ON SPECTRAL PROPERTIES OF THE MODIFIED CONVOLUTION OPERATOR

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Abstract. We investigated the s-number of the modified convolution operator and obtained the following results

$$c_1 \sup_{Q\in G} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(x)dx \right| \leqslant \|\varphi\|_{M_p^q} \leqslant c_3 \sup_{Q\in F} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(s)ds \right|,$$

where $1 < p < 2 < q < \infty$, $p' = \frac{p}{p-1}$, G be a set of all segments Q from $[0,1]$, F be a set of all compacts from $[0,1]$, $|Q|$ is the measure of a set Q.

1. Introduction

Let $1 \leqslant p < \infty,\ 0 < q \leqslant \infty$. We denote by $\mathfrak{S}_{p,q}$ the space of all compact operators A, acting in the space $L_2[0,1]$ of all 1-periodic functions square integrable on $[0,1]$ for s-numbers such that the following quasinorm is finite

$$\|A\|_{\mathfrak{S}_{p,q}} = \left(\sum_{m=1}^{\infty} s_m^q(A) m^{q/p-1} \right)^{1/q},$$

if $q < \infty$, and

$$\|A\|_{\mathfrak{S}_{p,\infty}} = \sup_m m^{\frac{1}{p}} s_m \quad \texttt{if} \ q = \infty.$$

Recall that the sequence $s_m(A)$ (the $s-$ numbers of operator A) are numerated eigenvalues of the operator $\sqrt{A^*A}$.

We consider the convolution operator

$$(Af)(y) = \int_0^1 K(x-y)f(x)dx$$

2010 *Mathematics Subject Classification.* 42A45; 44A35.

Key words and phrases. Fourier series multipliers; convolution operator; s-number; Lorentz space; Besov space.

acting in $L_2[0,1]$.Given a function $\varphi \in L_1[0,1]$ we consider also the modified convolution operator

$$(A_\varphi f)(y) = \int_0^1 (K\varphi)(x-y)f(x)dx.$$

We say that φ belongs to the space $M_{p_0,q_0}^{p_1,q_1}$, if for $A \in \mathfrak{S}_{p_0,q_0}$ $A_\varphi \in \mathfrak{S}_{p_1,q_1}$ and

$$\|A_\varphi\|_{\mathfrak{S}_{p_1,q_1}} \leqslant c\|A\|_{\mathfrak{S}_{p_0,q_0}},$$

where $c > 0$ depends only on p_0, q_0, p_1, q_1.

This means that the linear operator R_φ defined by the equality $R_\varphi(A) = A_\varphi$ is bounded from $\mathfrak{S}_{p_0,q_0}$ to $\mathfrak{S}_{p_1,q_1}$. Let

$$\|\varphi\|_{M_{p_0,q_0}^{p_1,q_1}} = \|R_\varphi\|_{\mathfrak{S}_{p_0,q_0}\to\mathfrak{S}_{p_1,q_1}}.$$

Given that the eigenvalues of the operator $K * f$ coincide with the Fourier coefficients of the kernel K with respect to the trigonometric system, in the case $p_0 = p_1 = q_0 = q_1 = p$ this problem reduces to the well-known problem of Fourier series multipliers. Let $K \in L_1([0,1])$ and $\{a_m(K)\}_{m\in Z}$ be the sequence of its Fourier coefficients with respect to the trigonometric system $\{e^{2\pi ikx}\}_{k\in Z}$. It is assumed that K is such that $\{a_m(K)\}_{m\in Z} \in l_p$, $1 \leqslant p \leqslant \infty$. Let $T_\varphi = \{a_m(K\varphi)\}_{m\in Z} \in l_p$. The problem is to determine conditions on the function φ, ensuring the boundedness of the operator $T_\varphi \colon l_p \longrightarrow l_p$.

This problem was considered in works of S.B. Stechkin [1], I.I. Hirschman [2], S.L. Edelstein [3], M.Sh. Birman and M.Z. Solomyak [4], G.E. Karadzhov [5] and others.

We obtain sufficient conditions on a multiplier φ ensuring that it belongs to space $M_{p_0,q_0}^{p_1,q_1}$. These conditions are expressed in terms of Lorentz and Besov spaces. We also construct examples showing the sharpness of the obtained constants for corresponding embedding theorems.

2. Main results

Let f be a μ measurable function which takes finite values almost everywhere and let

$$m(\sigma, f) = \mu(\{x : \; x \in [0,1], \;\; |f| > \sigma\})$$

be its distribution function. The function

$$f^*(t) = \inf\{\sigma : m(\sigma, f) \leqslant t\}$$

is a nonincreasing rearrangement of f.

We say that a function f belongs to the Lorentz space $L_{p,q}$ if f is measurable and

$$\|f\|_{L_{p,q}} = \left(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty,$$

for $1 \leqslant q < \infty$ and

$$\|f\|_{L_{p,\infty}} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty,$$

for $q = \infty$.

Theorem 1. *Let* $1 < p_0 < 2 \leqslant p_1, 1 \leqslant q_1 \leqslant q_0 \leqslant \infty, \frac{1}{r} = \frac{1}{p_0} - \frac{1}{p_1}, \frac{1}{s} = \frac{1}{q_1} - \frac{1}{q_0}$ *and* $\varphi \in L_{r,s}[0,1]$. *If* $A \in \mathfrak{S}_{p_0,q_0}$, *then* $A_\varphi \in \mathfrak{S}_{p_1,q_1}$ *and*

$$\|A_\varphi\|_{\mathfrak{S}_{p_1,q_1}} \leqslant c\|\varphi\|_{L_{r,s}}\|A\|_{\mathfrak{S}_{p_0,q_0}},$$

i.e. $L_{r,s}[0,1] \hookrightarrow M^{p_1,q_1}_{p_0,q_0}$.

