CHEBYSHEV-TYPE QUADRATURE FORMULAS FOR NEW WEIGHT CLASSES

ARMEN VAGHARSHAKYAN

Abstract. We give Chebyshev-type quadrature formulas for certain new weight classes. These formulas are of highest possible degree when the number of nodes is a power of 2. We also describe the nodes in a constructive way, which is important for applications. One of our motivations to consider these type of problems is the Faraday cage phenomenon for discrete charges as discussed by J. Korevaar and his colleagues.

1. Introduction

Let us denote by $Pol(m)$ the family of polynomials of degree $\leq m$.

Definition 1.1. Let $\rho(x)$ be a nonnegative function on $[-1, 1]$. Denote by:

$$M_n(\rho)$$

the biggest natural number for which there exists a choice of distinct points $\{x_k\}_{k=1}^n \subset [-1, 1]$ so that the equality:

$$\int_{-1}^{1} P(x)\rho(x)dx = \frac{1}{n}\sum_{k=1}^{n} P(x_k) \tag{1.2}$$

is valid for an arbitrary polynomial $P \in Pol(M_n(\rho))$.

Remark 1.3. A quadrature formula of the form (1.2) is called Chebyshev-type quadrature formula, and the points $\{x_k\}_{k=1}^n \subset [-1, 1]$ are called the nodes of the quadrature formula. One can consider quadrature formulas of a more general form:

$$\int_{-1}^{1} P(x)\rho(x)dx = \frac{1}{n}\sum_{k=1}^{n} A_k P(x_k) \tag{1.4}$$

where A_k are certain nonnegative numbers. These are known as Gauss type quadrature formulas.

It is well known (see [3], pp. 97) that a Gauss type quadrature formula with n nodes cannot be valid for an arbitrary $P \in Pol(2n)$. For completeness, we provide the proof of that fact here:

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Remark 1.5. Let $\rho(x) \geq 0$ be a nontrivial function on $-1 \leq x \leq 1$. Then there are no points $x_k \in [-1,1]$, $k = 1,\ldots,n$ and numbers A_k, $k = 1,2,\ldots,n$ such that the formula (1.4) is valid for an arbitrary $P \in Pol(2n)$.

Proof. Assume the converse, for some points $x_k \in [-1,1]$, $k = 1,2,\ldots,n$ and numbers A_k, $k = 1,2,\ldots,n$ the formula (1.4) is true. Then for the polynomial:

$$P_0(x) = \prod_{k=1}^{n} (x - x_k)^2$$

of degree $2n$ we must have:

$$0 < \int_{-1}^{1} P_0(x)\rho(x)dx = \frac{1}{n}\sum_{k=1}^{n} A_k P(x_k) = 0. \qquad \square$$

Remark 1.6. Remark (1.5) implies that $M_n(\rho) < 2n$.

The first Chebyshev-type formula was first discovered by Mehler in 1864 (see [11], pp. 185):

Theorem 1.7 (Mehler)**.** *The formula*

$$\frac{1}{\pi}\int_{-1}^{1} \frac{P(x)}{\sqrt{1-x^2}}dx = \frac{1}{n}\sum_{k=1}^{n} P\left(\cos\frac{2k-1}{2n}\pi\right)$$

is valid for an arbitrary polynomial $P \in Pol(2n-1)$.

On the other hand (see [11], pp. 186):

Theorem 1.8. *[Posse, Sonin] If for a weight function*

$$\rho(x), \quad x \in [-1,1],$$

we have that $M_n(\rho) = 2n-1$ for all $n \in N$ then:

$$\rho(x) = \frac{1}{\pi\sqrt{1-x^2}}. \tag{1.9}$$

The first weight function different from (1.9) for which $M_n(\rho) \geq n$ was given by Ullman (see [6]):

$$\mu(x) = \frac{2}{\pi\sqrt{1-x^2}} \cdot \frac{1+bx}{1+b^2+2bx}, \qquad |b| < \frac{1}{2}$$

where

$$M_n(\mu) = n.$$

Later new examples of weights for whom $M_n(\rho) = n$ were found (see the survey article [7]). These examples of weight functions are mainly of the following form:

$$\rho(x) = \frac{w(x)}{\sqrt{1-x^2}},$$

where $w(x)$ is a certain positive analytic function on $[-1, 1]$.

