varphi_1(x) = \varphi(x)$ for $x > 0$ and vanishes otherwise and $\varphi_2 = \varphi - \varphi_1$. Clearly, for $f \in H^1(\mathbb{R})$ we have $\mathcal{H}_\varphi f = \mathcal{H}_{\varphi_1} f + \mathcal{H}_{\varphi_2} f$, and the above particular cases apply separately to $\mathcal{H}_{\varphi_1}$ and $\mathcal{H}_{\varphi_2}$, respectively. Thus, we have justified our claim that $\mathcal{H}_\varphi f \in H^1(\mathbb{R})$ whenever $f \in H^1(\mathbb{R})$.

We are now going to apply the closed graph theorem, which says that a linear operator mapping a Banach space into another Banach space is bounded if and only if it is closed. We note that the closed graph theorem holds true under more general conditions but the above (classical) formulation is enough for our purposes. Since the Hausdorff operator is linear, it remains to check that it is closed. To this effect, assume that a sequence $\{f_n\}$, $n = 1, 2, ...$, is given in $H^1(\mathbb{R})$ such that $f_n \to f$ and $\mathcal{H}f_n \to g$ as $n \to \infty$, with some f and g both in $H^1(\mathbb{R})$, where convergence is meant in the norm of $H^1(\mathbb{R})$ in both cases. This statement can be taken as a definition of the closeness and, in any case, is very convenient for checking the closeness. It is plain that both limits hold in the norm of $L^1(\mathbb{R})$ as well. By Theorem 8, the Hausdorff operator is closed in $L^1(\mathbb{R})$, whence we conclude that $\mathcal{H}f = g$. This means that the operator is closed in $H^1(\mathbb{R})$ too. □

Once more, this proof does not seem to be the strongest. For example, it provides no bound for the norm of the operator, or more precisely, does not state a strong type boundedness inequality. Any multidimensional proof given below provides that in dimension one as well. However, the above proof leads to an interesting problem we will study in the next subsection.

4.4. **Commuting relations.**

The mentioned problem reads as follows. Two operators were studied above: the Hausdorff operator and the Hilbert transform, for what $\varphi \in L^1(\mathbb{R})$ the two operators commute. The next theorem answers this question.

Theorem 11. *Assume* $\varphi \in L^1(\mathbb{R})$.

(i) *We have*

$$(\mathcal{H}_\varphi f)^\sim = \mathcal{H}_\varphi \tilde{f} \quad \text{for all} \quad f \in H^1(\mathbb{R}) \tag{26}$$

if and only if

$$\varphi(x) = 0 \quad \text{for almost all} \quad x < 0. \tag{27}$$

(ii) *We have*

$$(\mathcal{H}_\varphi f)^\sim = -\mathcal{H}_\varphi \tilde{f} \quad \text{for all} \quad f \in H^1(\mathbb{R})$$

if and only if

$$\varphi(x) = 0 \quad \text{for almost all} \quad x > 0. \tag{28}$$

We will make use of Lemma 5; in addition we need one more auxiliary result.

Lemma 7. *Let* $0 < \delta < a/2$ *and let*

$$g_{\delta,a}(t) := \begin{cases} t/\delta & \text{for} \quad 0 \le t < \delta, \\ 1 & \text{for} \quad \delta \le t \le a - \delta, \\ (a-t)/\delta & \text{for} \quad a - \delta < t \le a, \\ 0 & \text{for} \quad t < 0 \quad \text{or} \quad t > a. \end{cases}$$

Then $g_{\delta,a} \in \hat{L}^1(\mathbb{R})$, *the Fourier transform of an integrable function.*

Proof. The statement is more or less clear from the form of the function: of compact support and continuous on $\mathbb{R}$, constant on $\delta, a-\delta$, and linear on $(0,\delta)$ and $(a-\delta, a)$. But we also can compute the Fourier transform of $g_{\delta,a}$. Integration by parts gives

$$\begin{aligned} \hat{g}_{\delta,a}(x) : &= \int_0^a g_{\delta,a}(t) e^{-ixt} dt \\ &= \frac{1}{\delta} \int_0^\delta t e^{-ixt} dt + \int_\delta^{a-\delta} e^{-ixt} dt + \frac{1}{\delta} \int_{a-\delta}^a (a-t) e^{-ixt} dt \end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{1}{-i\delta x} t e^{-ixt}\right]_{t=0}^{\delta} + \frac{1}{i\delta x}\int_0^\delta e^{-ixt}dt + \left[\frac{1}{-ix}e^{-ixt}\right]_{t=\delta}^{a-\delta} \\
&+ \left[\frac{1}{-i\delta x}(a-t)e^{-ixt}\right]_{t=a-\delta}^{a} - \frac{1}{i\delta x}\int_{a-\delta}^{a} e^{-ixt}dt \\
&= \delta^{-1}x^{-2}(e^{-ix\delta} - 1 - e^{-iax} + e^{-i(a-\delta)x}),
\end{aligned}
$$

for $x \neq 0$. In particular, we have

$$|\hat{g}_{\delta,a}(x)| \leq 4\delta^{-1}x^{-2}, \quad x \neq 0.$$

Since $\hat{g}_{\delta,a}$ is a continuous function and $\delta > 0$ is fixed, it follows from the above estimate that $\hat{g}_{\delta,a} \in L^1(\mathbb{R})$. Since $g_{\delta,a} \in L^1(\mathbb{R})$, as well, the inversion formula holds (see, e.g., [52, Ch.1]). This means that $g_{\delta,a}$ is the Fourier transform of a function (namely, $\hat{g}_{\delta,a}$) in $L^1(\mathbb{R})$, that is, $g_{\delta,a} \in \hat{L}^1(\mathbb{R})$. □

We are now in a position to prove Theorem 11.

