NEW PRE-DUAL SPACE OF MORREY SPACE

A. GOGATISHVILI AND R.CH. MUSTAFAYEV

Abstract. In this paper we give new characterization of the classical Morrey space. Complementary global Morrey-type spaces are introduced. It is proved that for particular values of parameters these spaces give new pre-dual space of the classical Morrey space. We also show that our new pre-dual space of the Morrey space coincides with known pre-dual spaces.

1. Introduction

The well-known Morrey spaces $\mathcal{M}_{p,\lambda}$ introduced by C.B. Morrey in 1938 [11] in relation to the study of partial differential equations, were widely investigated during last decades, including the study of classical operators of Harmonic Analysis - maximal, singular and potential operators - in generalizations of these spaces (the so-called Morrey-type spaces). In the theory of partial differential equations, along with the weighted Lebesgue spaces, Morrey-type spaces also play an important role. These spaces appeared to be quite useful in the study of the local behavior of the solutions to partial differential equations, apriori estimates and other topics in the theory of PDE.

In [5] local Morrey-type spaces $LM_{p\theta,\omega}$ and global Morrey-type spaces $GM_{p\theta,\omega}$ were defined and some properties of these spaces were studied. Authors investigated the boundedness of the Hardy-Littlewood maximal operator in these spaces. After this paper was intensive study of boundedness of other classical operators such as fractional maximal operator, Riesz potential and Calderón-Zygmund singular integral operator (see, for instance [7], for references).

Later in [6] "so-called" complementary local Morrey-type spaces ${}^{\complement}LM_{p\theta,\omega}$ were introduced and the boundedness of fractional maximal operator from complementary local Morrey-type space ${}^{\complement}LM_{p\theta,\omega}$ into local Morrey-type space $LM_{p\theta,\omega}$ was investigated. As in the definition of the space ${}^{\complement}LM_{p\theta,\omega}$ was used complement

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of ball instead of ball, it was named complementary local Morrey-type space and no relation between $LM_{p\theta,\omega}$ and ${}^{\complement}LM_{p\theta,\omega}$ was studied.

In [8] it is proved that the space ${}^{\complement}LM_{p'\theta',\widetilde{\omega}}$ is dual space of the space $LM_{p\theta,\omega}$, where $1 \le p, \theta < \infty$, p' and θ' are conjugate exponents of p and θ, respectively, and $\widetilde{\omega}(t) = \omega^{\theta-1}(t)\left(\int_t^\infty \omega^\theta(s)ds\right)^{-1}$ (see Theorem 3.6 below).

Our goal in this paper is to introduce global complementary Morrey-type space and show that this new space is pre-dual of classical Morrey space.

The paper is organized as follows. We start with notations and give some preliminaries in Section 2. In Section 3 we recall some results on associate spaces of local Morrey-type spaces and complementary local Morrey-type spaces. New characterization of the Morrey space was given in Section 4. In Section 5 we investigate some properties of intersection and union of complementary local Morrey-type spaces. New characterization of pre-dual space of Morrey space was given in Section 6. Finally, in Section 7 we recall known pre-dual spaces of the Morrey space and compare new one with known spaces.

2. Notations and Preliminaries

Now we make some conventions. Throughout the paper, we always denote by c and C a positive constant which is independent of main parameters, but it may vary from line to line. By $A \lesssim B$ we mean that $A \le cB$ with some positive constant c independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. Constant, with subscript such as c_1, does not change in different occurrences. For a measurable set E, χ_E denotes the characteristic function of E.

Given a function w defined on $(0,\infty)$, we say that w satisfies the doubling condition if there exists a constant $D > 0$ such that for any $t > 0$, we have $w(2t) \le Dw(t)$. When w satisfies this condition, we denote $w \in \triangle_2$, for short.

Unless a special remarks is made, the differential element dx is omitted when the integrals under consideration are the Lebesgue integrals.

Let Y be a Banach space and X its subspace. The closure of X in Y we will denote by $[X]_Y$.

Let X and Y be two Banach spaces. The symbol $X \hookrightarrow Y$ means that $X \subset Y$ and the natural embedding of X in Y is continuous. We say that X *coincides* with Y (and write $X = Y$) if X and Y are equivalent in the algebraic and topological sense (their norms are equivalent).

Definition 2.1. Banach spaces X_α, $\alpha \in A$, form a Banach family if there exists a Banach space W such that

$$X_\alpha \hookrightarrow W, \quad \alpha \in A.$$

If $(X_\alpha)_{\alpha \in A}$ is a Banach family, the concepts of its *sum* $\Sigma(X_\alpha)_{\alpha\in A}$ and *intersection* $\triangle(X_\alpha)_{\alpha\in A}$ will be introduced as follows.

Definition 2.2 ([4], Definition 2.1.35)**.** The sum of a family $(X_\alpha)_{\alpha\in A}$ is the term applied to a Banach space X such that

(a) $X_\alpha \hookrightarrow X$, $\alpha \in A$;

(b) If for certain Banach space Y we have

$$X_\alpha \hookrightarrow Y, \quad \alpha \in A, \quad \text{then} \quad X \hookrightarrow Y.$$

Changing the direction of embeddings, we obtain from here the definition of the *intersection* of the family $(X_\alpha)_{\alpha\in A}$.

Note that the sum and intersection of a Banach family exist ([4], Proposition 2.1.36).

For a fixed p with $p \in [1,\infty)$, p' denotes the conjugate exponent of p, namely,

$$p' := \begin{cases} \frac{p}{1-p} & \text{if} \quad 0 < p < 1, \\ +\infty & \text{if} \quad p = 1, \\ \frac{p}{p-1} & \text{if} \quad 1 < p < +\infty, \\ 1 & \text{if} \quad p = +\infty, \end{cases}$$

and $1/(+\infty) = 0$, $0/0 = 0$, $0 \cdot (\pm\infty) = 0$.

If E is a nonempty measurable subset on $\mathbb{R}^n$ and f is a measurable function on E, then we put

$$\|g\|_{L_p(E)} : = \left(\int_E |f(y)|^p dy \right)^{\frac{1}{p}}, \; 0 < p < +\infty,$$
$$\|f\|_{L_\infty(E)} : = \sup\{\alpha : |\{y \in E : |f(y)| \geq \alpha\}| > 0\}.$$

If I is a nonempty measurable subset on $(0,+\infty)$ and g is a measurable function on I, then we define $\|g\|_{L_p(I)}$ and $\|g\|_{L_\infty(I)}$, correspondingly.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ be the open ball centered at x of radius r and ${}^{\complement}B(x,r) := \mathbb{R}^n \backslash B(x,r)$.

Morrey spaces $\mathcal{M}_{p,\lambda}$ were introduced by C. Morrey in 1938 [11] and defined as follows: *for* $0 \leq \lambda \leq n$, $1 \leq p \leq \infty$, $f \in \mathcal{M}_{p,\lambda}$ *if* $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,r))} < \infty,$$

where $B(x,r)$ is the open ball centered at x of radius r.