In the following theorem the cases $p = p_o = q_0, q = p_1 = q_1$ are considered. The upper and the lower estimates of the norm $\|\varphi\|_{M^q_p}$ ($M^q_p := M^{q,q}_{p,p}$) are obtained.

Theorem 2. *Let* $1 < p < 2 < q < \infty,\ p' = \frac{p}{p-1}$. *Let* G *be a set of all segments* Q *from* $[0,1]$, F *be a set of all compacts from* $[0,1]$, *then*

$$c_1 \sup_{Q\in G} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left|\int_Q \varphi(x)dx\right| \leqslant \|\varphi\|_{M^q_p} \leqslant c_3 \sup_{Q\in F} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left|\int_Q \varphi(s)ds\right|,$$

where $|Q|$ *is the measure of a set* Q.

We shall define the class of generalized monotone functions for which the upper and the lower estimates coincide.

We say that function f is a generalized monotone function, if there exists a constant $c > 0$ such that for every $x \in (0,1]$ the inequality

$$|f(x)| \leqslant \frac{c}{x} \left|\int_0^x f(y)dy\right|$$

holds. The class of such functions is denoted by $\mathfrak{N}$.

Corollary 1. *Let* $1 < p < 2 < q < \infty$. *If* $\varphi \in \mathfrak{N}$, *then* $\varphi \in M^q_p$ *if and only if*

$$\sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}} \varphi^*(t) < \infty.$$

Moreover, $\|\varphi\|_{M^q_p} \sim \sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}} \varphi^*(t)$.

In the case parameters p_0, p_1 are both either less or greater then 2, we use the space of smooth functions.

Let $1 \leqslant p < \infty,\ \ \alpha > 0$. We denote by $B^\alpha_{p,q}[0,1]$ the space of all measurable functions such that

$$\|f\|_{B^\alpha_{p,q}} = \left(\sum_{k=0}^{\infty} (2^{\alpha k}\| \triangle_k f\|_p)^q\right)^{\frac{1}{q}} < \infty$$

for $1 \leqslant q < \infty$, and

$$\|f\|_{B^{\alpha}_{p,\infty}} = \sup_k 2^{\alpha k} \| \triangle_k f\|_p < \infty$$

for $q = \infty$. Here $\triangle_k f = \sum\limits_{[2^{k-1}] \leqslant |m| < 2^k} a_m(f) e^{2\pi i m x}$, $\{a_m(f)\}_{m \in N}$ are Fourier coefficients of function f by trigonometric system $\{e^{2\pi i k x}\}_{k \in \mathbb{Z}}$.

This class is called Nikol'skii-Besov space.

Theorem 3. *Let* $1 < p_0 \leqslant p_1 < \infty,\ 2 \notin [p_0, p_1],\ 1 < q_0 \leqslant q_1 \leqslant \infty$,

$$\alpha = \min_{x \in [\frac{1}{p_1},\ \frac{1}{p_0}]} \left| \frac{1}{2} - x \right|,\ \ \frac{1}{r} = \max_{x \in [\frac{1}{p_1}, \frac{1}{p_0}]} \left| \frac{1}{2} - x \right|,\ \ 1 - \frac{1}{s} = \frac{1}{q_0} - \frac{1}{q_1}$$

and $\varphi \in B^{\alpha}_{r,s}[0,1]$.

If $A \in \mathfrak{S}_{p_0,q_0}$*, then* $A_\varphi \in \mathfrak{S}_{p_1,q_1}$ *and*

$$\|A_\varphi\|_{\mathfrak{S}_{p_1,q_1}} \leqslant c \|\varphi\|_{B^{\alpha}_{r,s}} \|A\|_{\mathfrak{S}_{p_0,q_0}},$$

i.e. $B^{\alpha}_{r,s} \hookrightarrow M^{p_1,q_1}_{p_0,q_0}$.

In the case $p_0 = p_1 = q_0 = q_1$, Karadzhov's result (see [5]) follows from Theorem 3:

$$B^{\frac{1}{r}}_{r,1} \hookrightarrow M_p = M^{p,p}_{p,p}, \quad \frac{1}{r} = \left| \frac{1}{p} - \frac{1}{2} \right|.$$

Now consider the case $1 \leqslant q_1 < q_0 \leqslant \infty$.

Theorem 4. *Let* $1 < p_0 < p_1 < \infty,\ \ 1 \leqslant q_1 < q_0 \leqslant \infty,\ \ 2 \notin (p_0, p_1)\ \ \frac{1}{r} - \alpha = \frac{1}{p_0} - \frac{1}{p_1},\ \ \frac{1}{s} = \frac{1}{q_1} - \frac{1}{q_0},\ \ \alpha > \min\limits_{x \in [\frac{1}{p_1}, \frac{1}{p_0}]} |\frac{1}{2} - x|$.

Then $B^{\alpha}_{r,s}[0,1] \hookrightarrow M^{p_1,q_1}_{p_0,q_0}$.

3. Properties of $M^{p_1,q_1}_{p_0,q_0}$ class

To prove the properties of $M^{p_1,q_1}_{p_0,q_0}$ class we need the following lemma. We first define a discrete Lorentz space. l_{pq} is called discrete Lorentz space whose elements are sequences of numbers $\xi = \{\xi_k\}_{k=\infty}^{\infty}$ with the only limit point 0 , such that

$$\|\xi\|_{l_{pq}} = \left(\sum_{m=1}^{\infty} |\xi^*_m|^q m^{\frac{q}{p} - 1} \right)^{\frac{1}{q}}, \qquad 1 \leqslant q < \infty$$

where $\{\xi^*_m\}_{m=1}^{\infty}$ nonincreasing rearrangement of the sequence $\{|\xi_k|\}_{k=\infty}^{\infty}$.