Proof. Part **(i)**. Let us begin with the sufficiency part. We first set $g = \mathcal{H}\tilde{f}$. By definition and (16), we have for $t \in \mathbb{R}$

$$
\begin{aligned}
\hat{g}(t) &= \int_{\mathbb{R}} (\tilde{f})^\wedge(tx)\varphi(x)\,dx = \int_{\mathbb{R}} (-i\,\mathrm{sign} tx)\hat{f}(tx)\varphi(x)\,dx \\
&= (-i\,\mathrm{sign} t)\int_0^\infty \hat{f}(tx)\varphi(x)\,dx + (i\,\mathrm{sign} t)\int_{-\infty}^0 \hat{f}(tx)\varphi(x)\,dx. \qquad (29)
\end{aligned}
$$

We then set $h = (\mathcal{H}f)^\sim$. By (17), this is equivalent to the following relation: $\mathcal{H}f = -\tilde{h}$. By (16),

$$(\mathcal{H}f)^\wedge(t) = (-\tilde{h})^\wedge(t) = (i\,\mathrm{sign} t)\hat{h}(t),$$

whence for $t \in \mathbb{R}$

$$\hat{h}(t) = (-i\,\mathrm{sign} t)(\mathcal{H}f)^\wedge(t) = (-i\,\mathrm{sign} t)\int_{\mathbb{R}} \hat{f}(tx)\varphi(x)\,dx. \qquad (30)$$

Comparing (29) and (30) and taking into account (27) yields $\hat{g}(t) = \hat{h}(t)$, while taking into account (28) yields $\hat{g}(t) = -\hat{h}(t)$, both for all $t \in \mathbb{R}$. By uniqueness of the Fourier transform, we obtain the sufficiency part of both items of the theorem.

Let us now proceed to the necessity part. To this end, we suppose that (26) is satisfied and prove (27).

Let f be an arbitrary function in $H^1(\mathbb{R})$. By (16) and (18), we have

$$
\begin{aligned}
(\mathcal{H}_\varphi\tilde{f})^\wedge(t) :&= \int_{\mathbb{R}} (\tilde{f})^\wedge(tx)\varphi(x)\,dx \\
&= \frac{1}{|t|}\int_{\mathbb{R}} (\tilde{f})^\wedge(u)\varphi\left(\frac{u}{t}\right)du \\
&= \frac{1}{|t|}\int_{\mathbb{R}} (-i\,\mathrm{sign} u)\hat{f}(u)\varphi\left(\frac{u}{t}\right)du,
\end{aligned}
$$

and analogously,

$$\begin{aligned}((\mathcal{H}_\varphi f)^\sim)^\wedge(t) &= (-i\,\mathrm{sign}t)(\mathcal{H}_\varphi f)^\wedge(t)\\ &= (-i\,\mathrm{sign}t)\frac{1}{|t|}\int_{\mathbb{R}}\hat f(u)\varphi\left(\frac{u}{t}\right)du, \quad t\neq 0.\end{aligned}$$

Substituting $t := -1$ in the last two equalities, we derive from (26) that

$$\int_{\mathbb{R}}\hat f(u)\varphi(-u)\,du = -\int_{\mathbb{R}}(\mathrm{sign}u)\,\hat f(u)\varphi(-u)\,du. \tag{31}$$

Now, given $x > 0$, let $0 < \delta < x/2$ and

$$\hat f(u) := g_{\delta,x}(u), \quad u\in\mathbb{R},$$

where $g_{\delta,x}$ is defined in Lemma 7 above. Since $g_{\delta,x}(u) = 0$ for $u < 0$ and $\hat g_{\delta,x}\in L^1(\mathbb{R})$ (see the proof of Lemma 7), Lemma 5 yields $f\in H^1(\mathbb{R})$. For this f, equality (31) is of the form

$$\int_0^\infty g_{\delta,x}(u)\varphi(-u)\,du = -\int_0^\infty g_{\delta,x}(u)\varphi(-u)\,du,$$

which means that

$$\int_0^\infty g_{\delta,x}(u)\varphi(-u)du = 0 \quad\text{for all}\quad x>0 \quad\text{and}\quad 0<\delta<x/2.$$

Since $\varphi\in L^1(\mathbb{R})$ and

$$|g_{\delta,x}(u)|\le 1 \quad\text{for all}\quad u\in\mathbb{R}, \tag{32}$$

Lebesgue's dominated convergence theorem applies:

$$\int_0^x \varphi(-u)\,du = \lim_{\delta\downarrow 0}\int_0^\infty g_{\delta,x}(u)\varphi(-u)\,du = 0 \quad\text{for all}\quad x>0.$$

It is obvious that this equality is equivalent to the fact that the support of φ lies on the positive half of $\mathbb{R}$ (except perhaps a set of measure zero lies on the negative half of $\mathbb{R}$). Thus, the proof of the necessity of condition (27) is complete.

Part **(ii)** is a skew-symmetric counterpart of **(i)**, and it can be proved in an analogous way. □

4.5. **The case $p < 1$.**

There is a rather simple result for the Hausdorff operator in L^p, $1\le p\le\infty$. For these p, Minkowski's inequality in the integral form gives

$$\begin{aligned}\left\|\int_0^\infty |t^{-1}f(t^{-1}x)\varphi(t)|\,dt\right\|_{L^p_x} &\le \int_0^\infty t^{-1}\|f(t^{-1}x)\|_{L^p_x}|\varphi(t)|\,dt\\ = \int_0^\infty t^{-1+1/p}\|f\|_{L^p}|\varphi(t)|\,dt &= A_{\varphi,p}\|f\|_{L^p},\end{aligned}$$

where

$$A_{\varphi,p} = \int_0^\infty t^{-1+1/p}|\varphi(t)|\,dt. \tag{33}$$

From this inequality, we see that, if $1 \le p \le \infty$ and $A_{\varphi,p} < \infty$, then (23) gives a well-defined bounded linear operator $\mathcal{H}_\varphi$ in L^p.

If $f \in H^p$ with $0 < p < 1$, then $\widehat{f}$ is a continuous function satisfying

$$|\widehat{f}(\xi)| \le C_p \|f\|_{H^p} |\xi|^{1/p-1}$$

(see, e.g., [51, Chapt. III, §5.4, p.128]), and hence

$$\int_0^\infty |\widehat{f}(t\xi)\varphi(t)|\,dt \le \int_0^\infty C_p \|f\|_{H^p} |t\xi|^{1/p-1} |\varphi(t)|\,dt \tag{34}$$

$$= C_p\, A_{\varphi,p} \|f\|_{H^p} |\xi|^{1/p-1},$$

where $A_{\varphi,p}$ for $0 < p < 1$ is given by (33) as well. Thus, if $0 < p < 1$, $A_{\varphi,p} < \infty$, and $f \in H^p$, then the right hand side of (18) gives a continuous function of $\xi \in \mathbb{R}$ that is uniformly of $O(|\xi|^{1/p-1})$ and, hence, the tempered distribution $\mathcal{H}_\varphi f$ is well-defined through (18). Thus, including also the case $p = 1$ as mentioned above, we give the following definition.