Examples of unbounded weights have been found, too (see [5]).
The main result of this article - theorem (4.2) provides Chebyshev-type quadrature formulas for certain new weight classes $\mathbf{W}_n$. All these classes include not only the weight:

$$\rho(x) = \frac{1}{\pi\sqrt{1-x^2}} \tag{1.10}$$

(see remark (3.18)) but also many others (see remark (3.13)). The Chebyshev-type quadrature formulas for $\mathbf{W}_n$ are of highest possible degree when the number of nodes is a power of 2 (see remark (4.3)). Thus, even though (1.10) is the only weight for which the Chabyshev type quadrature formula is of the highest possible degree $M_n(\rho) = 2n - 1$ (see theorem (1.8)), yet we prove that for every n which is a power of 2 there are many more weights for whom there exists a Chebyshev-type quadrature formula of the highest possible degree.

Also note that the nodes in theorem (4.2) are described in a constructive way, which is important for applications.

2. The Case $\rho \equiv 1/2$

Let us consider the case $\rho \equiv 1/2$ separately. S. Bernstein (see [11], page 193) proved that if $n = 8$ or $n \geq 10$ then one cannot choose points $x_{k,n}$, $k = 1, 2, \ldots, n$ in the segment $[-1, 1]$ so that the following formula:

$$\frac{1}{2}\int_{-1}^{1} P(x)dx = \frac{1}{n}\sum_{k=1}^{n} P(x_{k,n})$$

is valid for all polynomials of degree n.

S. Bernstein (see [10]) proved the following inequality $M_n(1/2)$:

$$M_n(1/2) < \pi\sqrt{2n}.$$

A. Kuijlaars [4] (using the methods of S. Bernstein [9]) proved that:

Theorem 2.1. *There exists an absolute constant $c > 0$ such that:*

$$M_n(1/2) > c\sqrt{n}.$$

Today the interest into Chebyshev-type quadrature formulas is explained not only by numerous nontrivial conjectures relating to the issue (see [1, 5] for a list of open problems) but also a connection between the Faraday cage phenomenon for discrete charges and Chebyshev-type quadrature formulas, as explained by J. Korevaar and his colleagues (see [3]). For a more detailed explanation we refer the reader to the article [3].

This problem is also related to the discrepancy theory (see [8]).

3. Auxiliary Constructions

Let us denote by $x_0 = -1$ and

$$x_{n+1} = \sqrt{\frac{1+x_n}{2}}, \quad n = 0, 1, 2, \ldots$$

This sequence is increasing and

$$\lim_{n\to\infty} x_n = 1.$$

Definition 3.1. For an arbitrary natural n let us denote by $P_n(x)$ the polynomial of degree n, defined in the following way:
1. we put:

$$P_0(x) = 1, \quad P_2(x) = 2x^2 - 1,$$

2. for an odd number q let:

$$P_q(x) = x^q,$$

3. for a natural number $n = 2^p$ let:

$$P_{2^p}(x) = P_2\left(P_{2^{p-1}}(x)\right),$$

4. for an arbitrary natural number $n = 2^p q$, where $q = 1 (\mathrm{mod}\ 2)$ and $3 \leq q$ let:

$$P_n(x) = \left(P_{2^p}(x)\right)^q.$$

For $n = 2^p q$, where $q = 1 (\mathrm{mod}\ 2)$ denote:

$$|P_n| = p.$$

Example 3.2. *We have:*

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = 2x^2 - 1, \quad P_3(x) = x^3,$$
$$P_4(x) = 2\left(2x^2 - 1\right)^2 - 1, \quad P_5(x) = x^5, \quad P_6(x) = \left(2x^2 - 1\right)^3,$$
$$P_7(x) = x^7, \quad P_8(x) = 2\left(2\left(2x^2 - 1\right)^2 - 1\right)^2 - 1$$

