MODULI OF SMOOTHNESS AND RATE OF A.E. CONVERGENCE FOR SOME CONVOLUTION OPERATORS

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ABSTRACT. One purpose of this article is to establish results on the rate of almost everywhere convergence of approximation processes of convolution type in $L^p(\mathbb{R}^n)$, where instead of a particular rate (like t^μ, $\mu > 0$, $t \to 0+$) fractional moduli of smoothness are employed. An essential tool is a modified K-functional. Away from saturation orders these results are nearly optimal. A second purpose is to illustrate that the methods applied also work in other settings which feature a convolution/multiplier structure.

1. INTRODUCTION

Starting point are the two papers [21, 13] on the rate of a.e. convergence of certain integral means of convolution type on $L^p(\mathbb{R})$, $p \geq 1$. One application for the Weierstrass means of the general results there reads as follows.

Let $\varepsilon > 0$ and $f \in L^p(\mathbb{R})$, $1 < p < \infty$, satisfy $\sup_{|h|<t} \|f(\cdot + h) - f(\cdot)\|_p = O(t^\mu)$, $0 < \mu < 1$, then

$$\frac{1}{2\pi t} \int_{\mathbb{R}} f(x - y)\, e^{-(|y|/t)^2}\, dy - f(x) = o_x(t^\mu |\log t|^{1/p+\varepsilon}) \quad a.e. \quad as \ t \to 0 + .$$

It is impossible to choose $\varepsilon = 0$.

Here we considerably generalize the results of [21] in the multivariate framework $L^p(\mathbb{R}^n)$, $1 < p < \infty$, by relating the intrinsic smoothness of f, given by its norm modulus of smoothness, with the rate of a.e. convergence directly; in particular, moduli of smoothness of fractional order are needed to describe the rate of a.e. convergence for saturated approximation processes like the Riesz or the general Weierstrass/Abel-Cartwright means adequately. Further we indicate that the method employed also works for summability methods of certain orthogonal expansions. The methods of proof themselves were motivated by a programmatic remark due to H. S. Shapiro [17, p. 120]: "Another interesting area for study is how far *pointwise* (rather than norm) approximation theorems can be inferred from the Fourier transform of the kernel." This means to look at this area from a multiplier point of view.

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Here and in the next two sections we deal with the $L^p(\mathbb{R}^n)$-setting. First define the modulus of smoothness $\omega_\lambda(\cdot, f)_p$ of order $\lambda > 0$ of $f \in L^p$, $1 < p < \infty$, by

$$\omega_\lambda(t, f)_p := \sup_{|h| < t} \|\Delta_h^\lambda f\|_p, \qquad \Delta_h^\lambda f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\lambda}{j} f(x + jh), \quad \lambda > 0.$$

Approximation processes $(T_t f)_{t>0}$ of convolution type on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, are considered, where

$$(1.1) \quad T_t f := t^{-n} \int_{\mathbb{R}^n} K(y/t) f(x - y) \, dy, \qquad K \in L^1(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} K(y) \, dy = 1.$$

It is well-known (see e.g. [18, p.62]) that $\sup_{|y| \geq |x|} |K(y)| \in L^1$ implies

$$\lim_{t \to 0+} T_t f(x) - f(x) = 0 \quad \text{a. e.}$$

Here we are looking for a refinement of this result in the sense of

$$(1.2) \qquad T_t f(x) - f(x) = o(1) \, \omega_\lambda(t, f)_p F(\omega_\lambda(t, f)_p) \qquad \text{a. e.}$$

Also interesting is the question how the rate of norm convergence directly entails a certain rate of a.e. convergence. The central problem is to find a "good" (decreasing) adjusting function $F \colon (0, 1) \to (0, \infty)$ ensuring (1.2). It turns out advantageously to replace $\omega_\lambda(t, f)$ by Peetre's (modified) K-functional. This we define as follows.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\operatorname{supp} \varphi = \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$, $\varphi(\xi) > 0$ for $1/2 < |\xi| < 2$, and $\sum_{k=-\infty}^{\infty} \varphi_k(\xi) = 1$, $\xi \neq 0$, where $\varphi_k(\xi) = \varphi(2^{-k}\xi)$. Then, for $1 \leq p \leq \infty$ and $\lambda > 0$, define the Riesz space $H_\lambda^p(\mathbb{R}^n)$ by (cf. [2, p. 147])

$$H_\lambda^p(\mathbb{R}^n) := \{f \in L^p : \|f\|_p + |f|_{H_\lambda^p} < \infty\}, \quad |f|_{H_\lambda^p} := \left\| \sum_{k=-\infty}^{\infty} \mathcal{F}^{-1}[|\xi|^\lambda \varphi_k] * f \right\|_p.$$

Here $\mathcal{F}$ denotes the Fourier transform ($\mathcal{F}^{-1}$ its inverse), defined on the Schwarz test function space $\mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}[f](\xi) \equiv \widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-i\xi x} \, dx, \qquad f \in \mathcal{S}.$$

If one introduces an associated K-functional by

$$(1.3) \qquad K(t, f) \equiv K(t, f; L^p, H_\lambda^p) := \inf_{g \in H_\lambda^p} (\|f - g\|_p + t|g|_{H_\lambda^p}), \quad t > 0,$$

(note that it is increasing), then Wilmes [29] has shown that

$$(1.4) \qquad K(t^\lambda, f; L^p(\mathbb{R}^n), H_\lambda^p) \approx \omega_\lambda(t, f)_p, \quad 1 < p < \infty, \lambda > 0.$$

Observing that for $\lambda = N \in \mathbb{N}$ and $1 < p < \infty$ the Riesz space H_N^p can be identified with the Sobolev space W_N^p : $H_N^p = W_N^p$, Wilmes' result is not surprising in view of the well-known characterization [1, p. 341].