Definition 12. If $0 < p \le 1$ and φ is a measurable function on $(0, \infty)$ with $A_{\varphi,p} < \infty$, then we define the continuous linear mapping $\mathcal{H}_\varphi : H^p \to \mathcal{S}'$ by (18).

Kanjin [35] proved the following theorem.

Theorem 13. *Let $0 < p < 1$ and $M = [1/p - 1/2] + 1$. Suppose $A_{\varphi,1} < \infty$, $A_{\varphi,2} < \infty$, and suppose $\widehat{\varphi}$ is a function of class C^{2M} on $\mathbb{R}$ with $\sup_{\xi\in\mathbb{R}} |\xi|^M |\widehat{\varphi}^{(M)}(\xi)| < \infty$ and $\sup_{\xi\in\mathbb{R}} |\xi|^M |\widehat{\varphi}^{(2M)}(\xi)| < \infty$. Then the Hausdorff operator $\mathcal{H}_\varphi$ is a bounded operator in H^p.*

This theorem contains assumptions on $\widehat{\varphi}$. The proof is based on a more or less regular atomic decomposition.

In the case where $\varphi(t) = \alpha(1-t)^{\alpha-1}$ for $0 < t < 1$ and $\varphi(t) = 0$ otherwise, the operator $\mathcal{H}_\varphi = \mathcal{C}_\alpha$ is called the *Cesàro operator* of order α. Giang and Móricz [19] proved that the Cesàro operator $\mathcal{C}_1$ is bounded in the Hardy space H^1. Kanjin [35] proved that the Cesáro operator $\mathcal{C}_\alpha$ is a bounded operator in H^p provided α is a positive integer and $2/(2\alpha + 1) < p < 1$. Kanjin proved this result by using Theorem 13. Later on, it was proved in [45] that the Cesáro operator $\mathcal{C}_\alpha$ is a bounded operator in H^p for every $\alpha > 0$ and every $0 < p < 1$. The proof is based on the ideas elaborated in [45] and uses the one-dimensional version of the modified atomic decomposition for H^p given in [44].

Definition 14. Let $0 < p \le 1$ and let M be a positive integer. For $0 < s < \infty$, we define $\mathcal{A}_{p,M}(s)$ as the set of all those $f \in L^2$ for which $\widehat{f}(\xi) = 0$ for $|\xi| \le s^{-1}$ and $\|\widehat{f}^{(k)}\|_{L^2} \le s^{k-1/p+1/2}$ for $k = 0, 1, \cdots, M$. We define $\mathcal{A}_{p,M}$ as the union of $\mathcal{A}_{p,M}(s)$ over all $0 < s < \infty$.

Lemma 8. *Let $0 < p \le 1$ and M be a positive integer satisfying $M > 1/p - 1/2$. Then there exists a constant $c_{p,M}$, depending only on p and M, such that the following hold.*
(1) $\|f(\cdot - x_0)\|_{H^p} \le c_{p,M}$ *for all* $f \in \mathcal{A}_{p,M}$ *and all* $x_0 \in \mathbb{R}$;
(2) *Every* $f \in H^p$ *can be decomposed as*

$$f=\sum_{j=1}^{\infty}\lambda_j f_j(\cdot-x_j), \tag{35}$$

where $f_j\in\mathcal{A}_{p,M}$, $x_j\in\mathbb{R}$, $0\le\lambda_j<\infty$, *and*

$$\Big(\sum_{j=1}^{\infty}\lambda_j^p\Big)^{1/p}\le c_{p,M}\|f\|_{H^p},$$

and the series in (35) *converges in* H^p. *If* $f\in H^p\cap L^2$, *then this decomposition can be made so that the series in* (35) *converges in* L^2 *as well.*

This lemma is given in [44, Lemma 2] except for the assertion on the L^2 convergence. A complete proof of part (2) of the lemma can be found in [45, §3].

In a recent paper [39] the following generalizations of [45] are obtained.

Theorem 15. *Let* $0<p<1$, $M=[1/p-1/2]+1$, *and let* ϵ *be a positive real number. Suppose* φ *is a function of class* C^M *on* $(0,\infty)$ *such that*

$$|\varphi^{(k)}(t)|\le\min\{t^{\epsilon},t^{-\epsilon}\}t^{-1/p-k}\quad for\quad k=0,\,1,\,\dots,\,M.$$

Then $\mathcal{H}_\varphi$ *is a bounded linear operator in* H^p.

Theorem 16. *Let* $0<p<1$, $M=[1/p-1/2]+1$, *and let* ϵ *and* a *be positive real numbers. Suppose* φ *is a function on* $(0,\infty)$ *such that* supp φ *is a compact subset of* $(0,\infty)$, φ *is of class* C^M *on* $(0,a)\cup(a,\infty)$, *and*

$$|\varphi^{(k)}(t)|\le|t-a|^{\epsilon-1-k}\quad for\quad k=0,\,1,\,\dots,\,M.$$

Then $\mathcal{H}_\varphi$ *is a bounded linear operator in* H^p.

In this section we have given not all possible results and arguments. On the contrary, we present here only those specific for dimension one and not taking place or unclear for higher dimension. One of the reasons is that more general approaches in the next section lead to results which, being taken in dimension one, clearly supply those from the present section. No doubt that any interested reader can easily recognize such results.

5. Hausdorff operators in Euclidean spaces

In the multidimensional case the situation is, as usual, more complicated. The Cesàro means in [21] and the Hausdorff means in [41] were considered in dimension two only for the so-called product Hardy space $H^{11}(\mathbb{R}\times\mathbb{R})$ (the simplest partial case, see the corresponding subsection below):

$$(\mathcal{H}_\varphi f)(x)=\int_{\mathbb{R}^2}\frac{\varphi(u)}{|u_1u_2|}f(\frac{x_1}{u_1},\frac{x_2}{u_2})\,du.$$

In [55] these and related results were slightly extended. Necessary and sufficient conditions for fulfillment of commuting relations were also obtained in [42] for this simple situation. The case of usual Hardy space $H^1(\mathbb{R}^2)$ and moreover $H^1(\mathbb{R}^n)$ seemed to be unsolvable by the used method.

