KREIN'S STRINGS WHOSE SPECTRAL FUNCTIONS ARE OF POLYNOMIAL GROWTH

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Abstract. In case Krein's strings with spectral functions of polynomial growth a necessary and sufficient condition for the Krein's correspondence to be continuous is given.

1. Entrance type

Let $m(x)$ be a non-decreasing and right continuous function on $(-\infty,\infty)$ satisfying

$$m(-\infty)=0, \qquad m(\infty)\leq\infty.$$

Set

$$l=\sup\left\{x>-\infty,\ m(x)<+\infty\right\},\ \ l_{+}=\sup\operatorname{supp}dm,\ l_{-}=\inf\operatorname{supp}dm.$$

Note $m(l)=\infty$ if $l<\infty$. Assume

$$\int_{-\infty}^{a}|x|\,dm(x)<\infty \tag{1.1}$$

with some $a\in(l_{-},l_{+})$. One can regard dm as a distribution of weight and in this case m works as a string. On the other hand, one can associate a generalized diffusion process with generator L

$$L=\frac{d}{dm}\frac{d}{dx}$$

if we impose a suitable boundary condition if necessary. The condition (1.1) is called as entrance condition in 1D diffusion theory developed by W.Feller, so we call m satisfying (1.1) a string of entrance type. For an entrance type m, it is easy to show that for $\lambda\in\mathbf{C}$ an integral equation

$$\varphi(x)=1-\lambda\int_{-\infty}^{x}(x-y)\varphi(y)dm(y)$$

has a unique solution. Introduce a subspace

$$L_{0}^{2}(dm)=\left\{f\in L^{2}(dm);\ \operatorname{supp}f\subset(-\infty,l)\right\},$$

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and for $f \in L_0^2(dm)$ define a generalized Fourier transform by

$$\widehat{f}(\lambda) = \int_{-\infty}^{l} f(x)\varphi_\lambda(x)dm(x).$$

Krein's spectral theory implies there exists a measure σ on $[0,\infty)$ satisfying

$$\int_{-\infty}^{l} |f(x)|^2 \, dm(x) = \int_0^\infty \left|\widehat{f}(\xi)\right|^2 d\sigma(\xi) \quad \text{for any } f \in L_0^2(dm). \tag{1.2}$$

σ is called a spectral measure for the string m. The non-uniqueness of such σ occurs if and only if

$$l_+ + m\left(l_+\right) < \infty. \tag{1.3}$$

The number $l \,(\geq l_+)$ possesses its meaning only when (1.3) is satisfied, and in this case there exists a σ satisfying (1.2) with the boundary condition

$$f\left(l_+\right) + \left(l - l_+\right) f^+\left(l_+\right) = 0$$

at l_+. Here f^+ is the derivative from the right hand side. If $l = \infty$, this should be interpreted as

$$f^+\left(l_+\right) = 0.$$

At the left boundary l_- no boundary condition is necessary if $l_- = -\infty$, and if $l_- > -\infty$ we impose the reflective boundary condition, namely

$$f^-\left(l_-\right) = 0 \quad \text{the derivative from left.}$$

Generally, for a string m of entrance type it is known that for $\lambda < 0$ there exists uniquely f such that

$$\begin{cases} -Lf = \lambda f, \quad f > 0, \quad f^+ \leq 0, \quad f\left(l-\right) = 0 \\ f(x)\varphi_\lambda^+(x) - f^+(x)\varphi_\lambda(x) = 1. \end{cases}$$

This unique f is denoted by f_λ and contains information of the boundary condition we are imposing on $-L$ at the right boundary l_+, and f_λ can be represented by φ_λ as

$$f_\lambda(x) = \varphi_\lambda(x)\int_x^l \frac{dy}{\varphi_\lambda(y)^2}. \tag{1.4}$$

$\varphi_\lambda(x)$ is an entire function of minimal exponential type as a function of λ and the zeroes of $\varphi_\lambda(x)$ coincide with the eigenvalues of $-L$ defined as a self-adjoint operator on $L^2(dm, (-\infty, x])$ with the Dirichlet boundary condition at x, which means that $\varphi_\lambda(x)$ has simple zeroes only on $(0,\infty)$. The Green function g_λ for $-L$ on $L^2(dm)$ is given by

$$g_\lambda(x,y) = g_\lambda(y,x) = f_\lambda(y)$$

for $x \leq y$. The relationship between σ and g_λ is described by an identity

$$\int_{-\infty}^{l}\int_{-\infty}^{l} g_\lambda(x,y)f(x)\overline{f(y)}dm(x)dm(y) = \int_0^\infty \frac{\left|\widehat{f}(\xi)\right|^2}{\xi-\lambda}\sigma(d\xi)$$

for any $f \in L^2(dm)$, and

$$g_\lambda(x,y) = \int_0^\infty \frac{\varphi_\xi(x)\varphi_\xi(y)}{\xi-\lambda}d\sigma\left(\xi\right),$$

through which σ is determined uniquely from the string m. Distinct ms may give an equal σ, namely for $a \in \mathbf{R}$ a new string

$$m_a\left(x\right) = m\left(x+a\right)$$

defines the same σ, because

$$\varphi_\lambda^a(x) = \varphi_\lambda(x+a), \quad f_\lambda^a(x) = f_\lambda(x+a),$$

hence

$$g_\lambda^a(x,x) = \varphi_\lambda^a(x)f_\lambda^a(x) = g_\lambda(x+a,x+a) = \int_0^\infty \frac{\varphi_\xi(x+a)^2}{\xi-\lambda}d\sigma\left(\xi\right).$$

On the other hand

$$g_\lambda^a(x,x) = \int_0^\infty \frac{\varphi_\xi^a(x)\varphi_\xi^a(y)}{\xi-\lambda}d\sigma_a\left(\xi\right) = \int_0^\infty \frac{\varphi_\xi(x+a)^2}{\xi-\lambda}d\sigma_a\left(\xi\right),$$

hence an identity

$$\sigma_a\left(\xi\right) = \sigma\left(\xi\right)$$

should be held. Conversely we have

Theorem 1 (Kotani[2, 3])**.** *If two strings m_1 and m_2 of $\mathcal{E}$ have the same spectral measure σ, then $m_1(x+c) = m_2(x)$ for an $c \in \mathbf{R}$.*

