SPECTRAL PROPERTIES OF DEGENERATE ELLIPTIC OPERATORS WITH MATRIX COEFFICIENTS

M.G. GADOEV, S.A. ISKHOKOV

Introduction

The paper is continuation of the work [1] and it is devoted to spectral properties of a class nonselfadjoint degenerate elliptic operators A in the space $\mathcal{H}^l = L_2(0,1)^l$ associated with noncoercive bilinear forms.

Questions as completeness of the system of root vector-functions of the operator A in $\mathcal{H}^l$, description of the domain of operator A, estimation of the resolvent of operator A, asymptotical distribution of eigenvalues of operator A are considered.

Spectral asymptotics of degenerate elliptic operators far from selfadjoint ones were studied in [2-7] in a case when eigenvalues of an operator are divided into two series, one lies out of the angle $|\,arg\,z| \le \varphi$, $\varphi < \pi$ and another localizes to the ray $R_+ = (0, +\infty)$. This paper as [1] sides with [2,3,7], among them more general results were obtained in [7] where it was assumed that a leading coefficient of operator A

$$a(t) \in C^m([0,1]; End\mathbf{C}^l) \tag{0.1}$$

and has simple different eigenvalues (e.v.) for any $t \in [0,1]$.

In stead of (0.1) we require only that $a(t) \in C([0,1]; End\mathbf{C}^l)$.

1. Statement of main results

1. A nonselfadjoint operator A given in a Hilbert space H is called far from selfadjoint ones if it is impossible to express it in the form

$$A = B(E+S), \quad B = B^*, \quad S \in \sigma_\infty(H). \tag{1.1}$$

Here, and in the sequel, the symbol $\sigma_\infty(H)$ stands for the class of completely continuous linear operators in H and B^* - adjoint operator of B.

Spectral properties of elliptic differential and pseudodifferential operators near to selfadjoint ones, i.e. those are expressible in the form (1.1), have been studied in a literature in greater details (see [8, 9]). Also, spectral properties of elliptic differential operators (d.o.) and pseudodifferential operators (p.d.o.) far from selfadjoint ones were investigated in the case when those are given in a compact manifold without edge (see [7, 10-12], and references cited therein). In the case of domains with a boundary d.o. and p.d.o. that are far from selfadjoint were studied in [3, 4, 13-18]; only [3, 4, 13] are devoted to degenerate elliptic problems.

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2. In this work we study spectral properties nonselfadjoint operator in $L_2(0,1)^l$ generated by bilinear form

$$\mathcal{A}[u,v] = \sum_{i,j=0}^{m} \int_0^1 < p_i(t)a_{ij}(t)u^{(i)}(t), p_j(t)v^{(j)}(t) >_{\mathbf{C}^l} dt. \quad (1.2)$$

Here

$$p_i(t) = \{t(1-t)\}^{\theta+i-m} \quad (i = \overline{0,m}),\ \theta < m,\ u^{(i)}(t) = \frac{d^i u(t)}{dt^i},$$

$$a_{ij} \in L_\infty(J;\ End\,\mathbf{C}^l) \quad (i,j = \overline{0,m}),$$

where $J = (0,1)$. Symbol $<\,,\,>_{\mathbf{C}^l}$ stands for scalar product in $\mathbf{C}^l$.

Let $\mathcal{H}_+$ to be the closer of linear manifold $C_0^\infty(J)$ with respect to the norm

$$|\varphi|_+ = (\int_J p_m^2(t)|\varphi^{(m)}(t)|^2 dt + \int_J |\varphi(t)|^2 dt)^{1/2}.$$

We put:

$$\mathcal{H} = L_2(J), \quad \mathcal{H}^l = \mathcal{H} \oplus \cdots \oplus \mathcal{H} \quad (l-),$$

$$\mathcal{H}_+^l = \mathcal{H}_+ \oplus \cdots \oplus \mathcal{H}_+ \quad (l-).$$

Here and in the sequel we denote the scalar products in the spaces $\mathcal{H}, \mathcal{H}^l$ by the same symbol $(\,,\,)$. Analogously the norms in the spaces $\mathcal{H}_+, \mathcal{H}_+^l$ and $\mathcal{H}, \mathcal{H}^l, \mathbf{C}^l$ will be denoted by $|\ |_+, |\ |$ respectively.

