A SUBSPACE CORRECTION METHOD FOR DISCONTINUOUS GALERKIN DISCRETIZATIONS OF LINEAR ELASTICITY EQUATIONS

BLANCA AYUSO DE DIOS, IVAN GEORGIEV, JOHANNES KRAUS, AND LUDMIL ZIKATANOV

ABSTRACT. We study preconditioning techniques for discontinuous Galerkin discretizations of isotropic linear elasticity problems in primal (displacement) formulation. We propose subspace correction methods based on a splitting of the vector valued piecewise linear discontinuous finite element space, that are optimal with respect to the mesh size and the Lamé parameters. The pure displacement, the mixed and the traction free problems are discussed in detail. We present a convergence analysis of the proposed preconditioners and include numerical examples that validate the theory and assess the performance of the preconditioners.

1. Introduction

The finite element approximation of the equations of isotropic linear elasticity may be accomplished in various ways. The most straightforward approach is to use the primal formulation and conforming finite elements. It is well known that such a method, in general, does not provide approximation to the displacement field when the material is nearly incompressible (the Poisson ratio is close to 1/2). This phenomenon is called *volume locking*. To alleviate locking, several approaches exist. Among the possible solutions, we mention the use of mixed methods, reduced integration techniques, stabilization techniques, nonconforming methods, and the use of discontinuous Galerkin methods. We refer to [11, 14] for further discussions on such difficulties and their remedies. In this work we focus on the Symmetric Interior Penalty discontinuous Galerkin (SIPG) methods introduced in [14, 15, 19, 20] for the approximation of isotropic linear elasticity. We have chosen to work with these DG discretizations, since we have in mind a method that is simple but still applicable to different types of boundary conditions. In fact, unlike classical low order non-conforming methods (see [11]), the Interior Penalty (IP) stabilization methods introduced in [14, 15] can be shown to be stable in the case of essential (Dirichlet or pure displacement) boundary conditions, or natural (Neumann type, or traction free) boundary conditions. As

2010 *Mathematics Subject Classification.* 65F10, 65N20, 65N30.

Key words and phrases. linear elasticity equations, locking free discretizations, preconditioning.

a consequence, these IP methods provide a robust approximation to the displacement field and avoid the volume locking regardless the boundary conditions of the problem.

For the design of the preconditioners we follow the ideas introduced in [4] for second order elliptic problems. However, such extensions are not straightforward, since we aim at constructing preconditioners that work well for three different types of boundary conditions: essential, natural and mixed boundary conditions, used in linear elasticity. This complicates the matters quite a bit. We consider a splitting of the vector valued, piecewise linear, discontinuous finite element space, into two subspaces: the vector valued Crouzeix-Raviart space and a space complementary to it which consists of functions whose averages are L^2 orthogonal to the constants on every edge/face of the partition. This space decomposition is direct and the spaces are orthogonal with respect to a bilinear form obtained via using "reduced integration" to calculate the contributions of the penalty terms in SIPG.

In the pure displacement case (essential boundary conditions), the restriction of the bilinear form based on reduced integration is coercive on the Crouzeix-Raviart space and is spectrally equivalent to the SIPG bilinear form. The space decomposition mentioned above is then orthogonal in this reduced integration bilinear form. Thus, in case of essential boundary conditions we have a natural block diagonal preconditioner for the linear elasticity problem: (1) a solution of a problem arising from discretization by nonconforming Crouzeix-Raviart elements; (2) solution of a well-conditioned problem on the complementary space.

For traction free problems or problems with Dirichlet conditions only on part of the boundary, the situation is quite different. On one hand the reduced integration bilinear form when restricted to the Crouzeix-Raviart space has a null space whose dimension depends on the size of the problem (see [11]). On the other hand in the full SIPG bilinear form (without reduced integration) the space splitting discussed above is no longer orthogonal. Our approach in resolving these issues is based on a delicate estimate given in §3.1 which shows a uniform bound on the angle between the Crouzeix-Raviart and its complementary space in the SIPG bilinear form for all types of boundary conditions. Once such a bound is available we show that a uniform block diagonal preconditioner can be constructed.

The rest of the paper is organized as follows. We present the linear elasticity problem, the basic notation and discuss the DG discretizations considered in §2. Next, in §3 we introduce the splitting of the vector valued piecewise linear DG space and discuss some properties of the related subspaces. In section §4, we introduce the subspace correction methods, and we prove that they give rise to a uniform preconditioner for the symmetric IP method. The last section §5 contains several numerical tests that support the theoretical results.

2. Interior Penalty Discontinuous Galerkin methods for linear elasticity equations

In this section, we introduce the linear elasticity problem together with the basic notation and the derivation of the Interior Penalty (IP) methods and we discuss the stability of these methods.

2.1. **Linear Elasticity: Problem formulation and notation.** Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polygon or polyhedron (not necessarily convex) and let $\boldsymbol{u}$ be a vector field in $\mathbb{R}^d$, defined on Ω such that $\boldsymbol{u} \in [H^1(\Omega)]^d$. The elasticity tensor, which we denote by $\mathcal{C}$, is a linear operator, i.e., $\mathcal{C}\colon \mathbb{R}^{d\times d}_{\text{sym}} \mapsto \mathbb{R}^{d\times d}_{\text{sym}}$, acting on a symmetric matrix $A \in \mathbb{R}^{d\times d}_{\text{sym}}$, in the following way:

$$\mathcal{C}\,A = 2\mu A + \lambda\,\mathrm{trace}(A) I,$$

where μ and λ are the Lamé parameters and satisfy $0 < \mu_1 < \mu < \mu_2$ and $0 \leq \lambda < \infty$. In terms of the modulus of elasticity (Young's modulus), $\mathfrak{E}$, and Poisson's ratio, ν, the Lamè parameters can be rewritten in the case of plane strain as: $\mu = \mathfrak{E}/(2(1+\nu))$ and $\lambda = \nu\mathfrak{E}/((1+\nu)(1-2\nu)$. The material tends to the incompressible limit (becomes incompressible) when the Lamé parameter $\lambda \to \infty$ or equivalently when the Poisson's ratio $\nu \to 1/2$.

One can show that the linear operator $\mathcal{C}$ is selfadjoint and has two eigenvalues: (1) a simple eigenvalue equal to $(2\mu + d\lambda)$ corresponding to the identity matrix; (2) an eigenvalue equal to 2μ, corresponding to the $\frac{d(d+1)}{2} - 1$ dimensional space of traceless, symmetric, real matrices. Thus for $d = 2, 3$, we always have that

$$2\mu\langle A : A\rangle \leq \langle \mathcal{C}A : A\rangle \leq (2\mu + d\lambda)\langle A : A\rangle, \tag{2.1}$$

where $\langle\cdot : \cdot\rangle$ denotes the Frobenius scalar product of two tensors in $\mathbb{R}^{d\times d}$. We also denote by $\langle\cdot,\cdot\rangle$ the Euclidean scalar product of two vectors in $\mathbb{R}^d$, i.e.,

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \sum_{k=1}^{d} v_k w_k, \qquad \langle \boldsymbol{v} : \boldsymbol{w}\rangle = \sum_{j=1}^{d}\sum_{k=1}^{d} v_{jk} w_{jk}.$$

The corresponding inner products in $[L^2(\Omega)]^d$ and $[L^2(\Omega)]^{d\times d}$ are denoted by

$$(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \langle \boldsymbol{v}, \boldsymbol{w}\rangle, \qquad (\boldsymbol{v} : \boldsymbol{w}) = \int_\Omega \langle \boldsymbol{v} : \boldsymbol{w}\rangle.$$

We write $\partial\Omega = \Gamma_N \cup \Gamma_D$ with Γ_N and Γ_D referring respectively to the subsets of the $\partial\Omega$ where Neumann and Dirichlet boundary conditions are imposed.

Let $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T)$ be the symmetric part of the gradient of a vector valued function $\boldsymbol{u}$. The elasticity problem in primal formulation then is: Find $\boldsymbol{u} \in [H^{1+\alpha}_{\Gamma_D}(\Omega)]^d$, $\alpha > 0$, which is the unique minimizer of the energy functional $\mathcal{J}(\boldsymbol{u})$, given by

$$\mathcal{J}(\boldsymbol{u}) := \frac{1}{2}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u})) - (\boldsymbol{f}, \boldsymbol{u}) - (\boldsymbol{g}_N, \boldsymbol{v})_{\Gamma_N} \tag{2.2}$$

Here $\boldsymbol{f} \in [L^2(\Omega)]^d$ is a given volume force and $\boldsymbol{g}_N \in [H^{3/2}(\Gamma_N)]^d$ is a given surface force acting on $\Gamma_N \subset \partial\Omega$. The Euler-Lagrange equations corresponding to the minimization problem (2.2) give the following well known system of linear PDEs for the unknown displacement field $\boldsymbol{u}$:

$$
\begin{aligned}
-\mathrm{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) &= \boldsymbol{f}, && \text{on } \Omega,\\
(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n} &= \boldsymbol{g}_N, && \text{on } \Gamma_N,\\
\boldsymbol{u} &= \boldsymbol{0}, && \text{on } \Gamma_D.
\end{aligned}
\tag{2.3}
$$

In the above equations, $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$. The solution $\boldsymbol{u}$ vanishes on a closed part of the boundary Γ_D (Dirichlet boundary) and the normal stresses are prescribed on Γ_N (Neumann part of the boundary). In the traction free case ($\Gamma_N = \partial\Omega$), the existence of a unique solution to (2.3) is guaranteed if the data satisfy the following compatibility condition:

$$
\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx + \int_{\partial\Omega} \boldsymbol{g}_N \cdot \boldsymbol{v} ds = 0 \quad \forall\, \boldsymbol{v} \in \mathbf{RM}(\Omega),
$$

where $\mathbf{RM}(\Omega)$ is the space of rigid motions, defined by:

$$
\mathbf{RM}(\Omega) := \left\{ \boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b}\boldsymbol{x} \quad : \quad \boldsymbol{a} \in \mathbb{R}^d \quad \boldsymbol{b} \in \mathfrak{so}(d) \ \right\}
\tag{2.4}
$$

where $\boldsymbol{x}$ is the position vector function in Ω and $\mathfrak{so}(d)$ is the Lie algebra of skew-symmetric $d \times d$ matrices. In this case, the uniqueness of solution is guaranteed up to a rigid motion (and is unique, if we require that the solution is orthogonal to any element from $\mathbf{RM}(\Omega)$). In the case of $\Gamma_D \neq \emptyset$ and closed with respect to $\partial\Omega$ no extra conditions are required to guarantee uniqueness. By considering the variational formulation of (2.3), the issue of solvability and uniqueness of the problem reduces to show coercivity of the associated bilinear form. As it is well known, for linear elasticity, this hinges on the classical Korn's inequality [10] which guarantees the existence of a generic positive constant $C_\Omega > 0$ such that:

$$
\|\nabla \boldsymbol{v}\|_{0,\Omega}^2 \leq C_\Omega \left(\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega}^2 + \|\boldsymbol{v}\|_{0,\Omega}^2 \right), \qquad \forall\, \boldsymbol{v} \in [H^1(\Omega)]^d \,.
\tag{2.5}
$$

The second term on the right hand side can be omitted as follows from the Poincaré or Poincaré-Friedrich's inequality, obtaining thus first Korn's inequality for $\boldsymbol{v} \in [H^1_{0,\Gamma_D}(\Omega)]^d$ and second Korn's inequality for $\boldsymbol{v} \in [H^1(\Omega)]^d/\mathbf{RM}(\Omega)$.

2.2. **Interior penalty methods: Preliminaries and notation.** We now introduce the basic notations and tools needed for the derivation of the DG methods.