In [4] the problem was solved but for a “strange” Hausdorff type operator

$$(\mathcal{H}_\mu f)(x) = \int_{\mathbb{R}} |u|^{-n} f\Big(\frac{x}{u}\Big)\, d\mu(u), \tag{36}$$

where $x \in \mathbb{R}^n$, defined by *one-dimensional* averaging. This does not look to be natural for the multivariate case.

We are going to generalize (23) in a “normal” way, and then obtain conditions sufficient for the boundedness of a naturally defined Hausdorff type operator taking $H^1(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$.

5.1. **Definition and basic properties.**

We define the Hausdorff type operator by

$$(\mathcal{H}f)(x) = (\mathcal{H}_\Phi f)(x) = \int_{\mathbb{R}^n} \Phi(u) f\big(xA(u)\big) du,$$

where $A = A(u) = (a_{ij})_{i,j=1}^n = \big(a_{ij}(u)\big)_{i,j=1}^n$ is the $n \times n$ matrix with the coefficients $a_{ij}(u)$ being measurable functions of u. This matrix may be singular at most on a set of measure zero; $xA(u)$ is the row n-vector we obtain by multiplying the row n-vector x by the matrix. A similar definition was given in [8] (the only difference is that in [8] as well as in [46] $\mathcal{H}$ is considered to be a row vector and thus $f(A(u)x)$ stands in place of $f(xA(u))$; moreover, in [46] only diagonal matrices A with the diagonal entries equal to one another), along with the following basic properties. Comparing the introduced definition with (36), one sees that it is possible to take $u \in \mathbb{R}^m$ for any $1 \le m \le n$, with subsequent m-dimensional averaging.

Let Φ satisfy the condition

$$\|\Phi\|_{L_A} = \int_{\mathbb{R}^n} |\Phi(u)| |\det A^{-1}(u)|\, du < \infty,$$

or, for similarity with the one-dimensional case, $\varphi(u) = \Phi(u) \det A^{-1}(u) \in L^1(\mathbb{R}^n)$.

Lemma 9. *The operator $\mathcal{H}f$ is bounded taking $L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ provided $\Phi \in L_A$. Furthermore, there holds*

$$\|\mathcal{H}f\|_{L^1(\mathbb{R}^n)} \le \|\Phi\|_{L_A} \|f\|_{L^1(\mathbb{R}^n)}.$$

Proof. Applying Fubini's theorem and substituting $xA(u) = v$ (or $x = vA^{-1}(u)$), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |\mathcal{H}f(x)|\, dx &\le \int_{\mathbb{R}^n} |\Phi(u)|\, du \int_{\mathbb{R}^n} |f(xA(u))|\, dx \\ &\le \int_{\mathbb{R}^n} |\Phi(u)|\, du \int_{\mathbb{R}^n} |\det A^{-1}(u)|\, |f(v)|\, dv \\ &= \|\Phi\|_{L_A} \|f\|_{L^1},\end{aligned}$$

and the result follows. □

If we again recall the discussion around (19), we shall realize that it is quite important to have an explicit formula for the Fourier transform of $\mathcal{H}f$ via the Fourier transform of f. First, let us define the latter as

$$\hat{f}(u)=\int_{\mathbb{R}^n} f(x)e^{-i<u,x>}dx,$$

where $<u,x>=u_1x_1+...+u_nx_n$. For an integrable function f its Fourier transform is well defined. Let B^T be transposed to the matrix B.

Lemma 10. *Let $f\in L^1(\mathbb{R}^n)$ and $\Phi\in L_A$. The Fourier transform of $\mathcal{H}f$ is represented by the formula*

$$(\mathcal{H}f)\widehat{\ }(y)=\int_{\mathbb{R}^n}\Phi(u)\,|\det A^{-1}(u)|\,\widehat{f}\big(y(A^{-1})^T(u)\big)\,du.$$

Proof. It follows from the previous lemma that $\mathcal{H}f\in L^1(\mathbb{R}^n)$, and hence its Fourier transform is well defined. By Fubini's theorem,

$$\begin{aligned}(\mathcal{H}f)\widehat{\ }(y) &= \int_{\mathbb{R}^n} e^{-i<x,y>}dx\int_{\mathbb{R}^n}\Phi(u)f(xA(u))\,du\\ &= \int_{\mathbb{R}^n}\Phi(u)\,du\int_{\mathbb{R}^n} f(xA(u))e^{-i<x,y>}dx.\end{aligned}$$

Substituting again $xA(u)=v$, or equivalently $x=vA^{-1}(u)$, we have the following simple relations

$$<x,y>=<xAA^{-1},y>=<vA^{-1},y>=<v,y(A^{-1})^T>.$$

We then obtain

$$\begin{aligned}(\mathcal{H}f)\widehat{\ }(y) &= \int_{\mathbb{R}^n}\Phi(u)\,du\int_{\mathbb{R}^n}|\det A^{-1}(u)|f(v)e^{-i<v,y(A^{-1})^T(u)>}dv\\ &= \int_{\mathbb{R}^n}\Phi(u)\,|\det A^{-1}(u)|\,\widehat{f}\big(y(A^{-1})^T(u)\big)\,du,\end{aligned}$$

as required. □

As an example, we mention that the corresponding Cesàro operator is given by

$$\Phi(u)|\det A^{-1}(u)|=\chi_{\{|\det A^{-1}(u)|\le 1\}}(u).$$

It is worth mentioning that in dimension one a sort of symmetry takes place for f and φ in various representations of the Hausdorff operator from the previous section, see Lemma 4. It turns out that this is accidental circumstance, meaningless for several dimensions.