For $q = \infty$,

$$\|\xi\|_{l_{p\infty}} = \sup_m m^{\frac{1}{p}} \xi^*_m.$$

Lemma 1. *(see* [6]*) Let* $1 < r, p_0, p_1 < \infty\ \ 1 \leqslant q_0, q_1, s \leqslant \infty$*. Then*

$$\|a * b\|_{l_{p_1,q_1}} \leqslant c \|b\|_{l_{r,s}} \|a\|_{l_{p_0,q_0}},$$

where $\frac{1}{p_1} + 1 = \frac{1}{r} + \frac{1}{p_0},\ \ \frac{1}{q_1} = \frac{1}{s} + \frac{1}{q_0}$.

Let $\bar{A} = (A_0, A_1)$ where A_0, A_1 are Banach spaces be compatible pair. We define the functional $K(t,a)$ for $t > 0$ and $a \in A_0 + A_1$ by the following formula

$$K(t,a) = \inf_{a=a_0+a_1} \left(\|a_0\|_{A_0} + t\|a_1\|_{A_1}\right).$$

We denote by $\bar{A}_{\theta,q,k}$ the space $\{a \in A_0 + A_1 : \|a\|_{\theta,q,k} = \Phi_{\theta,q}(K(t,a))\}$, where $\Phi_{\theta,q}$ is functional defined on the nonnegative functions φ by formula

$$\Phi_{\theta,q}(\varphi(t)) = \left(\int_0^\infty \left(t^{-\theta}\varphi(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, \quad 1 \leqslant q < \infty.$$

and

$$\Phi_{\theta,\infty}(\varphi(t)) = \sup_{t>0} t^{-\theta}\varphi(t), \quad q = \infty.$$

Let $X_{\alpha_1^0,p_1^0}$ and $X_{\alpha_2^1,p_2^1}$ are the spaces obtained by the method of real interpolation of Banach pairs of spaces (X_0^1, X_1^1), (X_0^2, X_1^2) respectively.

Lemma 2. *(see [7]) Let* $0 < \alpha_i, \beta_i < 1,\ 1 \leqslant p_i, q_i \leqslant \infty,\ i = 0,1,\ \alpha_0 \neq \alpha_1,\ \beta_0 \neq \beta_1$. *If* T *be a bilinear operator:*

$$T : X_{\alpha_0,p_0} \times Y_0 \to Z_{\beta_0,q_0}$$

and

$$T : X_{\alpha_1,p_1} \times Y_1 \to Z_{\beta_1,q_1}$$

then

$$T : X_{\alpha,p} \times Y_{\theta,r} \to Z_{\beta,q}.$$

Here $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\beta = (1-\theta)\beta_0 + \theta\beta_1$, $\frac{1}{p} + \frac{1}{r} > 1$, $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} + (1-\theta)(\frac{1}{q_0} - \frac{1}{p_0}) + \theta(\frac{1}{q_1} - \frac{1}{p_1})_+$, $x_+ = \max(x,0)$

Remark. Since the s-numbers of convolution operator A coincide with the modules of the Fourier coefficients of the kernel K, the problem of estimating of s-numbers of "transformed" operator A_φ can be reduced to the study of the following inequality

(1) $$\|a * b\|_{l_{p_1,q_1}} \leqslant c\|a\|_{l{p_0,q_0}},$$

and we have to decribe the class of those functions φ with Fourier coefficients $b = \{b_m\}_{m\in Z}$, for which inequality (1) holds.

Theorem 5. 1) *Let* $1 \leqslant p_0, p_1 < \infty$, $1 \leqslant q_0, q_1 \leqslant \infty$, $\frac{1}{p_i} + \frac{1}{p_i'} = \frac{1}{q_i} + \frac{1}{q_i'} = 1$, $i = 0,1$. *Then*

$$M_{p_0,q_0}^{p_1,q_1} = M_{p_1',q_1'}^{p_0',q_0'}.$$

2) *Let* $1 < p_0 < r_0 < p_1' < \infty$, $\frac{1}{p_1} + \frac{1}{p_1'} = 1$ $\frac{1}{p_1} - \frac{1}{p_0} = \frac{1}{r_1} - \frac{1}{r_0}$, *then*

$$M_{p_0,q_0}^{p_1,q_1} \hookrightarrow M_{r_0,s}^{r_1,t},$$

where $\frac{1}{t} - \frac{1}{s} = (\frac{1}{q_1} - \frac{1}{q_0})_+$.

Proof. The proof of the first statement follows from Remark and from the fact that $\|(T_\varphi)^*\| = \|T_{\bar\varphi}\|$, where $\bar\varphi$ is complex conjugate of function φ. Now we prove 2).

Let $\varphi \in M^{p_1,q_1}_{p_0,q_0}$, then by 1) it follows that $\varphi \in M^{p_0',q_0'}_{p_1',q_1'}$, and

$$\|A_\varphi\|_{\mathfrak{S}_{p_1,q_1}} \leqslant \|\varphi\|_{M^{p_1,q_1}_{p_0,q_0}} \|A\|_{\mathfrak{S}_{p_0,q_0}}, \quad \forall A \in \mathfrak{S}_{p_0,q_0},$$

$$\|A_\varphi\|_{\mathfrak{S}_{p_0',q_0'}} \leqslant \|\varphi\|_{M^{p_1,q_1}_{p_0,q_0}} \|A\|_{\mathfrak{S}_{p_1',q_1'}}, \quad \forall A \in \mathfrak{S}_{p_1,q_1},$$

where $\frac{1}{p_i} + \frac{1}{p_i'} = \frac{1}{q_i} + \frac{1}{q_i'} = 1$. According to Lemma 1 the operator $T(a,\varphi) = a * b$

$$T\colon l_{p_0,q_0} \times M^{p_1,q_1}_{p_0,q_0} \longrightarrow l_{p_1,q_1}$$

is bounded. Using 1) we have

$$T\colon l_{p_1',q_1'} \times M^{p_1,q_1}_{p_0,q_0} \longrightarrow l_{p_0',q_0'}.$$