If we hope to obtain the continuity of the correspondence between m and σ, we have to keep the non-uniqueness in mind. Namely, for m of $\mathcal{E}$ a sequence $\{m_n\}_{n\geq 1}$ of $\mathcal{E}$ defined by

$$m_n\left(x\right) = m\left(x-n\right)$$

converges to the trivial function 0 as $n \to \infty$. However the associated σs are independent of n. Therefore we can not expect the continuity unless we change the definition of the convergence or impose a condition on m so that the uniqueness holds. In this paper we take the latter way. Set

$$M(x) = \int_{-\infty}^{x} (x-y)\, dm(y) = \int_{-\infty}^{x} m(y)dy.$$

Then, the condition (1.1) is equivalent to

$$M(x) < \infty$$

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for $x < l$. Using a convention

$$[-\infty, a) = (-\infty, a)\,,\ (a, \infty] = (a, \infty) \text{ and so on,}$$

we can see M is a non-decreasing convex function on $(-\infty, \infty)$ satisfying

$$\begin{cases} M(x) = 0 \text{ on } (-\infty, l_-], \\ \text{continuous and strictly increasing on } [l_-, l), \\ M(x) = \infty \text{ on } (l, \infty)\,. \end{cases}$$

For a fixed positive number c, we assume

$$0 \in (l_-, l] \text{ and } M(l) \geq c, \tag{1.5}$$

and normalize such an m by

$$M(0) = c. \tag{1.6}$$

Denote by $\mathcal{E}^{(c)}$ the set of all entrance type strings satisfying (1.5), (1.6) and set

$$\mathcal{E} = \bigcup_{c>0} \mathcal{E}^{(c)}.$$

In this definition of $\mathcal{E}$ among functions satisfying (1.1) any function m defined by

$$m(x) = \begin{cases} 0 & \text{on } (-\infty, l) \\ \infty & \text{on } (l, \infty) \end{cases} \tag{1.7}$$

for some $l \leq \infty$ is excluded from $\mathcal{E}$. The uniqueness of the correspondence between m and σ holds under this normalization. Set

$$\mathcal{S} = \text{the set of all spectral measures for strings of } \mathcal{E}.$$

Any suitable characterization of $\mathcal{S}$ is not known yet, however, any measure on $[0, \infty)$ with polynomial growth at ∞ belongs to $\mathcal{S}$.

2. Splitting $\mathcal{E}$ into $\mathcal{M}^+$ and $\mathcal{M}^-$

In this section we split any element m of $\mathcal{E}$ into $m_- = m|_{(-\infty,0)}$ and

$$m_+(x) = m(x) - m(0-).$$

$\mathcal{M}^-$ denotes the image of the map $m \in \mathcal{E} \rightarrow m_-$, hence $\mathcal{M}^-$ is the set of all non-decreasing functions m on $(-\infty, 0)$ satisfying

$$\int_{-\infty}^{0} (-x)\, dm(x) < \infty, \quad m(-\infty) = 0.$$

Similarly $\mathcal{M}^+$ denotes the image of the map $m \in \mathcal{E} \rightarrow m_+$. In this case $\mathcal{M}^+$ coincides with the set of Krein's strings on $[0, \infty)$. $m \in \mathcal{E}$ can be reconstructed from $m_\pm$ in a natural manner. From now on we have to consider the convergence of sequences of elements of $\mathcal{E}$ and for this purpose it is convenient to employ this splitting.

Here we clarify the convergence of sequences of monotone functions taking value ∞. For non-negative and non-decreasing function m which may take ∞, set

$$\widehat{m}\,(x) = \frac{2}{\pi} \tan^{-1} m(x), \qquad x \in \boldsymbol{R}.$$

Then

$$\widehat{m}\,(x) \in [0, 1]$$

and right continuous non-decreasing function satisfying

$$0 \leq \widehat{m}\,(-\infty) \leq \widehat{m}\,(x) \leq \widehat{m}\,(l-) \leq \widehat{m}\,(l) = 1,$$

if $l < \infty$. We say a sequence of non-negative and non-decreasing functions m_n converges to m as $n \to \infty$

$$\widehat{m_n}\,(x) \to \widehat{m}\,(x) \tag{2.1}$$

at any point of continuity of $\widehat{m}\,(x)$.

Lemma 1. *Suppose $m_n \in \mathcal{E}$ converges to $m \in \mathcal{E}$ as $n \to \infty$. Then it holds that*

$$\varliminf_{n\to\infty} l_n \geq l.$$

Proof. Let $x < l$ be a point of continuity for $\widehat{m}$. Then

$$\widehat{m_n}\,(x) \to \widehat{m}\,(x) < 1,$$

hence

$$\widehat{m_n}\,(x) < 1$$

for every sufficiently large n, which implies $x < l_n$ and completes the proof. □

Let us recall several facts from Krein [1] and Kasahara [6]. Denoting $\varphi_\lambda^{(0)}\,(x)$, $\psi_\lambda^{(0)}\,(x)$ the solutions to

$$\varphi_\lambda^{(0)}\,(x) = 1 - \lambda \int_0^x (x-y)\,\varphi_\lambda^{(0)}\,(y)\,dm(y), \quad \psi_\lambda^{(0)}\,(x) = 1 - \lambda \int_0^x (x-y)\,\psi_\lambda^{(0)}\,(y)\,dm(y),$$

the characteristic function for m is defined by

$$h^{(0)}\,(\lambda) = \lim_{x\to l} \frac{\psi_\lambda^{(0)}\,(x)}{\varphi_\lambda^{(0)}\,(x)} = \int_0^l \varphi_\lambda^{(0)}\,(x)^{-2}\,dx.$$

If we include m_∞ which is identically ∞ on $[0, \infty)$ in $\mathcal{M}^+$, then $\mathcal{M}^+$ is compact under the topology (2.1). It should be noted that for $m_\infty \in \mathcal{M}^+$ the characteristic function is given by

$$h^{(0)}\,(\lambda, m_\infty) = 0.$$

Let $m_0 \in \mathcal{M}^+$ be the element which vanishes identically on $[0, \infty)$. Then its characteristic function is

$$h^{(0)}\,(\lambda, m_0) = \infty.$$

Let $\mathcal{H}$ be the set of all holomorphic functions on $\boldsymbol{C}\backslash[0,\infty)$ whose element h is represented by

$$h(\lambda) = a + \int_0^\infty \frac{1}{\xi - \lambda} d\sigma(\xi)$$

with $a \geq 0$ and a measure on $[0, \infty)$

$$\int_0^\infty \frac{1}{\xi + 1} d\sigma(\xi) < \infty.$$

We add identically ∞ function to $\mathcal{H}$. Krein[1] proved that the characteristic function yields a bijection between $\mathcal{M}^+$ and $\mathcal{H}$. Kasahara[6] obtained the continuity of this bijection by defining the convergence of sequences of elements of $\mathcal{H}$ by

$$h_n(\lambda) \to h(\lambda)$$

for every $\lambda < 0$.