Theorem 3.3. *For an arbitrary natural n we have:*

$$\max_{0\leq k\leq n} |P_k| = [\log_2 n].$$

Proof. We present each number $1 \leq k \leq n$ n the form $k = 2^{p_k} q_k$, where $q_k = 1 (\mathrm{mod}\ 2)$. Then:

$$\max_{0\leq k\leq n} |P_k| = \max_{0\leq k\leq n} |p_k| = [\log_2 n].$$ □

Example 3.4. *We have:*

$$|P_{2^k}| = k, \quad |P_{2^k - 1}| = k - 1.$$

Theorem 3.5. *For an arbitrary natural $1 \leq p$ we have:*

$$P_{2^p}(x_p) = -1, \quad P_{2^p}(x_{p+1}) = 0, \quad P_{2^p}(1) = 1.$$

On the interval $[x_p,\ 1]$ the polynomial $P_{2^p}(x)$ is an increasing function.

Definition 3.6. Let

$$S_0 : [x_1, 1] \to [x_0, x_1]\,,$$

and for each $x \in [x_1, 1]$,

$$S_0(x) = -x.$$

For a natural n let us denote by:

$$S_n : [x_{n+1}, 1] \to [x_n, x_{n+1}]\,.$$

the one-to-one mapping, such that:

$$P_{2^n}(x) = -P_{2^n}(S_n(x))\,, \quad x_{n+1} \le x \le 1.$$

We have:

$$S_n(x_{n+1}) = x_{n+1}, \quad S_n(1) = x_n, \quad n = 1, 2, \ldots$$

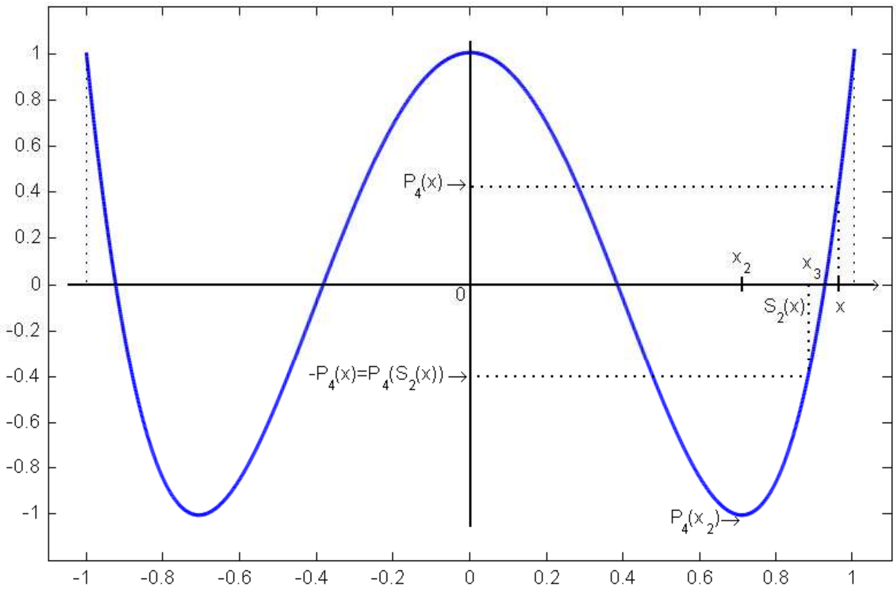

FIGURE 1. The mapping S_2

Example 3.7. *We have:*

$$S_1(y) = -y, \quad \frac{dS_1(y)}{dy} = -1, \quad x_2 < y < 1,$$

$$S_2(y) = \sqrt{1-y^2}, \quad \frac{dS_2(y)}{dy} = -\frac{y}{\sqrt{1-y^2}}, \quad x_3 < y < 1,$$

$$S_3(y) = \frac{y + \sqrt{1-y^2}}{\sqrt{2}}, \quad \frac{dS_3(y)}{dy} = -\frac{y - \sqrt{1-y^2}}{\sqrt{2(1-y^2)}}, \quad x_4 < y < 1.$$