Domain partitioning. Let $\mathcal{T}_h$ be a shape-regular of partition of Ω into d-dimensional simplices T (triangles if $d = 2$ and tetrahedrons if $d = 3$). We denote by h_T the diameter of T and we set $h = \max_{T \in \mathcal{T}_h} h_T$. We also assume that $\mathcal{T}_h$ is conforming in the sense that it does not contain hanging nodes. A face (shared by two neighboring elements or being part of the boundary) is denoted by E. Clearly, such a face is a $(d-1)$ dimensional simplex, that is, a line segment in two dimensions and a triangle in three dimensions. We denote the set of all faces by $\mathcal{E}_h$, and the collection of all interior faces and boundary faces by $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$, respectively. Further, the set of Dirichlet faces is denoted by $\mathcal{E}_h^D$, and the set of Neumann faces by $\mathcal{E}_h^N$. We thus have,

$$\mathcal{E}_h = \mathcal{E}_h^o \cup \mathcal{E}_h^\partial, \quad \mathcal{E}_h^D = \mathcal{E}_h^\partial \cap \Gamma_D, \quad \mathcal{E}_h^N = \mathcal{E}_h^\partial \cap \Gamma_N, \quad \mathcal{E}_h^\partial = \mathcal{E}_h^D \cup \mathcal{E}_h^N.$$

Trace operators (average and jump) on $E \in \mathcal{E}_h$. To define the average and jump trace operators for an interior face $E \in \mathcal{E}_h^o$, and any $T \in \mathcal{T}_h$, such that $E \in \partial T$ we set $\boldsymbol{n}_{E,T}$ to be the unit outward (with respect to T) normal vector to E. With every face $E \in \mathcal{E}_h^o$ we also associate a unit vector $\boldsymbol{n}_E$ which is orthogonal to the $(d-1)$ dimensional affine variety (line in 2D and plane in 3D) containing the face. For the boundary faces, we always set $\boldsymbol{n}_E = \boldsymbol{n}_{E,T}$, where T is the *unique* element for which we have $E \subset \partial T$. In our setting, for the interior faces, the particular direction of $\boldsymbol{n}_E$ is not important, although it is important that this direction is fixed. For every face $E \in \mathcal{E}_h$, we define $T^+(E)$ and $T^-(E)$ as follows:

$$\begin{aligned} T^+(E) &:= \{T \in \mathcal{T}_h \text{ such that } E \subset \partial T, \text{ and } \langle \boldsymbol{n}_E, \boldsymbol{n}_{E,T}\rangle > 0\}, \\ T^-(E) &:= \{T \in \mathcal{T}_h \text{ such that } E \subset \partial T, \text{ and } \langle \boldsymbol{n}_E, \boldsymbol{n}_{E,T}\rangle < 0\}. \end{aligned} \tag{2.6}$$

It is immediate to see that both sets defined above contain *no more than* one element, that is: for every face we have exactly one $T^+(E)$ and for the interior faces we also have exactly one $T^-(E)$. For the boundary faces we only have $T^+(E)$. In the following, we write $T^\pm$ instead of $T^\pm(E)$, when this does not cause confusion and ambiguity.

For a given function $\boldsymbol{w} \in [L^2(\Omega)]^d$ the average and jump trace operators for a fixed $E \in \mathcal{E}_h^o$ are as follows:

$$\{\!\{\boldsymbol{w}\}\!\} := \left(\frac{\boldsymbol{w}^+ + \boldsymbol{w}^-}{2}\right), \qquad [\![\boldsymbol{w}]\!] := (\boldsymbol{w}^+ - \boldsymbol{w}^-), \tag{2.7}$$

where $\boldsymbol{w}^+$ and $\boldsymbol{w}^-$ denote respectively, the traces of $\boldsymbol{w}$ onto E taken from within the interior of T^+ and T^-. On boundary faces $E \in \mathcal{E}_h^\partial$, we set $\{\!\{\boldsymbol{w}\}\!\} = \boldsymbol{w}$ and $[\![\boldsymbol{w}]\!] = \boldsymbol{w}$. We remark that our notation differs from the one used in [1], [3], [2] (which is considered a classical one for the IP methods). We have chosen a notation that is consistent with the one used in [15], where the IP method we consider was introduced for the pure displacement problem. In addition, it seems that such a choice leads to a shorter and simpler description of the preconditioners we propose here.

Finite Element Spaces. The piecewise linear DG space is defined by

$$V^{\mathrm{DG}} := \{u \in L^2(\Omega) \text{ such that } u\big|_T \in \mathbb{P}^1(T), \quad \forall\, T \in \mathcal{T}_h\,\},$$

where $\mathbb{P}^1(T)$ is the space of linear polynomials on T. The corresponding space of vector valued functions is defined as

$$\boldsymbol{V}^{\mathrm{DG}} := [V^{\mathrm{DG}}]^d.$$

For a given face E, we denote by $\mathcal{P}_E^0 : L^2(E) \mapsto \mathbb{P}^0(E)$ the L^2-projection onto the constant (vector valued or scalar valued) functions on E defined by

$$\mathcal{P}_E^0 w = \frac{1}{|E|}\int_E w \quad \text{for all} \quad w \in L^2(E), \tag{2.8}$$

$$\mathcal{P}_E^0 \boldsymbol{w} = \frac{1}{|E|}\int_E \boldsymbol{w} \quad \text{for all} \quad \boldsymbol{w} \in [L^2(E)]^d. \tag{2.9}$$

Observe that for $\boldsymbol{w} \in \boldsymbol{V}^{\mathrm{DG}}$ the mid-point integration rule implies that $\mathcal{P}_E^0 \boldsymbol{w} = \boldsymbol{w}(m_E)$ for all $E \in \mathcal{E}_h$, with m_E denoting the barycenter of the edge or face E.

The classical Crouzeix-Raviart finite element space can be defined as a subspace of V^{DG}, as follows:

$$V^{\mathrm{CR}} = \left\{ v \in V^{\mathrm{DG}} \quad : \quad \mathcal{P}_E^0 [\![v]\!] = 0, \ \forall\, E \in \mathcal{E}_h^o \right\}. \tag{2.10}$$

The corresponding space of vector valued functions is

$$\boldsymbol{V}^{\mathrm{CR}} := [V^{\mathrm{CR}}]^d \tag{2.11}$$

2.3. **Weighted residual derivation of the IP methods.** In [15] the authors introduced a symmetric interior penalty method for the problem of linear elasticity (2.3) in the pure displacement case (i.e, $\Gamma_D = \partial\Omega, \ \Gamma_N = \emptyset$). We define the function space

$$[H^2(\mathcal{T}_h)]^d = \left\{ \boldsymbol{u} \in [L^2(\Omega)]^d \text{ such that } \boldsymbol{u}\big|_T \in [H^2(T)]^d, \quad \forall\, T \in \mathcal{T}_h \right\}.$$

For any pair of vector fields (or tensors) $\boldsymbol{v}$ and $\boldsymbol{w}$, we denote

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \int_T \langle \boldsymbol{v}, \boldsymbol{w} \rangle.$$

For scalar and vector valued functions we also use the notation

$$(v, w)_{\mathcal{E}} = \sum_{E \in \mathcal{E}} \int_E vw, \quad \text{and} \quad (\boldsymbol{v}, \boldsymbol{w})_{\mathcal{E}} = \sum_{E \in \mathcal{E}} \int_E \langle \boldsymbol{v}, \boldsymbol{w} \rangle. \tag{2.12}$$

We now derive, using the weighted residual framework [8], the IP methods for the more general case of mixed boundary conditions. To present a short derivation of the methods, we assume $\boldsymbol{u} \in [H^2(\Omega)]^d$. Such assumption is not required for the methods to work. We present the derivation under such assumption in order to avoid unnecessary details which would shift the focus of our presentation on preconditioners.

By assuming that the solution of (2.3) is a priori discontinuous, $\boldsymbol{u} \in [H^2(\mathcal{T}_h)]^d$, we may rewrite the continuous problem (2.3) as follows: Find $\boldsymbol{u} \in [H^2(\mathcal{T}_h)]^d$ such that

$$
\begin{cases}
-\operatorname{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) = \boldsymbol{f} & \text{on } T \in \mathcal{T}_h\,, \\
[\![(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}]\!]_E = \boldsymbol{0} & \text{on } E \in \mathcal{E}_h^o\,, \\
[\![\boldsymbol{u}]\!]_E = \boldsymbol{0} & \text{on } E \in \mathcal{E}_h^o\,, \\
[\![\boldsymbol{u}]\!]_E = \boldsymbol{0} & \text{on } E \in \mathcal{E}_h^D\,, \\
[\![(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n} - \boldsymbol{g}_N]\!]_E = \boldsymbol{0} & \text{on } E \in \mathcal{E}_h^N\,.
\end{cases} \tag{2.13}
$$

where we recall that $\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\,\mathrm{trace}(\boldsymbol{\varepsilon}(\boldsymbol{u}))I$. Following [8], we next introduce a variational formulation of (2.13) by considering the following five operators

$$
\begin{aligned}
&\mathcal{B}_0\colon [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{T}_h)]^d, \\
&\mathcal{B}_1\colon [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{E}_h^o)]^d, \qquad && \mathcal{B}_1^\partial\colon [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{E}_h^D)]^d \\
&\mathcal{B}_2\colon [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{E}_h^o)]^d, && \mathcal{B}_2^\partial\colon [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{E}_h^N)]^d,
\end{aligned}
$$

and weighting each equation in (2.13) appropriately. This then amounts to considering the following problem: Find $\boldsymbol{u} \in [H^2(\mathcal{T}_h)]^d$ such that for all $\boldsymbol{v} \in [H^2(\mathcal{T}_h)]^d$

$$
\begin{aligned}
(-\operatorname{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) - \boldsymbol{f}, \mathcal{B}_0(\boldsymbol{v}))_{\mathcal{T}_h} + ([\![(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}]\!], \mathcal{B}_2(\boldsymbol{v}))_{\mathcal{E}_h^o} + ([\![\boldsymbol{u}]\!], \mathcal{B}_1(\boldsymbol{v}))_{\mathcal{E}_h^o} \\
+ ([\![\boldsymbol{u}]\!], \mathcal{B}_1^\partial(\boldsymbol{v}))_{\mathcal{E}_h^D} + ([\![(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n} - \boldsymbol{g}_N]\!], \mathcal{B}_2^\partial(\boldsymbol{v}))_{\mathcal{E}_h^N} = \boldsymbol{0}.
\end{aligned} \tag{2.14}
$$

Different choices of the operators $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_1^\partial$ and $\mathcal{B}_2^\partial$ above give rise to different variational formulations and, consequently to different DG methods. We refer to [8, Theorem 6] for sufficient conditions on the operators $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_1^\partial$ and $\mathcal{B}_2^\partial$ to guarantee[1] the uniqueness of the solution of (2.14).

To derive the IP method of interest, we take $\boldsymbol{v}$ piecewise smooth and we set $\mathcal{B}_0(\boldsymbol{v}) = \boldsymbol{v}$, $\mathcal{B}_2(\boldsymbol{v}) = \{\!\{\boldsymbol{v}\}\!\}$ and $\mathcal{B}_2^\partial(\boldsymbol{v}) = \boldsymbol{v}$ in (2.14), to obtain that

$$
\begin{aligned}
(-\operatorname{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{v})_{\mathcal{T}_h} + ([\![(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}]\!], \{\!\{\boldsymbol{v}\}\!\})_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} + ([\![\boldsymbol{u}]\!], \mathcal{B}_1(\boldsymbol{v}))_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} \\
= (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h} + (\boldsymbol{g}_N, \boldsymbol{v})_{\mathcal{E}_h^N}\,.
\end{aligned} \tag{2.15}
$$

Defining

$$
\mathcal{F}(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h} + ([\![\boldsymbol{g}]\!], \mathcal{B}_1^\partial(\boldsymbol{v}))_{\mathcal{E}_h^D} + (\boldsymbol{g}_N, \boldsymbol{v})_{\mathcal{E}_h^N}, \tag{2.16}
$$

and integrating by parts the first term on the left side of (2.15) then leads to

$$
(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{T}_h} - (\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}\}\!\}, [\![\boldsymbol{v}]\!])_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} + ([\![\boldsymbol{u}]\!], \mathcal{B}_1(\boldsymbol{v}))_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} = \mathcal{F}(\boldsymbol{v}). \tag{2.17}
$$

For a fixed edge $E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D$ the operator $\mathcal{B}_1(\boldsymbol{v})$ is defined by

$$
\mathcal{B}_1(\boldsymbol{v}) := -\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}))\boldsymbol{n}\}\!\} + \alpha_0\beta_0\mathcal{P}_E^0[\![\boldsymbol{v}]\!] + \alpha_1\beta_1[\![\boldsymbol{v}]\!], \tag{2.18}
$$

[1]We note that in [8] the focus is on the scalar Laplace equation. The arguments for the elasticity problem, are basically the same.

where, following [15], the parameters β_0 and β_1 are chosen depending on the Lamé constants λ and μ:

$$\beta_0 := d\lambda + 2\mu, \qquad \beta_1 := 2\mu\ . \tag{2.19}$$

The remaining two parameters, α_0 and α_1, are still at our disposal to ensure (later on) stability and to avoid locking of the resulting method.