In the following we will only consider approximation processes of type (1.1) with radial kernels, i.e., $K(y) := k(|y|)$. Since $\widehat{K}$ is also radial, we write $\widehat{K}(\xi) := m(|\xi|)$. Thus the approximation processes to be discussed may be written in the form

$$(1.5) \qquad T_{m_t} f = \mathcal{F}^{-1}[m(t|\xi|)] * f, \quad \mathcal{F}^{-1}[m(|\xi|)] \in L^1(\mathbb{R}^n), \quad m(0) = 1.$$

Also we restrict ourselves to two cases: the saturated means which satisfy

$$(1.6) \qquad \lim_{s \to 0+} \frac{1 - m(s)}{s^\gamma} = c, \qquad c \neq 0,\ c \in \mathbb{C},\ \text{ some } \gamma > 0,$$

and the non-saturated ones for which $c = 0$ for all $\gamma > 0$. This automatically leads to the Riesz spaces H_γ^p, $\gamma > 0$ as saturation classes. Typical examples of saturated processes with saturation class H_γ^p are the general Riesz means $(R_t^{\delta,\gamma})_{t>o}$ and the general Weierstrass means $(W_t^\gamma)_{t>o}$. In the first case there is $m_{\delta,\gamma}(u) = (1 - u^\gamma)_+^\delta$, where $\gamma > 0$ and $\delta > (n-1)/2$, in the second $m_\gamma(u) = e^{-u^\gamma}$, $\gamma > 0$. Both means can be used [25] to characterize the K-functional

$$(1.7) \qquad K(t^\gamma, f; L^p(\mathbb{R}^n), H_\gamma^p) \approx \|R_t^{\delta,\gamma} f - f\|_p \approx \|W_t^\gamma f - f\|_p, \qquad 1 \le p < \infty.$$

Before formulating a first result we quote two lemmas concerning multipliers useful when estimating norms of convolution operators and maximal functions. By purpose, we restrict ourselves to derivatives of integer order – for fractional derivatives see the corresponding references. We employ the following notation $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$, where the former (the latter) is the shortcut of $A \le c\,B$ ($A \ge c\,B$) for some constant $c > 0$ independent of essential quantities.

Lemma 1.1. [24] *Let m be a measurable, bounded, sufficiently smooth function on $(0, +\infty)$ which vanishes at infinity and which satisfies for some integer $N > (n-1)/2$*

$$A_N(m) \equiv \int_0^\infty u^N \, |m^{(N+1)}(u)| \, du < \infty \,.$$

Then $\mathcal{F}^{-1}[m(|\xi|)] \in L^1(\mathbb{R}^n)$ and $\|\mathcal{F}^{-1}[m]\|_1 \lesssim A_N(m)$.

Lemma 1.2. [5, 6] *Let m be a measurable, bounded, sufficiently smooth function on $(0, +\infty)$ which vanishes at infinity and which satisfies for some $N \in \mathbb{N}$, $N > n|1/p - 1/2| + 1/2$,*

$$(1.8) \qquad B_N(m) \equiv \left(\int_0^\infty |u^N m^{(N)}(u)|^2 \frac{du}{u} \right)^{1/2} + \int_0^\infty u^{N-1} |m^{(N)}(u)| \, du < \infty \,.$$

Define the maximal operator T_m^* on $L^2(\mathbb{R}^n)$ by

$$T_m^* f(x) = \sup_{t>0} |T_{m_t} f(x)|, \quad T_{m_t} f(x) = \mathcal{F}^{-1}[\, m(t|\xi|)\, \widehat{f}\,](x).$$

Then T_m^* is of strong type (p, p), $1 < p < \infty$, with the operator norm estimate $\|T_m^*\|_{p \to p} \lesssim B_N(m)$; also T_m^* is of weak type $(1, 1)$.

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Our main result concerning the problem (1.2) reads as follows.

Theorem 1.3. *Let* $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. *For* $\lambda > 0$, *choose a sequence* $(\theta_\ell)_{\ell \in \mathbb{N}_0}$ *via* $K(\theta_\ell^\lambda, f) \equiv K(\theta_\ell^\lambda, f; L^p(\mathbb{R}^n), H_\lambda^p(\mathbb{R}^n)) = 2^{-\ell}$, *and a decreasing function* $F: (0, 1) \to (0, \infty)$ *with* $\sum_{\ell=0}^{\infty} 1/F^p(2^{-\ell}) < \infty$. *Further assume that the function* $tF(t)$ *is increasing. Let* $\psi \geq 0$ *be a smooth cut-off function with* $\psi(t) = 1$ *if* $0 \leq t \leq 1/2$, *and* $= 0$ *if* $t \geq 3/4$. *Let* $h \geq 0$ *be a bump function with* $\operatorname{supp} h \subset [1/2, 2]$ *such that* $\sum_{j=0}^{\infty} h(2^j s) = 1$, $0 < s < 1$, *and let* m *be sufficiently smooth with finite*

$$B_N(m(1-\psi)), \quad B_N((1-m(2^{-j}\cdot))\,h) =: B_j, \quad j \in \mathbb{N}_0, \quad N \in \mathbb{N}, \quad N \geq (n+1)/2.$$

(a) *If*

$$(1.9) \qquad \sum_{j=0}^{\infty} B_j \, 2^{j\lambda} \left(\sum_{\ell \,:\, \theta_\ell < 2^{-j}}^{\infty} \left(\frac{1}{F(2^{-\ell-1})} \right)^p \right)^{1/p} < \infty,$$

then we have

$$\mathcal{F}^{-1}[m(t|\xi|)] * f(x) - f(x) = o_x(1)K(t^\lambda, f)\, F(K(t^\lambda, f)) \quad \text{a. e.,} \quad t \to 0+,$$

where $o_x(1) \in L^p(\mathbb{R}^n)$.

(b) *If* f *additionally satisfies* $K(t^\lambda, f) \lesssim t^\eta$ *for some fixed* η, $0 < \eta < \lambda$, *and* $F(t) := 1 + |\log t|^{1+1/p+\varepsilon}$, $\varepsilon > 0$, *with* $\sum_{j=0}^{\infty} (j+1)^{-1-\varepsilon} B_j \, 2^{j\lambda} < \infty$, $\varepsilon > 0$, *then, for* $t \to 0+$,

$$\mathcal{F}^{-1}[m(t|\xi|)] * f(x) - f(x) = o_x(1)K(t^\lambda, f)\,|\log K(t^\lambda, f)\,|^{1+1/p+\varepsilon} \quad \text{a. e.,}$$

where again $o_x(1) \in L^p(\mathbb{R}^n)$.

In view of (1.4), the assumption $K(t^\lambda, f) \lesssim t^\eta$ means that f should have some Hölder smoothness.

Examples 1.4. (a) Standard choices for the adjusting function F are $F(t) = 1 + |\log t|^{1/p+\varepsilon}$, $0 < t < 1$, or $F(t) = 1 + |\log t|^{1/p}|\log |\log t||^{1/p+\varepsilon}$, where $0 < t < 1/2$, $\varepsilon > 0$, etc.; in these cases, certainly $t F(t)$ is increasing.

(b) We start by giving two examples of non-saturated approximation processes.