Further, applying the theorem on bilinear interpolation (Lemma 2) we find that the operator

$$T\colon l_{r_0,s} \times M^{p_1,q_1}_{p_0,q_0} \longrightarrow l_{r_1,t}$$

is also bounded, i.e. $M^{p_1,q_1}_{p_0,q_0} \hookrightarrow M^{r_1,t}_{r_0,s}$, where

$$\frac{1}{r_1} = \frac{1-\theta}{p_1} + \frac{\theta}{p_0'}, \quad \frac{1}{r_0} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1'}, \quad \frac{1}{t} - \frac{1}{s} = \left(\frac{1}{q_1} - \frac{1}{q_0}\right)_+$$

for every $0 < \theta < 1$. Eliminating θ from this equations, we obtain that

$$\frac{1}{p_1} - \frac{1}{p_0} = \frac{1}{r_1} - \frac{1}{r_0},$$

and the condition $0 < \theta < 1$ implies the condition $1 < p_0 < r_0 < p_1' < \infty$, where $\frac{1}{p_1} + \frac{1}{p_1'} = 1$.

The proof is complete.

By 2), in particular, the following proposition follows.

Let $1 < p < r < p' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ then

$$M_{p,q} \hookrightarrow M_{r,t}$$

where $M_{p,q} = M^{p,q}_{p,q}$ and $q, t \in [1,\infty[$ are any.

4. Proof of main results

For a given pair $\bar A = (A_0, A_1)$ we consider the space $\mathit{\Gamma}(\bar A)$, consisting of all functions f bounded and continuous in the strip

$$S = \{z:\ 0 \leqslant Rez \leqslant 1\}$$

with values in $A_0 + A_1$. Moreover, f are analytic in the open strip

$$S_0 = \{z:\ 0 < Rez < 1\}$$

and such that the mapping $t \to f(j+it)$ $(j=0,1)$ is a continuous function on the real axis with values in A_j $(j=0,1)$ which tends to 0 for $|t| \to \infty$. Clearly that $\Gamma(\bar{A})$ is a vector space. We endow Γ with the norm

$$\|f\|_\Gamma = \max(\sup_t \|f(it)\|_{A_0}, \sup_t \|f(1+it)\|_{A_1}).$$

The space $\bar{A}_{[\theta]},\ 0 \leqslant \theta \leqslant 1$ consists of all elements $a \in A_0 + A_1$, such that $a = f(\theta)$ for some function $f \in \Gamma(\bar{A})$. The norm on $\bar{A}_{[\theta]}$ is equal to

$$\|a\|_{[\theta]} = \inf\{\|f\|_\Gamma :\ f(\theta) = a, f \in \Gamma.\}$$

In order to prove our main result, we need two lemmas in [8].

Lemma 3 (Bilinear interpolation, the complex method, see [8])**.** *Let T be a bilinear operator such that*

$$T\colon X_0 \times Y_0 \longrightarrow Z_0,$$

and

$$T\colon X_1 \times Y_1 \longrightarrow Z_1.$$

Then

$$T\colon X_{[\theta]} \times Y_{[\theta]} \longrightarrow Z_{[\theta]},$$

where $X_{[\theta]}, Y_{[\theta]}, Z_{[\theta]}$ are the spaces obtained by the method of complex interpolation of Banach pairs of spaces $(X_0, X_1, (Y_0, Y_1), (Z_0, Z_1))$ respectively.

Lemma 4. *(bilinear interpolation, the real method, see* [8]*) Let T be a bilinear operator such that*

$$T\colon X_0 \times Y_0 \longrightarrow Z_0,$$

and

$$T\colon X_1 \times Y_1 \longrightarrow Z_1$$

with the norms M_0, M_1 respectively. Then

$$T\colon X_{\theta,t_1} \times Y_{\theta,t_2} \longrightarrow Z_{\theta,s},$$

where $\frac{1}{s} + 1 = \frac{1}{t_1} + \frac{1}{t_2}$. Moreover

$$\|T\| \leqslant c M_0^{1-\theta} M_1^\theta.$$

Proof of Theorem 1. First we prove the inequality:

$$\|a * b\|_{l_{p_1,q_1}} \leqslant c\|\varphi\|_{L_r}\|a\|_{l_{p_0,q_0}}, \tag{2}$$

where $b = \{b_k\}_{k\in\mathbb{Z}}$ are Fourier coefficients of function φ.

If $r \leqslant 2$, inequality (2) follows by Lemma 1 (for $s = r$) and Hardy-Littlewood inequality

$$\|b\|_{l_{r',r}} \leqslant c\|\varphi\|_{L_r}.$$

Now let $2 < r < \infty$. Let $a \in l_2$, $f \sim \sum\limits_{k\in Z} a_k e^{2\pi ikx}$, then by Perseval's equality we get

$$\|a * b\|_{l_2} = \|f\varphi\|_{L_2} \leqslant \|f\|_{L_2}\|\varphi\|_{L_\infty} = \|\varphi\|_{L_\infty}\|a\|_{l_2},$$

i.e. $M_2 = L_\infty$. From Lemma 1, using the Parseval equality we have

$$\|a * b\|_{l_{p_1,q_1}} \leqslant c\|\varphi\|_{L_2}\|a\|_{l_{p_0,q_0}},$$

where $\frac{1}{p_1} + 1 = \frac{1}{2} + \frac{1}{p_0}$, $\frac{1}{p_1} + \frac{1}{2} = \frac{1}{p_0}$, $\frac{1}{q_1} = \frac{1}{q_0} + \frac{1}{2}$.