We define

$$\begin{aligned} a_{j,0}([\![\boldsymbol{u}]\!],[\![\boldsymbol{v}]\!]) &:= \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \int_E \langle h_E^{-1}[\![\boldsymbol{u}]\!], \mathcal{P}_E^0[\![\boldsymbol{v}]\!]\rangle, \\ a_{j,1}([\![\boldsymbol{u}]\!],[\![\boldsymbol{v}]\!]) &:= \alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \int_E \langle h_E^{-1}[\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]\rangle\ , \end{aligned} \tag{2.20}$$

and set

$$a_j([\![\boldsymbol{u}]\!],[\![\boldsymbol{v}]\!]) = a_{j,0}([\![\boldsymbol{u}]\!],[\![\boldsymbol{v}]\!]) + a_{j,1}([\![\boldsymbol{u}]\!],[\![\boldsymbol{v}]\!]).$$

Then, the weak formulation of Problem (2.13) reads: Find $\boldsymbol{u} \in [H^2(\mathcal{T}_h)]^d$ such that

$$\mathcal{A}(\boldsymbol{u},\boldsymbol{w}) = \mathcal{F}(\boldsymbol{w}), \qquad \forall\ \boldsymbol{w} \in [H^2(\mathcal{T}_h)]^d. \tag{2.21}$$

The bilinear form $\mathcal{A}(\cdot,\cdot)$ is given by

$$\mathcal{A}(\boldsymbol{u},\boldsymbol{w}) = \mathcal{A}_0(\boldsymbol{u},\boldsymbol{w}) + a_{j,1}([\![\boldsymbol{u}]\!],[\![\boldsymbol{w}]\!]), \tag{2.22}$$

where

$$\begin{aligned} \mathcal{A}_0(\boldsymbol{u},\boldsymbol{w}) = (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{T}_h} &- (\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}\}\!\}, [\![\boldsymbol{w}]\!])_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} \\ &- ([\![\boldsymbol{u}]\!], \{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{w}))\boldsymbol{n}\}\!\})_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} + a_{j,0}([\![\boldsymbol{u}]\!],[\![\boldsymbol{w}]\!]). \end{aligned} \tag{2.23}$$

It is straightforward to see that

$$\begin{aligned} \mathcal{A}(\boldsymbol{u},\boldsymbol{w}) = (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{T}_h} &- (\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}\}\!\}, [\![\boldsymbol{w}]\!])_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} \\ &+ \theta([\![\boldsymbol{u}]\!], \{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{w}))\boldsymbol{n}\}\!\})_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} + a_j([\![\boldsymbol{u}]\!],[\![\boldsymbol{w}]\!]). \end{aligned} \tag{2.24}$$

To obtain the discrete formulation, we replace the function space $[H^2(\mathcal{T}_h)]^d$ in (2.21) by $\boldsymbol{V}^{\mathrm{DG}}$, and we get the **IP-1 approximation** to the problem: Find $\boldsymbol{u}_h \in \boldsymbol{V}^{\mathrm{DG}}$ such that:

$$\mathcal{A}(\boldsymbol{u}_h,\boldsymbol{w}) = \mathcal{F}(\boldsymbol{w}), \qquad \forall\ \boldsymbol{w} \in \boldsymbol{V}^{\mathrm{DG}}. \tag{2.25}$$

We could also consider the approximation given by the **IP-0 method**: Find $\boldsymbol{u}_h \in \boldsymbol{V}^{\mathrm{DG}}$ such that:

$$\mathcal{A}_0(\boldsymbol{u}_h,\boldsymbol{w}) = \mathcal{F}(\boldsymbol{w}), \qquad \forall\ \boldsymbol{w} \in \boldsymbol{V}^{\mathrm{DG}}. \tag{2.26}$$

As we see next, the **IP-0** method provides a robust approximation to the problem (2.3) in the pure displacement problem $\Gamma_D = \partial\Omega$. As we mentioned earlier, for other types of boundary conditions such equivalence in general does not hold.

Remark 2.1. *Although we do not consider non-symmetric IP methods in this paper, let us remark that non-symmetric versions can be easily incorporated in the definition of $\mathcal{B}_1(\boldsymbol{v})$. For example, by setting:*

$$\mathcal{B}_1(\boldsymbol{v}) := \theta\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}))\boldsymbol{n}\}\!\} + \alpha_0\beta_0\mathcal{P}_E^0[\![\boldsymbol{v}]\!] + \alpha_1\beta_1[\![\boldsymbol{v}]\!],$$

we obtain a non-symmetric bilinear form for the values $\theta = 0$ or $\theta = 1$. Such values of θ correspond to the Incomplete Interior Penalty (IIPG, $\theta = 0$) and Non-symmetric Interior Penalty (NIPG, $\theta = 1$) discretizations, respectively.

2.4. **Stability Analysis.** We close this section presenting the stability and continuity results pertinent to our work. We start by introducing some norm notation. For $\boldsymbol{v} \in [H^2(\mathcal{T}_h)]^d$ we define the semi-norms

(2.27)
$$\|\nabla\boldsymbol{v}\|^2_{0,\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h}\|\nabla\boldsymbol{v}\|^2_{0,T} \qquad \|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h}\int_T\langle\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}):\boldsymbol{\varepsilon}(\boldsymbol{v})\rangle$$
$$|\mathcal{P}_E^0[\![\boldsymbol{v}]\!]|^2_* = \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{v}]\!]\|^2_{0,E} \qquad |[\![\boldsymbol{v}]\!]|^2_* = \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|[\![\boldsymbol{v}]\!]\|^2_{0,E}\,,$$

and norms:

(2.28)
$$\|\boldsymbol{v}\|_h^2 = \|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\mathcal{T}_h} + \beta_0|\mathcal{P}_E^0[\![\boldsymbol{v}]\!]|^2_* + \beta_1|[\![\boldsymbol{v}]\!]|^2_* + \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{v})\cdot\mathbf{n}\|^2_{0,E}\,.$$

For $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{DG}}$ we define the norms

$$\|\boldsymbol{v}\|^2_{DG0} = \|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\mathcal{T}_h} + \beta_0|\mathcal{P}_E^0[\![\boldsymbol{v}]\!]|^2_* \tag{2.29}$$

and

$$\|\boldsymbol{v}\|^2_{DG} = \|\boldsymbol{v}\|^2_{DG0} + \beta_1|[\![\boldsymbol{v}]\!]|^2_*\,. \tag{2.30}$$

Notice that for $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{DG}}$ the norms (2.28) and (2.30) are equivalent. We finally introduce the norm:

$$\|\boldsymbol{v}\|^2_{H^1(\mathcal{T}_h)} = \|\nabla\boldsymbol{v}\|^2_{0,\mathcal{T}_h} + \beta_0|\mathcal{P}_E^0[\![\boldsymbol{v}]\!]|^2_* + +\beta_1|[\![\boldsymbol{v}]\!]|^2_*\,. \tag{2.31}$$

Notice that continuity of the **IP-1** and **IP-0** bilinear forms with respect to the norm (2.28) follows easily from Cauchy-Schwarz inequality together with the bound on the maximum eigenvalue of $\mathcal{C}$, i.e., for all $\boldsymbol{u} \in [H^2(\mathcal{T}_h)]^d$ and all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{DG}}$ we have

$$\begin{aligned}
(\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n})\}\!\}, [\![\boldsymbol{v}]\!])_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} &= (\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n})\}\!\}, \mathcal{P}_E^0[\![\boldsymbol{v}]\!])_{\mathcal{E}_h^o\cup\mathcal{E}_h^D}\\
&\le \frac{1}{\alpha_0\beta_0}h_E^{1/2}\|\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\mathbf{n}\|_{0,\mathcal{E}_h^o\cup\mathcal{E}_h^D}\cdot\frac{\alpha_0\beta_0}{4}\|h_E^{-1/2}\mathcal{P}_E^0[\![\boldsymbol{v}]\!]\|_{0,\mathcal{E}_h^o\cup\Gamma_D}\\
&\le \frac{1}{\alpha_0}\|h_E^{1/2}\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\mathbf{n}\|_{0,\mathcal{E}_h^o\cup\mathcal{E}_h^D}\frac{\alpha_0\beta_0}{4}\|h_E^{-1/2}\mathcal{P}_E^0[\![\boldsymbol{v}]\!]\|_{0,\mathcal{E}_h^o\cup\Gamma_D}\,.
\end{aligned}$$

The equivalence of the norms (2.28) and (2.30) for any $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{DG}}$ guarantees therefore the continuity of the **IP-1** bilinear form with respect to the norm defined in (2.30) for finite element functions.

The solvability of the discrete methods (2.25) and (2.26) is guaranteed if and only if, a discrete version of the Korn's inequality holds on $\boldsymbol{V}^{\mathrm{DG}}$. In [7] the following discrete Korn inequality is shown for $[H^1(\mathcal{T}_h)]^d$-vector fields:

$$\|\boldsymbol{\nabla v}\|^2_{0,\mathcal{T}_h} \leq C\left(\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\mathcal{T}_h} + |\pi_1[\![\boldsymbol{v}]\!]|^2_* + \|\boldsymbol{\nabla}\times\boldsymbol{v}\|^2_{0,\mathcal{T}_h}\right) \tag{2.32}$$

where $\pi_1 : [L^2(\mathcal{E}_h)]^d \longrightarrow \mathbb{P}^1(\mathcal{E}_h)$ is the L^2-orthogonal projection onto the space of piecewise linear vector valued functions on $\mathcal{E}_h$ (or a subset of it).

Coercivity of the **IP-1** bilinear form with respect to the norm (2.30) can be easily shown by taking $\boldsymbol{u} = \boldsymbol{w} = \boldsymbol{v}$ in (2.24):

$$\begin{aligned}\mathcal{A}(\boldsymbol{v},\boldsymbol{v}) &= (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{T}_h} + \alpha_0\beta_0\|h_E^{-1/2}\mathcal{P}^0_E[\![\boldsymbol{v}]\!]\|^2_{0,\mathcal{E}^o_h\cup\Gamma_D} + \alpha_1\beta_1\|h_E^{-1/2}[\![\boldsymbol{v}]\!]\|^2_{0,\mathcal{E}^o_h\cup\Gamma_D}\\ &\quad - 2(\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v})\boldsymbol{n})\}\!\},[\![\boldsymbol{v}]\!])_{\mathcal{E}^o_h\cup\mathcal{E}^D_h}\ .\end{aligned}$$

Using Cauchy-Schwarz, trace and inverse inequalities together with the arithmetic-geometric inequality and the bound on the maximum eigenvalue of $\mathcal{C}$ it follows that

$$\begin{aligned}(\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v})\boldsymbol{n})\}\!\},[\![\boldsymbol{v}]\!])_{\mathcal{E}^o_h\cup\mathcal{E}^D_h} &= (\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v})\boldsymbol{n})\}\!\},\mathcal{P}^0_E[\![\boldsymbol{v}]\!])_{\mathcal{E}^o_h\cup\mathcal{E}^D_h}\\ &\leq \frac{C_t(1+C_{inv})}{\alpha_0\beta_0}\|\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\mathcal{T}_h} + \frac{\alpha_0\beta_0}{4}\|h_E^{-1/2}\mathcal{P}^0_E[\![\boldsymbol{v}]\!]\|^2_{0,\mathcal{E}^o_h\cup\Gamma_D}\\ &\leq \frac{C_t(1+C_{inv})}{\alpha_0}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\mathcal{T}_h} + \frac{\alpha_0\beta_0}{4}\|h_E^{-1/2}\mathcal{P}^0_E[\![\boldsymbol{v}]\!]\|^2_{0,\mathcal{E}^o_h\cup\Gamma_D}.\end{aligned} \tag{2.33}$$

Hence, we finally have

$$\begin{aligned}\mathcal{A}(\boldsymbol{v},\boldsymbol{v}) \geq (1 - \frac{2C_t(1+C_{inv})}{\alpha_0})\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\mathcal{T}_h} + \alpha_1\beta_1\|h_E^{-1/2}[\![\boldsymbol{v}]\!]\|^2_{0,\mathcal{E}^o_h\cup\Gamma_D}\\ + \frac{\alpha_0}{2}\beta_0\|h_E^{-1/2}\mathcal{P}^0_E[\![\boldsymbol{v}]\!]\|^2_{0,\mathcal{E}^o_h\cup\Gamma_D}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}^{\mathrm{DG}}\ ,\end{aligned}$$

and therefore by taking $\alpha_0 = \max(1, 4C_t(1+C_{inv}))$ (sufficiently large) we ensure the coercivity of $\mathcal{A}(\cdot,\cdot)$ with respect to the $\|\cdot\|_{DG}$-norm with constant independent of h, μ, and λ. Using now (2.32) (since the norm (2.30) contains the full jump) we conclude that $\mathcal{A}(\cdot,\cdot)$ is coercive with respect to the $\|\cdot\|_{H^1(\mathcal{T}_h)}$-norm (2.31). Therefore the **IP-1** method defined by (2.24) provides a robust approximation to (2.3) and does not lock as $\lambda \to \infty$.