3. Preliminary estimates

In this section we prepare several estimates for φ_λ. φ_λ is defined as

$$\varphi_\lambda(x) = \sum_{n=0}^\infty (-\lambda)^n \phi_n(x), \tag{3.1}$$

where $\{\phi_n\}_{n\geq 0}$ are

$$\phi_n(x) = \int_{-\infty}^x (x - y)\, \phi_{n-1}(y) dm(y), \qquad \phi_0(x) = 1.$$

Set

$$M(x) = \int_{-\infty}^x (x-y)\, dm(y) = \int_{-\infty}^x m(y) dy.$$

Then, the convergence of the above series can be seen by

Lemma 2. *For any $k \geq 0$ it holds that*

$$\phi_k(x) \leq \frac{M(x)^k}{k!}. \tag{3.2}$$

φ_λ is given by an absolute convergent series (3.1) and satisfies

$$\left| \frac{\partial^k \varphi_\lambda}{\partial \lambda^k}(x) \right| \leq M(x)^k \exp\left(|\lambda|\, M(x)\right) \quad \text{for } k \geq 0.$$

Proof. Observe

$$\phi_1(x) = \int_{-\infty}^x (x-y)\, dm(y) = M(x).$$

Assuming (3.2) for some k, we have

$$\phi_{k+1}(x) \leq \frac{1}{k!} \int_{-\infty}^x (x-y)\, M(y)^k dm(y)$$

$$= \frac{1}{k!}\int_{-\infty}^{x}\left(M(y) - k\,(x-y)\,M'(y)\right)M(y)^{k-1}M'(y)dy$$
$$\leq \frac{1}{k!}\int_{-\infty}^{x}M'(y)M(y)^{k}dy = \frac{M(x)^{k+1}}{(k+1)!}.$$

On the other hand

$$\frac{\partial^k\varphi_\lambda}{\partial\lambda^k}(x) = (-1)^k\sum_{j=k}^{\infty}\frac{j!}{(j-k)!}\phi_j(x)\,(-\lambda)^{j-k},$$

is valid, hence we obtain

$$\left|\frac{\partial^k\varphi_\lambda}{\partial\lambda^k}(x)\right| \leq \sum_{j=k}^{\infty}\frac{j!}{(j-k)!}\frac{1}{j!}M(x)^j\,|\lambda|^{j-k} = M(x)^k\exp\left(|\lambda|\,M(x)\right). \qquad \square$$

Since f_λ is represented by φ_λ through (1.4), to have estimates for f_λ we need to know estimates of $\varphi_\lambda(x)^{-2}$. A better way to investigate $\varphi_\lambda(x)^{-2}$ is to use probabilistic methods. Recall that $\varphi_\lambda(a)$ has simple zeroes $\{\mu_n\}_{n\geq 1}$ for each fixed a as a function of λ which are eigenvalues of $-L$ on $(-\infty, a]$ with Dirichlet boundary condition at $x = a$. Since the Green function for this operator is

$$a - (x \vee y),$$

we see

$$\sum_{n=1}^{\infty}\mu_n^{-1} = \operatorname{tr}(-L)^{-1} = \int_{-\infty}^{a}(a-x)\,dm(x) = M(a) < \infty.$$

Let $\{X_n\}_{n\geq 1}$ be independent random variables each of which has an exponential distribution of mean μ_n^{-1}. Then, an identity

$$E\exp\left(\lambda\sum_{n=1}^{\infty}X_n\right) = \prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\mu_n}\right)^{-1} = \varphi_\lambda(a)^{-1}$$

holds. Therefore, letting $\left\{\widetilde{X}_n\right\}_{n\geq 1}$ be independent copies of $\{X_n\}_{n\geq 1}$ and setting

$$Y_n = X_n + \widetilde{X}_n, \quad X = \sum_{n=1}^{\infty}Y_n,$$

we have

$$\varphi_\lambda(a)^{-2} = E\exp(\lambda X). \tag{3.3}$$

Lemma 3. *For each integer $k \geq 0$ we have*

$$\left(\sum_{j=1}^{\infty}\frac{1}{\mu_j}\right)^k \leq EX^k \leq (k+1)!\left(\sum_{j=1}^{\infty}\frac{1}{\mu_j}\right)^k.$$

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Proof. We start from the case that the number of $\{\mu_n\}_{n\geq 1}$ is finite N, which is grater than k. Then

$$E\left(X^k\right) = E\left(\sum_{n=1}^{N} Y_n\right)^k = \sum_{\substack{k_1+k_2+\cdots+k_N=k \\ k_j\geq 0 \text{ for any } j}} \frac{k!}{k_1!k_2!\cdots k_N!} EY_1^{k_1} EY_2^{k_2}\cdots EY_N^{k_N}$$

holds. From

$$Ee^{\lambda Y_j} = \frac{\mu_j^2}{\left(\mu_j-\lambda\right)^2}$$

we have

$$EY_j^{k_j} = \frac{(k_j+1)!\mu_j^2}{\mu_j^{k_j+2}} = \frac{(k_j+1)!}{\mu_j^{k_j}},$$

hence

$$E\left(X^k\right) = \sum_{\substack{k_1+k_2+\cdots+k_N=k \\ k_j\geq 0 \text{ for any } j}} \frac{k!}{k_1!k_2!\cdots k_N!} \frac{(k_1+1)!\,(k_2+1)!\cdots(k_N+1)!}{\mu_1^{k_1}\mu_2^{k_2}\cdots\mu_N^{k_N}}$$

From

$$\max_{\substack{k_1+k_2+\cdots+k_N=k \\ k_j\geq 0 \text{ for any } j}} (k_1+1)!\,(k_2+1)!\cdots(k_N+1)! = (k+1)!$$

it follows that

$$E\left(X^k\right) \leq (k+1)! \sum_{\substack{k_1+k_2+\cdots+k_N=k \\ k_j\geq 0 \text{ for any } j}} \frac{k!}{k_1!k_2!\cdots k_N!} \frac{1}{\mu_1^{k_1}\mu_2^{k_2}\cdots\mu_N^{k_N}} = (k+1)!\left(\sum_{j=1}^{N}\frac{1}{\mu_j}\right)^k.$$

Letting $N\to\infty$, we have the upper estimate. The lower inequality can be proved similarly. $\square$

This lemma yields

Lemma 4. *For each integer $k\geq 0$ and $\lambda<0$ inequalities below are valid*

$$\begin{cases} \dfrac{\partial^k}{\partial\lambda^k}\varphi_\lambda(a)^{-2} \leq (k+1)!\left(\int_{-\infty}^{a}\varphi_\lambda(x)^2\,dm(x)\int_x^a\varphi_\lambda(y)^{-2}\,dy\right)^k\varphi_\lambda(a)^{-2} \\ \dfrac{\partial^k}{\partial\lambda^k}\varphi_\lambda(a)^{-2} \geq \left(\int_{-\infty}^{a}\varphi_\lambda(x)^2\,dm(x)\int_x^a\varphi_\lambda(y)^{-2}\,dy\right)^k\varphi_\lambda(a)^{-2}. \end{cases}$$