Note that $\mathcal{M}_{p,0} = L_\infty(\mathbb{R}^n)$ and $\mathcal{M}_{p,n} = L_p(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

In [1] D.R.Adams introduced a variant of Morrey-type spaces as follows: For $0 \leq \lambda \leq n$, $1 \leq p, \theta \leq \infty$, $f \in \mathcal{M}_{p\theta,\lambda}$ if $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}_{p\theta,\lambda}} \equiv \|f\|_{\mathcal{M}_{p\theta,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \|r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))}\|_{L_\theta(0,\infty)} < \infty.$$

(If $\theta = \infty$, then $\mathcal{M}_{p\theta,\lambda} = \mathcal{M}_{p,\lambda}$.)

Let us recall definitions of local Morrey-type spaces and complementary local Morrey-type spaces.

Definition 2.3. ([5]) Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,\omega}$ the local Morrey-type space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p\theta,\omega}} \equiv \|f\|_{LM_{p\theta,\omega}(\mathbb{R}^n)} = \big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\theta(0,\infty)}.$$

Definition 2.4. ([6]) Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by ${}^{\complement}LM_{p\theta,\omega}$ the complementary local Morrey-type space, the space of all functions $f \in L_p({}^{\complement}B(0,t))$ for all $t > 0$ with finite quasinorm

$$\|f\|_{{}^{\complement}LM_{p\theta,\omega}} \equiv \|f\|_{{}^{\complement}LM_{p\theta,\omega}(\mathbb{R}^n)} = \Big\|w(r)\|f\|_{L_p({}^{\complement}B(0,r))}\Big\|_{L_\theta(0,\infty)}.$$

Definition 2.5. Let $0 < p, \theta \le \infty$. We denote by Ω_θ the set all non-negative measurable functions ω on $(0,\infty)$ such that

$$0 < \|\omega\|_{L_\theta(t,\infty)} < \infty, \; t > 0,$$

and by ${}^{\complement}\Omega_\theta$ the set all non-negative measurable functions ω on $(0,\infty)$ such that

$$0 < \|\omega\|_{L_\theta(0,t)} < \infty, \; t > 0.$$

It is convenient to define local Morrey-type spaces and complementary local Morrey-type spaces at any fixed point $x \in \mathbb{R}^n$.

Definition 2.6. Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. For any fixed $x \in \mathbb{R}^n$ we denote by $LM_{p\theta,\omega}^{\{x\}}$, the local Morrey-type space: the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p\theta,\omega}^{\{x\}}} \equiv \|f\|_{LM_{p\theta,\omega}^{\{x\}}(\mathbb{R}^n)} := \big\|w(r)\|f\|_{L_p(B(x,r))}\big\|_{L_\theta(0,\infty)} = \|f(x+\cdot)\|_{LM_{p\theta,\omega}}.$$

Definition 2.7. Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. For any fixed $x \in \mathbb{R}^n$ we denote by ${}^{\complement}LM_{p\theta,w}^{\{x\}}$ the complementary local Morrey-type space, the space of all functions $f \in L_p({}^{\complement}B(x,t))$ for all $t > 0$ with finite quasinorm

$$\|f\|_{{}^{\complement}LM_{p\theta,w}^{\{x\}}} \equiv \|f\|_{{}^{\complement}LM_{p\theta,w}^{\{x\}}(\mathbb{R}^n)} := \Big\|w(r)\|f\|_{L_p({}^{\complement}B(x,r))}\Big\|_{L_\theta(0,\infty)} = \|f(x+\cdot)\|_{{}^{\complement}LM_{p\theta,\omega}}.$$

Note by $LM_{p\theta,\omega} = LM_{p\theta,\omega}^{\{0\}}$ and ${}^{\complement}LM_{p\theta,\omega} = {}^{\complement}LM_{p\theta,w}^{\{0\}}$.

In [5] global Morrey-type spaces $GM_{p\theta,w}$ were defined.

Definition 2.8. ([5]) Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $GM_{p\theta,w}$, the global Morrey-type space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorms

$$\|f\|_{GM_{p\theta,w}} = \|f\|_{GM_{p\theta,w}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,\omega}} = \sup_{x\in\mathbb{R}^n} \|f\|_{LM_{p\theta,\omega}^{\{x\}}}.$$

Note that the space $GM_{p\theta,w}$ is the *intersection* space of the Banach family $(LM^{\{x\}}_{p\theta,\omega})_{x\in\mathbb{R}^n}$, that is, $GM_{p\theta,w} = \triangle(LM^{\{x\}}_{p\theta,\omega})$.

Note that

$$\|f\|_{LM_{p\infty,1}} = \|f\|_{GM_{p\infty,1}} = \|f\|_{L_p}.$$

Furthermore, $GM_{p\infty,r^{-\lambda/p}} \equiv \mathcal{M}_{p,\lambda}$, $0 < \lambda < n$. The interpolation properties of the spaces $GM_{p\infty,w}$ were studied by S. Spanne in [13]. The spaces $GM_{p\theta,r^{-\lambda}}$ were used by G. Lu [10] for studying the embedding theorems for vector fields of Hörmander type.

As mentioned in [6], the *intersection* $\triangle({}^{\complement}LM^{\{x\}}_{p\theta,w})$ of the Banach family $({}^{\complement}LM^{\{x\}}_{p\theta,w})_{x\in\mathbb{R}^n}$, defined by the finiteness of the quasi-norm

$$\|f\|_{\triangle({}^{\complement}LM^{\{x\}}_{p\theta,w})} = \sup_{x\in\mathbb{R}^n} \Big\|\omega(r)\|f\|_{L_p({}^{\complement}B(x,r))}\Big\|_{L_\theta(0,\infty)} = \sup_{x\in\mathbb{R}^n} \|f\|_{{}^{\complement}LM^{\{x\}}_{p\theta,w}}$$

is of no particular interest because this expression is equal to the product $\|f\|_{L_p(\mathbb{R}^n)} \times \|\omega\|_{L_\theta(0,\infty)}$. It means that

$$\triangle({}^{\complement}LM^{\{x\}}_{p\theta,w}) = \begin{cases} L_p(\mathbb{R}^n), & \text{if } \|w\|_{L_\theta(0,\infty)} < \infty \\ \Theta, & \text{if } \|w\|_{L_\theta(0,\infty)} = \infty. \end{cases}$$

It is natural to define global complementary Morrey-type space as a *sum* of a Banach family in the following way.

Definition 2.9. Let $0 < q, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by ${}^{\complement}GM_{q\theta,w} := \Sigma_{x\in\mathbb{R}^n}({}^{\complement}LM^{\{x\}}_{q\theta,w})$, the complementary global Morrey space, the set of all functions f such that $f = \sum_k f_k$ in the sense of distributions, where $f_k \in {}^{\complement}LM^{\{x_k\}}_{q\theta,w}$, $x_k \in \mathbb{R}^n$, and $\sum_k \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{q\theta,w}} < \infty$.

We define a quasi-norm in ${}^{\complement}GM_{q\theta,w}$

$$\|f\|_{{}^{\complement}GM_{q\theta,w}} := \inf_{f=\sum_k f_k} \sum_k \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{q\theta,w}},$$

where the infimum is taken over all representation of f of the form $\sum_k f_k$, $f_k \in {}^{\complement}LM^{\{x_k\}}_{q\theta,w}$, $\sum_k \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{q\theta,w}} < \infty$ and $x_k \in \mathbb{R}^n$.

Remark 2.10. Note that in view of Lemma 7.5 this definition is correct (see [4, p.110] and [3]).