As we mentioned earlier, in the pure displacement case ($\Gamma_D = \partial\Omega$, $\Gamma_N = \emptyset$) the bilinear form $\mathcal{A}_0(\cdot\cdot)$ defined in (2.23) is coercive. Indeed we may use the identity (which holds for $C_0^\infty(\Omega)$ functions):

$$\mathrm{div}\boldsymbol{\varepsilon}(\boldsymbol{v}) = \frac{1}{2}\left(\mathrm{div}\nabla\boldsymbol{v} + \nabla\mathrm{div}\boldsymbol{v}\right) \tag{2.34}$$

and rewrite the volume term in (2.23) (also in (2.24)) as follows:

$$\begin{aligned}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{T}_h} &= \sum_{T\in\mathcal{T}_h}\int_T \langle \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}):\boldsymbol{\varepsilon}(\boldsymbol{v})\rangle \\ &= \sum_{T\in\mathcal{T}_h}\int_T (2\mu\langle\nabla\boldsymbol{u}:\nabla\boldsymbol{v}\rangle + (\mu+\lambda)\langle \mathrm{div}\boldsymbol{u},\mathrm{div}\boldsymbol{v}\rangle)\,.\end{aligned}$$

Then, from the discrete Poincaré inequality [12, 6], the resulting modified bilinear form for $\mathcal{A}_0(\cdot,\cdot)$ is now coercive in $\boldsymbol{V}^{DG}$ with respect to the $\|\cdot\|_{H^1(\mathcal{T}_h)}$ norm, with coercivity constant independent of h and λ;

$$\mathcal{A}_0(\boldsymbol{v},\boldsymbol{v}) \geq C\|\boldsymbol{v}\|^2_{H^1(\mathcal{T}_h)} \quad \forall\, \boldsymbol{v}\in\boldsymbol{V}^{DG}. \tag{2.35}$$

Therefore, the discrete problem (2.26) is well posed and the **IP-0** method is stable and robust (locking free in the limit $\lambda\to\infty$). Notice that in (2.35) we are using the $\|\cdot\|_{H^1(\mathcal{T}_h)}$-norm which includes not only the norm $|P^0_E[\![\boldsymbol{v}]\!]|_*$, but also the norm $|[\![\boldsymbol{v}]\!]|_*$. This is a consequence of the vector valued counterpart of [4, Lemma 2.3]. The stability property given in (2.35) implies that the **IP-0** and **IP-1** methods are spectrally equivalent for the pure displacement problem. These observations are summarized in the next Lemma:

Lemma 2.2. *Let $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{A}_0(\cdot,\cdot)$ be the bilinear forms of the* **IP-1** *and* **IP-0** *methods for the linear elasticity problem, defined in* (2.24) *and* (2.23)*, respectively. For the pure displacement problem $\Gamma_D=\partial\Omega$, $\Gamma_N=\emptyset$, there exist a constant $c>0$ that depends only on the geometry of the domain Ω but is independent of the mesh size and the Lamé parameters μ and λ such that*

$$\mathcal{A}_0(\boldsymbol{v},\boldsymbol{v}) \leq \mathcal{A}(\boldsymbol{v},\boldsymbol{v}) \leq c\mathcal{A}_0(\boldsymbol{v},\boldsymbol{v}) \quad \forall\, \boldsymbol{v}\in\boldsymbol{V}^{DG}. \tag{2.36}$$

The above lemma guarantees that for the pure displacement problem, constructing a uniform preconditioner for the **IP-1** is equivalent to constructing a uniform preconditioner for the **IP-0** method (see [4]). For linear elasticity equations, unlike for scalar equations, this can be done only when $\Gamma_D=\partial\Omega$.

For a detailed derivation and error estimates, we refer to [15, Theorem 2.5].

3. Space decomposition

We present now a decomposition of the DG space of piecewise linear vector valued functions that plays a key role in the construction of iterative solvers. This decomposition was introduced in [4] for scalar functions and also in [9] in a different context. Its extension to vector valued functions is more or less straightforward. We omit those proofs which are just an easy modification of the corresponding proofs in the scalar case. However, we review the main ingredients and ideas behind such proofs, since they play an important role in the analysis of the preconditioner given later on. In the last part of the section we give some properties of the spaces entering in the and prove a result that is essential for showing that the proposed preconditioner is uniform.

Following [4] we introduce the space complementary to V^{CR} in V^{DG},

$$\mathcal{Z} = \left\{ z \in V^{\mathrm{DG}} \text{ and } \mathcal{P}^0_E \{\!\{ z \}\!\} = 0, \text{ for all } E \in \mathcal{E}_h^o \right\}. \tag{3.1}$$

The corresponding space of vector valued functions is

$$\boldsymbol{\mathcal{Z}} = [\mathcal{Z}]^d. \tag{3.2}$$

To describe the basis functions associated with the spaces (2.11) and (3.2), let $\varphi_{E,T}$ denote the scalar basis function on T, dual to the degree of freedom at the mass center of the face E, and extended by zero outside T. For $E \in \partial T$, $E' \in \partial T$, the function $\varphi_{E,T}$ satisfies

$$\varphi_{E,T}(m_{E'}) = \begin{cases} 1 & \text{if} \quad E = E', \\ 0 & \text{otherwise,} \end{cases}$$

and also we have

$$\varphi_{E,T} \in \mathbb{P}^1(T), \quad \varphi_{E,T}(x) = 0, \forall\, x \notin T.$$

For all $\boldsymbol{u} \in \boldsymbol{V}^{\mathrm{DG}}$ we then have

$$\boldsymbol{u}(x) = \sum_{T \in \mathcal{T}_h} \sum_{E \in \partial T} \boldsymbol{u}_T(m_E) \varphi_{E,T}(x) = \sum_{E \in \mathcal{E}_h} \boldsymbol{u}^+(m_E) \varphi_E^+(x) + \sum_{E \in \mathcal{E}_h^o} \boldsymbol{u}^-(m_E) \varphi_E^-(x), \tag{3.3}$$

where in the last identity we have just changed the order of summation and used the short hand notation $\varphi_E^{\pm}(x) := \varphi_{E,T^{\pm}}(x)$ together with

$$\begin{aligned} \boldsymbol{u}^{\pm}(m_E) := \boldsymbol{u}_{T^{\pm}}(m_E) = \frac{1}{|E|} \int_E \boldsymbol{u}_{T^{\pm}} ds, \qquad & \forall\, E \in \mathcal{E}_h^o, \ : \ E = \partial T^+ \cap \partial T^-, \\ \boldsymbol{u}(m_E) := \boldsymbol{u}_T(m_E) = \frac{1}{|E|} \int_E \boldsymbol{u}_T ds, \qquad & \forall\, E \in \mathcal{E}_h^\partial, \text{ such that } E = \partial T \cap \partial \Omega. \end{aligned}$$

Recalling now the definitions of $T^+(E)$ and $T^-(E)$ given in (2.6) we set

$$\begin{aligned} \varphi_E^{CR} &= \varphi_{E,T^+(E)} + \varphi_{E,T^-(E)}, \qquad && \forall\, E \in \mathcal{E}_h^o, \\ \varphi_E^{CR} &= \varphi_{E,T^+(E)}, && \forall\, E \in \mathcal{E}_h^N. \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \psi_E^z &= \frac{\varphi_{E,T^+(E)} - \varphi_{E,T^-(E)}}{2}, \qquad && \forall\, E \in \mathcal{E}_h^o, \\ \psi_E^z &= \varphi_{E,T^+(E)}, && \forall\, E \in \mathcal{E}_h^D. \end{aligned} \tag{3.5}$$

Some clarification is needed here. Note that from the definition of $\varphi_{E,T^+(E)}$ and $\varphi_{E,T^-(E)}$ for an interior edge $E \in \mathcal{E}_h^o$, it does not follow that their sum is even defined on the edge E, since it is just a sum of two functions from $L^2(\Omega)$. However, the sum $(\varphi_{E,T^+(E)} + \varphi_{E,T^-(E)})$ has a representative, which is continuous across E and this representative is denoted here with $\boldsymbol{\varphi}_E^{CR}$, see Figure 3.1.

Clearly, $\{\varphi_E^{CR}\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^N}$ are linearly independent, and $\{\boldsymbol{\psi}_E^z\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D}$ are linearly independent. A simple argument then shows that

$$\boldsymbol{V}^{\mathrm{CR}} = \mathrm{span}\left\{\{\varphi_E^{CR}\boldsymbol{e}_k\}_{k=1}^d\right\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^N}, \quad \boldsymbol{\mathcal{Z}} = \mathrm{span}\left\{\{\psi_E^{z}\boldsymbol{e}_k\}_{k=1}^d\right\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D}.$$

Here $\boldsymbol{e}_k$, $k = 1,\dots,d$ is the k-th canonical basis vector in $\mathbb{R}^d$. Hence by performing a change of basis in (3.3), we have obtained a "natural" splitting of

$$\boldsymbol{V}^{\mathrm{DG}} = \boldsymbol{V}^{\mathrm{CR}} \oplus \boldsymbol{\mathcal{Z}}$$

and the set

$$\{\boldsymbol{\psi}_E^z\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \cup \{\varphi_E^{CR}\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^N}, \tag{3.6}$$

provides a natural basis for the DG finite element space. This is summarized in the next proposition.

Proposition 3.1. *For any $\boldsymbol{u} \in \boldsymbol{V}^{DG}$ there exist unique $\boldsymbol{v} \in \boldsymbol{V}^{CR}$ and a unique $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$ such that*

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{z} \quad \text{and} \quad \begin{aligned} \boldsymbol{v} &= \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^N}\left(\frac{1}{|E|}\int_E \{\!\{\boldsymbol{u}\}\!\}ds\right)\varphi_E^{CR}(x) \in \boldsymbol{V}^{CR}, \\ \boldsymbol{z} &= \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D}\left(\frac{1}{|E|}\int_E [\![\boldsymbol{u}]\!]ds\right)\boldsymbol{\psi}_E^{z}(x) \in \boldsymbol{\mathcal{Z}}. \end{aligned} \tag{3.7}$$

The proof of the above result follows by arguing as for the scalar case in [4, Proposition 3.1], but proceeding componentwise. The next Lemma shows that the splitting we have proposed is orthogonal with respect to the inner product defined by $\mathcal{A}_0(\cdot,\cdot)$.

Lemma 3.2. *The splitting* (3.7) $\boldsymbol{V}^{DG} = \boldsymbol{V}^{CR} \oplus \boldsymbol{\mathcal{Z}}$ *is $\mathcal{A}_0$-orthogonal. That is*

$$\mathcal{A}_0(\boldsymbol{v},\boldsymbol{z}) = \mathcal{A}_0(\boldsymbol{z},\boldsymbol{v}) = 0 \qquad \forall\,\boldsymbol{v} \in \boldsymbol{V}^{CR}, \quad \forall\,\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}. \tag{3.8}$$

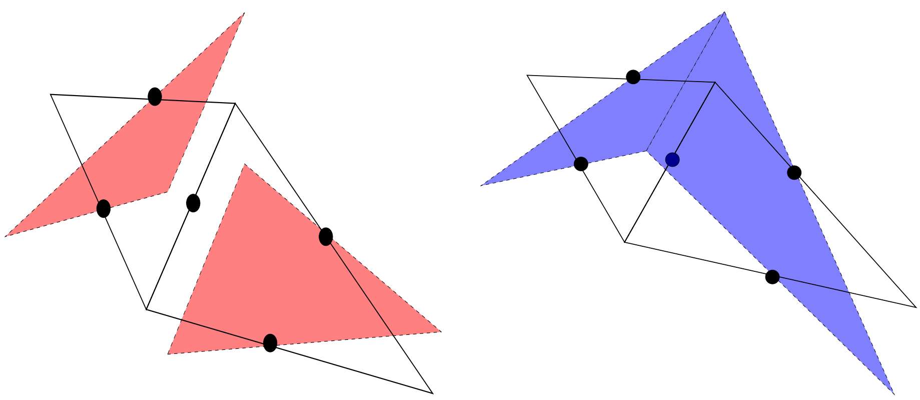

FIGURE 3.1. Basis functions associated with the face E: ψ_E^z (left) and φ_E^{CR} (right).

The proof follows straightforwardly by using the weighted residual formulation (2.15)-(2.23) and the definition of the spaces $\boldsymbol{V}^{\mathrm{CR}}$ and $\boldsymbol{\mathcal{Z}}$.