Proof. First note

$$\frac{\partial^k}{\partial\lambda^k}\varphi_\lambda(a)^{-2} = \varphi_\lambda(a)^{-2}\frac{\partial^k}{\partial\mu^k}\left(\varphi_{\lambda+\mu}(a)^{-2}\varphi_\lambda(a)^2\right)\Big|_{\mu=0}$$

and for each fixed $\lambda<0$

$$\frac{\varphi_\lambda(a)}{\varphi_{\lambda+\mu}(a)}$$

can be regarded as the Laplace transform of an infinite sum of independent random variables each of which has an exponential distribution with mean $(\mu_n - \lambda)^{-1}$. Therefore, Lemma 3 implies

$$\frac{\partial^k}{\partial \mu^k}\left(\varphi_{\lambda+\mu}(a)^{-2}\varphi_\lambda(a)^2\right)\Big|_{\mu=0} \leq (k+1)!\left(\sum_{j=1}^{\infty}\frac{1}{\mu_j-\lambda}\right)^k.$$

If we denote the Green operator for L on $(-\infty, a]$ with Dirichlet boundary condition at a by G_λ, then

$$G_\lambda(x,y) = \varphi_\lambda(y)\varphi_\lambda(x)\int_x^a \varphi_\lambda(z)^{-2}\,dz \quad \text{for } x \geq y$$

hence

$$\sum_{j=1}^{\infty}\frac{1}{\mu_j-\lambda} = \mathrm{tr}G_\lambda = \int_{-\infty}^a \varphi_\lambda(x)^2\,dm(x)\int_x^a \varphi_\lambda(y)^{-2}\,dy$$

holds, and we have the inequality in the statement. The second inequality can be obtained similarly. □

The right hand side of the inequalities in Lemma 4 can be estimated further.

Lemma 5. *For $\lambda < 0$ the following inequalities are valid.*

(1) $\dfrac{\partial^k}{\partial \lambda^k}\varphi_\lambda(a)^{-2} \leq (k+1)!\left\{M(a) \wedge \left(\dfrac{\log \varphi_\lambda(a)}{-\lambda}\right)\right\}^k \varphi_\lambda(a)^{-2}$

(2) $\dfrac{\partial^k}{\partial \lambda^k}\varphi_\lambda(a)^{-2} \geq M(a)^k\varphi_\lambda(a)^{-2-k}$

Proof. The first inequality of (1) and the inequality of (2) follow from the monotonicity of $\varphi_\lambda(z)$, namely we have

$$\int_y^a \varphi_\lambda(z)^{-2}\,dz \leq \varphi_\lambda(y)^{-2}(a-y), \quad \int_y^a \varphi_\lambda(z)^{-2}\,dz \geq \varphi_\lambda(a)^{-2}(a-y),$$

which implies

$$\int_{-\infty}^a \varphi_\lambda(y)^2\,dm(y)\int_y^a \varphi_\lambda(z)^{-2}\,dz \leq \int_{-\infty}^a (a-y)\,dm(y) \leq M(a)$$

and

$$\int_{-\infty}^a \varphi_\lambda(y)^2\,dm(y)\int_y^a \varphi_\lambda(z)^{-2}\,dz \geq \varphi_\lambda(a)^{-2}\int_{-\infty}^a (a-y)\,dm(y) = \varphi_\lambda(a)^{-2}M(a).$$

The second inequality of (1) follows by using the equation satisfied by $\varphi_\lambda(x)$

$$d\varphi_\lambda'(y) = -\lambda\varphi_\lambda(y)\,dm(y),$$

which yields

$$-\lambda\int_{-\infty}^a \varphi_\lambda(y)^2\,dm(y)\int_y^a \varphi_\lambda(z)^{-2}\,dz$$

$$
\begin{aligned}
&= \int_{-\infty}^{a} \varphi_\lambda(y)\, d\varphi_\lambda'(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz \\
&= \varphi_\lambda(y)\, \varphi_\lambda'(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz \Big|_{-\infty}^{a} \\
&\quad - \int_{-\infty}^{a} \varphi_\lambda'(y)^2\, dy \int_y^a \varphi_\lambda(z)^{-2}\, dz + \int_{-\infty}^{a} \frac{\varphi_\lambda'(y)}{\varphi_\lambda(y)} dy.
\end{aligned}
$$

Noting

$$
\varphi_\lambda(y)\, \varphi_\lambda'(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz \underset{y\to-\infty}{\sim} -\lambda m(y)\,(a-y) \underset{y\to-\infty}{\to} 0,
$$

we see

$$
\begin{aligned}
&-\lambda \int_{-\infty}^{a} \varphi_\lambda(y)^2\, dm(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz \\
&\qquad = \log \varphi_\lambda(a) - \int_{-\infty}^{a} \varphi_\lambda'(y)^2\, dy \int_y^x \varphi_\lambda(z)^{-2}\, dz \\
&\qquad \le \log \varphi_\lambda(a),
\end{aligned}
$$

which completes the proof. □

As the last lemma in this section we have

Lemma 6. *For any integer $k \ge 1$ it holds that*

$$
\int_a^l \frac{\partial^k}{\partial \lambda^k} \varphi_\lambda(y)^{-2}\, dy \le \left(\frac{1}{-\lambda}\right)^k \frac{k!}{\varphi_\lambda'(a)}.
$$

Proof. From (1) of Lemma 5 it follows that

$$
\begin{aligned}
\int_a^l \frac{\partial^k}{\partial \lambda^k} \varphi_\lambda(y)^{-2}\, dy &\le \int_a^l \left(\int_{-\infty}^{x} \varphi_\lambda(y)^2\, dm(y) \int_y^x \varphi_\lambda(z)^{-2}\, dz \right)^k \varphi_\lambda(x)^{-2}\, dx \\
&\le \left(\frac{1}{-\lambda}\right)^k \int_a^l (\log \varphi_\lambda(x))^k\, \varphi_\lambda(x)^{-2}\, dx
\end{aligned}
$$

Therefore, noting the monotonicity of $\varphi_\lambda(x), \varphi_\lambda'(x)$ and $\varphi_\lambda(a) \ge 1$, we see

$$
\begin{aligned}
&\left(\frac{1}{-\lambda}\right)^k \int_a^l (\log \varphi_\lambda(x))^k\, \varphi_\lambda(x)^{-2}\, dx \\
&\quad = \left(\frac{1}{-\lambda}\right)^k \int_{\varphi_\lambda(a)}^{\varphi_\lambda(l)} (\log z)^k \frac{1}{z^2 \varphi_\lambda'\left(\varphi_\lambda^{-1}(z)\right)} dz \\
&\quad \le \left(\frac{1}{-\lambda}\right)^k \frac{1}{\varphi_\lambda'\left(\varphi_\lambda^{-1}(\varphi_\lambda(a))\right)} \int_{\varphi_\lambda(a)}^{\varphi_\lambda(l)} (\log z)^k \frac{dz}{z^2} \le \left(\frac{1}{-\lambda}\right)^k \frac{k!}{\varphi_\lambda'(a)}.
\end{aligned}
$$