3. Associate and dual spaces of local Morrey-type and complementary local Morrey-type spaces

Let $(\mathcal{R},\mu)$ be a totally σ-finite non-atomic measure space. Let $\mathfrak{M}(\mathcal{R},\mu)$ be the set of all μ-measurable a.e. finite real functions on $\mathcal{R}$.

Definition 3.1. Let X be a set of functions from $\mathfrak{M}(\mathcal{R},\mu)$, endowed with a positively homogeneous functional $\|\cdot\|_X$, defined for every $f \in \mathfrak{M}(\mathcal{R},\mu)$ and such that $f \in X$ if and only if $\|f\|_X < \infty$. We define the associate space X' of X as the set of all functions $f \in \mathfrak{M}(\mathcal{R},\mu)$ such that $\|f\|_{X'} < \infty$, where

$$\|f\|_{X'} = \sup \left\{ \int_{\mathcal{R}} |fg| d\mu : \|g\|_X \le 1 \right\}.$$

In what follows we assume $\mathcal{R} = \mathbb{R}^n$ and $d\mu = dx$.

In [8] the associate spaces of local Morrey-type and complementary local Morrey-type spaces were calculated. Our method of construction of the predual space of the Morrey space mainly based on these results. For the sake of completeness we recall some statements from [8].

Theorem 3.2. ([8], Theorem 4.5) *Assume* $1 \le p < \infty$, $0 < \theta \le \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$. *Set* $X = {}^{\complement}LM_{p\theta,\omega}$.

(i) Let $0 < \theta \le 1$. Then

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}(B(0,t))} \|\omega\|^{-1}_{L_\theta(0,t)},$$

with the positive constant in equivalency independent of f.

(ii) Let $1 < \theta \le \infty$. Then

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}(B(0,t))} d\left(-\|\omega\|^{-\theta'}_{L_\theta(0,t+)}\right)\right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}},$$

with the positive constant in equivalency independent of f.

Theorem 3.3. ([8], Theorem 4.6) *Assume* $1 \le p < \infty$, $0 < \theta \le \infty$. *Let* $\omega \in \Omega_\theta$. *Set* $X = LM_{p\theta,\omega}$.

(i) Let $0 < \theta \le 1$. Then

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}({}^{\complement}B(0,t))} \|\omega\|^{-1}_{L_\theta(t,\infty)},$$

with the positive constant in equivalency independent of f.

(ii) Let $1 < \theta \le \infty$. Then

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}({}^{\complement}B(0,t))} d\|\omega\|^{-\theta'}_{L_\theta(t-,\infty)}\right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}},$$

with the positive constant in equivalency independent of f.

In fact more general results, which are important for our applications, are true.

Theorem 3.4. *Assume* $1 \le p < \infty$, $0 < \theta \le \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$. *For any fixed* $x \in \mathbb{R}^n$ *set* $X = {}^{\complement}LM^{\{x\}}_{p\theta,\omega}$.

(i) Let $0 < \theta \le 1$. Then

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}(B(x,t))} \|\omega\|^{-1}_{L_\theta(0,t)},$$

with the positive constant in equivalency independent of f and x.

(ii) Let $1 < \theta \le \infty$. Then

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|_{L_{p'}(B(x,t))}^{\theta'} d\left(-\|\omega\|_{L_\theta(0,t+)}^{-\theta'} \right) \right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}},$$

with the positive constant in equivalency independent of f and x.

Proof. Let x be any fixed point in $\mathbb{R}^n$. Then

$$\begin{aligned}
\|f\|_{X'} = \|f\|_{\left({}^{\complement}LM_{p\theta,\omega}^{\{x\}}\right)'} &= \sup\left\{ \int_{\mathbb{R}^n} |f(y)g(y)|dy : \|g\|_{{}^{\complement}LM_{p\theta,\omega}^{\{x\}}} \le 1 \right\} \\
&= \sup\left\{ \int_{\mathbb{R}^n} |f(x+y)g(x+y)|dy : \|g(x+\cdot)\|_{{}^{\complement}LM_{p\theta,\omega}} \le 1 \right\} \\
&= \sup\left\{ \int_{\mathbb{R}^n} |f(x+y)g(y)|dy : \|g\|_{{}^{\complement}LM_{p\theta,\omega}} \le 1 \right\} \\
&= \|f(x+\cdot)\|_{\left({}^{\complement}LM_{p\theta,\omega}\right)'}.
\end{aligned}$$

It remains to apply Theorem 3.2. $\square$

Theorem 3.5. *Assume $1 \le p < \infty$, $0 < \theta \le \infty$. Let $\omega \in \Omega_\theta$. For any fixed $x \in \mathbb{R}^n$ set $X = LM_{p\theta,\omega}^{\{x\}}$.*

(i) Let $0 < \theta \le 1$. Then

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}({}^{\complement}B(x,t))} \|\omega\|_{L_\theta(t,\infty)}^{-1},$$

with the positive constant in equivalency independent of f and x.

(ii) Let $1 < \theta \le \infty$. Then

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|_{L_{p'}({}^{\complement}B(x,t))}^{\theta'} d\|\omega\|_{L_\theta(t-,\infty)}^{-\theta'} \right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}},$$

with the positive constant in equivalency independent of f and x.

The proof of Theorem 3.5 is similar to that of Theorem 3.4 (we only need to apply Theorem 3.3 instead of Theorem 3.2) and we omit it.

It was shown in [8] that for some values of parameters the dual spaces coincide with the asssociated spaces. Namely, the following theorems were proved.

Theorem 3.6. ([8], Theorem 5.1) *Assume $1 \le p < \infty$, $1 \le \theta < \infty$. Let $\omega \in \Omega_\theta$ and $\|\omega\|_{L_\theta(0,\infty)} = \infty$. Then*

$$(LM_{p\theta,\omega})^* = {}^{\complement}LM_{p'\theta',\widetilde{\omega}}\,, \tag{3.1}$$

where $\widetilde{\omega}(t) = \omega^{\theta-1}(t) \left(\int_t^\infty \omega^\theta(s)ds \right)^{-1}$, under the following pairing:

$$< f, g >= \int_{\mathbb{R}^n} fg.$$

Moreover $\|f\|_{{}^{\complement}LM_{p'\theta',\widetilde{\omega}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, *where the supremum is taken over all functions* $g \in LM_{p\theta,\omega}$ *with* $\|g\|_{LM_{p\theta,\omega}} \le 1$.

Theorem 3.7. ([8], Theorem 5.2) *Assume* $1 \le p < \infty$, $1 \le \theta < \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$ *and* $\|\omega\|_{L_\theta(0,\infty)} = \infty$. *Then*

$$\left({}^{\complement}LM_{p\theta,\omega}\right)^* = LM_{p'\theta',\overline{\omega}}, \tag{3.2}$$

where $\overline{\omega}(t) = \omega^{\theta-1}(t)\left(\int_0^t \omega^\theta(s)ds\right)^{-1}$, *under the following pairing:*

$$< f, g >= \int_{\mathbb{R}^n} fg.$$

Moreover $\|f\|_{LM_{p'\theta',\overline{\omega}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, *where the supremum is taken over all functions* $g \in {}^{\complement}LM_{p\theta,\omega} : \|g\|_{{}^{\complement}LM_{p\theta,\omega}} \le 1$.

In fact more general results hold true.