Definition 3.8. For an arbitrary integral number $0 \le n$ and a function $f(x)$ defined on the interval $x_n \le x \le 1$, let us denote by $R_n(f)$ the function defined on the interval $x_{n+1} < y < 1$ by:

$$R_n(f)(y) = \frac{f\left(S_n(y)\right) + f(y)}{2}.$$

Remark 3.9. If the function f is constant on the interval $[-1, 1]$ then for an arbitrary $0 \le n$ we have:

$$R_n(f)(y) = f(y), \quad \text{on} \quad x_{n+1} < y < 1.$$

Remark 3.10. For any of the polynomials $P_k(x), \quad k = 0, 1, 2, \dots$ which we constructed, we have:

$$R_0(P_k)(x) = P_k(x), \quad k = 0 \pmod 2,$$

and

$$R_0(P_k)(x) = 0, \quad k = 1 \pmod 2.$$

Remark 3.11. For any of the polynomials $P_{2^pk}(x), \quad k, p = 1, 2, \dots,$ which we constructed, we have:

$$R_p(P_{2^pk})(x) = P_{2^pk}(x), \quad k = 0 \pmod 2,$$

and

$$R_p(P_{2^pk})(x) = 0, \quad k = 1 \pmod 2.$$

Definition 3.12. For natural n we'll denote by $\mathbf{W}_n$ the family of nonnegative functions $\rho(x), \ -1 < x < 1$ satisfying the following conditions:

$$\int_{-1}^{1} \rho(x)dx = 1$$

and

$$\rho(y) = -\rho(S_k(y))\frac{dS_k(y)}{dy}, \quad x_k < y \le 1,$$

for $k = 0, 1, 2, \dots, n-1$.

Remark 3.13. Let us note that for an arbitrary weight function $\rho(x) \in \mathbf{W}_n$ we have:

$$\int_{x_n}^{1} \rho(y)dy = 2^{-n}. \tag{3.14}$$

Moreover, an arbitrary nonnegative function defined on $[x_n, 1]$ satisfying the condition (3.14), coincides with a weight function from $\mathbf{W}_n$ on $[x_n, 1]$.

Example 3.15. *Note, that:*

$$S_0(y) = -y, \quad x_1 \leq y \leq 1,$$

therefore for the weight function $\rho \in \mathbf{W}_1$ *we get the condition:*

$$\rho(y) = \rho(-y), \quad x_0 < y \leq 1.$$

Example 3.16. *We have:*

$$S_1(y) = \sqrt{1-y^2}, \quad x_2 \leq y \leq 1,$$

therefore for the weight function $\rho \in \mathbf{W}_2$ *we get the conditions:*

$$\rho(y) = \rho(-y), \; x_0 = -1 < y \leq 1,$$

and

$$\rho(y) = \rho\left(\sqrt{1-y^2}\right)\frac{y}{\sqrt{1-y^2}}, \quad x_1 < y \leq 1,$$

In particular, the function:

$$\rho(y) = \frac{1}{\pi\sqrt{1-y^2}}$$

satisfies these conditions.

Example 3.17. *We have:*

$$S_2(y) = \frac{y+\sqrt{1-y^2}}{\sqrt{2}}, \quad x_3 \leq y \leq 1,$$

therefore for the weight function $\rho \in \mathbf{W}_3$ *we get the conditions:*

$$\rho(y) = \rho(-y), \quad x_0 < y \leq 1,$$

and

$$\rho(y) = \rho\left(\sqrt{1-y^2}\right)\frac{y}{\sqrt{1-y^2}}, \quad x_1 < y \leq 1,$$

and

$$\rho(y) = \rho\left(\frac{y+\sqrt{1-y^2}}{\sqrt{2}}\right)\frac{y-\sqrt{1-y^2}}{\sqrt{2(1-y^2)}}, \quad x_2 < y < 1.$$

In particular, the following function:

$$\rho(y) = \frac{1}{\pi\sqrt{1-y^2}}$$

satisfies all the three conditions.