Thus, for the bilinear operator $T(a, \varphi) = a * b$ we obtain

$$T\colon l_2 \times L_\infty \longrightarrow l_2,$$

$$T\colon l_{p_0,q_0} \times L_2 \longrightarrow l_{p_1,q_1}.$$

Applying the method of complex interpolation (Lemma 3), we obtain inequality (2). Now we shall prove the inequality

$$\|a * b\|_{l_{p_1,q_1}} \leqslant c\|\varphi\|_{L_{r,s}}\|a\|_{l_{p_0,q_0}}, \tag{3}$$

where $\frac{1}{s} = \frac{1}{q_1} - \frac{1}{q_0}$.

Let $q_0 = \infty$ and p_0 be fixed in inequality (2). Taking $\frac{1}{q_1^i} = \frac{1}{r_i}, i = 0, 1$, choose parameters r_0, r_1, p_1^0, p_1^1 such that

$$\frac{1}{p_0} = \frac{1}{p_1^i} + \frac{1}{r_i} \quad i = 0, 1. \tag{4}$$

Then from inequality (2)we have:

$$\|a * b\|_{l_{p_1^0,r_0}} \leqslant c_1\|a\|_{l_{p_0,\infty}}\|\varphi\|_{L_{r_0}},$$

$$\|a * b\|_{l_{p_1^1,r_1}} \leqslant c_2\|a\|_{l_{p_0,\infty}}\|\varphi\|_{L_{r_1}}.$$

Using Marcinkiewicz - Calderón interpolation theorem (see [8]) we get

$$\|a * b\|_{l_{p_1,s}} \leqslant (c_1\|a\|_{l_{p_0,\infty}})^\theta (c_2\|a\|_{l_{p_0,\infty}})^{1-\theta}\|\varphi\|_{L_{r,s}} = c\|a\|_{l_{p_0,\infty}}\|\varphi\|_{L_{r,s}} \tag{5}$$

where $\frac{1}{p_1} = \frac{1-\theta}{p_1^0} + \frac{\theta}{p_1^1}$ $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$ i.e. $\frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{r}$.

Now we apply Lemma 2 with fixed parameters r, s and parameters p_1^i, p_0^i, $i = 0, 1$ satisfing to (4) to the inequality of type (5). We have:

$$(L_{r,s}, L_{r,s})_{\theta,1} \times (l_{p_0^0,\infty}, l_{p_0^1,\infty})_{\theta,q_0} \to (l_{p_1^0,s}, l_{p_1^1,s})_{\theta,q_1}$$

or

$$T : L_{r,s} \times l_{p_0,q_0} \to l_{p_1,q_1}$$

where $\frac{1}{q_1} - \frac{1}{q_0} = \frac{1}{s} - \frac{1}{\infty}$, $\frac{1}{p_1} = \frac{1-\theta}{p_1^0} + \frac{\theta}{p_1^1}$, $\frac{1}{p_0} = \frac{1-\theta}{p_0^0} + \frac{\theta}{p_0^1}$ i.e. $\frac{1}{q_1} = \frac{1}{s} + \frac{1}{q_0}$, $\frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{r}$.

Since the parameters $p_1^i, p_0^i, i = 0, 1$ are arbitrary in inequality (5), it guarantees the arbitrary of the corresponding parameters in inequality (4).

Thus, the following inequality holds:

$$\|a * b\|_{l_{p_1,q_1}} \leqslant c\|a\|_{l_{p_0,q_0}}\|\varphi\|_{L_{r,s}},$$

where $b = \{b_m\}_{m\in Z}$ are Fourier coefficients of function φ and $\frac{1}{r} = \frac{1}{p_0} - \frac{1}{p_1}, \frac{1}{s} = \frac{1}{q_1} - \frac{1}{q_0}$. According to the Remark this inequality is equivalent to the statement of Theorem 1.

Proof of Theorem 2. Let $\varphi \in M_p^q$ and Q be an arbitrary segment in $[0,1]$,

$$f_0(x) = \begin{cases} 1, x \in Q \\ 0, x \notin Q \end{cases}$$

Note that by Boas theorem [9] (see also [10]) we get

$$\|\widehat{f_0}(k)\|_{l_p} \sim \|f_0\|_{L_{p',p}} = \left(\int_0^1 (t^{\frac{1}{p'}} f_0^*(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} = |Q|^{\frac{1}{p'}}. \tag{6}$$

Applying Theorem 5 from [11] and using (6) we obtain:

$$\|\varphi\|_{M_p^q} = \sup_{f \neq 0} \frac{\|\widehat{f\varphi}\|_{l_q}}{\|\widehat{f}\|_{l_p}} \geqslant \frac{\|\widehat{f_0\varphi}\|_{l_q}}{\|\widehat{f_0}\|_{l_p}} \geqslant$$

$$\geqslant \frac{c}{|Q|^{\frac{1}{p'}}} \int_0^1 \left(t^{\frac{1}{q'}} \left(\sup_{|W| \geqslant t, W \in G} \frac{1}{|W|} \left| \int_W f_0(x)\varphi(x)dx \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \geqslant$$

$$\geqslant \frac{c}{|Q|^{\frac{1}{p'}}} \sup_{t>0} t^{\frac{1}{q'}} \left(\sup_{|W| \geqslant t, W \in G} \frac{1}{|W|} \left| \int_{W \cap Q} \varphi(x)dx \right| \right) \geqslant$$

$$\geqslant \frac{c_1}{|Q|^{\frac{1}{p'}}} |Q|^{\frac{1}{q'}-1} \left| \int_Q \varphi(x)dx \right| = \frac{c_1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(x)dx \right|.$$

Since the interval Q is arbitrary, we get

$$\|\varphi\|_{M_p^q} \geqslant c_1 \sup_{Q \in G} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(x)dx \right|,$$

where constants c and c_1 depend only on parameters p and q.

The proof of obtaining an upper estimate follows from Theorem 1 and the embedding $l_{p,p} \hookrightarrow l_{p,q}$, for $p < q$.