Theorem 3.8. *Assume* $1 \le p < \infty$, $1 \le \theta < \infty$. *Let* $\omega \in \Omega_\theta$ *and* $\|\omega\|_{L_\theta(0,\infty)} = \infty$. *Then for any* $x \in \mathbb{R}^n$

$$\left(LM_{p\theta,\omega}^{\{x\}}\right)^* = {}^{\complement}LM_{p'\theta',\widetilde{\omega}}^{\{x\}}, \tag{3.3}$$

where $\widetilde{\omega}(t) = \omega^{\theta-1}(t)\left(\int_t^\infty \omega^\theta(s)ds\right)^{-1}$, *under the following pairing:*

$$< f, g >= \int_{\mathbb{R}^n} fg.$$

Moreover $\|f\|_{{}^{\complement}LM^{\{x\}}_{p'\theta',\widetilde{\omega}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, *where the supremum is taken over all functions* $g \in LM^{\{x\}}_{p\theta,\omega}$ *with* $\|g\|_{LM^{\{x\}}_{p\theta,\omega}} \le 1$.

Theorem 3.9. *Assume* $1 \le p < \infty$, $1 \le \theta < \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$ *and* $\|\omega\|_{L_\theta(0,\infty)} = \infty$. *Then for any* $x \in \mathbb{R}^n$

$$\left({}^{\complement}LM_{p\theta,\omega}^{\{x\}}\right)^* = LM_{p'\theta',\overline{\omega}}^{\{x\}}, \tag{3.4}$$

where $\overline{\omega}(t) = \omega^{\theta-1}(t)\left(\int_0^t \omega^\theta(s)ds\right)^{-1}$, *under the following pairing:*

$$< f, g >= \int_{\mathbb{R}^n} fg.$$

Moreover $\|f\|_{LM^{\{x\}}_{p'\theta',\overline{\omega}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, *where the supremum is taken over all functions* $g \in {}^{\complement}LM^{\{x\}}_{p\theta,\omega} : \|g\|_{{}^{\complement}LM^{\{x\}}_{p\theta,\omega}} \le 1$.

Proofs of Theorem 3.8 and Theorem 3.9 are analogous to proofs of Theorem 3.6 and Theorem 3.7, respectively and we omit them.

4. New characterization of Morrey space

In this section we give new characterization of classical Morrey space.

Note that

$$g \mapsto \inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(\complement B(x,r))} dr$$

is positively homogeneous functional on $\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$.

Denote by

$$\widetilde{\mathcal{M}}_{p,\lambda} := \left\{ f \in \mathfrak{M}(\mathbb{R}^n, dx) : \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} < \infty \right\} \tag{4.1}$$

the associate space of the set of functions $\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$, where

$$\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} := \sup \left\{ \int_{\mathbb{R}^n} |fg| : \inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(\complement B(x,r))} dr \le 1 \right\}. \tag{4.2}$$

To study properties of the space $\widetilde{\mathcal{M}}_{p,\lambda}$ following results are useful.

Lemma 4.1. *Let $1 \le p < \infty$ and $0 < \lambda < n$. Then the inequality*

$$\int_{\mathbb{R}^n} |fg| \le C \|f\|_{\mathcal{M}_{p,\lambda}} \inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(\complement B(x,r))} dr, \tag{4.3}$$

holds with positive constant C independent of functions f and g.

Proof. For $\theta = \infty$ and $w(t) = t^{\frac{\lambda-n}{n}}$ Corollary 3.5 (part (ii)) implies the following inequality

$$\int_{\mathbb{R}^n} |fg| \le C \sup_{t>0} t^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,t))} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(\complement B(x,r))} dr, \tag{4.4}$$

with constant C independent of f, g and $x \in \mathbb{R}^n$. Therefore

$$\begin{aligned} \int_{\mathbb{R}^n} |fg| &\le C \sup_{x\in\mathbb{R}^n,\, t>0} t^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,t))} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(\complement B(x,r))} dr \\ &= C \|f\|_{\mathcal{M}_{p,\lambda}} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(\complement B(x,r))} dr. \end{aligned} \tag{4.5}$$

In view of arbitrariness of x we arrive at (4.3). $\square$

Lemma 4.2. *Let $1 \le p < \infty$ and $0 < \lambda < n$. Then*

$$\inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(\complement B(x,r))} dr = 0 \tag{4.6}$$

if and only if $g = 0$ on $\mathbb{R}^n$.

Proof. Obviously, $\inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(\complement B(x,r))} dr = 0$, when $g = 0$ a.e. on $\mathbb{R}^n$.

Now assume that $\inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|g\|_{L_{p'}(\complement B(x,r))} dr = 0$. For any fixed $R > 0$ consider the function $f = \chi_{B(0,R)}$. Obviously, $f \in \mathcal{M}_{p,\lambda}$, since $\|\chi_{B(0,R)}\|_{\mathcal{M}_{p,\lambda}} \approx R^{\lambda/p}$. Then by the inequality (4.3), we have $\int_{B(0,R)} |f| = 0$, therefore, $f = 0$ a.e. on $B(0,R)$. From arbitrariness of R, we get that $f = 0$ a.e. on $\mathbb{R}^n$. $\square$

Lemma 4.3. *Let* $1 \le p < \infty$ *and* $0 < \lambda < n$. *Then*

$$\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1} \subset L_1^{\mathrm{loc}}(\mathbb{R}^n).$$

Proof. Let g be any function from $\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. Then there exists $x \in \mathbb{R}^n$ such that $g \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. Let R be any fixed positive number. Since the function $f = \chi_{B(x,R)} \in \mathcal{M}_{p,\lambda}$ and $\|f\|_{\mathcal{M}_{p,\lambda}} \approx R^{\lambda/p}$, by the inequality (4.5) we get

$$\int_{B(x,R)} |g(y)| dy \le C R^{\frac{\lambda}{p}} \|g\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} < \infty.$$

In view of arbitrariness of R we get that $g \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. $\square$

Lemma 4.4. *Assume* $1 \le p < \infty$ *and* $0 < \lambda < n$. *Moreover, let* $f \in L_{p'}^{\mathrm{loc}}(\mathbb{R}^n)$. *Then for any fixed* $x \in \mathbb{R}^n$ *and* $R > 0$

$$f\chi_{B(x,R)} \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}.$$

Proof. Indeed, for any fixed $x \in \mathbb{R}^n$ and $R:\ 0 < R < \infty$, we get

$$\begin{aligned}
\|f\chi_{B(x,R)}\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} &= \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|f\chi_{B(x,R)}\|_{L_{p'}(\complement B(x,r))} dr \\
&= \int_0^\infty r^{\frac{n-\lambda}{p}-1} \left(\int_{\complement B(x,r)\cap B(x,R)} |f|^{p'} \right)^{\frac{1}{p'}} dr \\
&= \int_0^R r^{\frac{n-\lambda}{p}-1} \left(\int_{\complement B(x,r)\cap B(x,R)} |f|^{p'} \right)^{\frac{1}{p'}} dr \\
&\le \left(\int_{B(x,R)} |f|^{p'} \right)^{\frac{1}{p'}} \int_0^R r^{\frac{n-\lambda}{p}-1} dr \\
&= c_1 R^{\frac{n-\lambda}{p}} \left(\int_{B(x,R)} |f|^{p'} \right)^{\frac{1}{p'}} < \infty. \qquad \square
\end{aligned}$$

Our main result in this section reads as follows.