(i) DE LA VALLÉE POUSSIN MEANS. Consider a C^∞-cut-off-function m: $[0, \infty) \to [0, 1]$, $m(u) = 1$ for $0 \leq u \leq 1$ and $m(u) = 0$ for $2 \leq u < \infty$. By Lemma 1 it is clear that $m(|\xi|)$ is the Fourier transform of an admissible approximation kernel. Further, condition (1.6) is satisfied. Now define $V_t f := \mathcal{F}^{-1}[m(t\,|\xi|)] * f$ as de La Vallée Poussin means and observe that, by Lemma 2, $B_j = 0$ for $j \geq 2$, the remaining B's are finite. Thus, by the characterization of the K-functional in [1, p. 341], we obtain for any fixed $k \in \mathbb{N}$, $\varepsilon > 0$,

$$V_t f(x) - f(x) = o_{x,k}(1)\, \omega_k(t, f)_p\, |\log \omega_k(t, f)_p|^{1/p+\varepsilon} \quad \text{a. e.,} \quad t \to 0+.$$

For comparison reasons we note the associated norm convergence result

$$\|V_t f - f\|_p \le C_k\, \omega_k(t, f)_p\,, \qquad f \in L^p,\ \ k \in \mathbb{N}.$$

(ii) Choose $m\colon (0,\infty) \to [0,1]$, $m(u) = 1 - e^{-1/u}$, $m(0) = 1$. Since $m \in C^\infty$ and $\lim_{u\to\infty} m(u) = 0$, by Lemma 1 we have $\mathcal{F}^{-1}[m(|\xi|)] \in L^1(\mathbb{R}^n)$. Further, $B_N((1-e^{-1/t})(1-\psi(t))) < \infty$ and $B_j = O(2^j e^{-2^{j-1}})$, $j \in \mathbb{N}_0$, so that by Lemma 2 we see that, as in the previous example, we have for any fixed $k \in \mathbb{N}$ and $t \to 0+$

$$\mathcal{F}^{-1}[1 - e^{-1/(t|\xi|)}] * f(x) - f(x) = o_{x,k}(1)\, \omega_k(t, f)_p\, |\log \omega_k(t, f)_p|^{1/p+\varepsilon} \quad \text{a.e.}$$

As already mentioned, typical examples of saturated approximation processes are the general Riesz means, the general Weierstrass means, but also the integral means occuring in Lebesgue's differentiation theorem. Theorem 1.3 now yields the following results.

(iii) GENERAL RIESZ MEANS. Consider the general Riesz means from (1.7), i.e., $m(u) \equiv m_{\delta,\gamma}(u) = (1 - u^\gamma)_+^\delta\,$, . Then the assumptions of Theorem 1.3 (a) are satisfied, since $B_j = O(2^{-j\gamma})$ and $B_N(m(u)(1-\psi)) < \infty$ when $0 < \lambda < \gamma$. Thus, we get with $F(t) = |\log t|^{1/p+\varepsilon}$ that

$$R_t^{\delta,\gamma} f(x) - f(x) = o_{x,\lambda,\gamma}(1)\, \omega_\lambda(t, f)_p\, |\log \omega_\lambda(t, f)_p|^{1/p+\varepsilon} \quad \text{a.e.}, \quad t \to 0+\,.$$

In view of (1.4) and (1.7) the associated norm convergence result reads

$$(1.10) \qquad \qquad \|R_t^{\delta,\gamma} f - f\|_p \approx \omega_\gamma(t, f)_p\,, \qquad f \in L^p.$$

When we choose $F(t) = |\log t|^{1+1/p+\varepsilon}$, then the hypotheses of Theorem 1.3 (b) are satisfied even when $\lambda = \gamma$ provided the additional smoothness assumption $\omega_\gamma(t, f)_p \lesssim t^\eta$ for some η, $0 < \eta < \gamma$, holds. Therefore, we have for almost all $x \in \mathbb{R}^n$ that

$$R_t^{\delta,\gamma} f(x) - f(x) = o_x(1)\, \omega_\gamma(t, f)_p\, |\log \omega_\gamma(t, f)_p|^{1+1/p+\varepsilon} \quad \text{a.e.}, \quad t \to 0+,$$

or, when inferring pointwise convergence from strong convergence,

$$R_t^{\delta,\gamma} f(x) - f(x) = o_x(1)\, \|R_t^{\delta,\gamma} f - f\|_p\, |\log \|R_t^{\delta,\gamma} f - f\|_p|^{1+1/p+\varepsilon} \quad \text{a.e.}, \ t \to 0+\,.$$

We point out that, probably caused by the method of proof, the above result involving $\omega_\gamma(t, f)_p$ does not look natural "near" the saturation order t^γ (like $t^\gamma |\log t|$), since in the saturation case $R_t^{\delta,\gamma} f(x) - f(x) = O_x(t^\gamma)$ – see [21].

(iv) GENERAL WEIERSTRASS MEANS. In this case the choice $m_\gamma(u) = e^{-u^\gamma}$ yields the same approximation results as in (iii) for the the general Riesz means.

(v) LEBESGUE'S DIFFERENTIATION THEOREM. This can also be looked at under the aspect of the rate of a.e.-convergence. Observe that (cf. [19, p. 171]

$$\frac{1}{|B_t(0)|} \int_{B_t(0)} f(x + y)\, dy = \mathcal{F}^{-1}[m(t|\xi|)] * f(x), \qquad m(u) = c_n\, u^{-n/2} J_{n/2}(u),$$

where J_α denotes the Bessel function of order α and c_n is given by $m(0)$ 1. Clearly, $T^*_{m(1-\psi)}$ essentially is the Hardy-Littlewood maximal function, thus $\|T^*_{m(1-\psi)}\|_{p\to p} < \infty$, which replaces the condition $B_N(m(1-\psi)) < \infty$ in Theorem 1.3. Using the power series representation of the Bessel function it immediately follows that $B_j = O(2^{-2j})$. Hence, for $0 < \lambda < 2$, we obtain almost everywhere

$$\frac{1}{|B_t(0)|} \int_{B_t(0)} f(x+y)\, dy = f(x) + o_x(1)\, \omega_\lambda(t,f)_p |\log \omega_\lambda(t,f)_p|^{1/p+\varepsilon}, \quad t \to 0+,$$

or, when f satisfies a Hölder condition in L^p-norm, we get for $t \to 0+$

$$\frac{1}{|B_t(0)|} \int_{B_t(0)} f(x+y)\, dy = f(x) + o_x(1)\, \omega_2(t,f)_p |\log \omega_2(t,f)_p|^{1+1/p+\varepsilon} \quad \text{a.e..}$$

2. Proof of Theorem 1.3.

Consider a smooth cut-off function ψ with $\psi(u) = 1$ for $u \le 1/2$ and $= 0$ for $u \ge 3/4$. Then

$$(2.1) \qquad T_{m_t} f - f = T_{(m_t-1)\psi_t} f + T_{(m_t-1)(1-\psi_t)} f.$$

First consider the contribution given by the multiplier $(m-1)(1-\psi)$. By the choice of ψ we have

$$\mathrm{supp}(m-1)(1-\psi) \subset \{\xi : |\xi| \ge 1/2\}.$$

Introduce $\chi \in C_0^\infty$ such that $\chi(\xi) = 1$ for $|\xi| \le 1/2$ and $\chi(\xi) = 0$ for $|\xi| \ge 1$, then

$$(2.2) \qquad \|T_{1-\chi_t} f\|_p \le \|T_{\chi_t} f - f\|_p \lesssim K(t^\lambda, f) \quad 0 < t < 1.$$