□

4. Characteristic function

Set

$$\mathcal{E}_k^{(c)} = \left\{ m \in \mathcal{E}^{(c)};\ \int_{-\infty}^{a} M(x)^k \, dx < \infty \ \text{ for } \exists a \in (l_-, l_+) \right\}, \quad \mathcal{E}_k = \bigcup_{c>0} \mathcal{E}_k^{(c)},$$

and

$$\mathcal{S}_k = \left\{ \sigma;\ \int_0^\infty \frac{\sigma(d\xi)}{\xi^{k+1}+1} < \infty \right\}.$$

It is known that $\mathcal{S}_k \subset \mathcal{S}$ in Kotani[2], and Theorem7 of Kotani[4] states that $m \in \mathcal{E}_k$ if and only if $\sigma \in \mathcal{S}_k$ for $k \geq 0$. For $m \in \mathcal{E}_k$ Lemma 5 and Lemma 6 imply

$$\begin{aligned}
\int_{-\infty}^{l} \frac{\partial^k \varphi_\lambda(y)^{-2}}{\partial \lambda^k} dy &= \int_{-\infty}^{0} \frac{\partial^k \varphi_\lambda(y)^{-2}}{\partial \lambda^k} dy + \int_0^l \frac{\partial^k \varphi_\lambda(y)^{-2}}{\partial \lambda^k} dy \\
&\leq (k+1)! \int_{-\infty}^{0} M(y)^k \varphi_\lambda(y)^{-2} \, dy + \left(\frac{1}{-\lambda} \right)^k \frac{k!}{\varphi_\lambda'(0)} \\
&\leq (k+1)! \int_{-\infty}^{0} M(y)^k dy + \left(\frac{1}{-\lambda} \right)^k \frac{k!}{\varphi_\lambda'(0)}.
\end{aligned}$$

The first term is finite and the second term is finite as well if $\varphi_\lambda'(0) > 0$. If $\varphi_\lambda'(0) = 0$, then $m(x) = 0$ identically on $(-\infty, 0)$. In this case, for $y > 0$

$$\varphi_\lambda(y) = \varphi_\lambda^{(0)}(y)$$

and

$$\int_{-\infty}^{l} \frac{\partial^k \varphi_\lambda(y)^{-2}}{\partial \lambda^k} dy = \frac{d^k}{d\lambda^k} h^{(0)}(\lambda),$$

which is finite even when $m(x) = 0$ identically on $(0, \infty)$, since in this case $\varphi_\lambda^{(0)}(y) = 1$ identically. Define

$$h(\lambda) = \int_{-\infty}^{l} \frac{\partial^k \varphi_\lambda(y)^{-2}}{\partial \lambda^k} dy$$

for $m \in \mathcal{E}_k$. If it is necessary, we denote $h(\lambda)$ by $h(\lambda, m)$. To show that h is a characteristics for $m \in \mathcal{E}_k$ we have to connect h to the spectral measure σ of m.

Lemma 7. *For $m \in \mathcal{E}_k$ and for $\lambda < 0$ we have*

$$\lim_{x \to -\infty} \frac{\partial^l}{\partial \lambda^l} \left(f_\lambda(x) \varphi_\lambda(x) \right) = \frac{d^{l-k} h(\lambda)}{d\lambda^{l-k}} \quad \text{for any } l \geq k. \tag{4.1}$$

Proof. We prove the lemma only for $l = k$, since the proof for $l > k$ is quite similar. From

$$\varphi_\lambda(x)^2 = \sum_{k=0}^{\infty} (-\lambda)^k \sum_{i=0}^{k} \phi_i(x) \phi_{k-i}(x)$$

it follows that

$$\frac{\partial^j}{\partial\lambda^j}\varphi_\lambda(x)^2 = \sum_{k=j}^{\infty}\frac{k!}{(k-j)!}(-\lambda)^{k-j}\sum_{i=0}^{k}\phi_i(x)\phi_{k-i}(x),$$

and, from Lemma 2

$$\begin{aligned}\left|\frac{\partial^j}{\partial\lambda^j}\varphi_\lambda(x)^2\right| &\le \sum_{k=j}^{\infty}\frac{k!}{(k-j)!}|\lambda|^{k-j}\sum_{i=0}^{k}\phi_i(x)\phi_{k-i}(x)\\ &\le \sum_{k=j}^{\infty}\frac{k!}{(k-j)!}|\lambda|^{k-j}\sum_{i=0}^{k}\frac{M(x)^k}{i!\,(k-i)!}\\ &= \sum_{k=j}^{\infty}|\lambda|^{k-j}\frac{(2M(x))^k}{(k-j)!} = (2M(x))^j e^{2M(x)|\lambda|}\end{aligned}$$

is valid. Moreover Lemma 4 and Lemma 5 imply

$$\left|\frac{\partial^i\varphi_\lambda(y)^{-2}}{\partial\lambda^i}\right| \le (i+1)!M(y)^i\varphi_\lambda(y)^{-2} \le (i+1)!M(y)^i,$$

which shows

$$\begin{aligned}\left|\frac{\partial^i}{\partial\lambda^i}\int_x^l\frac{dy}{\varphi_\lambda(y)^2}\right| &\le \int_b^l\left|\frac{\partial^i\varphi_\lambda(y)^{-2}}{\partial\lambda^i}\right|dy + \int_x^b\left|\frac{\partial^i\varphi_\lambda(y)^{-2}}{\partial\lambda^i}\right|dy\\ &\le \int_b^l\left|\frac{\partial^i\varphi_\lambda(y)^{-2}}{\partial\lambda^i}\right|dy + (i+1)!\int_x^b M(y)^i\,dy.\end{aligned}$$

On the other hand, we have from the identity (1.4)

$$\begin{aligned}&\frac{\partial^k}{\partial\lambda^k}\left(f_\lambda(x)\varphi_\lambda(x)\right)\\ &= \varphi_\lambda(x)^2\frac{\partial^k}{\partial\lambda^k}\int_x^l\frac{dy}{\varphi_\lambda(y)^2} + \sum_{j=1}^{k}\binom{k}{j}\left(\frac{\partial^j}{\partial\lambda^j}\varphi_\lambda(x)^2\right)\left(\frac{\partial^{k-j}}{\partial\lambda^{k-j}}\int_x^l\frac{dy}{\varphi_\lambda(y)^2}\right).\end{aligned}\tag{4.2}$$