Theorem 4.5. *Assume* $1 \le p < \infty$ *and* $0 < \lambda < n$. *Then*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \approx \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}. \tag{4.7}$$

Proof. By Lemma 4.1, it is easy to see that

$$\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} \lesssim \|f\|_{\mathcal{M}_{p,\lambda}}.$$

Let us to prove opposite estimate $\|f\|_{\mathcal{M}_{p,\lambda}} \lesssim \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}$. If $\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} = \infty$, then there is nothing to prove. Assume that $\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} < \infty$.

Observe that for $g \in L^{\mathrm{loc}}_{p'}(\mathbb{R}^n)$ the inequality

$$\int_{B(x,R)} |fg| \le CR^{\frac{n-\lambda}{p}} \left(\int_{B(x,R)} |g|^{p'}\right)^{\frac{1}{p'}} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}} \tag{4.8}$$

holds with constant $C > 0$ independent of f, g, x and R. Indeed, let x be any fixed point in $\mathbb{R}^n$ and $R > 0$. When $\int_{B(x,R)} |g|^{p'} = 0$ there is nothing to prove, since in this case $g = 0$ a.e. on $B(x,R)$. Assume that $\int_{B(x,R)} |g|^{p'} > 0$. Denote by

$$h(y) = \frac{g(y)\chi_{B(x,R)}(y)}{c_1 R^{\frac{n-\lambda}{p}} \left(\int_{B(x,R} |g|^{p'}\right)^{\frac{1}{p'}}}. \tag{4.9}$$

By Lemma 4.4

$$h \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1},$$

and moreover, $\|h\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} \le 1$. Consequently,

$$\inf_{x\in\mathbb{R}^n} \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|h\|_{L_{p'}({}^{\complement}B(x,r))} dr \le 1.$$

Therefore

$$\int_{\mathbb{R}^n} |hf| \le \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}, \tag{4.10}$$

and from (4.9), we get (4.8).

The inequality (4.8) implies that $f \in L^{\mathrm{loc}}_1(\mathbb{R}^n)$. By Theorem of Resonance (see [12, Lemma 27, p.283]) we get that $f \in L^{\mathrm{loc}}_p(\mathbb{R}^n)$. The function $g := |f|^{p-1}\chi_{B(x,R)} \in L^{\mathrm{loc}}_{p'}(\mathbb{R}^n)$, and if we put the function g into the inequality (4.8), we obtain

$$\int_{B(x,R)} |f|^p \le cR^{\frac{n-\lambda}{p}} \left(\int_{B(x,R)} |f|^p\right)^{\frac{1}{p'}} \|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}.$$

Therefore,

$$R^{\frac{\lambda-n}{p}} \left(\int_{B(x,R)} |f|^p\right)^{\frac{1}{p}} \le c\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}.$$

Since a constant c is independent of x and R, we get

$$\|f\|_{\mathcal{M}_{p,\lambda}} \le c\|f\|_{\widetilde{\mathcal{M}}_{p,\lambda}}.$$

□

5. Intersection and union of complementary local Morrey-type spaces

In this section we investigate some properties of intersection and union of complementary local Morrey-type spaces.

The following lemma is true.

Lemma 5.1. *Let $0 < p, \theta \leq \infty$ and $w \in {}^{\complement}\Omega_\theta \bigcap \triangle_2$. Then for any x_1, $x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$*

$$
{}^{\complement}LM_{p\theta,w}^{\{x_1\}}(\mathbb{R}^n) \bigcap {}^{\complement}LM_{p\theta,w}^{\{x_2\}}(\mathbb{R}^n) = {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n). \tag{5.1}
$$

Proof. In order to prove that

$$
{}^{\complement}LM_{p\theta,w}^{\{x_1\}}(\mathbb{R}^n) \bigcap {}^{\complement}LM_{p\theta,w}^{\{x_2\}}(\mathbb{R}^n) \subset {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n) \tag{5.2}
$$

observe that for $x_1 \neq x_2$

$$
{}^{\complement}LM_{p\theta,w}^{\{x_1\}}(\mathbb{R}^n) \bigcap {}^{\complement}LM_{p\theta,w}^{\{x_2\}}(\mathbb{R}^n) \subset L_p(\mathbb{R}^n). \tag{5.3}
$$

Indeed, let $f \in {}^{\complement}LM_{p\theta,w}^{\{x_1\}}(\mathbb{R}^n) \bigcap {}^{\complement}LM_{p\theta,w}^{\{x_2\}}(\mathbb{R}^n)$. Since for any $R > 0$ and $x \in \mathbb{R}^n$

$$
\begin{aligned}
\|f\|_{{}^{\complement}LM_{p\theta,\omega}^{\{x\}}} &= \left(\int_0^\infty w(r)^\theta \|f\|_{L_p({}^{\complement}B(x,r))}^\theta dr \right)^{\frac{1}{\theta}} \\
&\geq \left(\int_0^R w(r)^\theta \|f\|_{L_p({}^{\complement}B(x,r))}^\theta dr \right)^{\frac{1}{\theta}} \\
&\gtrsim \left(\int_0^R w(r)^\theta dr \right)^{\frac{1}{\theta}} \|f\|_{L_p({}^{\complement}B(x,R))}
\end{aligned} \tag{5.4}
$$

and $w \in {}^{\complement}\Omega_\theta$, then $f \in L_p({}^{\complement}B(x_i, R))$, $i = 1, 2$. Denote by $R_0 = |x_1 - x_2|/2$. Then

$$
\|f\|_{L_p(\mathbb{R}^n)} \leq \|f\|_{L_p({}^{\complement}B(x_1,R_0))} + \|f\|_{L_p({}^{\complement}B(x_2,R_0))} < \infty.
$$

It proves (5.3). Remains to show that

$$
{}^{\complement}LM_{p\theta,w}^{\{x_1\}}(\mathbb{R}^n) \bigcap {}^{\complement}LM_{p\theta,w}^{\{x_2\}}(\mathbb{R}^n) \subset {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n). \tag{5.5}
$$

Let $f \in {}^{\complement}LM_{p\theta,w}^{x}(\mathbb{R}^n)$, where x is a fixed point in $\mathbb{R}^n$. For any $r > 2|x|$ we have ${}^{\complement}B(0, r) \subset {}^{\complement}B(x, r/2)$. Indeed, for $y \in B(x, r/2)$ we get $|y| \leq |y - x| + |x| \leq$

$r/2 + r/2 = r$, that is, $B(x, r/2) \subset B(0, r)$. Using $w \in \triangle_2$,

(5.6)

$$\begin{aligned}
\|f\|_{{}^{\complement}LM_{p\theta,\omega}} &= \left(\int_0^\infty w(r)^\theta \|f\|^\theta_{L_p({}^{\complement}B(0,r))} dr \right)^{\frac{1}{\theta}} \\
&= \left(\left\{ \int_0^{2|x|} + \int_{2|x|}^\infty \right\} w(r)^\theta \|f\|^\theta_{L_p({}^{\complement}B(0,r))} dr \right)^{\frac{1}{\theta}} \\
&\lesssim \left(\int_0^{2|x|} w(r)^\theta dr \right)^{\frac{1}{\theta}} \|f\|_{L_p(\mathbb{R}^n)} + \left(\int_{2|x|}^\infty w(r)^\theta \|f\|^\theta_{L_p({}^{\complement}B(x,r/2))} dr \right)^{\frac{1}{\theta}} \\
&\lesssim \left(\int_0^{2|x|} w(r)^\theta dr \right)^{\frac{1}{\theta}} \|f\|_{L_p(\mathbb{R}^n)} + \left(\int_0^\infty w(r)^\theta \|f\|^\theta_{L_p({}^{\complement}B(x,r))} dr \right)^{\frac{1}{\theta}} \\
&\approx \left(\int_0^{2|x|} w(r)^\theta dr \right)^{\frac{1}{\theta}} \|f\|_{L_p(\mathbb{R}^n)} + \|f\|_{{}^{\complement}LM^{\{x\}}_{p\theta,\omega}}.
\end{aligned}$$