3.1. **Some properties of the space $\boldsymbol{\mathcal{Z}}$.** We now present some properties of the functions in the space $\boldsymbol{\mathcal{Z}}$. We start with a simple observation. From the definition of the spaces $\boldsymbol{V}^{\mathrm{CR}}$ and $\boldsymbol{\mathcal{Z}}$ it is easy to see that

$$\sum_{T\in\mathcal{T}_h} \|\nabla \boldsymbol{z}\|_{0,T}^2 = (\llbracket \boldsymbol{z} \rrbracket, \{\!\{\nabla \boldsymbol{z}\}\!\})_{\mathcal{E}_h^o\cup\mathcal{E}_h^D}.$$

Applying the Schwarz inequality, one then gets the following estimate

$$\sum_{T\in\mathcal{T}_h} \|\nabla \boldsymbol{z}\|_{0,T}^2 \leq C\|h^{-1/2}P_E^0\llbracket \boldsymbol{z} \rrbracket\|_{0,\mathcal{E}_h}^2,$$

which is a straightforward way to see that the restriction of the **IP-1** and **IP-0**-bilinear forms (even for $\theta = 0, 1$ as in Remark 2.1) to the space $\boldsymbol{\mathcal{Z}}$ are coercive in the $\|\cdot\|_{H^1(\mathcal{T}_h)}$-norm (2.31) (regardless whether the boundary conditions are Dirichlet, Neumann or mixed type). Therefore the resulting stiffness matrices are positive definite.

The next result provides bounds on the eigenvalues of $\mathcal{A}_0(\cdot,\cdot)$ and $\mathcal{A}(\cdot,\cdot)$, when restricted to $\boldsymbol{\mathcal{Z}}$.

Lemma 3.3. *Let $\boldsymbol{\mathcal{Z}}$ be the space defined in* (3.2). *Then for all $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$, the following estimates hold*

$$h^{-2}\|\boldsymbol{z}\|_0^2 \lesssim \mathcal{A}_0(\boldsymbol{z},\boldsymbol{z}) \lesssim h^{-2}\|\boldsymbol{z}\|_0^2\ , \tag{3.9}$$

and also,

$$[(\alpha_0)\beta_0 + \alpha_1\beta_1]h^{-2}\|\boldsymbol{z}\|_0^2 \lesssim \mathcal{A}(\boldsymbol{z},\boldsymbol{z}) \lesssim [\alpha_0\beta_0 + \alpha_1\beta_1]h^{-2}\|\boldsymbol{z}\|_0^2\ , \tag{3.10}$$

where β_0 and β_1 are as defined in (2.19).

Proof. Arguing as in [4, Lemma 5.3] (but now componentwise for vector valued functions) one can show that (due the special structure of the space $\boldsymbol{\mathcal{Z}}$).

$$h^{-2}\|\boldsymbol{z}\|_0^2 \lesssim \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|\mathcal{P}_E^0\llbracket \boldsymbol{z} \rrbracket\|_{0,E}^2 \lesssim h^{-2}\|\boldsymbol{z}\|_0^2\ . \tag{3.11}$$

From the coercivity of $\mathcal{A}_0$ it follows then

$$\alpha_0\beta_0 h^{-2}\|\boldsymbol{z}\|_0^2 \lesssim \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|\mathcal{P}_E^0\llbracket \boldsymbol{z} \rrbracket\|_{0,E}^2 \leq \mathcal{A}_0(\boldsymbol{z},\boldsymbol{z})\ .$$

Similarly, the $L^2(\mathcal{E}_h)$ stability of the projection $\mathcal{P}_E^0$ together with the coercivity of $\mathcal{A}$ gives

$$
\begin{aligned}
(\alpha_0\beta_0 + \alpha_1\beta_1)h^{-2}\|\boldsymbol{z}\|_0^2 &\lesssim \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 \\
&\qquad + \alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 \\
&\lesssim \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 \\
&\qquad + C\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|[\![\boldsymbol{z}]\!]\|_{0,E}^2 \\
&\leq \mathcal{A}(\boldsymbol{z},\boldsymbol{z}),
\end{aligned}
$$

and so, the lower bounds in (3.9) and (3.10) follow. We next show the upper bound in (3.9), and the upper bound in (3.10) is obtained in an analogous fashion. Using (2.33) together with (2.1) we get

$$
\begin{aligned}
\mathcal{A}_0(\boldsymbol{z},\boldsymbol{z}) &\leq \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h^o\cup\Gamma_D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 + \|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{z})\|_{0,\mathcal{T}_h}^2 \\
&\leq \beta_0\left(\alpha_0\|h_E^{-1/2}\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{\mathcal{E}_h^o\cup\Gamma_D}^2 + C\|\boldsymbol{\varepsilon}(\boldsymbol{z})\|_{0,\mathcal{T}_h}^2\right) .
\end{aligned}
$$

Hence, the upper bound in (3.9) follows in a straightforward fashion using the trace and inverse inequalities together with the obvious inequality $\|\boldsymbol{\varepsilon}(\boldsymbol{z})\|_{0,\mathcal{T}_h} \leq \|\boldsymbol{\nabla}\boldsymbol{z}\|_{0,\mathcal{T}_h}$. □

We close this section with establishing a uniform bound on the angle between $\boldsymbol{V}^{\mathrm{CR}}$ and $\boldsymbol{Z}$ in the inner product given by the bilinear form $\mathcal{A}(\cdot,\cdot)$. The estimate is given in Proposition 3.4. It plays a crucial role in bounding the condition number of the preconditioned system.

We remind that $E \in \mathcal{E}_h$ denotes a $(d-1)$-dimensional simplex (a face), which is either the intersection of two d-dimensional simplices $T \in \mathcal{T}_h$ or an intersection of a d-dimensional simplex $T \in \mathcal{T}_h$ and the complement of Ω, i.e., $E = T \cap (\mathbb{R}^d \setminus \Omega)$. In the former case, the face E is called an interior face and in the latter it is called a boundary face.

The proof of Proposition 3.4 requires arguments involving the incidence relations between simplices $T \in \mathcal{T}_h$ and faces $E \in \mathcal{E}_h$, and estimates on the cardinality of these incidence sets. For the readers' convenience, we provide a list of such estimates below.

- We define $\mathcal{N}_0(E)$ to be the set of d-dimensional $T \in \mathcal{T}_h$ simplices that contain E:

$$
\mathcal{N}_0(E) := \{T \in \mathcal{T}_h, \quad \text{such that} \quad E \in T\}
$$

By definition, for the cardinality of this set we have $|\mathcal{N}_0(E)| = 2$ for the interior faces and $|\mathcal{N}_0(E)| = 1$ for the boundary faces.

- We define the set of neighbor (or neighboring) faces $\mathcal{N}_1(E)$ to be the set of faces which share an element with E:

$$\mathcal{N}_1(E) := \{E' \in \mathcal{E}_h, \quad \text{such that} \quad \mathcal{N}_0(E) \cap \mathcal{N}_0(E') \neq \emptyset\}$$

From Proposition A.1 (see Appendix A) we have that $|\mathcal{N}_1(E)| \leq (2d+1)$.

- Next, we define $\mathcal{N}_2(E)$ to be the set of faces which share at least one neighboring face with E:

$$\mathcal{N}_2(E) := \{E' \in \mathcal{E}_h, \quad \text{such that} \quad \mathcal{N}_1(E) \cap \mathcal{N}_1(E') \neq \emptyset\}$$

From Proposition A.1 we have the estimate $|\mathcal{N}_2(E)| \leq (2d+1)^2$.

- For the basis functions $\{\psi_E^z\}_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D}$ we have the following relations:

$$\frac{1}{|E|} \int_E [\![\psi_{E'}^z]\!] = \delta_{EE'}, \quad \text{and} \quad [\![\psi_E^z]\!](x) = 1, \quad \text{for all} \quad x \in E, \tag{3.12}$$

$$|[\![\psi_E^z]\!](x)| \leq 1, \quad \text{for all} \quad x \in E', \quad \text{and all} \quad E' \in \mathcal{N}_2(E). \tag{3.13}$$

The above relations all follow from the definition of $\psi_E^z(x)$ and the fact that $[\![\psi_E^z]\!]$ is linear function on every face in $\mathcal{E}_h$, and therefore $\int_E [\![\psi_{E'}^z]\!] = |E| [\![\psi_{E'}^z]\!](m_E)$.

- Finally, for $E \in \mathcal{E}_h$, $E' \in \mathcal{E}_h$, and $E'' \in \mathcal{E}_h$ it is straightforward to see that we have:

$$\text{If} \quad E \notin \mathcal{N}_1(E') \cap \mathcal{N}_1(E'') \quad \text{then} \quad \int_E [\![\psi_{E'}^z]\!][\![\psi_{E''}^z]\!] = 0. \tag{3.14}$$

An easy consequence from the definitions then is the following:

$$\text{If} \quad E' \notin \mathcal{N}_2(E'') \quad \text{then} \quad \int_E [\![\psi_{E'}^z]\!][\![\psi_{E''}^z]\!] = 0, \quad \text{for all} \quad E \in \mathcal{E}_h. \tag{3.15}$$

We finally give Proposition 3.4. To avoid unnecessary complications with the notation, we state and prove the result for scalar valued functions. The proof for vector valued functions is easy to obtain, and with the same constant, by just applying the scalar valued result component-wise.

Proposition 3.4. *The following inequality holds for $z \in \mathcal{Z}$:*

$$\sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \|h_E^{-1/2}([\![z]\!] - \mathcal{P}_E^0 [\![\boldsymbol{z}]\!])\|_{0,E}^2 \leq \left(1 - \frac{1}{\rho}\right) \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \|h_E^{-1/2} [\![\boldsymbol{z}]\!]\|_{0,E}^2, \tag{3.16}$$

with a constant $\rho \geq 1$ which depends on the shape regularity of the mesh.

Proof. Since $\mathcal{P}_E^0$ is the L^2 orthogonal projection on the constants, we have that

$$\|h_E^{-1/2}([\![z]\!] - \mathcal{P}_E^0 [\![\boldsymbol{z}]\!])\|_{0,E}^2 = \|h_E^{-1/2} [\![z]\!]\|_{0,E}^2 - \|h_E^{-1/2} \mathcal{P}_E^0 [\![\boldsymbol{z}]\!]\|_{0,E}^2. \tag{3.17}$$

Let $z \in \mathcal{Z}$, i.e., $z = \sum_{E' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} z_{E'} \psi^z_{E'}$. From (3.12) we have that $\mathcal{P}^0_E [\![\psi^z_{E'}]\!] = \delta_{EE'}$, and hence, we may conclude that

$$\|h_E^{-1/2} \mathcal{P}^0_E [\![\boldsymbol{z}]\!]\|^2_{0,E} = \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \sum_{E' \in \mathcal{E}_h} \delta_{EE'} \frac{|E|}{h_E} z_E z_{E'}$$
$$= \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \mathbb{D}_{EE} z_E^2 = \langle \mathbb{D} \tilde{z}, \tilde{z} \rangle.$$

Here we have denoted by $\mathbb{D} : \mathbb{R}^{|\mathcal{E}_h|} \mapsto \mathbb{R}^{|\mathcal{E}_h|}$ a diagonal matrix with non-zero elements $\mathbb{D}_{EE} := \frac{|E|}{h_E}$ and by $\tilde{z} \in \mathbb{R}^{|\mathcal{E}_h|}$ the vector of coefficients $\tilde{z} = \{z_E\}_{E \in \mathcal{E}_h}$ in the expansion of $z \in \mathcal{Z}$ via the basis $\{\psi^z_E\}_{E \in \mathcal{E}_h}$.