We can easily find the adjoint operator $\mathcal{H}^*$ as the one satisfying, for appropriate ('good') functions f and g,

$$\int_{\mathbb{R}^n}(\mathcal{H}f)(x)g(x)\,dx=\int_{\mathbb{R}^n}(\mathcal{H}^*g)(x)f(x)\,dx. \tag{37}$$

Indeed, again substituting $xA(u)=v$, we obtain

$$\int_{\mathbb{R}^n}(\mathcal{H}f)(x)g(x)\,dx = \int_{\mathbb{R}^n}\Phi(u)f(xA(u))\,du\,g(x)\,dx$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u)f(v)|\det A^{-1}(u)|g(vA^{-1}(u))\,du\,dv$$
$$= \int_{\mathbb{R}^n} f(v)\int_{\mathbb{R}^n} \Phi(u)|\det A^{-1}(u)|g(vA^{-1}(u))\,du\,dv,$$

and the adjoint operator is defined (compare [8] and [46]) as

$$(\mathcal{H}^*f)(x) = (\mathcal{H}^*_{\Phi,A}f)(x) = \int_{\mathbb{R}^n} \Phi(u)|\det A^{-1}(u)|\,f(xA^{-1}(u))\,du.$$

This operator is also of Hausdorff type. Indeed, it is actually can be written as $\mathcal{H}_{\psi,B}f$, where $\psi(u) = \Phi(u)|\det A^{-1}(u)|$ and $B = A^{-1}(u)$. Therefore the conditions for its boundedness on $H^1(\mathbb{R}^n)$ readily follow from the main result below.

The key ingredient in one of the proofs is Lemma 12 on the behavior in u of the BMO-norm of $f(xA(u))$. This also allows us to get conditions for the boundedness of both operators in $BMO(\mathbb{R}^n)$.

A different approach directly yields the bound for the H^1 norm of $f(xA(u))$, see Lemma 11 below.

5.2. **Hardy spaces.**

There are various definitions of Hardy spaces. Since each may become the one which insures the sharpness, we will give many of them.

Let ψ be a real-valued differentiable function on $\mathbb{R}^n$ which satisfies:

$$(i) \qquad |\psi(x)| \ll (1+|x|)^{-n-1},\ |\nabla\psi(x)| \ll (1+|x|)^{-n-1},$$
$$(ii) \qquad \int_{\mathbb{R}^n} \psi(x)dx = 0.$$

Write $\psi_t(x) = \psi(x/t)t^{-n}$, $t > 0$. Given a function f with

$$\int_{\mathbb{R}^n} |f(x)|(1+|x|)^{-n-1}dx < \infty,$$

define the Lusin area integral $S_\psi f$ by

$$S_\psi f(x) = \left(\int_{\Gamma(x)} |f*\psi_t(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma(x)$ is the cone $\{(y,t) : |y-x| < t\}$.

Given any Schwartz function η with $\int \eta \neq 0$, define the non-tangential maximal function by

$$M_\eta f(x) = \sup_{t>0} |f*\eta_t(x)|.$$

Classical results due to Ch. Fefferman and Stein (see [12]) say that

$$\|f\|_{H^1} \asymp \|M_\eta f\|_{L^1} \asymp \|S_\psi f\|_{L^1}. \tag{38}$$

Let us outline how an eventual proof of the boundedness of the Hausdorff operator could make use of the above definitions. By Fubini's theorem,

$$\int_{\mathbb{R}^n} \eta_t(\xi)d\xi \int_{\mathbb{R}^n} \Phi(u)f\big((y-\xi)A(u)\big)du$$

$$\leq \int_{\mathbb{R}^n} |\Phi(u)|du \Big| \int_{\mathbb{R}^n} f\big((y-\xi)A(u)\big)\eta_t(\xi)d\xi\Big|.$$

Substituting $\xi A(u) = v$, we obtain

$$\int_{\mathbb{R}^n} f\big((y-\xi)A(u)\big)\eta_t(\xi)d\xi$$
$$= \int_{\mathbb{R}^n} f(yA(u)-v)\mu_{\frac{t}{|\det A^{-1}(u)|^{1/n}}}\left(\frac{vA^{-1}(u)}{|\det A^{-1}(u)|^{1/n}}\right)dv.$$

Hence, if

$$\frac{vA^{-1}(u)}{|\det A^{-1}(u)|^{1/n}} = v$$

for all v, then

$$M_\eta(\mathcal{H}_\Phi f)(y) \leq \mathcal{H}_{|\Phi|}(M_\eta f)(y).$$

All this looks strait-forward and natural, nevertheless it is not clear how to continue these calculations to derive something essential. We only wish to mention that in dimension one we can definitely obtain in this way a proof of the boundedness of the Hausdorff operator on the Hardy space as well as the bound for the norm.

To overcome the difficulties of the multivariate case, we will make use of the other three definitions.

1. Very natural is that via the Riesz transforms, n singular integral operators, as an analog of the one-dimensional definition based on the unique singular operator - the Hilbert transform. The jth, $j = 1, 2, ..., n$, Riesz transform can be defined either explicitly

$$R_j f(x) = \frac{\Gamma(n/2+1/2)}{\pi^{n/2+1/2}} \int_{\mathbb{R}^n} \frac{u_j}{|u|^{n+1}} f(x-u)\,du,$$

or via the Fourier transform

$$\widehat{R_j f}(x) = i\frac{x_j}{|x|}\hat{f}(x).$$

One can see that taking $n = 1$ gives just the Hilbert transform. It is well-known (see, e.g., [50, Ch.VII, §3]) that

$$||f||_{H^1(\mathbb{R}^n)} \sim ||f||_{L^1(\mathbb{R}^n)} + \sum_{j=1}^n ||R_j f||_{L^1(\mathbb{R}^n)} = \sum_{p=0}^n \int_{\mathbb{R}^n} |R_p f(x)|\,dx,$$

where $R_0 f \equiv f$. This definition could be perfect if clear conditions for commuting of the Hausdorff operator and Riesz transforms existed. Unfortunately, no such results exist unless A is a diagonal matrix. Nevertheless, we will use this definition along with the others.

2. The definition using atomic decomposition of the Hardy space proved to be very effective (see, e.g., [10]). We will give and apply the simplest version of atomic decomposition. Let $a(x)$ denote an atom (a $(1,\infty,0)$-atom), a function of *compact support*:

$$\operatorname{supp} a \subset B(x_0, r); \tag{39}$$

and satisfies the following *size condition* (L^∞ normalization)

(40) $$||a||_\infty \le \frac{1}{|B(x_0,r)|};$$

and the *cancellation condition*

(41) $$\int_{\mathbb{R}^n} a(x)\,dx = 0.$$

Here $B(x_0,r)$ denotes the ball of radius r centered at x_0.

It is well known that

(42) $$||f||_{H^1} \sim \inf\{\sum_k |c_k| : f(x) = \sum_k c_k a_k(x)\},$$

where a_k are the above described atoms.