This follows from the standard estimate: For all $g \in S$ there holds

$$\|T_{1-\chi_t} f\|_p \le \|T_{1-\chi_t}(f-g)\|_p + \|T_{\chi_t} g - g\|_p \lesssim \|f-g\|_p + t^\lambda \|\mathcal{F}^{-1}[\,|\xi|^\lambda \widehat{g}\,]\|_p$$

and hence also for the infimum over all $g \in S$. Set

$$w(t) := K(t^\lambda, f) F(K(t^\lambda, f))$$

and assume $\theta_{\ell+1} < t \le \theta_\ell$. Since

$$\mathrm{supp}(m_t-1)(1-\psi_t) \subset \{\xi : |\xi| \ge \frac{1}{2\theta_\ell}\}, \quad 1 - \chi_{2\theta_\ell}(\xi) = 1 \ \text{ for } |\xi| \ge \frac{1}{2\theta_\ell}\,,$$

we have

$$T_{(m_t-1)(1-\psi_t)} f = T_{(m_t-1)(1-\psi_t)(1-\chi_{2\theta_\ell})} f = T_{(m_t-1)(1-\psi_t)}(T_{1-\chi_{2\theta_\ell}} f).$$

This implies the key relation

$$(2.3) \qquad |T_{(m_t-1)(1-\psi_t)} f(x)| \le C \sup_{\ell \ge 0} \frac{T^*_{(m-1)(1-\psi)}(T_{1-\chi_{2\theta_\ell}} f)(x)}{w(\theta_{\ell+1})} w(t)\,, \quad t > 0,$$

for almost all $x \in \mathbb{R}^n$. But

$$T_{(m_t-1)(1-\psi_t)} f = T_{m_t(1-\psi_t)} f + T_{\psi_t} f - f,$$

so

$$T^*_{(m-1)(1-\psi)}f \leq T^*_{m(1-\psi)}f + T^*_\psi f + |f|.$$

Now T^*_ψ, $T^*_{m(1-\psi)}\colon L^p \to L^p$ are bounded maximal operators by Lemma 1.2 and the hypothesis. So we may conclude that $||T^*_{(m-1)(1-\psi)}||_{p\to p} < \infty$ and, therefore,

$$\left|\left|\sup_{\ell\geq 0}\frac{T^*_{(m-1)(1-\psi)}(T_{1-\chi_{2\theta_\ell}}f)}{w(\theta_{\ell+1})}\right|\right|^p_p \lesssim \sum_{\ell\geq 0}\frac{||T^*_{(m-1)(1-\psi)}(T_{1-\chi_{2\theta_\ell}}f)||^p_p}{w(\theta_{\ell+1})^p}$$

$$\lesssim ||T^*_{(m-1)(1-\psi)}||^p_{p\to p}\sum_{\ell\geq 0}\frac{||T_{1-\chi_{2\theta_\ell}}f||^p_p}{w(\theta_{\ell+1})^p}.$$

The estimate (2.2) and the definition of $w(t)$ give

$$(2.4) \qquad \left|\left|\sup_{\ell\geq 0}\frac{T^*_{(m-1)(1-\psi)}(T_{1-\chi_{2\theta_\ell}}f)}{w(\theta_{\ell+1})}\right|\right|^p_p \leq C_p(m,\psi)\sum_{\ell=1}^\infty \left(\frac{1}{F(K(\theta_\ell^\lambda,f))}\right)^p$$

which is finite by hypothesis (1.9) (only the contribution $j=0$ is needed here).

Now we estimate the contribution caused by $T_{(m_t-1)\psi_t}f(x)$.

Set $\mu \equiv (m-1)\psi$ and remember that $\mu = 0$ for $|\xi| \geq 3/4$ and $\mu(\xi) = m(\xi) - 1$ for $|\xi| \leq 1/2$. Recall the partition of unity from the assumptions of Theorem 1.3

$$\sum_{j=0}^\infty h(2^j t|\xi|) = 1 \quad \text{for} \quad |t\xi| < 1, \ \ 0 < t < 1.$$

Without loss of generality we may assume $\theta_0 = 1$ and choose $\ell_0 = \ell_0(j) \in \mathbb{N}$ such that $\theta_{\ell_0} \geq 2^{-j}$ but $\theta_{\ell_0+1} < 2^{-j}$. Then

$$\sup_{0<t<1}\frac{|T_{\mu_t}f|}{w(t)} \leq \sum_{j=0}^\infty \sup_{0<t<1}\frac{|T_{\mu_t h_{t\,2^j}}f|}{w(t)}$$

$$(2.5) \qquad \leq \sum_{j=0}^\infty \sup_{\ell\geq\ell_0}\sup_{\theta_{\ell+1}<t\leq\theta_\ell}\frac{|T_{\mu_t}h_{t\,2^j}f|}{w(t)} + \sum_{j=0}^\infty \sup_{2^{-j}<t\leq 1}\frac{|T_{\mu_t}h_{t\,2^j}f|}{w(2^{-j})}.$$

In the last term we used that w is increasing. We begin to discuss the critical first sum on the right side of (2.5). It is clear that $\theta_{\ell+1} < t \leq \theta_\ell$ implies supp $(h_{t\,2^j}) \subset [(2^{j+1}\theta_\ell)^{-1}, (2^{j-1}\theta_{\ell+1})^{-1}]$. Now choose a non-decreasing function $\Phi \in C^\infty[0,\infty)$ such that $\Phi(|\xi|) = 1$ for $|\xi| \geq 1/2$ and $\Phi(|\xi|) = 0$ for $|\xi| \leq 1/4$ and set $\Phi_{\ell,j}(|\xi|) = \Phi(2^j\theta_\ell|\xi|)$. Hence $\Phi_{\ell,j}(|\xi|) = 1$ for $|\xi| \geq (2^{j+1}\theta_\ell)^{-1}$ and, therefore, we have