Hence the second term is dominated by

$$\sum_{j=1}^{k}\binom{k}{j}\left|\frac{\partial^j}{\partial\lambda^j}\varphi_\lambda(x)^2\right|\left|\frac{\partial^{k-j}}{\partial\lambda^{k-j}}\int_x^l\frac{dy}{\varphi_\lambda(y)^2}\right| \le I_1 + I_2,$$

where

$$\begin{cases} I_1 = \sum_{j=1}^{k}\binom{k}{j}(2M(x))^j e^{2M(x)|\lambda|}\int_b^l\left|\frac{\partial^{k-j}\varphi_\lambda(y)^{-2}}{\partial\lambda^{k-j}}\right|dy\\ I_2 = \sum_{j=1}^{k}\binom{k}{j}(2M(x))^j e^{2M(x)|\lambda|}(k-j+1)!\int_x^b M(y)^{k-j}\,dy.\end{cases}$$

Clearly I_1 term tends to 0 as $x \to -\infty$. As for I_2, if $1 \le j \le k$, then

$$M(x)^j M(y)^{k-j} \le M(y)^k$$

and the condition

$$\int_{-\infty}^{b} M(y)^k \, dy < \infty$$

imply

$$\lim_{x \to -\infty} M(x)^j \int_x^b M(y)^{k-j} \, dy = 0,$$

which shows that I_2 converges to 0 as $x \to -\infty$. From Lemma 4 and Lemma 5 we know

$$\int_{-\infty}^{l} \left| \frac{\partial^k \varphi_\lambda(y)^{-2}}{\partial \lambda^k} \right| dy < \infty,$$

hence the first term of (4.2) tends to

$$\int_{-\infty}^{l} \frac{\partial^k \varphi_\lambda(y)^{-2}}{\partial \lambda^k} dy$$

as $x \to -\infty$. □

The lemma below connects h with σ.

Lemma 8. *For $m \in \mathcal{E}_k$ and for $\lambda < 0$ we have*

$$h(\lambda) = k! \int_0^\infty \frac{1}{(\xi - \lambda)^{k+1}} d\sigma(\xi).$$

Proof. Lemma 7 for $l = k+1$ and (4.1) imply

$$\frac{dh(\lambda)}{d\lambda} = (k+1)! \lim_{x \to -\infty} \int_0^\infty \frac{\varphi_\xi(x)^2}{(\xi - \lambda)^{k+2}} d\sigma(\xi).$$

Decompose the integral as follows.

$$\int_0^\infty \frac{\varphi_\xi(x)^2}{(\xi - \lambda)^{k+2}} d\sigma(\xi) = \int_0^N \frac{\varphi_\xi(x)^2}{(\xi - \lambda)^{k+2}} d\sigma(\xi) + \int_N^\infty \frac{\varphi_\xi(x)^2}{(\xi - \lambda)^{k+2}} d\sigma(\xi).$$

Since the second term is dominated as

$$\int_N^\infty \frac{\varphi_\xi(x)^2}{(\xi - \lambda)^{k+2}} d\sigma(\xi) \le \frac{1}{N - \lambda} \int_0^\infty \frac{\varphi_\xi(x)^2}{(\xi - \lambda)^{k+1}} d\sigma(\xi) = \frac{1}{N - \lambda} \frac{\partial^k}{\partial \lambda^k} \left(f_\lambda(x) \varphi_\lambda(x) \right),$$

letting $N \to \infty$, we easily see

$$\frac{dh(\lambda)}{d\lambda} = (k+1)! \int_0^\infty \frac{1}{(\xi - \lambda)^{k+2}} d\sigma(\xi),$$

which shows

$$h(\lambda) = \alpha + k! \int_0^\infty \frac{1}{(\xi - \lambda)^{k+1}} d\sigma(\xi)$$

with some constant $\alpha \in \mathbb{R}$. Note here

$$h(\lambda) = \int_{-\infty}^{l} \frac{\partial^k}{\partial \lambda^k} \frac{1}{\varphi_\lambda(y)^2} dy.$$

The identity (3.3) shows that for each $y < l$

$$\varphi_\lambda(y)^{-2} = E \exp(\lambda X)$$

with some positive random variable X. Therefore we have

$$\frac{\partial^k \varphi_\lambda(y)^{-2}}{\partial \lambda^k} = E X^k \exp(\lambda X) \to 0 \quad \text{as} \quad \lambda \to -\infty$$

monotonously, which shows $\alpha = 0$ and completes the proof. □

Now the theorem below is valid.

Theorem 2. *If two strings m_1 and m_2 of $\mathcal{E}_k^{(c)}$ have the same $h(\lambda)$, then $m_1 = m_2$ holds.*

Proof. Under the condition Lemma 8 implies their spectral measures are equal. Then Theorem 1 shows that there exists a constant x_0 such that $m_1(x + x_0) = m_2(x)$, which is equivalent to $M_1(x + x_0) = M_2(x)$. However, the normalization (1.6) implies $x_0 = 0$. □

This theorem justifies for us to call h to be a characteristics for m of $\mathcal{E}_k$.

5. Continuity of the correspondence between $\mathcal{E}_k$ and $\mathcal{S}_k$

In this section we consider the continuity of the correspondence between $\mathcal{E}_k$ and $\mathcal{S}_k$ through h. For this purpose we remark the estimates for h from above and below.

Lemma 9. *For a string m of $\mathcal{E}_k$ and for $\lambda < 0$ the followings hold.*

$$h(\lambda) \geq \int_{-\infty}^{l} M(x)^k \varphi_\lambda(x)^{-2-k} \, dx.$$

Proof. The inequality follows from Lemma 5 easily. □

Let m_n, m be strings of $\mathcal{E}_k$ and define the convergence of m_n to m by

(A) $m_n(x) \to m(x)$ for every point of continuity of m.

(B) $\lim_{x \to -\infty} \sup_{n \geq 1} \int_{-\infty}^{x} M_n(y)^k dy \to 0.$

Then we have

Theorem 3. *Suppose a sequence* $\{m_n\}_{n\geq 1}$ *of* $\mathcal{E}_k^{(c)}$ *converges to* m *of* $\mathcal{E}_k^{(c)}$. *Then their characteristic functions* h_n, h *satisfy*

$$h_n(\lambda) \to h(\lambda)$$

for every $\lambda < 0$.