Indeed, from Theorem 1 it follows

$$L_{r,\infty} \hookrightarrow M_p^q,$$

i. e.

$$\|\varphi\|_{M_p^q} \leqslant c_2 \sup_{t>0} t^{\frac{1}{r}} \varphi^*(t) \leqslant c_3 \sup_{t>0} \frac{1}{t^{\frac{1}{p'}+\frac{1}{q}}} \int_0^t \varphi^*(s)ds =$$

$$= c_3 \sup_{Q \in F} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \int_Q |\varphi(x)|dx \sim \sup_{Q \in F} \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}} \left| \int_Q \varphi(x)dx \right|.$$

Proof of Corollary1. Let Q be an arbitrary compact from F.

From the condition of generalized monotonicity of the function φ we have

$$\frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}}\left|\int\limits_Q \varphi(y)dy\right| \leqslant \frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}}\int\limits_Q \frac{c}{y}\left|\int\limits_0^y \varphi(x)dx\right|dy \leqslant$$

$$\leqslant \frac{c}{|Q|^{\frac{1}{p'}+\frac{1}{q}}}\sup_{A\in G}\frac{1}{|A|^{\frac{1}{p'}+\frac{1}{q}}}\left|\int\limits_A \varphi(x)dx\right|\int\limits_Q \frac{dy}{y^{1-\frac{1}{q}-\frac{1}{p'}}} \leqslant c_1 \sup_{A\in G}\frac{1}{|A|^{\frac{1}{p'}+\frac{1}{q}}}\left|\int\limits_A \varphi(x)dx\right|.$$

Taking into account that $Q\in F$ is arbitrary, we have

$$\sup_{Q\in F}\frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}}\left|\int\limits_Q \varphi(x)dx\right| \leqslant c\sup_{Q\in G}\frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}}\left|\int\limits_Q \varphi(x)dx\right|.$$

Thus, from Theorem 2 we get

$$\|\varphi\|_{M_p^q} \sim \sup_{Q\in F}\frac{1}{|Q|^{\frac{1}{p'}+\frac{1}{q}}}\left|\int\limits_Q \varphi(x)dx\right| \sim \sup_{t>0} t^{\frac{1}{p}-\frac{1}{q}}\varphi^*(t).$$

Proof of Theorem 3. Let $2 < p_0 \leqslant p_1 < \infty$. For a sequence of numbers $a = \{a_m\}_{m\in Z}$ and a function $\varphi \in L_1[0,1]$ we consider the mapping T of the form $T(a,\varphi) = a * b$, where $b = \{b_m\}_{m\in Z}$ is the sequence of Fourier coefficients on trigonometric system of functions φ. This map is bilinear and from Karadzhov theorem ([5]) and Remark it follows that it is bounded from $l_1 \times B^{\frac{1}{2}}_{2,1}$ to l_1.

Since $M_2 = L_\infty$, the mapping

$$T_\varphi \colon l_2 \times L_\infty \longrightarrow l_2$$

is also bounded. By the Theorem on bilinear interpolation (Lemma 4) the operator T is bounded from $l_{p,q} \times (B^{\frac{1}{2}}_{2,1}, L_\infty)_{\theta,1}$ to $l_{p,q}$. In paper ([5]) it is shown that $B^{\frac{1}{r}}_{r,1} \hookrightarrow (B^{\frac{1}{2}}_{2,1}, L_\infty)_{\theta,1}$, where $\frac{1}{r} = \frac{1-\theta}{2}$. Thus, taking into account Theorem 5, we will get

$$T\colon l_{p,q} \times B^{\frac{1}{r}}_{r,1} \longrightarrow l_{p,q}, \tag{7}$$

for $2 < p < \infty,\ 1 \leqslant q \leqslant \infty,\ \frac{1}{r} = \frac{1}{2} - \frac{1}{p}$.

From Minkowski inequality and Parseval equality we get

$$T\colon l_1 \times L_2 \longrightarrow l_2.$$

Further, from Lemma 3 on bilinear interpolation (complex method) taking into account that $(L_2, B^{1/r}_{r,1})_{[\theta]} = B^\alpha_{r,s}$, it follows that T is a bounded mapping from $l_{p_0,q_0} \times B^\alpha_{r,s}$ to l_{p_1,q_1}, where $2 < p_0 \leqslant p_1 < \infty,\ \frac{1}{r} = \frac{1}{2} - \frac{1}{p_1},\ \alpha = \frac{1}{2} - \frac{1}{p_0}$. The arbitrary choice of parameters guarantees the arbitrary of the parameters available in the theorem.

The case $1 < p_0 < p_1 < 2$ follows from the statements proved above and the property $M^{p_1,q_1}_{p_0,q_0} = M^{p_0',q_0'}_{p_1',q_1'}$.

Proof of Theorem 4. Let $1 < p_0 < p_1 \leqslant 2$. Let us consider the bilinear mapping $T(a,\varphi) = a * b$, where $b = \{b_m\}_{m\in Z}$ is the sequence of Fourier coefficients of function φ. The mapping

$$T\colon l_{p_0,q_0} \times L_2 \longrightarrow l_{p_1,q_1}. \tag{8}$$

is bounded according to Theorem 1. Here $\frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{2}$, $\frac{1}{q_1} - \frac{1}{q_0} = \frac{1}{2}$, $1 < q_1 < 2 < q_0$, $1 < p_0 < 2 < p_1$. The result of Theorem 3, in the case $q_0 = q_1 = 1$, $p_0 = p_1 = p$ can be written as

$$T\colon l_{p,1} \times B^{1/t}_{t,1} \longrightarrow l_{p,1}, \quad \frac{1}{t} = \frac{1}{p} - \frac{1}{2}. \tag{9}$$