3. The dual space approach (see, e.g., [12]) is also one of the most effective and important. It employs the standard (well, very often, like in [12], to prove that no other linear functionals exist except those 'standard" becomes a challenging task) procedure of linearizing

$$\|h\|_{H^1(\mathbb{R}^n)} = \sup_{\|g\|_*\le 1}\Big|\int_{\mathbb{R}^n} h(x)g(x)\,dx\Big|,$$

where g is taken to be infinitely smooth and of compact support (such family of functions endowed with the BMO norm is known as VMO; however some authors call VMO a different space while the present one they call CMO, see, e.g., [6]) and the semi-norm $\|g\|_*$ is that in BMO :

$$\|g\|_* = \sup_Q \inf_c \frac{1}{|Q|}\int_Q |g(x)-c|\,dx,$$

where Q is a ball, $|Q|$ is its Lebesgue measure, and the supremum is taken over all such balls.

5.3. **Main result and proof.**

We denote

$$\|B\|_1 = \|B(u)\|_1 = \max_j(|b_{1j}(u)| + ... + |b_{nj}(u)|),$$

where b_{nj} are the entries of the matrix B, to be the operator ℓ-norm. We will say that $\Phi \in L^1_B$ if

$$\|\Phi\|_{L^1_B} = \int_{\mathbb{R}^n} |\Phi(u)|\,\|B(u)\|_1^n du < \infty.$$

The following result in [36] ensures the boundedness of Hausdorff type operators in $H^1(\mathbb{R}^n)$ for general matrices A.

Theorem 17. *The Hausdorff operator $\mathcal{H}f$ is bounded on the real Hardy space $H^1(\mathbb{R}^n)$ provided $\Phi \in L^1_{A^{-1}}$, with*

$$\|\mathcal{H}f\|_{H^1(\mathbb{R}^n)} \le \|\Phi\|_{L^1_{A^{-1}}} \|f\|_{H^1(\mathbb{R}^n)}. \tag{43}$$

The proof used duality argument outlined in the end of previous subsection. The difference in conditions $\Phi \in L_{A^{-1}}$ and $\Phi \in L^1_{A^{-1}}$ seems to be quite natural. The main case when they coincide is that where A is a diagonal matrix with equal entries on the diagonal - this is the subject of [46]. In [36] and then in [37] the problem of the sharpness of Theorem 17 was posed. We will prove that a slightly weaker condition provides the boundedness of Hausdorff type operators on $H^1(\mathbb{R}^n)$. The proof is based on atomic decomposition of $H^1(\mathbb{R}^n)$. We will discuss and compare both results afterwards.

Let $\|B\|_2 = \max_{|x|=1} |Bx^T|$, where $|\cdot|$ denotes the Euclidean norm. It is known (see, e.g., [32, Ch.5, 5.6.35]) that this norm does not exceed any other matrix norm. We will say that $\Phi \in L^2_B$ if

$$\|\Phi\|_{L^2_B} = \int_{\mathbb{R}^n} |\Phi(u)| \, \|B(u)\|_2^n du < \infty.$$

The following result is true.

Theorem 18. *The Hausdorff operator $\mathcal{H}f$ is bounded on the real Hardy space $H^1(\mathbb{R}^n)$ provided $\Phi \in L^2_{A^{-1}}$. Furthermore,*

$$\|\mathcal{H}f\|_{H^1(\mathbb{R}^n)} \ll \|\Phi\|_{L^2_{A^{-1}}} \|f\|_{H^1(\mathbb{R}^n)}. \tag{44}$$

Proof. We have

$$\begin{aligned} ||\mathcal{H}f||_{H^1} &= || \int_{\mathbb{R}^n} \Phi(u) f(\cdot A(u)) \, du||_{H^1} \\ &\ll \sum_{p=0}^{n} \int_{\mathbb{R}^n} |R_p \mathcal{H}f(x)| \, dx \\ &\le \int_{\mathbb{R}^n} |\Phi(u)| \sum_{p=0}^{n} ||R_p f(\cdot A(u))||_{L^1} \, du \\ &\ll \int_{\mathbb{R}^n} |\Phi(u)| \, ||f(\cdot A(u)||_{H^1} \, du. \end{aligned}$$

We wish to estimate the right-hand side from above by using (42). Let

$$f(xA(u)) = \sum_k c_k a_k(xA(u)).$$

We will show that multiplying $a_k(xA(u))$ by a constant depending on u (actually on $A(u)$) we get an atomic decomposition of f itself, with no composition in the argument. Since we analyze all such decompositions for f, the upper bound will be $||f||_{H^1}$ times the mentioned constant, which completes the proof.

Thus, let us figure out when, or under what conditions on the linear transformation $A(u)$ each function $a_k(xA(u))$ becomes an atom. We have

$$\int_{\mathbb{R}^n} a_k(xA(u)) \, dx = \int_{a_k(xA(u))\ne 0} a_k(xA(u)) \, dx,$$

and under substitution $xA(u) = v$ the integral becomes $\int_{\mathbb{R}^n} a_k(v)\,dv$ times a Jacobian depending only on u. This integral vanishes because of (41).

The support of $a_k(xA(u))$ is

$$< xA, xA > \le\; r^2,$$

an ellipsoid. To use known results, let us represent it in the transposed form

$$< A^T x^T, A^T x^T > \le\; r^2.$$

Let us solve the following extremal problem. We are looking for the minimal value of the quadratic form $< Bx^T, Bx^T >$, where B is a non-singular $n \times n$ real-valued matrix - we denote the linear transformation and its matrix with the same symbol - on the unit sphere $< x^T, x^T >= 1$.

Denoting by B^* the adjoint matrix to B, we arrive at the equivalent problem for the form $< B^*Bx^T, x^T >$. Since $(B^*)^* = B$, the operator B^*B is *self-adjoint*:

$$< B^*Bx, y >=< Bx, By >=< x, B^*By >,$$

and thus *positive definite*. Since B is *non-singular*, the same B^*B is.

If a transformation is positive definite, all its *eigenvalues* are *non-negative*; if it is also non-singular all the eigenvalues are *strictly positive*. Define the minimal eigenvalue of B^*B by l_1.

By Theorem 1 from Ch.II, §17 of [17], for self-adjoint C the form $< Cx, x >$ on the unit sphere does attain its minimum, which is equal to the least eigenvalue of C.