Proof. Let $\varphi_\lambda^{(n)}(x)$ be the φ_λ for m_n. Then, under the condition it is easy to see that

$$\varphi_\lambda^{(n)}(x) \to \varphi_\lambda(x)$$

for every λ and $x < l$ together with their derivatives with respect to λ compact uniformly, and for $x > l$

$$\varphi_\lambda^{(n)}(x) \to \infty.$$

Since

$$\begin{aligned} h_n(\lambda) &= \int_{-\infty}^{0} \frac{\partial^k \varphi_\lambda^{(n)}(y)^{-2}}{\partial \lambda^k} dy + \int_0^{l_n} \frac{\partial^k \varphi_\lambda^{(n)}(y)^{-2}}{\partial \lambda^k} dy \\ &= \int_{-\infty}^{0} \frac{\partial^k \varphi_\lambda^{(n)}(y)^{-2}}{\partial \lambda^k} dy + \frac{d^k}{d\lambda^k} \int_0^{l_n} \varphi_\lambda^{(n)}(y)^{-2} dy \end{aligned}$$

due to Lemma 5 we easily see the convergence of the first term. To treat the second term we use inequalities

$$\varphi_\lambda^{(n)}(y) \geq 1 - \lambda M_n(y)$$

and

$$M_n(y) = \int_{-\infty}^{y} m_n(z)\, dz = c + \int_0^y m_n(z)\, dz \geq c + m_n(0)\, y.$$

If $c > 0$, then

$$\varliminf_{n\to\infty} m_n(0) \geq m(0-) > 0$$

holds and the dominated convergence theorem shows

$$\lim_{n\to\infty} \int_0^{l_n} \varphi_\lambda^{(n)}(y)^{-2} dy = \int_0^l \varphi_\lambda(y)^{-2} dy, \tag{5.1}$$

which completes the proof. $\square$

A converse statement is possible and the main theorem of this article is as follows.

Theorem 4. *Let* $\{\sigma_n\}_{n\geq 1}$ *be elements of* $\mathcal{S}_k$ *and define*

$$h_n(\lambda) = k! \int_0^\infty \frac{1}{(\xi - \lambda)^{k+1}} d\sigma_n(\xi).$$

Suppose there exists a function h *on* $(-\infty, 0)$ *satisfying*

$$h_n(\lambda) \to h(\lambda)$$

and

$$\lim_{\lambda \to -\infty} h(\lambda) = 0. \tag{5.2}$$

Suppose there exists $m_n \in \mathcal{E}_k^{(c)}$ corresponding to h_n for some $c > 0$. Then, there exists a unique $m \in \mathcal{E}_k^{(c)}$ converging to m, and h becomes the characteristic function of m.

Proof. First we show the second condition (B) of the convergence in $\mathcal{E}_k$. Applying Lemma 9, we have

$$h_n(\lambda) \geq \int_{-\infty}^{l_n} M_n(x)^k \varphi_\lambda^{(n)}(x)^{-2-k}\,dx \geq \int_{-\infty}^{0} M_n(x)^k \varphi_\lambda^{(n)}(x)^{-2-k}\,dx. \tag{5.3}$$

Since

$$M_n(x) \leq c$$

is valid for any $x \leq 0$, due to Lemma 2

$$\varphi_\lambda^{(n)}(x) \leq e^{-\lambda c}$$

holds, which implies

$$h_n(\lambda) \geq e^{(2+k)\lambda c} \int_{-\infty}^{0} M_n(x)^k dx.$$

Suppose for any $n = 1, 2, \cdots$

$$\int_{-\infty}^{0} M_n(x)^k\,dx \leq C.$$

Then

$$\int_{-\infty}^{0} M_n(x)^k\,dx \geq \int_{x}^{0} M_n(y)^k\,dy \geq M_n(x)^k(-x),$$

hence

$$M_n(x) \leq C^{1/k}(-x)^{-1/k} \tag{5.4}$$

for every $n = 1, 2, \cdots$ and $x < 0$, which shows

$$1 \leq \varphi_{\lambda,n}(x) \leq e^{-\lambda M_n(x)} \leq \exp\left(-\lambda C^{1/k}(-x)^{-1/k}\right). \tag{5.5}$$

Now choose any $\epsilon > 0$ such that

$$h(\lambda) < \epsilon$$

holds for some $\lambda < 0$, which is possible due to the condition (5.2). Then, for every sufficiently large n we have

$$h_n(\lambda) < 2\epsilon.$$

On the other hand, (5.5) guarantees the existence of $a < 0$ such that

$$\varphi_{\lambda,n}(a) \leq 2$$

for every $n \geq 1$, which together with (5.3) shows

$$\int_{-\infty}^{a} M_n(x)^k dx \leq 2^{2+k} h_n(\lambda) < 2^{3+k}\epsilon.$$

This concludes the condition (B) is satisfied. From this result and a uniform estimate (5.4) it follows that a set of non-decreasing and convex functions $\{M_n\}_{n\geq 1}$ on $(-\infty, 0]$ is weakly compact. As was pointed out in the second section, the set $\mathcal{M}_+$ is also weakly compact. Therefore, by connecting $m_n^{\pm}$ we see that $\{m_n\}_{n\geq 1}$ on $(-\infty, \infty)$ turns out to be weakly compact in $\mathcal{E}_k^{(c)}$. Consequently Theorem 2 and Theorem 4 complete the proof. □

In order to investigate the case where there does not exist $c > 0$ such that $m_n \in \mathcal{E}_k^{(c)}$ for any $n = 1, 2, \cdots$, we need

Lemma 10. *Suppose $m_n \in \mathcal{E}_k$ satisfy the condition (B). Assume further*
(1) $M_n(0) \to 0$ *as* $n \to \infty$.
(2) $M_n(x) = \infty$ *for any* $x > 0$ *and* $n \geq 1$.
Then, it holds that

$$h_n(\lambda) \to 0$$

as $n \to \infty$ for any $\lambda < 0$.

Proof. From (1) of Lemma 5

$$h_n(\lambda) = \int_{-\infty}^{0} \frac{\partial^k \varphi_\lambda^{(n)}(y)^{-2}}{\partial \lambda^k} dy \leq (k+1)! \int_{-\infty}^{0} M_n(y)^k dy$$

follows. Since m_n satisfy the condition (B), for any $\epsilon > 0$ there exists $a < 0$ such that

$$(k+1)! \int_{-\infty}^{a} M_n(y)^k dy < \epsilon$$

holds for any $n = 1, 2, \cdots$. Therefore

$$h_n(\lambda) \leq \epsilon + (k+1)! \int_{a}^{0} M_n(y)^k dy \leq \epsilon + (k+1)! \, (-a) \, M_n(0)^k$$

is valid and the condition (1) implies the conclusion. □

Theorem 5. *Suppose h_n of Theorem4 satisfy*

$$h_n(\lambda) \to h(\lambda)$$

and

$$\lim_{\lambda \to -\infty} h(\lambda) = 0, \quad h(\lambda) \neq 0.$$

Then, there exist a sequence $\{a_n\}_{n\geq 1}$ in $\mathbb{R}$ with $m_n(\cdot + a_n) \in \mathcal{E}_k$ and an $m \in \mathcal{E}_k$ with characteristic function h such that

$$m_n(\cdot + a_n) \to m$$

holds in $\mathcal{E}_k$.