Let $f \in {}^{\complement}LM^{\{x_1\}}_{p\theta,w}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p\theta,w}(\mathbb{R}^n)$. By (5.3) $f \in L_p(\mathbb{R}^n)$. Then by (5.6), we get that $f \in {}^{\complement}LM_{p\theta,\omega}$, since $w \in {}^{\complement}\Omega_\theta$. $\square$

Corollary 5.2. *Let $0 < p, \theta \leq \infty$ and $w \in {}^{\complement}\Omega_\theta \bigcap \triangle_2$. Then*

$$\bigcap_{x \in \mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n) = {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n). \tag{5.7}$$

Lemma 5.3. *Let $0 < p, \theta \leq \infty$ and $w \in {}^{\complement}\Omega_\theta \bigcap \triangle_2$. Then for any $x \in \mathbb{R}^n$*

$$\left[{}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n) \right]_{{}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n)} = {}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n), \tag{5.8}$$

that is, the set ${}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n)$ is dense in ${}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n)$.

Proof. Let x be any fixed point $\mathbb{R}^n$. For $f \in {}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n)$ and any $k \in \mathbb{N}$, denote by $f_k = f\chi_{B(x,k)\setminus B(x,\frac{1}{k})}$. It is evident that $f_k \to f$, $k \to \infty$ a.e in $\mathbb{R}^n$. By Lebesgue's Dominated Convergence Theorem, we get that

$$\|f - f_k\|_{{}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n)} \to 0, \qquad k \to \infty.$$

On the other hand, it is evident that $f_k \in L_p(\mathbb{R}^n)$, $k \in \mathbb{N}$. Since $f_k \in {}^{\complement}LM^{\{x\}}_{p\theta,w}(\mathbb{R}^n)$, by (5.6), we get that $f_k \in {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n)$. Finally, we arrive at
$f_k \in {}^{\complement}LM_{p\theta,w}(\mathbb{R}^n) \bigcap L_p(\mathbb{R}^n)$. $\square$

Lemma 5.4. *Let $0 < p, \theta \le \infty$ and $w \in {}^{\complement}\Omega_\theta \bigcap \triangle_2$. Then for any x_1, $x_2 \in \mathbb{R}^n$ such that $x_1 \ne x_2$*

$$\left[{}^{\complement}LM_{p\theta,w}^{\{x_1\}}(\mathbb{R}^n) \bigcap {}^{\complement}LM_{p\theta,w}^{\{x_2\}}(\mathbb{R}^n) \right]_{{}^{\complement}LM_{p\theta,w}^{\{x_i\}}(\mathbb{R}^n)} = {}^{\complement}LM_{p\theta,w}^{\{x_i\}}(\mathbb{R}^n), \quad i = 1, 2. \tag{5.9}$$

Proof. The statement immediately follows from Lemma 5.1 and Lemma 5.3. □

6. New characterization of pre-dual space of Morrey space

In this section we prove that the space ${}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$ is pre-dual space of the Morrey space $GM_{p\infty,\frac{\lambda-n}{p}}$.

Theorem 6.1. *Let $1 \le p < \infty$ and $0 < \lambda < n$. Then*

$$\left({}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1} \right)^* = GM_{p\infty,\frac{\lambda-n}{p}}.$$

Proof. Let $f \in GM_{p\infty,\frac{\lambda-n}{p}}$ and $g \in {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$. For any representation of $g = \sum_k g_k$ with $g_k \in {}^{\complement}LM_{p'1,\frac{n-\lambda}{p}-1}^{\{x_k\}}$ and $\sum_k \|g_k\|_{{}^{\complement}LM_{p'1,\frac{n-\lambda}{p}-1}^{\{x_k\}}} < \infty$, where $x_k \in \mathbb{R}^n$, we have

$$\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| = \left| \int_{\mathbb{R}^n} f(x) \sum_k g_k(x)dx \right| \le \sum_k \int_{\mathbb{R}^n} |f(x)g_k(x)|dx.$$

Applying (4.4), we get

$$\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \lesssim \|f\|_{GM_{p\infty,\frac{\lambda-n}{p}}} \sum_k \|g_k\|_{{}^{\complement}LM_{p'1,\frac{n-\lambda}{p}-1}^{\{x_k\}}} \tag{6.1}$$

Since (6.1) holds for any representation of g, then

$$\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \lesssim \|f\|_{GM_{p\infty,\frac{\lambda-n}{p}}} \|g\|_{{}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}}. \tag{6.2}$$

It proves that $GM_{p\infty,\frac{\lambda-n}{p}} \subset \left({}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1} \right)^*$.

Let us show that $\left({}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1} \right)^* \subset GM_{p\infty,\frac{\lambda-n}{p}}$. It follows from the definition of the space ${}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$ that for any fixed $x \in \mathbb{R}^n$

$${}^{\complement}LM_{p'1,\frac{n-\lambda}{p}-1}^{\{x\}} \subset {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1},$$

and

$$\|f\|_{{}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}} \le \|f\|_{{}^{\complement}LM_{p'1,\frac{n-\lambda}{p}-1}^{\{x\}}}.$$

If $L \in \left({}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}\right)^*$, then for any $x \in \mathbb{R}^n$

$$|L(f)| \le C\|f\|_{{}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}} \le C\|f\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} \tag{6.3}$$

for every $f \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. Thus $L \in \left({}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}\right)^*$. By Theorem 3.7,

$$\left({}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}\right)^* = LM^{\{x\}}_{p\infty,\frac{\lambda-n}{p}},$$

and there exists unique $g_x \in LM^{\{x\}}_{p\infty,\frac{\lambda-n}{p}}$ such that

$$L(f) = \int_{\mathbb{R}^n} f(z)g_x(z)dz \tag{6.4}$$

for any $f \in {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. It is easy to see that if $x_1,\, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$, then $g_{x_1} = g_{x_2}$.