Further we consider the right hand side of (3.16) and we have

$$\sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \|h_E^{-1/2} [\![\boldsymbol{z}]\!]\|^2_{0,E} = \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} h_E^{-1} \left\| \sum_{E' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} z_{E'} [\![\psi^z_{E'}]\!] \right\|^2_{0,E}$$
$$= \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \int_E h_E^{-1} \sum_{E' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \sum_{E'' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} z_{E'} z_{E''} [\![\psi^z_{E'}]\!] [\![\psi^z_{E''}]\!]$$
$$= \sum_{E' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \sum_{E'' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} z_{E'} z_{E''} \left(\sum_{E \in \mathcal{E}_h} \int_E h_E^{-1} [\![\psi^z_{E'}]\!] [\![\psi^z_{E''}]\!] \right)$$
$$= \sum_{E' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \sum_{E'' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} z_{E'} z_{E''} \mathbb{S}_{E'E''} = \langle \mathbb{S} \tilde{z}, \tilde{z} \rangle.$$

Here, $\mathbb{S} : \mathbb{R}^{|\mathcal{E}_h|} \mapsto \mathbb{R}^{|\mathcal{E}_h|}$ denotes the symmetric real matrix with elements

$$\mathbb{S}_{E'E''} = \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \int_E h_E^{-1} [\![\psi^z_{E'}]\!] [\![\psi^z_{E''}]\!] = \sum_{E \in \mathcal{N}_1(E') \cap \mathcal{N}_1(E'')} \int_E h_E^{-1} [\![\psi^z_{E'}]\!] [\![\psi^z_{E''}]\!]. \tag{3.18}$$

In the last identity above, we have used (3.14). Note that according to (3.15), if $E' \notin \mathcal{N}_2(E'')$ then $\mathbb{S}_{E'E''} = 0$. Thus,

$$\langle \mathbb{S} \tilde{z}, \tilde{z} \rangle = \sum_{E' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \sum_{E'' \in \mathcal{N}_2(E)} z_{E'} z_{E''} \mathbb{S}_{E'E''}.$$

From this identity and (3.12) and (3.13), we obtain that

$$|\mathbb{S}_{E'E''}| \leq |\mathcal{N}_1(E') \cap \mathcal{N}_1(E'')| \max_{E \in \mathcal{N}_1(E') \cap \mathcal{N}_1(E'')} \frac{|E|}{h_E} \leq (2d+1) \max_{E \in \mathcal{N}_1(E') \cap \mathcal{N}_1(E'')} \frac{|E|}{h_E}.$$

Introducing

$$\rho = \sup_{\tilde{w} \in \mathbb{R}^{|\mathcal{E}_h|}} \frac{\langle \mathbb{S} \tilde{w}, \tilde{w} \rangle}{\langle \mathbb{D} \tilde{w}, \tilde{w} \rangle}, = \sup_{\tilde{w} \in \mathbb{R}^{|\mathcal{E}_h|}} \frac{\langle \mathbb{D}^{-1/2} \mathbb{S} \mathbb{D}^{-1/2} \tilde{w}, \tilde{w} \rangle}{\langle \tilde{w}, \tilde{w} \rangle},$$

we obtain that

$$\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle = \langle \mathbb{D}^{-1/2}\mathbb{S}\mathbb{D}^{-1/2}\mathbb{D}^{1/2}\tilde{z}, \mathbb{D}^{1/2}\tilde{z}\rangle \leq \rho\langle \mathbb{D}\tilde{z}, \tilde{z}\rangle. \tag{3.19}$$

This inequality can be rewritten as $\frac{1}{\rho}\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle \leq \langle \mathbb{D}\tilde{z}, \tilde{z}\rangle$ and hence

$$\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle - \langle \mathbb{D}\tilde{z}, \tilde{z}\rangle \leq \langle \mathbb{S}\tilde{z}, \tilde{z}\rangle - \frac{1}{\rho}\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle = (1 - \frac{1}{\rho})\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle.$$

Note that (3.17) implies that

$$\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle = \langle \mathbb{D}\tilde{z}, \tilde{z}\rangle + \sum_{E\in\mathcal{E}_h} \|h_E^{-1/2}([\![z]\!] - \mathcal{P}_E^0[\![\boldsymbol{z}]\!])\|_{0,E}^2, \tag{3.20}$$

and thus $\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle \geq \langle \mathbb{D}\tilde{z}, \tilde{z}\rangle$. This shows that $\rho \geq 1$ in (3.19).

It remains to show that ρ can be bounded by quantities depending only on the shape regularity of the mesh. Again, by (3.15) we have that: if $E' \notin \mathcal{N}_2(E'')$ then $\mathbb{S}_{E'E''} = 0$. Hence:

$$\begin{aligned}
\rho &\leq \|\mathbb{D}^{-1/2}\mathbb{S}\mathbb{D}^{-1/2}\|_{\ell^\infty} \leq \max_{E''\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \sum_{E'\in\mathcal{N}_2(E'')} \frac{|\mathbb{S}_{E''E'}|}{\sqrt{\mathbb{D}_{E'E'}\mathbb{D}_{E''E''}}} \\
&\leq \max_{E''\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \left[|\mathcal{N}_2(E'')| \max_{E'\in\mathcal{N}_2(E'')} \frac{|\mathbb{S}_{E'E''}|}{\sqrt{\mathbb{D}_{E'E'}\mathbb{D}_{E''E''}}}\right] \\
&\leq (2d+1)^3 \max_{E''\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \max_{E'\in\mathcal{N}_2(E'')} \max_{E\in\mathcal{N}_1(E')\cap\mathcal{N}_1(E'')} \frac{|E|}{h_E}\sqrt{\frac{h_{E'}h_{E''}}{|E'||E''|}}.
\end{aligned}$$

The quantity on the right side of this estimate only depends on the shape regularity of the mesh and the proof is complete. □

Remark 3.5. *We remark that the constants in Proposition 3.4 can be sharpened, at the price of further complicating the proof. The result given above is sufficient for our purposes, and we do not further comment on the possible "optimal" value of the constant ρ above. Another relevant observation is that the inequality in Proposition 3.4 holds true, with the same or even smaller ρ, if we replace $\mathcal{E}_h^o \cup \mathcal{E}_h^D$ with a subset of edges $\mathcal{E} \subset (\mathcal{E}_h^o \cup \mathcal{E}_h^D)$ in* (3.16)*. The proof is completely analogous (just $\mathcal{E}_h^o \cup \mathcal{E}_h^D$ is replaced by $\mathcal{E}$).*

4. Preconditioning

In this section, we present the construction and convergence analysis of the preconditioners we propose for the considered IP-methods.

To construct the preconditioners, we use the subspace splitting given in Proposition 3.1, which suggests a simple change of basis. We have that for any $\boldsymbol{u}, \boldsymbol{w} \in \boldsymbol{V}^{\mathrm{DG}}$, we can write $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{v}$, and $\boldsymbol{w} = \boldsymbol{\zeta} + \boldsymbol{\varphi}$, where $\boldsymbol{z}, \boldsymbol{\zeta} \in \boldsymbol{\mathcal{Z}}$ and $\boldsymbol{v}, \boldsymbol{\varphi} \in \boldsymbol{V}^{\mathrm{CR}}$. Therefore, by performing this change of basis we can write

$\mathcal{A}(\boldsymbol{u},\boldsymbol{w}) = \mathcal{A}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{\zeta},\boldsymbol{\phi}))$. The $\mathcal{A}_0$-orthogonality (3.8) of the subspaces in the splitting gives

$$\mathcal{A}_0((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{\zeta},\boldsymbol{\phi})) = \mathcal{A}_0(\boldsymbol{z},\boldsymbol{\zeta}) + \mathcal{A}_0(\boldsymbol{v},\boldsymbol{\phi}).$$

which implies that the resulting stiffness matrix of $\mathcal{A}_0$ in this new basis is block diagonal. For the pure displacement problem ($\Gamma_N = \emptyset$), as discussed in Section 2.4, the spectral equivalence given in Lemma 2.2, guarantees that an optimal preconditioner for $\mathcal{A}_0$ is also optimal for $\mathcal{A}$. Therefore it is enough to study how to efficiently solve each of the blocks in the above block diagonal structure of $\mathcal{A}_0$: the subproblem resulting from the restriction of $\mathcal{A}_0$ to $\boldsymbol{\mathcal{Z}}$ and the subproblem on the space $\boldsymbol{V}^{\mathrm{CR}}$.

For traction free or mixed type of boundary conditions, although a preconditioner for $\mathcal{A}_0$ does not result in an optimal solution method. However, the block structure of $\mathcal{A}_0$ in the new basis already suggests that a reasonable choice for an approximation of $\mathcal{A}(\cdot,\cdot)$ is

$$\mathcal{B}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{\zeta},\boldsymbol{\phi})) = \mathcal{A}(\boldsymbol{z},\boldsymbol{\zeta}) + \mathcal{A}(\boldsymbol{v},\boldsymbol{\phi}). \tag{4.1}$$

The following algorithm describes the application of a preconditioner, which is based on the bilinear form in the equation (4.1).

Algorithm 4.1. *Let $\boldsymbol{r} \in [L^2(\Omega)]^d$ be given. Then the action of the preconditioner on $\boldsymbol{r}$ is the function $\boldsymbol{u} \in \boldsymbol{V}^{DG}$ which is obtained from the following three steps.*

1. *Find $z \in \boldsymbol{\mathcal{Z}}$ such that*
$$\mathcal{A}(\boldsymbol{z},\boldsymbol{\zeta}) = (\boldsymbol{r},\boldsymbol{\zeta})_{\mathcal{T}_h} \quad \textit{for all} \quad \boldsymbol{\zeta} \in \boldsymbol{\mathcal{Z}}.$$
2. *Find $\boldsymbol{v} \in \boldsymbol{V}^{CR}$ such that*
$$\mathcal{A}(\boldsymbol{v},\boldsymbol{\varphi}) = (\boldsymbol{r},\boldsymbol{\varphi})_{\mathcal{T}_h} \quad \textit{for all} \quad \boldsymbol{\varphi} \in \boldsymbol{V}^{CR}.$$
3. *Set $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{v}$.*

As before, the application of this preconditioner corresponds to solving the subproblem of the restriction of $\mathcal{A}(\cdot,\cdot)$ to $\boldsymbol{\mathcal{Z}}$ and the subproblem of the restriction of $\mathcal{A}(\cdot,\cdot)$ to $\boldsymbol{V}^{\mathrm{CR}}$.

We now briefly discuss how the two smaller sub-problems can be efficiently solved in both cases: (1) the case of Dirichlet boundary conditions on all of $\partial\Omega$; and (2) the case of Neumann or mixed boundary conditions.

Solution in the subspace $\mathcal{Z}$. Lemma 3.3 guarantees that the restriction of $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{A}_0(\cdot,\cdot)$ to $\boldsymbol{\mathcal{Z}}$ is well-conditioned with respect to both, the mesh size and the Lamé constants λ,μ. Therefore, the linear system corresponding to the subproblem of the restriction to $\boldsymbol{\mathcal{Z}}$ can be efficiently solved by the method of Conjugate Gradients (CG). A simple consequence of the well known estimate on the convergence of CG (see, e.g., [18, 16]) shows that the number of CG iterations required to achieve a fixed error tolerance is uniformly bounded, independently of the size of the problem and the parameters.

Solution in $\boldsymbol{V}^{\mathrm{CR}}$. We now briefly discuss how to construct a uniform preconditioner for the corresponding subproblem on the space $\boldsymbol{V}^{\mathrm{CR}}$. Rather than developing a completely new method, the idea is to use the optimal preconditioners that have already been studied in literature, and modify them if needed so that they fit in the present framework. For our discussion, we distinguish two cases: the pure displacement problem ($\Gamma_N = \emptyset$) and the case with mixed or traction free boundary conditions ($\Gamma_N \neq \emptyset$).

- For the case of Dirichlet boundary conditions on the entire boundary–the so-called pure displacement problem–it is known how to construct optimal order multilevel preconditioners that are robust with respect to the parameter λ, see e.g. [5, 17, 13] and the references therein.
- The traction free problem or the case of mixed boundary conditions is more difficult to handle because the (discrete) Korn inequality is not satisfied for the standard discretization by Crouzeix-Raviart elements without additional stabilization, as was shown in [11]. The design of optimal and robust solution methods for stabilized discretizations is still an open problem, however, auxiliary space techniques might bridge this gap soon.

4.1. **Convergence Analysis.** We now prove that the proposed block preconditioners are indeed optimal so that their convergence is uniform with respect to mesh size and the Lamé parameters. This result is given in Theorem 4.3. The following Lemma is crucial for this proof, since it gives estimates on the norm of the off-diagonal blocks in the 2×2 block form of the stiffness matrix associated to $\mathcal{A}(\cdot,\cdot)$, corresponding to the space splitting $\boldsymbol{V}^{\mathrm{DG}} = \boldsymbol{V}^{\mathrm{CR}} \oplus \boldsymbol{\mathcal{Z}}$. The result provides a measure of the angle between the subspaces $\boldsymbol{V}^{\mathrm{CR}}$ and $\boldsymbol{\mathcal{Z}}$, with respect to the $\mathcal{A}$-norm. The proof of this result uses Proposition 3.4.

Lemma 4.2. Strengthened Cauchy-Schwarz inequality. *The following inequality holds for any $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$ and any $\boldsymbol{v} \in \boldsymbol{V}^{CR}$*

$$\mathcal{A}(\boldsymbol{z}, \boldsymbol{v})^2 \leq \gamma^2 \mathcal{A}(\boldsymbol{z}, \boldsymbol{z}) \mathcal{A}(\boldsymbol{v}, \boldsymbol{v})$$

where $\gamma < 1$ and γ depends only on α_0, α_1 and the constant from Proposition 3.4.