Proof. First we show

$$\underline{\lim}_{n\to\infty} M_n\left(l_n\right) > 0. \tag{5.6}$$

We assume (5.6) is false. Then, there exists a subsequence $\{n_j\}$ of $\{n\}$ along which

$$\lim_{j\to\infty} M_{n_j}\left(l_{n_j}\right) = 0$$

holds. Set

$$\widetilde{m}_j\left(x\right) = m_{n_j}\left(x + l_{n_j}\right).$$

Then $\widetilde{m}_j$ satisfy the conditions (1),(2) of Lemma 10. Similarly as in the proof of Theorem 3, we see the condition (B) is satisfied by this $\{\widetilde{m}_j\}$. Therefore, Lemma 10 implies

$$h_{n_j}\left(\lambda\right) = \widetilde{h}_j\left(\lambda\right) \to 0$$

for any $\lambda < 0$, which contradicts the condition of the present theorem. Consequently we have (5.6). Fix $c > 0$ such that

$$c < \underline{\lim}_{n\to\infty} M_n\left(l_n\right)$$

and define $a_n \in \mathbb{R}$ by the unique solution of

$$M_n\left(a_n\right) = c,$$

and set

$$\widetilde{m}_n\left(x\right) = m_n\left(x + a_n\right).$$

Then the rest of the proof is quite similar to that of Theorem 3. □

6. Several remarks

Probabilistic interpretation of the results obtained in the last sections is available in Yano[5],Kasahara-Watanabe[7, 8, 9].

6.1. **Transition probability.** For $\sigma \in \mathcal{S}_k$ set

$$h\left(\lambda\right) = \int_0^\infty \frac{1}{\left(\xi - \lambda\right)^{k+1}} d\sigma\left(\xi\right).$$

Lemma 11. *Let $\sigma, \sigma_n \in \mathcal{S}_k$. Then*

$$h_n\left(\lambda\right) \to h\left(\lambda\right) \tag{6.1}$$

holds for any $\lambda < 0$ if and only if the two conditions
(1) $\sigma_n\left(\xi\right) \to \sigma\left(\xi\right)$ at every point of continuity of σ.
(2)

$$\lim_{a\to\infty} \sup_{n\geq 1} \int_a^\infty \xi^{-k-1} d\sigma_n\left(\xi\right)$$

hold.

Proof. Assume (6.1) and note estimates

$$\begin{aligned} h(\lambda) &= \int_0^\infty \frac{1}{(\xi-\lambda)^{k+1}} d\sigma(\xi) \geq \int_a^\infty \left(\frac{\xi}{\xi-\lambda}\right)^{k+1} \xi^{-k-1} d\sigma(\xi) \\ &\geq \left(\frac{a}{a-\lambda}\right)^{k+1} \int_a^\infty \xi^{-k-1} d\sigma(\xi). \end{aligned}$$

Then taking $a = -2\lambda$ we easily see the condition (2) holds. The condition (1) is clearly valid in this case. The converse statement is also clear and the proof is omitted. □

For $\sigma \in \mathcal{S}_k$ set

$$p(t) = \int_0^\infty e^{-t\xi} d\sigma(\xi)$$

for $t > 0$, which is certainly convergent.

Lemma 12. *Let $\sigma, \sigma_n \in \mathcal{S}_k$. Then*

$$h_n(\lambda) \to h(\lambda)$$

holds for any $\lambda < 0$ if and only if the two conditions
(1) $p_n(t) \to p(t)$ for every $t > 0$.
(2)

$$\lim_{\epsilon \to 0} \sup_{n \geq 1} \int_0^\epsilon t^k p_n(t) dt = 0$$

hold.

Proof. Set

$$\rho(t) = \int_0^t s^k e^{-s} ds.$$

Then

$$\int_0^\epsilon t^k p_n(t) dt = \int_0^\infty d\sigma_n(\xi) \int_0^\epsilon t^k e^{-t\xi} dt = \int_0^\infty \xi^{-k-1} \rho(\epsilon\xi) d\sigma_n(\xi)$$

Note

$$\rho(t) \leq \left(\frac{t^{k+1}}{k+1}\right) \wedge k!.$$

Therefore

$$\begin{aligned} \int_0^\epsilon t^k p_n(t) dt &= \int_0^{1/\epsilon} \xi^{-k-1} \rho(\epsilon\xi) d\sigma_n(\xi) + \int_{1/\epsilon}^\infty \xi^{-k-1} \rho(\epsilon\xi) d\sigma_n(\xi) \\ &\leq \frac{\epsilon^{k+1}}{k+1} \int_0^{1/\epsilon} d\sigma_n(\xi) + k! \int_{1/\epsilon}^\infty \xi^{-k-1} d\sigma_n(\xi) \end{aligned}$$

is valid. The rest of the proof is a routine. □

Then Theorem 3 and Theorem 4 can be restated as follows:

Theorem 6. *Let $m, m_n \in \mathcal{E}_k^{(c)}$. Then $m_n \to m$ in $\mathcal{E}_k$ if and only if the conditions below are satisfied.*
(1) *$p_n(t) \to p(t)$ for every $t > 0$.*
(2)

$$\lim_{\epsilon\to 0}\sup_{n\geq 1}\int_0^{\epsilon} t^k p_n(t)dt = 0$$

hold.

6.2. **Fractional moment.** In the previous sections the fundamental identity was

$$\int_{-\infty}^{l} EX^k e^{\lambda X}dx = k!\int_0^{\infty}\frac{1}{(\xi-\lambda)^{k+1}}d\sigma(\xi).$$

However

$$k!\int_0^{\infty}\frac{1}{(\xi-\lambda)^{k+1}}d\sigma(\xi) = \int_0^{\infty}d\sigma(\xi)\int_0^{\infty}e^{-t\xi}e^{t\lambda}t^k dt = \int_0^{\infty}p(t)e^{t\lambda}t^k dt$$

is valid, hence

$$\int_{-\infty}^{l} EX^k e^{\lambda X}dx = \int_0^{\infty}p(t)e^{t\lambda}t^k dt$$

holds. Therefore it can be expected that an identity

$$\int_{-\infty}^{l} Ef(X)\,dx = \int_0^{\infty}p(t)f(t)dt$$

holds for most of functions f. Let

$$f(t) = e^{t\lambda}t^{\alpha}$$

with $\alpha \geq k$. Then

$$\int_{-\infty}^{l} E\left(X^{\alpha}e^{\lambda X}\right)dx = \int_0^{\infty}d\sigma(\xi)\int_0^{\infty}e^{-t\xi}e^{t\lambda}t^{\alpha}dt = \Gamma(\alpha+1)\int_0^{\infty}\frac{1}{(\xi-\lambda)^{\alpha+1}}d\sigma(\xi).$$

It is not clear if one can obtain an estimate like

$$(EX)^{\alpha} \leq E(X^{\alpha}) \leq C_{\alpha}(EX)^{\alpha},$$

although the first inequality is immediate from Jensen's inequality if $\alpha \geq 1$. Moreover in this article we have treated only the case where spectral measures are of polynomial growth, however a similar problem for more rapidly increasing spectral measures remains open.

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