$\Phi_{\ell,j} h_{t\,2^j} = h_{t\,2^j}$ provided $\theta_{\ell+1} < t \le \theta_\ell$. Thus

$$\left\| \sum_{j=0}^{\infty} \sup_{\ell \ge \ell_0} \frac{\sup_{\theta_{\ell+1} < t \le \theta_\ell} |T_{\mu_t h_{t\,2^j}} f|}{w(\theta_{\ell+1})} \right\|_p \le \sum_{j=0}^{\infty} \left\| \sup_{\ell \ge \ell_0} \frac{T^*_{\mu h_{2^j}} (T_{\Phi_{\ell,j}} f)}{w(\theta_{\ell+1})} \right\|_p$$

$$\le \sum_{j=0}^{\infty} \left(\sum_{\ell \ge \ell_0} \left(\frac{||T^*_{\mu h_{2^j}}(T_{\Phi_{\ell,j}} f)||_p}{w(\theta_{\ell+1})} \right)^p \right)^{1/p}$$

$$\le \sum_{j=0}^{\infty} ||T^*_{\mu h_{2^j}}||_{p \to p} \left(\sum_{\ell \ge \ell_0} \left(\frac{||T_{\Phi_{\ell,j}} f||_p}{w(\theta_{\ell+1})} \right)^p \right)^{1/p}$$

Now observe that $||T_{\Phi_{\ell,j}} f||_p = ||T_{1-\Phi_{\ell,j}} f - f||_p \lesssim K((2^j \theta_\ell)^\lambda, f)$ analogous to (2.2). Therefore, we can continue the preceding estimate by

$$\lesssim \sum_{j=0}^{\infty} ||T^*_{\mu h_{2^j}}||_{p \to p} \left(\sum_{\ell \ge \ell_0} \left(\frac{K((2^j \theta_\ell)^\lambda, f)}{w(\theta_{\ell+1})} \right)^p \right)^{1/p}$$

$$(2.6) \quad \lesssim \sum_{j=0}^{\infty} B_j 2^{j\lambda} \left(\sum_{\ell \ge \ell_0} \left(\frac{K(\theta_\ell^\lambda, f)}{w(\theta_{\ell+1})} \right)^p \right)^{1/p} \lesssim \sum_{j=0}^{\infty} B_j 2^{j\lambda} \left(\sum_{\ell \ge \ell_0} \left(\frac{1}{F(2^{-\ell})} \right)^p \right)^{1/p}.$$

For the last estimates we used the dilation invariance of the integral conditions in (1.8), namely $B_N(m_t) = B_N(m)$, $t > 0$, and $w(\theta_{\ell+1}) \approx 2^{-\ell\lambda} F(2^{-\ell})$.

The second series on the right side of (2.5) is even easier to estimate, since

$$\left\| \sum_{j=0}^{\infty} \sup_{2^{-j} < t < 1} \frac{|T_{\mu_t h_{t\,2^j}} f|}{w(2^{-j})} \right\|_p \lesssim \sum_{j=0}^{\infty} ||T^*_{\mu h_{2^j}}||_{p \to p} ||f||_p \frac{1}{w(2^{-j})}$$

$$(2.7) \qquad\qquad\qquad \lesssim \sum_{j=0}^{\infty} B_j 2^{j\lambda} \frac{||f||_p}{F(2^{-j\lambda})}$$

which is finite by (1.9) (for the simplification of the denominator we used that $K(2^{-j\lambda}, f) \ge 2^{-j\lambda} K(1, f)$).

Summarizing, in view of (2.1), we see that by the hypothesis (1.9) the estimates (2.5), (2.6) (clearly $\ell_0 \ge 0$) in combination with (2.2), (2.4) show that

$$\sup_{0 < t < 1} \frac{T_{m_t} f(x) - f(x)}{K(t^\lambda, f) F(K(t^\lambda, f))} < \infty \qquad \text{a.e., } t \to 0+,$$

which proves Theorem 1.3 (a).

Concerning Part (b) observe that, in view of the above proof, we have only to estimate the last double sum in (2.6) conveniently. By the the smoothness of f the choice of $\theta_{\ell_0(j)}$ yields $\ell_0(j) \gtrsim j + 1$ since

$$2^{-\ell_0(j)} = K(\theta_{\ell_0(j)}^\lambda, f) = 2K(\theta_{\ell_0(j)+1}^\lambda, f) \lesssim \theta_{\ell_0(j)+1}^\eta < 2^{-j\eta}.$$

Therefore, with $F(t) = |\log t|^{1+1/p+\varepsilon}$,

$$\left(\sum_{\ell \geq \ell_0}\left(\frac{1}{F(2^{-\ell})}\right)^p\right)^{1/p} \lesssim \left(\sum_{\ell \geq \ell_0}\ell^{-p-1-\varepsilon p}\right)^{1/p} \lesssim \ell_0^{-1-\varepsilon} \lesssim (j+1)^{-1-\varepsilon},$$

which gives the assertion (b). $\qquad\square$

Remark 2.1. (a) On account of [8, Theorem 3.14] the results in Section 1 also hold in the framework of Lorentz-Zygmund spaces discussed there.

(b) Further, these results easily carry over to the $L^p(\mathbb{T}^n)$-setting, $1 < p < \infty$, in view of the results in [19, Chap. VII, §3], [14], and [30].

3. Generalizations

We want to show that the above method to obtain rates for a.e. convergence also works in other settings. For this purpose note that recalling the methods of Section 2 one realizes that we have only needed control over norms of operators given via multipliers, some norm estimates for the rate of convergence of de La Vallée Poussin type means via the K-functional and characterizations of the K-functional. Concerning a characterization via moduli of smoothness based on (generalized) translations we have needed the special structure of $\mathbb{R}^n$. Apart from that $\mathbb{R}^n$ did not appear once one has the required multiplier criteria.

We start by briefly describing the abstract setting concerning eigenfunction expansions given in [4]. For a Lebesgue measurable set $E \subset \mathbb{R}^n$ and a non-negative measure μ on E define the space $L^p(E, d\mu), 1 < p < \infty$, as the Banach space of all (Lebesgue) measurable functions on E with finite norm

$$\|f\|_p := \left(\int_E |f(x)|^p \, d\mu(x)\right)^{1/p}.$$

Assume that there exists a sequence of projections $(P_k)_{k \in \mathbb{N}_0}$, $P_k \colon L^p \to L^p$, being linear, bounded, and satisfying

(i) $P_j P_k = \delta_{j,k} P_k$, $\delta_{j,k}$ being Kronecker's symbol (mutual orthogonality);

(ii) the sequence (P_k) is total, i.e., $P_k f = 0$ for all $k \in \mathbb{N}_0$ implies $f = 0$;

(iii) the sequence (P_k) is fundamental, i.e., the linear span of the ranges $P_k(L^p)$ is dense: $\overline{\bigcup_{k \in \mathbb{N}_0} P_k(L^p)} = L^p$.