Hence the solution of the initial problem is just l_1. We mention also that the matrix of the adjoint real transformation is the initial one transposed. Therefore, the desired minimum is the least eigenvalue l_1 of $B^T B$.

It follows from this by taking $B = A^T$ that

$$l_1 < x^T, x^T > \le\; < A^T x^T, A^T x^T > \le\; r^2,$$

and every point of the ellipsoid

$$< A^T x^T, A^T x^T > \le\; r^2$$

lies in the ball

$$< x^T, x^T > \le\; \frac{r^2}{l_1},$$

where, in our specific notation, l_1 is the minimal eigenvalue of the matrix $A^T A$.

It remains to check the ∞ norm. Instead of the measure of the ball in (40) we must have the measure of the ball of radius $r/\sqrt{l_1}$. This is achieved by multiplying $a_k(xA(u))$ by $l_1^{n/2}$ and hence $l_1^{n/2} a_k(xA(u))$ is an atom. Correspondingly,

$$||f(\cdot A(u))||_{H^1} \ll l_1^{-n/2} ||f||_{H^1},$$

and finally $\mathcal{H}f$ belongs to H^1 provided

$$\int_{\mathbb{R}^n} |\Phi(u)|\, l_1^{-n/2}(u)\,du < \infty. \tag{45}$$

Further, $l_1^{-n/2} = L_n^{n/2}$, where L_n is the maximal eigenvalue of the matrix $(A^TA)^{-1} = A^{-1}(A^T)^{-1}$. But it is known that such L_n is equal to the spectral radius of the corresponding matrix $A^{-1}(A^T)^{-1}$ and, in turn, to $||A^{-1}||_2^2$. Replacing $l_1^{-n/2}(u)$ in (45) with the obtained bound completes the proof. □

As is mentioned above, the obtained condition (44) is weaker that (43) but of course still more restrictive than (9). It is weaker in the sense that the $\|\cdot\|_2$ matrix norm is smaller than any other matrix norm. On the other hand, all norms in the finite-dimensional space are equivalent. However, having the $\|\cdot\|_2$ matrix norm as a bound is not meaningless, since otherwise the problem of a sharp constant, more precisely, its dependance on dimension, the so-called Goldberg's problem (see, e.g., [22]) may appear.

5.4. *BMO* estimates.

Analyzing the above proof, one can see that the main point is the following result.

Lemma 11. *Let $F(x,u) = f(xB(u))$. Then*

$$||F(\cdot,u)||_{H^1} \ll ||B^{-1}(u)||_2^n\, ||f||_{H^1}, \quad u \in \mathbb{R}^n.$$

One of the advantages of the duality proof in [36] is the following lemma interesting by itself.

Lemma 12. *Let $F(x,u) = f(xB(u))$. Then*

$$\|F(\cdot,u)\|_* \le \|B(u)\|^n |\det B^{-1}(u)| \|f\|_*, \quad u \in \mathbb{R}^n.$$

Proof. Changing variables, we have

$$\begin{aligned} \|F\|_* &= \sup_Q \inf_c \frac{1}{|Q|}\int_Q |f(xB) - c|\,dx \\ &= \sup_Q \inf_c \frac{|\det B^{-1}|}{|Q|}\int_{QB} |f(x) - c|\,dx, \end{aligned}$$

where $QB = QB(u)$ is the image of Q (ellipsoid) after right multiplying by B. We now enlarge the domain of integration, QB, up to the least circumscribed ball Q_B. Repeating now the argument from the proof of Theorem 18, we arrive at the desired result. □

With this lemma as a tool in hand, we can obtain results on the boundedness of Hausdorff type operators in $BMO(\mathbb{R}^n)$, that is, both Hausdorff operator and its adjoint. The interested reader can easily formulate corresponding results.

5.5. **Product Hardy spaces.**

We will now try to obtain sufficient conditions for the boundedness of Hausdorff type operators in the product Hardy space $H_m^1 = H^1(\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_m})$. These spaces are of interest and importance in certain questions of Fourier Analysis (see, e.g., [13], [20]).

Unfortunately, we are not able to obtain conditions for the boundedness of Hausdorff operators on such spaces as general as those for $H^1(\mathbb{R}^n)$. The main restriction will be posed on matrices A. Let

$$A(u) = A_1(u) \oplus ... \oplus A_m(u)$$

be *block diagonal* with square matrices (blocks) A_j be (almost everywhere) non-singular $n_j \times n_j$ matrices and with zero entries off the blocks. Such matrices are by no means artificial and are of importance in various subjects (see, e.g., [32]).

All the information we need on H^1_m one can find in one source [54]. We give it immediately. For an appropriate function f, first of all for a tempered distribution f, the Riesz operators $R_{j_1,\dots,j_m}$, $j_p = 0, 1, \dots, n_p$; $p = 1, 2, \dots, m$, are defined at $x \in \mathbb{R}^n$ by

$$(R_{j_1,\dots,j_m} f)^\wedge(x) = \left(\prod_{p=1}^{m} \left(i \frac{x_{n_1+\dots+n_{p-1}+j_p}}{|x^{(p)}|} \right) \right) \hat{f}(x),$$

where

$$x^{(p)} = (x_{n_1+\dots+n_{p-1}+1}, \dots, x_{n_p}) \in \mathbb{R}^{n_p}.$$

Of course, $R_{0,\dots,0} f = f$. There hold

$$||R_{j_1,\dots,j_m} f||_{H^1_m} \ll ||f||_{H^1_m} \tag{46}$$

and

$$||f||_{H^1_m} \sim \sum_{p=1}^{m} \sum_{j_p=0}^{n_p} ||R_{j_1,\dots,j_m} f||_{L^1}. \tag{47}$$

We are now in a position to derive from Theorem 18 the following

Theorem 19. *If the matrix A is block diagonal as above, then the Hausdorff operator $\mathcal{H}f$ is bounded on the real Hardy space H^1_m provided*

$$\|\mathcal{H}f\|_{H^1_m} \le C_{n,m} \int_{\mathbb{R}^n} |\Phi(u)| \prod_{p=1}^{m} \sum_{j_p=0}^{1} \Delta_p^{j_p}(u)\, du < \infty,$$

where

$$\Delta_p^{j_p}(u) = \begin{cases} |\det A_p^{-1}(u)|, & j_p = 0, \\ ||A_p^{-1}(u)||_2^{n_p}, & j_p = 1. \end{cases}$$