Indeed, by (6.4) we get that

$$\int_{\mathbb{R}^n} (g_{x_1} - g_{x_2})f = 0 \tag{6.5}$$

for any $f \in {}^{\complement}LM^{\{x_1\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n)$. By Lemma 5.4, we have

$$\left[{}^{\complement}LM^{\{x_1\}}_{p\theta,w}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p\theta,w}(\mathbb{R}^n)\right]_{{}^{\complement}LM^{\{x_i\}}_{p\theta,w}(\mathbb{R}^n)} = {}^{\complement}LM^{\{x_i\}}_{p\theta,w}(\mathbb{R}^n), \quad i = 1,2.$$

In view of fact that ${}^{\complement}LM^{\{x_i\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n)$, $i = 1,2$ are Banach spaces and the intersection

$${}^{\complement}LM^{\{x_1\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n) \bigcap {}^{\complement}LM^{\{x_2\}}_{p'1,\frac{n-\lambda}{p}-1}(\mathbb{R}^n)$$

is subspace both of them, we get that $g_{x_1} = g_{x_2}$. By (6.3)

$$\int_{\mathbb{R}^n} f(z)g(z)dz \le C\|f\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}}.$$

Thus

$$\|g\|_{LM^{\{x\}}_{p\infty,\frac{\lambda-n}{p}}} \le C.$$

Since constant C does not depend on x, we get that

$$\|g\|_{GM_{p\infty,\frac{\lambda-n}{p}}} < \infty.$$

Summarizing, we have already proved that there exists unique $g \in GM_{p\infty,\frac{\lambda-n}{p}}$

$$L(f) = \int_{\mathbb{R}^n} fg \tag{6.6}$$

for every $f \in \bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. Let us prove that (6.6) holds for any $f \in {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$. Let f be any function from $GM_{p'1,\frac{n-\lambda}{p}-1}$ and $f = \sum_k f_k$ be any representation of f with $f_k \in {}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}$ and $\sum_k \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} < \infty$, where $x_k \in \mathbb{R}^n$. For finite representation there is nothing to prove. Assume that the representation is infinite, that is, $f = \sum_{k=1}^{\infty} f_k$. Since

$$\begin{aligned}\left|L(f) - L\left(\sum_{k=1}^{m} f_k\right)\right| &= \left|L\left(\sum_{k=m+1}^{\infty} f_k\right)\right| \\ &\le \sum_{k=m+1}^{\infty} |L(f_k)| \lesssim \sum_{k=m+1}^{\infty} \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} \to 0,\ m\to\infty,\end{aligned}$$

and, using (4.4),

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} fg - \int_{\mathbb{R}^n}\left(\sum_{k=1}^{m} f_k\right) g\right| \\ &\qquad = \left|\int_{\mathbb{R}^n}\left(f - \sum_{k=1}^{m} f_k\right) g\right| = \left|\int_{\mathbb{R}^n}\left(\sum_{k=m+1}^{\infty} f_k\right) g\right| \\ &\qquad \le \sum_{k=m+1}^{\infty} \int_{\mathbb{R}^n} |f_k g| \lesssim \|g\|_{GM_{p\infty,\lambda}} \sum_{k=m+1}^{\infty} \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} \to 0,\ m\to\infty,\end{aligned}$$

we arrive at

$$L(f) = \int_{\mathbb{R}^n} fg$$

for any $f \in {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$. □

7. Equivalent Predual Spaces

For $p > 1$ there are already three characterization of the predual space of a Morrey space in the literature. First, in 1986, C.T. Zorko proved the following theorem.

Theorem 7.1 ([14], Theorem 5). *Let $p \in (1,\infty)$ and $\lambda \in (0,n)$. Then a predual space of $\mathcal{M}_{p,\lambda}$ is $\mathcal{Z}_{p',\lambda}$ in the following sense: if $g \in \mathcal{M}_{p,\lambda}$, then $\int_{\mathbb{R}^n} fg$ is an element of $(\mathcal{Z}_{p',\lambda})^*$. Moreover, for any $L \in (\mathcal{Z}_{p',\lambda})^*$, there exists $g \in \mathcal{M}_{p,\lambda}$ such that*

$$L(f) = \int_{\mathbb{R}^n} fg, \qquad f \in \mathcal{Z}_{p',\lambda}.$$

The space $\mathcal{Z}_{p',\lambda}$ is defined by the set of all functions f on $\mathbb{R}^n$ with the norm

$$\|f\|_{\mathcal{Z}_{p',\lambda}} = \inf\left\{ \|\{c_k\}\|_{\ell^1} : f = \sum_k c_k a_k \right\} < \infty,$$

where a_k is a $(p', n-\lambda)$-atom and $\|\{c_k\}\| = \sum_k |c_k| < \infty$, and the infimum is taken over all possible atomic decompositions of f. Additionally, we say that a function a on $\mathbb{R}^n$ is an $(p', n-\lambda)$-atom provided that a is supported on a ball $B \subset \mathbb{R}^n$ and satisfies

$$\|a\|_{p'} \leq \frac{1}{|B|^{\frac{n-\lambda}{np}}}.$$

Second, in 1998, E.A. Kalita obtained another description of the predual space of a Morrey space as follows.

Theorem 7.2 ([9], Theorem 1)**.** *Let $p \in (1, \infty)$ and $\lambda \in (0, n)$. Then a predual space of $\mathcal{M}_{p,\lambda}$ is $\mathcal{K}_{p',\lambda}$ in the following sense: if $g \in \mathcal{M}_{p,\lambda}$, then $\int_{\mathbb{R}^n} fg$ is an element of $(\mathcal{K}_{p',\lambda})^*$. Moreover, for any $L \in (\mathcal{K}_{p',\lambda})^*$, there exists $g \in \mathcal{M}_{p,\lambda}$ such that*

$$L(f) = \int_{\mathbb{R}^n} fg, \qquad f \in \mathcal{K}_{p',\lambda}.$$

The $\mathcal{K}_{p',\lambda}$ consists of all functions f on $\mathbb{R}^n$ with the quasi-norm

$$\|f\|_{\mathcal{K}_{p',\lambda}} = \inf_{\sigma} \left(\int_{\mathbb{R}^n} |f|^{p'} \omega_\sigma^{1-p'} \right)^{\frac{1}{p'}},$$

where

$$\omega_\sigma(x) = \int_{\mathbb{R}^{n+1}_+} r^{-(n-\lambda)} 1_{\mathbb{R}^1_+}(r - |x-y|) d\sigma(y, r),$$

and where the infimum is taken over all $\sigma \in \mathrm{M}^+(\mathbb{R}^{n+1}_+)$ (the class of all nonnegative Radon measures on the upper half space $\mathbb{R}^{n+1}_+ = \{(x, r) : x \in \mathbb{R}^n, \, r > 0\}$) with normalization $\sigma(\mathbb{R}^{n+1}_+) = 1$.

Third, in 2004, D.R. Adams and J. Xiao obtained another description of the predual space of a Morrey space as follows.

Theorem 7.3 ([2], Theorem 2.3)**.** *Let $p \in (1, \infty)$ and $\lambda \in (0, n)$. Then the pre-dual space of $\mathcal{M}_{p,\lambda}$ is $\mathcal{H}_{p',\lambda}$ under the following pairing:*

$$< f, g >= \int_{\mathbb{R}^n} fg.$$

Moreover,

$$\|f\|_{\mathcal{M}_{p,\lambda}} = \sup_g \left| \int_{\mathbb{R}^n} fg \right|,$$

where the supremum is taken over all functions $g \in \mathcal{H}_{p',\lambda}$ with $\|g\|_{\mathcal{H}_{p',\lambda}} \leq 1$.