Then, with each $f \in L^p$ one may associate its (formal) Fourier series expansion $f \sim \sum_{k=0}^{\infty} P_k f$. Denote by $\mathfrak{P} \subset L^p$ the set of all "polynomials", i.e., those $f \in X$, for which only finitely many $P_k f \neq 0$. Given a scalar-valued sequence $m = (m_k)_{k \in \mathbb{N}_0}$, we define the linear operator $T_m \colon \mathfrak{P} \to \mathfrak{P}$ by $T_m f = \sum_{k=0}^{\infty} m_k P_k f$.

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In particular, employing the notation $\phi_c^\gamma \colon [0, \infty) \to \mathbb{R}$, $\phi_c^\gamma(u) = (u(u+c))^{\gamma/2}$, we define an abstract derivative $D_c^\gamma \colon D_p(D_c^\gamma) \to L^p$, first on $\mathfrak{P}$ by

$$D_c^\gamma f := \sum_{k=0}^\infty \phi_c^\gamma(k) P_k f, \qquad \gamma > 0, \; c \geq 0, \; f \in \mathfrak{P},$$

and then take its closure in L^p, i.e., to each $f \in D_p(D_c^\gamma)$ there exist $f_j \in \mathfrak{P}$ with $\|f - f_j\|_p \to 0$ and $(D_c^\gamma f_j)_j$ is a Cauchy sequence in L^p. We call m a multiplier on L^p, notation $m \in M^p$, if

$$\|m\|_M := \sup\{\|T_m f\|_p : \|f\|_p \leq 1, \, f \in L^p\}$$

is finite (on account of (iii) there is a unique extension of T_m from $\mathfrak{P}$ to L^p).

To ensure control over maximal functions, which were essentially used in the $L^p(\mathbb{R}^n)$-case, we assume for some $N \in \mathbb{N}_0$ the following property

(3.1) $\| \sup_k (C, N)_k f\|_p \leq C_N \|f\|_p$ for all $f \in L^p$, $1 < p < \infty$,

where the $(C, N)_k f$ denote the Cesàro means of order N of $f \in L^p$. These are given by

(3.2) $(C, N)_k f := (A_k^N)^{-1} \sum_{i=0}^k A_{k-i}^N P_i f, \quad A_k^N = \binom{k+N}{k}.$

Clearly, (3.1) implies the boundedness of the Cesàro means

(3.3) $\|(C, N)_k f\|_p \leq C_N \|f\|_p$ for all $f \in L^p$, $1 < p < \infty$,

which immediately leads to a substitute of Lemma 1.1 – for the idea of the proof see the proof of Lemma 4.2 below (cf. [4, II]).

Lemma 3.1. (a) *Let the projections* $(P_k)_{k \in \mathbb{N}_0} \subset \mathcal{L}(L^p)$ *be as above and* (3.3) *hold. Then*

$$\|T_m f\| \leq C_N \sum_{k=0}^\infty \binom{k+N}{k} |\Delta^{N+1} m_k| \|f\|_p \quad \text{for all } f \in L^p.$$

(b) *If one defines for continuous* $m \colon [0, \infty) \to \mathbb{C}$ *with* $\lim_{t \to \infty} m(t) = 0$ *an operator* $T_{m_{\phi;t}}$ *via the family of sequences* $m_{\phi;t} := \left(m(\rho(t) \phi_c^\gamma(k)) \right)_{k \in \mathbb{N}_0}$, $\gamma > 0$, *with* $\rho \colon [0, \infty) \to [0, \infty)$, $\rho(0) = 0$, *strictly increasing then, for sufficiently smooth* m,

$$\|T_{m_{\phi;t}} f\|_p \lesssim \int_0^\infty s^N |m^{(N+1)}(s)| \, ds \, \|f\|_p$$

uniformly in t.

The latter estimate follows by a variant of the Second Theorem of Consistency – see [23, Lemma 3.8]. Recall $\phi_c^\gamma(u) = (u(u+c))^{\gamma/2}$ and note that $\rho(t)$ will, e.g., be specialized to $2^j \phi_c^\gamma(t)$.

One substitute of Lemma 1.2 reads as follows.

Lemma 3.2. *Let the projections* $(P_k)_{k \in \mathbb{N}_0} \subset \mathcal{L}(L^p)$ *be as above, let* (3.1) *hold, and* m *be as in* Lemma 3.1 (b). *Define the maximal operator* $T^*_{m_\phi}$ *by*

$$T^*_{m_\phi} f := \sup_{t > 0} |T_{m_{\phi;t}} f|.$$

Then, for sufficiently smooth m *and all* $f \in L^p$,

$$\|T^*_{m_\phi} f\|_p \lesssim B^*_N(m) \|f\|_p, \qquad B^*_N(m) := \int_0^\infty s^N |m^{(N+1)}(s)| \, ds.$$

In the special case $\rho(t) = \phi^\gamma_c(t)$, $c = 0$, $\gamma = 1$, the maximal function is described on $\mathfrak{P}$ by $\sup_{t>0} | \sum_{k=0}^\infty m(tk) \, P_k f|$.

Proof. As is well-known (cf. Part II of [4])

$$|T_{m_{\phi;t}} f| = \left| \sum_{k=0}^\infty \binom{k+N}{k} \Delta^{N+1} m(\rho(t)\phi^\gamma_c(k)) \, (C,N)_k f \right|$$

$$\leq |\sup_k (C,N)_k f| \sum_{k=0}^\infty \binom{k+N}{k} |\Delta^{N+1} m(\rho(t)\phi^\gamma_c(k))|$$

$$\lesssim |\sup_k (C,N)_k f| \int_0^\infty s^N |(d/ds)^{(N+1)} m(\rho(t)\phi^\gamma_c(s))| \, ds$$

$$\lesssim |\sup_k (C,N)_k f| \int_0^\infty s^N |m^{(N+1)}(s)| \, ds.$$

Now take the supremum over $t > 0$ on the left hand side, then the L^p-norm of this inequality to get the assertion by the hypothesis (3.1). $\qquad\square$

We point out that this simple derivation costs a certain trade-off concerning the size of the parameter N. But here we do not care for this aspect.