Applying Lemma 3 on the bilinear interpolation to (8) and (9), and taking into account the properties of the embedding of the spaces $l_{p,q}$ and $B^{\alpha}_{p,q}$ we have:

$$T\colon l_{p_0,1} \times B^{\alpha}_{r,1} \longrightarrow l_{p_1,\infty}, \tag{10}$$

where parameters r, α, p_0, p_1 satisfy the following conditions:

$$1 < p_0 < p_1 \leqslant 2, \quad \frac{1}{r} - \alpha = \frac{1}{p_0} - \frac{1}{p_1}, \quad \alpha > \frac{1}{p_1} - \frac{1}{2} \tag{11}$$

Let in (11) parameter r be fixed. Using Lemma 4 on bilinear interpolation and taking into account that

$$(B^{\alpha_0}_{r,1}, B^{\alpha_1}_{r,1})_{\theta,h} = B^{\alpha}_{r,h}, \ \texttt{with} \ \alpha = (1-\theta)\alpha^0 + \alpha^1,$$

we get

$$T\colon l_{p_0,h_1} \times B^{\alpha}_{r,h_2} \longrightarrow l_{p,h_3},$$

where $\frac{1}{h_1} + 1 = \frac{1}{h_2} + \frac{1}{h_3}$, $\alpha > \frac{1}{p_1} - \frac{1}{2} = \min\limits_{x\in[\frac{1}{p_1},\frac{1}{p_0}]} |\frac{1}{2} - x|$, $\frac{1}{r} - \alpha = \frac{1}{p_0} - \frac{1}{p_1}$.

Therefore, with fixed $a \in l_{p_0,\infty}$ and r we obtain that

$$P_a\colon B^{\alpha_i}_{r,1} \longrightarrow l_{p_1^i,\infty},$$

and

$$\|P_a\|_{B^{\alpha_i}_{r,1} \to l_{p_1^i,\infty}} \leqslant c_i \|a\|_{l_{p_0,\infty}},$$

where $\frac{1}{r} - \alpha_i = \frac{1}{p_0} - \frac{1}{p_1^i}, \alpha^i > \frac{1}{p_1^i} - \frac{1}{2}, i = 0,1$.

Using Marcinkiewicz - Calderón interpolation theorem we have

$$P_a\colon B^{\alpha}_{r,s} \longrightarrow l_{p_1,s},$$

and

$$\|P_a\|_{B^{\alpha}_{r,s} \to l_{p_1,s}} \leqslant c\|a\|_{l_{p_0,\infty}}.$$

Thus

$$T\colon l_{p_0,\infty} \times B^{\alpha}_{r,s} \longrightarrow l_{p_1,s}.$$

To complete the proof we fix a function φ and the parameters r, s, α and we choose the parameters $p_0^i, p_1^i, i = 0, 1$ satisfying (11).We use Lemma 2 to get$B^\alpha_{r,s}[0,1] \hookrightarrow M^{p_1,q_1}_{p_0,q_0}$.

The case $2 \leqslant p_0 < p_1 < \infty$, as in the proof of Theorem 3 will follow from $M^{p_1,q_1}_{p_0,q_0} = M^{p_0',q_0'}_{p_1',q_1'}$.

5. Examples demonstrating the sharpness of the results

Proposition 1. *Let* $1 < p_0 < 2 \leqslant p_1$, $\frac{1}{r} = \frac{1}{p_0} - \frac{1}{p_1}$, $\frac{1}{s} = (\frac{1}{q_1} - \frac{1}{q_0})_+$. *If* $q_1 < q_0$, *then for any* $\varepsilon > 0$ *there exists* $f_1 \in L_{r,s+\varepsilon}$ *such that* $f_1 \notin M^{p_1,q_1}_{p_0,q_0}$ *, If* $q_1 \geqslant q_0$ *there exists* $f_2 \in L_{r-\varepsilon,\infty}$, *such that* $f_2 \notin M^{p_1,q_1}_{p_0,q_0}$.

Proof. Let ε be an arbitrary positive number, and numbers β_1, β_2 be such that

$$\beta_1 > \frac{1}{s+\varepsilon}, \quad \beta_2 > \frac{1}{q_0}, \quad \beta_1 + \beta_2 < \frac{1}{s} + \frac{1}{q_0} = \frac{1}{q_1}.$$

Let

$$b_k = \frac{1}{(|k|+1)^{1/r'} \ln^{\beta_1}(|k|+1)},$$

$$a_k = \frac{1}{(|k|+1)^{1/p_0} \ln^{\beta_2}(|k|+1)},$$

and

$$f_1 \sim \sum_{k=-\infty}^{+\infty} b_k e^{2\pi i k x}.$$

Then for $m \neq 0$

$$(a*b)_m = \sum_{k=-\infty}^{+\infty} \frac{1}{(|k|+1)^{1/r'} \ln^{\beta_1}(|k|+1)(|k-m|+1)^{1/p_0} \ln^{\beta_2}(|k-m|+1)} \sim$$

$$\sim \int_{-\infty}^{+\infty} \frac{dx}{|x|^{1/r'} |\ln|x||^{\beta_1} |x-m|^{1/p_0} |\ln|x-m||^{\beta_2}}.$$

$$\int_{-\infty}^{+\infty} \frac{dx}{|x|^{1/r'} |\ln|x||^{\beta_1} |x-m|^{1/p_0} |\ln|x-m||^{\beta_2}} =$$

$$= |m|^{-(\frac{1}{r'}+\frac{1}{p_0})+1} \cdot \int_{-\infty}^{+\infty} \frac{dy}{|y|^{1/r'} |\ln|y| + \ln|m||^{\beta_1} |y-1|^{1/p_0} |\ln|y-1| + \ln|m||^{\beta_2}} =$$

$$= |m|^{-(\frac{1}{r'}+\frac{1}{p_0})+1} |\ln|m||^{-\beta_1-\beta_2} \cdot \int_{-\infty}^{+\infty} \frac{dy}{|y|^{1/r'} \left|\frac{\ln|y|}{\ln|m|}+1\right|^{\beta_1} |y-1|^{1/p_0} \left|\frac{\ln|y-1|}{\ln|m|}+1\right|^{\beta_2}} \geqslant$$

$$\geqslant |m|^{-(\frac{1}{r'}+\frac{1}{p_0})+1}|\ln|m||^{-\beta_1-\beta_2}\cdot\int\limits_{-\infty}^{+\infty}\frac{dy}{|y|^{1/r'}|\ln|y|+1|^{\beta_1}|y-1|^{1/p_0}|\ln|y-1|+1|^{\beta_2}}.$$