Remark 3.18. By a reasoning similar to (3.16) and (3.17) one can prove that:

$$\rho(y) = \frac{1}{\pi\sqrt{1-y^2}} \in \mathbf{W}_n \text{ for } n = 1, 2, \ldots.$$

4. **Main Results**

Theorem 4.1. *Let $n \geq 1$ be a natural number. Let $\rho(x) \in \mathbf{W}_n$. Then for an arbitrary function $f(x)$ we have:*

$$\int_{-1}^{1} f(x)\rho(x)dx = 2^n \int_{x_n}^{1} (R_{n-1} \circ \cdots \circ R_0)(f)(x) \cdot \rho(x)dx.$$

Proof. If $\rho(x) \in \mathbf{W}_1$ we have $\rho(y) = \rho(-y)$ so,

$$\begin{aligned}\int_{-1}^{1} f(x)\rho(x)dx &= \int_{-1}^{0} f(x)\rho(x)dx + \int_{0}^{1} f(x)\rho(x)dx = \\ &= \int_{0}^{1} f(x)\rho(x)dx - \int_{0}^{1} f\left(S_0(y)\right)\rho\left(S_0(y)\right)\frac{dS_0(y)}{dy}dy = \\ &= \int_{0}^{1} (f(x)+f(-x))\rho(x)dx = 2\int_{x_1}^{1} R_0(f)(y)\rho(y)dy.\end{aligned}$$

If $\rho(x) \in \mathbf{W}_2$ then we have $\rho(y) = \rho(-y)$ and

$$\rho(y) = -\rho\left(S_1(y)\right)\frac{dS_1(y)}{dy}, \quad x_2 \leq y \leq 1.$$

Thus, for an arbitrary function $f(x)$ we have:

$$\begin{aligned}\int_{-1}^{1} f(x)\rho(x)dx &= 2\int_{x_1}^{1} R_0(f)(x)\rho(x)dx = \\ &= 2\int_{x_1}^{x_2} R_0(f)(x)\rho(x)dx + 2\int_{x_2}^{1} R_0(f)(x)\rho(x)dx = \\ &= 2\int_{x_2}^{1} R_0(f)(x)\rho(x)dx - 2\int_{x_2}^{1} R_0(f)\left(S_1(y)\right)\rho\left(S_1(y)\right)\frac{dS_1(y)}{dy}dy = \\ &= 2\int_{x_2}^{1} \left(R_0(f)\left(S_1(y)\right) + R_0(f)(y)\right)\rho(y)dy = \\ &= 2^2\int_{x_2}^{1} (R_1 \circ R_0)(f)(y)\rho(y)dy.\end{aligned}$$

We prove the theorem for $\rho \in \mathbf{W}_n, \quad n = 1, 2, \ldots$ in an analogous way. □

Theorem 4.2. *Let $\rho(x) \in \mathbf{W}_n,\ n = 2, 3, \ldots$. Let $X = \{t_j;\ j = 1, 2, \ldots, m\}$, where $1 \leq m$. Let the points:*

$$-1 < t_1 < t_2 < \cdots < t_m < 1,$$

satisfy the conditions:

$$S_k\left(X \cap [x_{k+1}, 1]\right) = X \cap [x_k, x_{k+1}], \quad k = 0, 1, 2, \ldots, [\log_2 n] - 1.$$

Then for an arbitrary polynomial $P(x) \in Pol(n-1)$ we have:

$$\int_{-1}^{1} P(x)\rho(x)dx = \frac{1}{m}\sum_{k=1}^{m} P(t_k).$$

Remark 4.3. Theorem (4.2) implies that for any $\rho \in \mathbf{W_n}$ we have:

$$m = |X| \geq 2^{[\log_2 n]-1}.$$

If we choose n to be a power of 2: $n = 2^p$ where p is a natural number, and $t_m = x_p$, then we get the nodes $X = \{t_j;\ \ j = 1, 2, \ldots, m\}$ where $2m = n$. Hence,

$$M_m(\rho) \geq 2m - 1.$$

Taking into account the remark (1.6) we get:

$$M_m(\rho) = 2m - 1.$$

Thus we found a Chebyshev-type quadrature formula for $\rho \in \mathbf{W_n}$ of highest possible degree, where the number of nodes $m = n/2$ is a power of 2.

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A. Vagharshakyan
Mathematics Department
Brown University
151 Thayer St
Providence, RI 02912 USA
E-mail address: armen@math.brown.edu