Similarly to [27] we define the (modified) K-functional in the present setting by

$$K_\gamma(t, f; L^p, D_p(D^\gamma_c)) := \inf_{g \in D_p(D^\gamma_c)} \{\|f - g\|_p + (t(t+c))^{\gamma/2} \|D^\gamma_c g\|_p\},$$

where $\gamma > 0$ and $c \geq 0$. In essential it is shown in [26] that

$$(3.4) \qquad K_\gamma(t, f; L^p, D_p(D^\gamma_c)) \approx \left\| f - \sum_{k=0}^\infty e^{-\phi^\gamma_c(t)\phi^\gamma_c(k)} P_k f \right\|_p$$

$$(3.5) \qquad \approx \left\| f - \sum_{k=0}^\infty \left(1 - \phi^\gamma_c(t)\phi^\gamma_c(k) \right)^\kappa_+ P_k f \right\|_p,$$

where $\kappa > N$. Also norm estimates of the de La Vallée Poussin means by the K-functional easily follow. Let m be the smooth cut-off function from Example 1.4 (b) (i). Set

$$V_{\phi;t}f := \sum_{k=0}^{\infty} m(\phi_c^{\gamma}(t)\phi_c^{\gamma}(k))\, P_k f\,, \qquad f \in L^p\,.$$

Then $\|V_{\phi;t}f - f\|_p \lesssim \|f\|_p$ and $\|V_{\phi;t}g - g\|_p \lesssim \phi_c^{\gamma}(t)\|D_c^{\gamma}g\|_p$ by Lemma 3.1 (b), just as in the $L^p(\mathbb{R}^n)$-case. Therefore,

(3.6) $$\|V_{\phi;t}f - f\|_p \lesssim K_{\gamma}(t, f; L^p, D_p(D_c^{\gamma}))\,, \qquad f \in L^p\,.$$

Thus all ingredients of Sections 1 and 2 are available up to a characterization of the K-functional involving moduli of smoothness with respect to generalized translations.

Theorem 3.3 *Let* $f \in L^p$, $1 < p < \infty$. *Choose a positive sequence* $(\theta_\ell)_{\ell \in \mathbb{N}_0}$ *via*

$$K_{\gamma}(\theta_\ell, f; L^p, D_p(D_c^{\gamma})) \equiv K(\phi_c^{\gamma}(\theta_\ell), f; L^p, D_p(D_c^{\gamma})) = 2^{-\ell}, \quad 1 < p < \infty,$$

and a decreasing function $F: (0,1) \to (0,\infty)$ *with* $\sum_{\ell=0}^{\infty} 1/F^p(2^{-\ell}) < \infty$. *Let* $\psi \geq 0$ *be a smooth cut-off function with* $\psi(t) = 1$, $0 \leq t \leq 1/2$, *and* $= 0$, $t \geq 3/4$. *Let* $h \geq 0$ *be a bump function with* $\operatorname{supp} h \subset [1/2, 2]$ *such that*

$$\sum_{j=0}^{\infty} h(2^j \phi_c^{\gamma}(t)\phi_c^{\gamma}(k)) = 1 \quad if \quad 0 < \phi_c^{\gamma}(t)\phi_c^{\gamma}(k) < 1.$$

Let m *be sufficiently smooth and decaying with finite*

$$B_N^*(m(1-\psi)), \quad B_N^*((1-m)\, h(2^j\cdot)) =: B_j^*\,, \quad j \in \mathbb{N}_0\,, \quad N \text{ as in (3.1)}.$$

(a) *If* $tF(t)$ *is increasing and*

(3.7) $$\sum_{j=0}^{\infty} B_j^* \, 2^j \left(\sum_{\ell\,:\,\theta_\ell < 2^{-j}}^{\infty} \left(\frac{1}{F(2^{-\ell-1})} \right)^p \right)^{1/p} < \infty\,,$$

then, for every $f \in L^p$, $1 < p < \infty$,

$$T_{m_{\phi,t}}f(x) - f(x) = o_x(1)K_{\gamma}(t, f)\, F(K_{\gamma}(t, f)) \quad \text{a.e.,} \quad t \to 0+,$$

where $o_x(1) \in L^p$ *(recall* $T_{m_{\phi,t}}f = \sum_{k=0}^{\infty} m(\phi_c^{\gamma}(t)\phi_c^{\gamma}(k))P_k f$ *on* $\mathfrak{P}$*).*

(b) *Let* f *satisfy* $K_{\gamma}(t, f) \lesssim t^{\eta}$ *for some fixed* η, $0 < \eta < 1$. *If* $F(t) = 1 + |\log t|^{1+1/p+\varepsilon}$, $\varepsilon > 0$, *and* $\sum_{j=1}^{\infty}(j+1)^{-1-\varepsilon}B_j^* \, 2^j < \infty$, *then, for every* $f \in L^p$, $1 < p < \infty$, *we have*

$$T_{m_{\phi,t}}f(x) - f(x) = o_x(1)K_{\gamma}(t, f)\, |\log K_{\gamma}(t, f)|^{1+1/p+\varepsilon} \quad \text{a.e.,} \quad t \to 0+,$$

where again $o_x(1) \in L^p$.

As already suggested above, the proof of Theorem 1.3 carries over without difficulties. Replace $t|\phi|$ there by $\phi_c^\gamma(t)\phi_c^\gamma(k)$ here; then the analogs of (2.4) and (2.7) immediately follow. Concerning the critical contribution, analogous to the left term in (2.5), one gets the estimate

$$\lesssim \sum_{j=0}^\infty \|T^*_{((m-1)\psi)_\phi h_{2^j\phi}}\|_{p\to p} \left(\sum_{\ell\geq\ell_0} \Big(\frac{K(2^j\phi_c^\gamma(\theta_\ell),f)}{K(\phi_c^\gamma(\theta_{\ell+1}),f)F(K(\phi_c^\gamma(\theta_{\ell+1}),f))} \Big)^p \right)^{1/p}$$

and can reason as in Section 2 to arrive at the assertions of Theorem 3.3.

Example 3.4. (a) Expansions into spherical harmonics.
Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$, $n \in \mathbb{N}$, with the origin as center and elements $\xi, \theta, \ldots$, where $\xi = (\xi_1, \ldots, \xi_{n+1})$, $|\xi| = 1$. We denote by $L^p(\mathbb{S}^n)$, $1 \leq p < \infty$, the Banach space of all functions $f(\xi)$ which are p-th power integrable on $\mathbb{S}^n$, i.e., for which

$$\|f\|_p = \Big(\frac{1}{|\mathbb{S}^n|} \int_{\mathbb{S}^n} |f(\xi)|^p \, d\sigma(\xi) \Big)^{1/p} < \infty, \qquad |\mathbb{S}^n| = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)},$$

where $d\sigma$ is the surface element of $\mathbb{S}^n$. Let

$$f(\xi) \sim \sum_{k=0}^\infty Y_k(f;\xi), \qquad Y_k(f;\xi) = \sum_{m=1}^{H(k,n)} a_{k,m} Y_{k,m}(\xi), \quad f \in L^p(\mathbb{S}^n),$$

be the expansion in a series of spherical harmonics. Here $H(k,n)$ is the number of linear independent spherical harmonics of degree k on $\mathbb{S}^n$, $H(k,n) = (2k + n - 1)(k + n - 2)!/(k!\,(n-1)!)$. For the Beltrami-Laplace operator δ one has $\delta Y_{k,m} = -k(k + n - 1)Y_{k,m}$, thus set $D_{n-1}^\gamma := (-\delta)^\gamma$ – for this setting see, e.g., [28]. The basic property (3.1) with $N > (n-1)/2$ for spherical harmonics expansions can be found in Bonami and Clerc [3, §3].