Proof. We know that we can always choose α_0 large enough, such that for all $\boldsymbol{u} \in \boldsymbol{V}^{\mathrm{DG}}$ we have

$$\mathcal{A}_0(\boldsymbol{u}, \boldsymbol{u}) = (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{T}_h} - 2(\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}\}\!\}, [\![\boldsymbol{u}]\!])_{\mathcal{E}_h^o \cup \mathcal{E}_h^D} + \alpha_0 a_{j,0}([\![\boldsymbol{u}]\!], [\![\boldsymbol{u}]\!]) \geq 0.$$

Then it is sufficient to prove that there exists $\gamma = \gamma(\alpha_1) < 1$ such that for all $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$ and for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{CR}}$ the inequality

$$[a_{j,1}([\![\boldsymbol{z}]\!], [\![\boldsymbol{v}]\!])]^2 \leq \gamma^2 a_{j,1}([\![\boldsymbol{z}]\!], [\![\boldsymbol{z}]\!]) a_{j,1}([\![\boldsymbol{v}]\!], [\![\boldsymbol{v}]\!]),$$

holds. By the definition of the spaces $\boldsymbol{\mathcal{Z}}$ and $\boldsymbol{V}^{\mathrm{CR}}$, on the boundary edges $E \in \mathcal{E}_h^{\partial}$ we have either $\mathcal{P}_E^0 [\![\boldsymbol{z}]\!] = 0$ (if $E \in \mathcal{E}_h^N$) or $\mathcal{P}_E^0 [\![\boldsymbol{v}]\!] = 0$ (if $E \in \mathcal{E}_h^D$). Hence, from the symmetry of $\mathcal{P}_E^0$ we conclude that

$$\int_E \langle [\![\boldsymbol{z}]\!], \mathcal{P}_E^0 [\![\boldsymbol{v}]\!] \rangle = \int_E \langle \mathcal{P}_E^0 [\![\boldsymbol{z}]\!], [\![\boldsymbol{v}]\!] \rangle = 0, \text{ for all } E \in \mathcal{E}_h^\partial, \text{ and all } \boldsymbol{z} \in \boldsymbol{\mathcal{Z}},\ \boldsymbol{v} \in \boldsymbol{V}^{\mathrm{CR}}.$$

Since for the interior edges $E \in \mathcal{E}_h^o$ we also have $\mathcal{P}_E^0 [\![\boldsymbol{v}]\!] = 0$, the above relation and the definition of $\mathcal{P}_E^0$ altogether imply that for all $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$, and $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{CR}}$

$$\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \int_E \langle h_E^{-1}\mathcal{P}_E^0 [\![\boldsymbol{z}]\!], [\![\boldsymbol{v}]\!] \rangle = \alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \int_E \langle h_E^{-1} [\![\boldsymbol{z}]\!], \mathcal{P}_E^0 [\![\boldsymbol{v}]\!] \rangle = 0. \tag{4.2}$$

The equation (4.2) and the Schwarz inequality then lead to

$$\begin{aligned}
[a_{j,1}([\![\boldsymbol{z}]\!], [\![\boldsymbol{v}]\!])]^2 &= \left[\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \int_E \langle h_E^{-1} [\![\boldsymbol{z}]\!], [\![\boldsymbol{v}]\!] \rangle\right]^2 \\
&= \left[\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \int_E \langle h_E^{-1} ([\![\boldsymbol{z}]\!] - \mathcal{P}_E^0 [\![\boldsymbol{z}]\!]), [\![\boldsymbol{v}]\!] \rangle\right]^2 \\
&\le a_{j,1}([\![\boldsymbol{v}]\!], [\![\boldsymbol{v}]\!]) \left[\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \|h_E^{-1/2}([\![\boldsymbol{z}]\!] - \mathcal{P}_E^0 [\![\boldsymbol{z}]\!])\|_{0,E}^2\right].
\end{aligned}$$

Next, the result in Proposition 3.4 (more precisely its vector valued form) implies that

$$\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \|h_E^{-1/2}([\![\boldsymbol{z}]\!] - \mathcal{P}_E^0 [\![\boldsymbol{z}]\!])\|_{0,E}^2 \le \left(1 - \frac{1}{\rho}\right) \alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \|h_E^{-1/2} [\![\boldsymbol{z}]\!]\|_{0,E}^2$$

Therefore, we have

$$\begin{aligned}
[a_{j,1}([\![\boldsymbol{z}]\!], [\![\boldsymbol{v}]\!])]^2 &\le \left(1 - \frac{1}{\rho}\right) a_{j,1}([\![\boldsymbol{v}]\!], [\![\boldsymbol{v}]\!]) \left[\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \|h_E^{-1/2} [\![\boldsymbol{z}]\!]\|_{0,E}^2\right] \\
&\le \left(1 - \frac{1}{\rho}\right) a_{j,1}([\![\boldsymbol{z}]\!], [\![\boldsymbol{z}]\!]) a_{j,1}([\![\boldsymbol{v}]\!], [\![\boldsymbol{v}]\!]),
\end{aligned}$$

which shows the desired inequality. □

We are now in a position to prove that the preconditioner given by Algorithm 4.1 is uniform with respect to the mesh size and the problem parameters.

Theorem 4.3. *Let $\mathcal{A}(\cdot,\cdot)$ be the symmetric bilinear form defined by (2.24) where $\theta = -1$ and $\mathcal{B}(\cdot,\cdot)$ be the bilinear form defined by (4.1). Then the following estimates hold for all $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$ and for all $\boldsymbol{v} \in \boldsymbol{V}^{CR}$*

$$\frac{1}{1+\gamma}\mathcal{A}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})) \le \mathcal{B}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})) \le \frac{1}{1-\gamma}\mathcal{A}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})). \tag{4.3}$$

The constant $\gamma < 1$ is the constant from Lemma 4.2.

Proof. Using Lemma 4.2 we have

$$-2\gamma\sqrt{\mathcal{A}(\boldsymbol{z},\boldsymbol{z})\,\mathcal{A}(\boldsymbol{v},\boldsymbol{v})} \le 2\mathcal{A}(\boldsymbol{z},\boldsymbol{v}) \le 2\gamma\sqrt{\mathcal{A}(\boldsymbol{z},\boldsymbol{z})\,\mathcal{A}(\boldsymbol{v},\boldsymbol{v})}$$

and since $-a^2 - b^2 \le 2ab \le a^2 + b^2$ for any real numbers a and b we obtain

$$\begin{aligned}(1-\gamma)\,(\mathcal{A}(\boldsymbol{z},\boldsymbol{z}) + \mathcal{A}(\boldsymbol{v},\boldsymbol{v})) &\le \mathcal{A}(\boldsymbol{z},\boldsymbol{z}) + \mathcal{A}(\boldsymbol{v},\boldsymbol{v}) + 2\mathcal{A}(\boldsymbol{z},\boldsymbol{v}) \\ &\le (1+\gamma)\,(\mathcal{A}(\boldsymbol{z},\boldsymbol{z}) + \mathcal{A}(\boldsymbol{v},\boldsymbol{v}))\end{aligned}$$

which is the same as

$$(1-\gamma)\mathcal{B}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})) \le \mathcal{A}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})) \le (1+\gamma)\mathcal{B}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v}))$$

and thus (4.3) holds with the same constant $\gamma < 1$ as used in the estimate of Lemma 4.2. $\square$

Remark 4.4. *Note that $\gamma \le q < 1$ is uniformly bounded away from 1 and this bound holds independently of the parameters h, λ, and μ.*

5. Numerical experiments

In this section we present a set of numerical tests that illustrate our theoretical results. We consider the SIPG discretization of the model problem (2.3) on the unit square in $\mathbb{R}^2$ with mixed boundary conditions. For the penalty parameters in (2.20) we choose the values $\alpha_0 = 4$ and $\alpha_1 = 1$. The coarsest mesh (at level 0) consists of eight triangles and is refined four times. Each refined mesh at level ℓ, $\ell = 1, 2, 3, 4$ is obtained by subdividing every triangle at level $(\ell - 1)$ into four congruent triangles. The CBS constants and the spectral condition numbers summarized in the tables below have been computed using MATLAB.

In Table 5.1 we list the values of the constant γ^2 in the inequality stated in Lemma 4.2 for different levels of refinement. Evidently, γ is uniformly bounded with respect to the mesh size (or the number of refinement levels) and also with respect to the material parameters, Young's modulus $\mathfrak{E}$ and Poisson ratio ν (see Remark 4.4). It can be seen from Table 5.2 that the two subspaces $\boldsymbol{V}^{\mathrm{CR}}$ and $\boldsymbol{\mathcal{Z}}$

Table 5.1. Observed CBS constant γ^2 for $\Omega = (0,1)^2$.

γ^2	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.49999$
$\ell = 1$	0.0664	0.025	0.0024	2.4024×10^{-4}	2.4015×10^{-6}
$\ell = 2$	0.0678	0.0255	0.0025	2.4567×10^{-4}	2.4559×10^{-6}
$\ell = 3$	0.0684	0.0258	0.0025	2.4866×10^{-4}	2.4857×10^{-6}
$\ell = 4$	0.0686	0.0259	0.0025	2.4974×10^{-4}	2.4966×10^{-6}

remain nearly $\mathcal{A}$-orthogonal when we introduce a jump in the Poisson ratio (on the coarsest mesh); In our experiment we set $\nu = \nu_1 = 0.3$ (and $E = E_1 = 1$) in the subdomain $\Omega_1 = [0, 0.5] \times [0, 0.5] \cup [0.5, 1] \times [0.5, 1]$, and $\nu = \nu_2$ (and $E_2 = 1$) in the subdomain $\Omega_2 = \Omega \setminus \Omega_2$, respectively.

Next we consider an L-shaped domain $\Omega = [0, 1] \times [0, 1] \setminus (0.5, 1] \times (0.5, 1]$ with Neumann boundary conditions on the sides $y = 0$ and $y = 1$ and Dirichlet boundary conditions on the remaining part of the boundary. The initial triangulation (level 0) consists of 4 similar triangles. The angle is almost the same as for the square domain, see Table 5.3.

Furthermore, we computed the relative condition number of the preconditioner B corresponding to the bilinear form (4.1) for the model problem on the L-shaped domain. The results of this experiment, which are listed in Table 5.4, confirm the uniform bound provided by Theorem 4.3.

Table 5.2. Observed CBS constant γ^2 for $\Omega = (0, 1)^2$ and jumps in ν.

γ^2	$\nu_2 = 0.3$	$\nu_2 = 0.4$	$\nu_2 = 0.49$	$\nu_2 = 0.499$	$\nu_2 = 0.49999$
$\ell = 1$	0.0451	0.0177	0.0442	0.0509	0.0517
$\ell = 2$	0.0460	0.0180	0.0689	0.0803	0.0816
$\ell = 3$	0.0464	0.0182	0.0689	0.0802	0.0816
$\ell = 4$	0.0466	0.0182	0.0689	0.0802	0.0816

Table 5.3. Observed CBS constant γ^2 for L-shaped domain.

γ^2	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.49999$
$\ell = 1$	0.0561	0.0202	0.0019	1.8918×10^{-4}	1.8906×10^{-6}
$\ell = 2$	0.0631	0.0233	0.0022	2.2118×10^{-4}	2.2106×10^{-6}
$\ell = 3$	0.0672	0.0252	0.0024	2.4216×10^{-4}	2.4207×10^{-6}
$\ell = 4$	0.0682	0.0257	0.0025	2.4810×10^{-4}	2.4801×10^{-6}

Table 5.4. Tabulated values of $\kappa(B^{-1}A)$ for L-shaped domain.

$\kappa(B^{-1}A)$	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.49999$
$\ell = 1$	1.6204	1.3314	1.0912	1.0279	1.0028
$\ell = 2$	1.6713	1.3606	1.0990	1.0302	1.0030
$\ell = 3$	1.6997	1.3774	1.1037	1.0316	1.0031
$\ell = 4$	1.7073	1.3820	1.1050	1.0320	1.0032

Table 5.5. Values of $\kappa(A_{zz})$ for L-shaped domain.

$\kappa(A_{zz})$	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.49999$
$\ell = 1$	8.9067	7.1484	6.4788	6.4220	6.4158
$\ell = 2$	9.0875	7.1932	6.4829	6.4229	6.4164
$\ell = 3$	9.1577	7.2080	6.4841	6.4230	6.4164
$\ell = 4$	9.1794	7.2118	6.4844	6.4230	6.4164

Finally, we computed the condition number $\kappa(A_{zz})$ of the matrix A_{zz} related to the restriction of $\mathcal{A}(\cdot,\cdot)$ to the space $\boldsymbol{\mathcal{Z}}$, again for the model problem on the L-shaped domain. In view of Lemma 3.3 we already know that A_{zz} is well-conditioned, and this is clearly seen in Table 5.5 where the values of $\kappa(A_{zz})$ are listed.