Proof. Using (47), we obtain

$$\sum_{p=1}^{m} \sum_{j_p=0}^{n_p} ||R_{j_1,\dots,j_m} \mathcal{H}f||_{L^1} \le \int_{\mathbb{R}^n} |\Phi(u)| \sum_{p=1}^{m} \sum_{j_p=0}^{n_p} ||R_{j_1,\dots,j_m} f(\cdot A(u))||_{L^1} du.$$

Then, taking (46) into account, we just apply either Lemma 9 or (44) step by step, 2^m times all together. □

Of course, this theorem covers the main result in [41]; the latter is just the simplest partial case. Unfortunately, no condition exists for general matrix A to insure the boundedness of the Hausdorff operator on the product Hardy space. However, block diagonal matrices have the advantage that each block naturally transform only the corresponding group of variables not touching the others.

6. Open problems

In this section we overview open problems on Hausdorff operators in various settings. Some of them were, to certain extent, mentioned in the text, while the others appear as completely new.

6.1. **Power series.**

We formulate certain open problems that naturally arise from the above scrutiny in Section 3.

a) *Find multivariate versions of the results from* Section 3.

Till recently the only known direct generalization was given [9], where the above results were extended in the simplest, product-wise way to the case of the polydisk in $\mathbb{C}^n$. In [2] much more is obtained by new characterization of Hardy spaces on Reinhardt domains. However, much is still can be done.

b) *Study partial sums of (10) as well as of its generalization for Hausdorff and maybe quasi-Hausdorff matrices.*

c) *Find an example of a function (power series) NOT in H^1 for which Hausdorff means or even Cesàro means is in H^1; the same, of course, for quasi-Hausdorff means.*

6.2. **Fourier transform setting.**

In this subsection we overview open problems on Hausdorff operators in the Fourier transform setting both in one and several dimensions.

a) The main one reads as follows:

Construct a counterexample of a function in L^1 but not in H^1 whose value taken by a Hausdorff operator (or even the Cesàro means) is in H^1 (compare with **c)** *from the previous subsection.*

b) *Prove (or disprove) the sharpness of the obtained condition for the boundedness of the Hausdorff operator on $H^1(\mathbb{R}^n)$.*

c) *The same - similarly or instead - for* BMO.

d) *Find other (based on different definitions of $H^1(\mathbb{R}^n)$) proofs of the boundedness of the Hausdorff operator on $H^1(\mathbb{R}^n)$; with the same condition or maybe BETTER one.*

e) As we have seen above, the scale of spaces very similar to H^1 is that of H^p when $p < 1$. However, they differ much both in results and methods. The boundedness of Cesàro means on $H^p(\mathbb{R})$ for all $0 < p < 1$ was proved in [45].

Hausdorff operators in H^p when $p < 1$ in the multidimensional case.

6.3. **Partial integrals.**

We know from Lemma 10 that

$$(\mathcal{H}f)\widehat{\ }(y) = \int_{\mathbb{R}^n} \Phi(u)\,|\det A^{-1}(u)|\,\widehat{f}\big(y(A^{-1})^T(u)\big)\,du,$$

where $(A^{-1})^T$ is the transpose of A^{-1}.

Taking

$$\int_{|x|\le N} (\mathcal{H}f)\widehat{\ }(y) e^{i\langle x,y\rangle} dy,$$

we arrive at

$$\mathcal{H}_N f(x) = \int_{\mathbb{R}^n} \Phi(u)|\det A^{-1}|\,du \int_{\mathbb{R}^n} f(v)\,dv \int_{|y|\le N} e^{-i\langle x - vA^{-1}(u), y\rangle} dy.$$

The last integral is well-known (see, e.g., [52, Ch.IV, §3]), and we get

$$(2\pi N)^{-n/2}\mathcal{H}_N f(x)$$

$$= \int\limits_{\mathbb{R}^{2n}} \Phi(u)|\det A^{-1}|\, f(v)\, |x - vA^{-1}(u)|^{-n/2} J_{n/2}(N|x - vA^{-1}(u)|)\,du\,dv,$$

where $J_{n/2}$ is the Bessel function of order $n/2$, with f either in L^1 or in H^1. In dimension one it looks extremely simple: with $\varphi \in L^1$

$$\mathcal{H}_N^{\varphi} f(x) = \int_{\mathbb{R}^2} \varphi(u) f(t) \frac{\sin N(x - ut)}{x - ut}\,du\,dt.$$

A group of problems related to this is as follows.

a) *Study* $\mathcal{H}_N f$ *or maybe* $\mathcal{H}_N^* f = \sup_N |\mathcal{H}_N f|$.

b) *Find the rate - in* N *- of approximation to* f *by* $\mathcal{H}_N f$*, almost everywhere, or in* L^1 *or* H^1 *norm.*

6.4. **More problems.**

Certain possible problems worth being studied become apparent in discussions during the conference in Istanbul, 2006, first of all with O. Martio.

One of them is to consider general singular operators rather than those defined by the Hilbert or Riesz transforms.

We mention also the question the author was asked after his talk on that conference about *compactness* of Hausdorff operators.

Back to the BMO proof of the boundedness of the Hausdorff operator on $H^1(\mathbb{R}^n)$, instead of studying relation between the BMO norms of $f(xA(u))$ and $f(x)$ (see Lemma 12) one can try the same for $f(F_x(u))$ where $F_x(u)$ is a general family of mappings. A natural assumption on this family is to preserve BMO. In this context $f(xA(u))$ is a partial case when the mapping is linear. The latter definitely preserves BMO, and the only point in the above study was to figure out the bounds.

But the problem of preserving BMO is already solved in general case: necessary and sufficient conditions are given by P. Jones [34] (see also [3]) in dimension one and by Gotoh [25] in the multivariate setting; see also [6].

This leads to the study of a very general Hausdorff operator

$$(\mathcal{H}_{\Phi,F} f)(x) = \int_{\mathbb{R}^n} \Phi(u) f\big(F_x(u)\big)\,du.$$

All the above problems for such operators not only were never studied but even simplest initial results for them, such as Lemma 10, face essential difficulties. Considerable amount of them come from the unavoidable necessity to make use of implicit function theorems.

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