Concerning the characterization of the K-functional in the present situation, introduce the shift operator S_t,

$$(S_t f)(\xi) = \frac{1}{|\mathbb{S}^{n-1}|\,(\sin t)^{n-1}} \int_{\xi\cdot\theta=\cos t} f(\theta)\,dt(\theta), \quad 0 < t < \pi,$$

where $dt(\theta)$ is the element of the section $\{\theta : \xi\cdot\theta = \cos t\}$. The shift operator may equivalently be defined by the generalized translation operator for ultraspherical expansions, since $Y_k((S_t f);\xi) = (P_k^{(n-1)/2}(\cos t)/P_k^{(n-1)/2}(1))Y_k(f;\xi)$. If I is the identity operator, introduce a difference operator Δ_t^γ by $\Delta_t^\gamma = (I - S_t)^{\gamma/2}$ and an γ-th order modulus of smoothness by [28, §4.5]

$$\omega_{\gamma,\mathbb{S}^n}(t,f)_p = \sup_{0<s\leq t} \|\Delta_t^\gamma f\|_{L^p(\mathbb{S}^n)}, \qquad \Delta_t^\gamma = \sum_{k=0}^\infty (-1)^k \binom{\gamma/2}{k} (S_t)^k, \;\; \gamma > 0.$$

Then, by Wang and Li [28, Theorem 5.1.2] (see Kaljabin [12] for $\gamma \in \mathbb{N}$),

$$(3.8) \qquad K_\gamma(t,f;L^p(\mathbb{S}^n, D(D_{n-1}^\gamma))) \approx \omega_{\gamma,\mathbb{S}^n}(t,f)_p, \quad 1 < p < \infty, \;\; \gamma > 0.$$

Hence we can formulate results analogous to Example 1.4 (b). E.g., in the case of the de La Vallée Poussin means one has for each fixed $L \in \mathbb{N}$

$$V_{\phi;t}f(x) - f(x) = o_x(1)\,\omega_{L,\mathbb{S}^n}(t,f)_p\,|\log \omega_{L,\mathbb{S}^n}(t,f)_p|^{1/p+\varepsilon} \quad \text{a.e.}, \quad t \to 0+,$$

or for the Riesz means $R_t^{\kappa,\lambda}f(\xi) = \sum_{k=0}^{\infty}(1 - \phi_c^\lambda(t)\phi_c^\lambda(k))_+^\kappa Y_k(f;\xi)$, $\kappa > N$, when $0 < \gamma < \lambda$

$$R_t^{\kappa,\lambda}f(\xi) - f(\xi) = o_\xi(1)\,\omega_{\gamma,\mathbb{S}^n}(t,f)_p|\log \omega_{\gamma,\mathbb{S}^n}(t,f)_p|^{1/p+\varepsilon} \quad \text{a.e.}, \quad t \to 0+.$$

(Observe that we replace $2^j t|\xi|$ of Section 2 here by $2^j \phi_c^\gamma(t)\phi_c^\gamma(k)$ which leads to $B_j^* \lesssim 2^{-j\lambda/\gamma}$ in the application of Theorem 3.1 (a).)

For every $f \in L^p(\mathbb{S}^n)$, $1 < p < \infty$, with $\omega_{\gamma,\mathbb{S}^n}(t,f)_p \lesssim \phi_c^\eta(t)$ for some fixed η, $0 < \eta < \gamma$, we obtain

$$R_t^{\kappa,\gamma}f(\xi) - f(\xi) = o_\xi(1)\,\omega_{\gamma,\mathbb{S}^n}(t,f)_p|\log \omega_{\gamma,\mathbb{S}^n}(t,f)_p|^{1+1/p+\varepsilon} \quad \text{a.e.}, \quad t \to 0+.$$

In view of (3.8) and (3.5) a reformulation of this gives a direct estimate of the rate of a.e. pointwise convergence by the rate of strong convergence, i.e., for $t \to 0+$

$$R_t^{\kappa,\gamma}f(\xi) - f(\xi) = o(1)\|R_t^{\kappa,\gamma}f - f\|_{L^p(\mathbb{S}^n)}\,|\log \|R_t^{\kappa,\gamma}f - f\|_{L^p(\mathbb{S}^n)}|^{1+1/p+\varepsilon} \quad \text{a.e.}.$$

This estimate has the advantage that it does not need a characterization of the K-functional via a modulus of smoothness based on generalized translations. We mention that for even integers $2r$, $r \in \mathbb{N}$, one can find in [7] the following estimate for the K-functional by a different type of modulus of smoothness

$$K(t^{2r}, f) \lesssim \sup_{\rho \in O_t} \|\Delta_\rho^{2r}f\|_{L^p(\mathbb{S}^n)}, \quad \Delta_\rho f(\xi) = f(\rho\xi) - f(\xi), \quad \Delta_\rho^r = \Delta_\rho\Delta_\rho^{r-1}.$$

Here $\rho \in O_t$ are described by those orthogonal matrices with $n \times n$ real entries and determinant 1, for which $(\rho\theta \cdot \theta) \geq \cos t$ for all $\theta \in \mathbb{S}^n$.

(b) Gegenbauer expansions.
For the definitions see, e.g., [3, §6] where in particular the crucial property concerning the maximal function of the Cesàro means (3.1) with $N > \lambda + 1/2$ is proved. For the characterizations (3.4), (3.5), and the estimate (3.6) see [26] ($\alpha = \beta = \lambda - 1/2$); here the differential operator is

$$D = -(1 - x^2)\frac{d^2}{dx^2} + (2\lambda + 1)x\frac{d}{dx}.$$

For the characterization of the K-functional based on generalized translations we refer to [16].

(c) Laguerre and Hermite expansions on $\mathbb{R}_+$ and $\mathbb{R}$, resp.
Concerning the basic property (3.1) for Laguerre expansions see [22, p. 126], for Hermite expansions [22, p. 113].

Remark 3.5. (a) Since our intention in Section 3 is only to indicate the method, we don't care for "smallest" admissible N's (which would have to be

substituted by some critical $\kappa_{crit} \in \mathbb{R}_+$ and would require a fractional calculus – cf. [23]).

(b) Estimates for the maximal function of the Cesàro or Riesz means have been established in many settings, in particular in connection with a.e. summability: E.g., for the Fourier-Bessel transform in [20], for compact symmetric spaces of rank 1 in [3, §7], for compact Riemannian manifolds in [9], [10], for some Lie groups in [11].

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