6. Acknowledgments

Part of this work was completed while the fourth author was visiting RICAM, Austrian Academy of Sciences in Linz. Thanks go to the RICAM for the kind hospitality and support. The work of the first author was partially supported by the Spanish MEC under projects MTM2008-03541 and HI2008-0173. The work of the second author has been partially supported by the Bulgarian NSF, Grant DO 02-338/08. We also gratefully acknowledge the support by the Austrian Science Fund, Grants P19170-N18 and P22989-N18. The work of the fourth author has been supported in part by the US National Science Foundation, Grants DMS-0810982, and DMS-0749202.

Appendix A. Auxiliary results

A.1. **Bounds on the cardinality of $\mathcal{N}_1(E)$ and $\mathcal{N}_2(E)$.** We first recall the definitions of $\mathcal{N}_0(E)$, $\mathcal{N}_1(E)$ and $\mathcal{N}_2(E)$, already given in §3.1:

$$\begin{aligned}
\mathcal{N}_0(E) &:= \{T \in \mathcal{T}_h, \quad \text{such that} \quad E \in T\},\\
\mathcal{N}_1(E) &:= \{E' \in \mathcal{E}_h, \quad \text{such that} \quad \mathcal{N}_0(E) \cap \mathcal{N}_0(E') \neq \emptyset\},\\
\mathcal{N}_2(E) &:= \{E' \in \mathcal{E}_h, \quad \text{such that} \quad \mathcal{N}_1(E) \cap \mathcal{N}_1(E') \neq \emptyset\}.
\end{aligned}$$

In the proof of the strengthened Cauchy-Schwarz inequality §3.1 we needed several estimates on the cardinality of these sets and these estimates are given in the proposition below. We remind the reader that we have $|\mathcal{N}_0(E)| \leq 2$.

Proposition A.1. *The following inequalities hold:*

$$|\mathcal{N}_1(E)| \leq (2d+1) \quad \textit{and} \quad |\mathcal{N}_2(E)| \leq (2d+1)^2. \tag{A.1}$$

Proof. Let $E \in \mathcal{E}_h$ be fixed. To prove the bound on $|\mathcal{N}_1(E)|$ we consider the elements $T \in \mathcal{T}_h$, such that $E \in T$. In each such element T, there are exactly d faces $E' \in T$, $E' \neq E$. Since there are at most two elements $T \in \mathcal{T}_h$ containing E we have at most $2d$ faces $E' \in \mathcal{E}_h$ such that $E' \in \mathcal{N}_1(E)$, and $E' \neq E$. Adding E itself to the total count gives $|\mathcal{N}_1(E)| \leq (2d+1)$.

The second bound given in (A.1) follows from the first and the following inclusion:

$$\mathcal{N}_2(E) \subset \bigcup_{E' \in \mathcal{N}_1(E)} \mathcal{N}_1(E').$$

To show the above inclusion, we consider an arbitrary $E'' \in \mathcal{N}_2(E)$. By the definition of $\mathcal{N}_2(E)$, the intersection of $\mathcal{N}_1(E'')$ and $\mathcal{N}_1(E)$ is not empty. Equivalently, there exists $E' \in \mathcal{E}_h$ such that $E' \in \mathcal{N}_1(E'')$ and $E' \in \mathcal{N}_1(E)$. On the other hand, from the definition of $\mathcal{N}_1(E'')$, we have that $E' \in \mathcal{N}_1(E'')$ implies that $E'' \in \mathcal{N}_1(E')$, i.e., if E' is a neighbor of E'', then E'' is a neighbor of E'.

Putting this together, we conclude that: if $E'' \in \mathcal{N}_2(E)$, then there exists $E' \in \mathcal{N}_1(E)$, such that $E'' \in \mathcal{N}_1(E')$, and this is exactly the inclusion we wanted to show.

To prove the desired bound is then straightforward:

$$\begin{aligned}\left|\bigcup_{E' \in \mathcal{N}_1(E)} \mathcal{N}_1(E')\right| &\leq \sum_{E' \in \mathcal{N}_1(E)} |\mathcal{N}_1(E')| \leq \sum_{E' \in \mathcal{N}_1(E)} (2d+1) \\ &= (2d+1)|\mathcal{N}_1(E)| \leq (2d+1)^2.\end{aligned}$$

□

A.2. **A multiplicative relation.** This is to prove a basic relation used to derive (3.3) as well as (2.15). Let $\odot$ be a map $V \times W \mapsto U$, where U, V, and W are linear vector spaces over the real numbers. We assume that $\odot$ satisfies the following distributive laws:

$$a \odot (b+c) = a \odot b + a \odot c, \qquad (a+b) \odot c = a \odot c + b \odot c,$$

and we assume that for all $\xi \in \mathbb{R}$ and all $\eta \in \mathbb{R}$, we have:

$$(\xi a) \odot (\eta b) = (\xi\eta)(a \odot b). \tag{A.2}$$

We have the following identities, based on the definitions (2.7):

$$a^+ \odot b^+ - a^- \odot b^- = [\![a]\!] \odot \{\!\{b\}\!\} + \{\!\{a\}\!\} \odot [\![b]\!]. \tag{A.3}$$

Proving this relation is indeed trivial. Some examples for which the reader should verify these identities are: (1) For real numbers a and b one may take as $\odot$ the usual multiplication of real numbers; (2) a and b elements of a real Hilbert space and $\odot$ inner product; (3) a and b are linear operators, and $\odot$ is then the multiplication of linear operators. Note that in such case $\odot$ is not necessarily commutative; (4) a is a matrix and b is a vector, or more generally, a is a linear operator and b is an element of a Hilbert space.

From (2.7), we have that the right side of the identity (A.3) is

$$[\![a]\!] \odot \{\!\{b\}\!\} + \{\!\{a\}\!\} \odot [\![b]\!] = (a^+ - a^-) \odot \left(\frac{b^+ + b^-}{2}\right) + \left(\frac{a^+ + a^-}{2}\right) \odot (b^+ - b^-)$$

Using the distributive law, and (A.2) (linearity of $\odot$ with respect to scalar multiplication), we have

$$\begin{aligned}
&(a^+ - a^-) \odot \left(\frac{b^+ + b^-}{2}\right) + \left(\frac{a^+ + a^-}{2}\right) \odot (b^+ - b^-) \\
= \;& \frac{1}{2}(a^+ - a^-) \odot (b^+ + b^-) + \frac{1}{2}(a^+ + a^-) \odot (b^+ - b^-) \\
= \;& \frac{1}{2}a^+ \odot (b^+ + b^-) - \frac{1}{2}a^- \odot (b^+ + b^-) + \frac{1}{2}a^+ \odot (b^+ - b^-) + \frac{1}{2}a^- \odot (b^+ - b^-) \\
= \;& \frac{1}{2}a^+ \odot b^+ + \frac{1}{2}a^+ \odot b^- - \frac{1}{2}a^- \odot b^+ - \frac{1}{2}a^- \odot b^- \\
& + \frac{1}{2}a^+ \odot b^+ - \frac{1}{2}a^+ \odot b^- + \frac{1}{2}a^- \odot b^+ - \frac{1}{2}a^- \odot b^- \\
= \;& \frac{1}{2}a^+ \odot b^+ - \frac{1}{2}a^- \odot b^- + \frac{1}{2}a^+ \odot b^+ - \frac{1}{2}a^- \odot b^- = a^+ \odot b^+ - a^- \odot b^-.
\end{aligned}$$

References

[1] Douglas N. Arnold, Franco Brezzi, Bernardo Cockburn, and L. Donatella Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779 (electronic), 2001/02.

[2] Douglas N. Arnold, Franco Brezzi, Richard Falk, and L. Donatella Marini. Locking-free reissner-mindlin elements without reduced integration. *Comput. Methods Appl. Mech. Engrg.*, 196(37-40):3660–3671, 2007.

[3] Douglas N. Arnold, Franco Brezzi, and L. Donatella Marini. A family of discontinuous Galerkin finite elements for the Reissner-Mindlin plate. *J. Sci. Comput.*, 22/23:25–45, 2005.

[4] Blanca Ayuso de Dios and Ludmil Zikatanov. Uniformly convergent iterative methods for discontinuous Galerkin discretizations. *J. Sci. Comput.*, 40(1-3):4–36, 2009.

[5] Radim Blaheta, Svetozar Margenov, and Maya Neytcheva. Aggregation-based multilevel preconditioning of non-conforming fem elasticity problems. In Jack Dongarra, Kaj Madsen, and Jerzy Wasniewski, editors, *Applied Parallel Computing. State of the Art in Scientific Computing*, volume 3732 of *Lecture Notes in Computer Science*, pages 847–856. Springer Berlin / Heidelberg, 2006.

[6] Susanne C. Brenner. Poincaré-Friedrichs inequalities for piecewise H^1 functions. *SIAM J. Numer. Anal.*, 41(1):306–324 (electronic), 2003.

[7] Susanne C. Brenner. Korn's inequalities for piecewise H^1 vector fields. *Math. Comp.*, 73(247):1067–1087 (electronic), 2004.

[8] F. Brezzi, B. Cockburn, L. D. Marini, and E. Süli. Stabilization mechanisms in discontinuous Galerkin finite element methods. *Comput. Methods Appl. Mech. Engrg.*, 195(25-28):3293–3310, 2006.

[9] Erik Burman and Benjamin Stamm. Low order discontinuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.*, 47(1):508–533, 2008.

[10] G. Duvaut and J.-L. Lions. *Inequalities in mechanics and physics.* Springer-Verlag, Berlin, 1976. Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.

[11] Richard S. Falk. Nonconforming finite element methods for the equations of linear elasticity. *Math. Comp.*, 57(196):529–550, 1991.

[12] Xiaobing Feng and Ohannes A. Karakashian. Two-level additive Schwarz methods for a discontinuous Galerkin approximation of second order elliptic problems. *SIAM J. Numer. Anal.*, 39(4):1343–1365 (electronic), 2001.

[13] I. Georgiev, J. K. Kraus, and Margenov S. Multilevel preconditioning of Crouzeix-Raviart 3D pure displacement elasticity problems. In I. Lirkov, S. Margenov, and J. Wasniewski, editors, *Large Scale Scientific Computing*, volume 5910 of *Lecture Notes in Computer Science (LNCS)*, pages 103–110. Springer, Berlin, Heidelberg, 2010.

[14] Peter Hansbo and Mats G. Larson. Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method. *Comput. Methods Appl. Mech. Engrg.*, 191(17-18):1895–1908, 2002.

[15] Peter Hansbo and Mats G. Larson. Discontinuous Galerkin and the Crouzeix-Raviart element: application to elasticity. *M2AN Math. Model. Numer. Anal.*, 37(1):63–72, 2003.

[16] Magnus R. Hestenes and Eduard Stiefel. Methods of conjugate gradients for solving linear systems. *J. Research Nat. Bur. Standards*, 49:409–436 (1953), 1952.

[17] Johannes Kraus and Svetozar Margenov. *Robust algebraic multilevel methods and algorithms*, volume 5 of *Radon Series on Computational and Applied Mathematics.* Walter de Gruyter GmbH & Co. KG, Berlin, 2009.

[18] Yousef Saad. *Iterative methods for sparse linear systems.* Society for Industrial and Applied Mathematics, Philadelphia, PA, second edition, 2003.

[19] Thomas P. Wihler. Locking-free DGFEM for elasticity problems in polygons. *IMA J. Numer. Anal.*, 24(1):45–75, 2004.

[20] Thomas P. Wihler. Locking-free adaptive discontinuous Galerkin FEM for linear elasticity problems. *Math. Comp.*, 75(255):1087–1102 (electronic), 2006.

B. Ayuso de Dios
Centre de Recerca Matemàtica
Campus de Bellaterra
08193 Bellaterra, (Barcelona), Spain
E-mail address: bayuso@crm.cat

I. Georgiev
Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Sciences
Altenberger Str. 69
4040 Linz, Austria
E-mail address: ivan.georgiev@oeaw.ac.at

J. Kraus
Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Sciences
Altenberger Str. 69
4040 Linz, Austria
E-mail address: johannes.kraus@oeaw.ac.at

L. Zikatanov
Department of Mathematics
The Pennsylvania State University
University Park
PA 16802, USA
E-mail address: ltz@math.psu.edu