We say that g is in $\mathcal{H}_{p',\lambda}$ if

$$\|g\|_{\mathcal{H}_{p',\lambda}} = \inf_{\omega} \left(\int_{\mathbb{R}^n} |g|^{p'} \omega^{1-p'} \right)^{\frac{1}{p'}} < \infty, \tag{7.1}$$

where the infimum is over all nonnegative function ω on $\mathbb{R}^n$ satisfying

$$\|\omega\|_{L^1(\Lambda^{(\infty)}_{n-\lambda})} \le 1.$$

Here $\Lambda^{(\infty)}_d$, $0 < d \le n$, denotes the d-dimensional Hausdorff capasity, that is,

$$\Lambda^{(\infty)}_d(E) = \inf \sum r_j^d,$$

where the infimum is taken over all countable coverings of $E \subset \mathbb{R}^n$ by open balls of radius r_j.

The following relationship obtained in [2].

Theorem 7.4 ([2], Theorem 3.3). *Let $p \in (1,\infty)$ and $\lambda \in (0,n)$. Then $\mathcal{Z}_{p',\lambda} = \mathcal{K}_{p',\lambda} = \mathcal{H}_{p',\lambda}$ with*

$$\|\cdot\|_{\mathcal{Z}_{p',\lambda}} \approx \|\cdot\|_{\mathcal{K}_{p',\lambda}} \approx \|\cdot\|_{\mathcal{H}_{p',\lambda}}.$$

Let us compare ${}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$ with known pre-dual spaces. The following Lemma is true.

Lemma 7.5. *Let $1 \le p < \infty$ and $0 < \lambda < n$. Then*

$$\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1} \subset \mathcal{Z}_{p',\lambda}.$$

Proof. Let x be any point in $\mathbb{R}^n$ and let f be any function from ${}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}$. It is possible to decompose f in the following way:

$$f = \sum_k 2^{k\frac{n-\lambda}{p}} \|f\|_{L_{p'}(B(x,2^k)\setminus B(x,2^{k-1}))} \frac{f\chi_{B(x,2^k)\setminus B(x,2^{k-1})}}{2^{k\frac{n-\lambda}{p}} \|f\|_{L_{p'}(B(x,2^k)\setminus B(x,2^{k-1}))}}.$$

Denote by $\lambda_k = 2^{k\frac{n-\lambda}{p}} \|f\|_{L_{p'}(B(x,2^k)\setminus B(x,2^{k-1}))}$ and $a_k = \dfrac{f\chi_{B(x,2^k)\setminus B(x,2^{k-1})}}{2^{k\frac{n-\lambda}{p}} \|f\|_{L_{p'}(B(x,2^k)\setminus B(x,2^{k-1}))}}$.
Note that a_k is $(p', n-\lambda)$-atom. Indeed, it is obvious that $\operatorname{supp} a_k \subset B(x,2^k)$. On the other hand,

$$\|a_k\|_{L_{p'}(\mathbb{R}^n)} = \frac{1}{2^{k\frac{n-\lambda}{p}}} \approx \frac{1}{|B(x,2^k)|^{\frac{n-\lambda}{np}}}.$$

Note that $\{\lambda_k\} \in \ell^1$. Indeed,

$$\begin{aligned}\sum_k |\lambda_k| &= \sum_k 2^{k\frac{n-\lambda}{p}} \|f\|_{L_{p'}(B(x,2^k)\setminus B(x,2^{k-1}))} \\ &\lesssim \sum_k \int_{2^{k-2}}^{2^{k-1}} r^{\frac{n-\lambda}{p}-1} dr \|f\|_{L_{p'}({}^{\complement}B(x,2^{k-1}))} \\ &\lesssim \sum_k \int_{2^{k-2}}^{2^{k-1}} r^{\frac{n-\lambda}{p}-1} \|f\|_{L_{p'}({}^{\complement}B(x,r))} dr \\ &\lesssim \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|f\|_{L_{p'}({}^{\complement}B(x,r))} dr = \|f\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}}.\end{aligned}$$

Recall the following fact: If ψ is a testing function supported in $B(x_1, r_1)$ and a is an atom, we have

$$\left| \int a(x)\psi(x)dx \right| \le r_1^{-\frac{\lambda}{p}} \|\psi\|_\infty$$

(see, for instance, [14]).

Therefore $f = \sum_k \lambda_k a_k$ in the sense of distributions. Thus $f \in \mathcal{Z}_{p',\lambda}$ and

$$\|f\|_{\mathcal{Z}_{p',\lambda}} \lesssim \|f\|_{{}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1}} < \infty. \qquad \square$$

Finally, we prove that the space ${}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$ coincide with known pre-dual spaces, namely, the following Theorem is true.

Theorem 7.6. *Let* $1 \le p < \infty$ *and* $0 < \lambda < n$. *Then*

$${}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1} = \mathcal{Z}_{p',\lambda}.$$

Proof. At first prove that

$$\mathcal{Z}_{p',\lambda} \subset {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}.$$

Let $f \in \mathcal{Z}_{p',\lambda}$. Suppose $f = \sum_k c_k a_k$, where each a_k is $(p', n-\lambda)$-atom supported in some ball $B(x_k, r_k)$ and $\sum_k |\lambda_k| < \infty$. Observe that $a_k \in {}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}$. Indeed,

$$\begin{aligned}\|a\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} &= \int_0^\infty r^{\frac{n-\lambda}{p}-1} \|a_k\|_{L_{p'}({}^{\complement}B(x_k,r))} dr = \int_0^{r_k} r^{\frac{n-\lambda}{p}-1} \|a_k\|_{L_{p'}({}^{\complement}B(x_k,r))} dr \\ &\lesssim r_k^{\frac{n-\lambda}{p}} \|a_k\|_{L_{p'}(\mathbb{R}^n)} \lesssim \frac{r_k^{\frac{n-\lambda}{p}}}{|B(x_k,r_k)|^{\frac{n-\lambda}{np}}} = c_1 < \infty.\end{aligned}$$

Then

$$\sum_k \|c_k a_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} \lesssim c_1 \sum_k |c_k| < \infty,$$

that is, $f \in {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$.

Conversely, by Lemma 7.5

$$\bigcup_{x\in\mathbb{R}^n} {}^{\complement}LM^{\{x\}}_{p'1,\frac{n-\lambda}{p}-1} \subset \mathcal{Z}_{p',\lambda}.$$

Assume that $f \in {}^{\complement}GM_{p'1,\frac{n-\lambda}{p}-1}$ and $f = \sum_k f_k$ be any representation of f with $f_k \in {}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}$ and $\sum_k \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} < \infty$, where $x_k \in \mathbb{R}^n$. Then

$$\|f\|_{\mathcal{Z}_{p',\lambda}} = \|\sum_k f_k\|_{\mathcal{Z}_{p',\lambda}} \le \sum_k \|f_k\|_{\mathcal{Z}_{p',\lambda}} \lesssim \sum_k \|f_k\|_{{}^{\complement}LM^{\{x_k\}}_{p'1,\frac{n-\lambda}{p}-1}} < \infty. \qquad \square$$

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A. Gogatishvili
Institute of Mathematics of the Academy
of Sciences of the Czech Republic
Žitna 25, 115 67 Prague 1
Czech Republic
E-mail address: gogatish@math.cas.cz

R.Ch. Mustafayev
Department of Mathematics
Faculty of Science and Arts
Kirikkale University
71450 Yahsihan
Kirikkale, Turkey

Institute of Mathematics and Mechanics
Academy of Sciences of Azerbaijan
F. Agayev 9
Baku, AZ 1141, Azerbaijan
E-mail address: rzamustafayev@gmail.com