Thus $(a*b)_m \geqslant c(|m|+1)^{-(\frac{1}{r'}+\frac{1}{p_0})+1}|\ln(|m|+2)|^{-\beta_1-\beta_2}$. Since

$$\sum_{m=0}^{+\infty}\Big((m+1)^{-(\frac{1}{r'}+\frac{1}{p_0})+1}|\ln(|m|+2)|^{-\beta_1-\beta_2})^{q_1}(|m|+1)^{(\frac{q_1}{p_1}-1)}\Big)=\infty,$$

$a*b\notin l_{p_1,q_1}$ and therefore $f_1\notin M^{p_1,q_1}_{p_0,q_0}$. Since Fourier coefficients of f_1 is the sequence $\{b_k\}_{k\in Z}$ it follows that $f_1\in L_{r,s}$

To prove the second part of proposition we take $s=\infty$. Let numbers α_1 and α_2 be such that

$$\alpha_1>\frac{1}{(r-\varepsilon)'}=1-\frac{1}{r-\varepsilon},\quad \alpha_2>\frac{1}{p_0},\quad \alpha_1+\alpha_2<1-\frac{1}{r}+\frac{1}{p_0}$$

(note that last inequality does not contradict the previous two). Choosing

$$b_k=\frac{1}{(|k|+1)^{\alpha_1}},\quad a_k=\frac{1}{(|k|+1)^{\alpha_2}},\quad f_2\sim\sum_{k=-\infty}^{+\infty}b_ke^{2\pi ikx},$$

we can show that

$$a*b\sim\left\{(|k|+1)^{-\alpha_1-\alpha_2+1}\right\}_{k\in Z}.$$

Hence $a*b\notin l_{p_1,q_1}$ and therefore $f_2\notin M^{p_1q_1}_{p_0q_0}$. At the same time taking into account the monotonicity of the sequence $\{b_k\}_{k\in\mathbb{Z}}$ and Hardy - Littlewood theorem, we have that $f_2\in L_{r-\varepsilon,\infty}$. The statement is proved.

Theorem 6. *Let* $1<p_0<p_1<2,\ 1<q_1\leqslant q_0,\ \frac{1}{r}-\alpha=\frac{1}{p_0}-\frac{1}{p_1},\ \frac{1}{s}=\frac{1}{q_1}-\frac{1}{q_0}$. *Then for any* $\varepsilon>0$ *there exist* $f_1\in B^{\alpha-\varepsilon}_{r,\infty}\bigcap B^{\alpha}_{r-\varepsilon,\infty}$ *and* $f_2\in B^{\alpha}_{r,s+\varepsilon}$ *such that* $f_1\notin M^{p_1,q_1}_{p_0,q_0}$, $f_2\notin M^{p_1,q_1}_{p_0,q_0}$.

Proof. Let $s<\infty$ and numbers β_1,β_2 be such that

$$\beta_1>\frac{1}{s+\varepsilon},\quad \beta_2>\frac{1}{q_0},\quad \beta_1+\beta_2<\frac{1}{s}+\frac{1}{q_0}=\frac{1}{q_1}.$$

Let $b=\{b_k\}_{k\in Z}$ and $a=\{a_k\}_{k\in Z}$, where

$$b_k=\frac{1}{(|k|+1)^{\alpha-\frac{1}{r}+1}\ln^{\beta_1}(|k|+2)^{\beta_1}},$$

$$a_k=\frac{1}{(|k|+1)^{k/p_0}\ln^{\beta_2}(|k|+2)^{\beta_2}},$$

It is obvious that $a\in l_{p_0,q_0}$, and $f_2\sim\sum\limits_{k=-\infty}^{+\infty}b_ke^{2\pi ikx}$ belongs to $B^{\alpha}_{r,s+\varepsilon}$.

It is easy to show that

$$(a*b)_m\geqslant c(|m|+1)^{\alpha-\frac{1}{r}+\frac{1}{p_0}}(\ln(|m|+2))^{\beta_1+\beta_2}$$

and consequently $a * b \notin l_{p_1,q_1}$. Therefore, $f_2 \notin M_{p_0,q_0}^{p_1,q_1}$

To construct the function f_1 it is sufficient to consider the sequences

$$b = \left\{ \frac{1}{(|m|+1)^{\gamma_1}} \right\}_{m \in Z}, \quad a = \left\{ \frac{1}{(|m|+1)^{\gamma_2}} \right\}_{m \in Z},$$

where

$$\gamma_1 > \max(\alpha - \varepsilon - \frac{1}{r} + 1, \alpha - \frac{1}{r+\varepsilon} + 1),$$

$$\gamma_2 > \frac{1}{p_0}$$

and

$$\gamma_1 + \gamma_2 < \alpha - \frac{1}{r} + \frac{1}{p_0}.$$

$f_1 \sim \sum_{k=-\infty}^{+\infty} b_k e^{2\pi i k x}$. The proof that $f_1 \in B_{r,\infty}^{\alpha-\varepsilon} \bigcap B_{r-\varepsilon,\infty}^{\alpha}$, $f_1 \notin M_{p_0,q_0}^{p_1,q_1}$ is similar to the proof of the first part.

Acknowledgments. This paper was supported by the grant MTM 2011-27637. This work was done as part of the research program Approximation Theory and Fourier Analysis at the Centre de Recerca Matemàtica, in the Fall semester of 2011. The the first named author is grateful to CRM for its support and hospitality.

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CRM Preprint Series number 1077