Denote by $\|T\|$ the norm of a bounded operator T given in $\mathcal{H}$ or $\mathcal{H}^l$.

As a domain of the bilinear form $\mathcal{A}[u,v]$ (1.2) we accept the space $\mathcal{H}_+^l$.

Suppose that $a_{mm}(t) \in C^m(\overline{J};\ End\,\mathbf{C}^l)$ and matrix $a(t) = a_{mm}(t)$ for any $t \in \overline{J}$ has l different nonzero eigenvalues $\mu_1(t), ..., \mu_l(t)$. Thus the eigenvalues of the matrix $a(t)$ can be numerated such that $\mu_j(t), \mu_j^{-1}(t) \in C^m(\overline{J}), j = \overline{1,l}$.

Let the following conditions are fulfilled:

$$|a_{ij}(t)| \le Mt^\delta(1-t)^\delta \quad (i+j < 2m), \quad \delta > 0, \quad (1.3)$$

$$\mu_j(t) \notin S \qquad (j = \overline{1,l},\ t \in \bar{J}), \quad (1.3')$$

where $S \subset \mathbf{C}$ - some closed angle with vertex at the origin and $\mu_j(t)$ - eigenvalue (e.v.) of the matrix $a(t)$.

Under above stated conditions the following theorems are valid (see [1]):

Theorem 1.1. *There is a unique closed operator A in $\mathcal{H}^l$ enjoying the following properties:*

(i) $\quad D(A) \subset \mathcal{H}_+^l$, $(Au,v) = \mathcal{A}[u,v] \quad (\forall u \in D(A),\ v \in \mathcal{H}_+^l)$,

(ii) *for some $z_0 \in \mathbf{C}$ there is a continuous inverse operator*

$$(A - z_0E)^{-1} : \mathcal{H}^l \to \mathcal{H}^l.$$

Let A be the same operator as one in the conditions $(i), (ii)$.

Theorem 1.2. *The operator A has discreet spectrum. The system of root vector-functions of the operator A is complete in $\mathcal{H}^l$, i.e. the set of its finite linear combinations is dense in $\mathcal{H}^l$. The order of the resolvent of the operator A is not greater then $\frac{1}{2m}$. For $N(\lambda)$ – the number of eigenvalues of the operator A whose modules are less or equal to λ with the multiplicity counted, the bound $N(\lambda) \leq M\lambda^{1/2m}$, $(\lambda \geq 1)$ is valid.*

3. Denote by $\mathcal{H}_-$ the completion of $\mathcal{H}$ with respect to the norm

$$|u|_- = \sup_{0 \neq \varphi \in \mathcal{H}_+} \frac{|(u, \varphi)|}{|\varphi|_+}.$$

We put $\mathcal{H}^l_- = \mathcal{H}_- \oplus \cdots \oplus \mathcal{H}_-$ (l - times). An element $F = (F_1, \ldots, F_l) \in \mathcal{H}^l_-$ generates antilinear continuous functional on $\mathcal{H}^l_+$ by the formula

$$< F, v >= \lim_{i \to +\infty} (u_i, v), \qquad v \in \mathcal{H}^l_+,$$

where a sequence of vector-functions $u_1, u_2, \ldots \in \mathcal{H}^l$ is selected such that $u_i \to F\,(i \to +\infty)$ $\mathcal{H}^l_-$.

Note that if $v = (v_1, \ldots, v_l) \in \mathcal{H}^l_+$ then

$$< F, v >= \sum_{i=1}^{l} < F_i, v_i >, \quad |F|_- = (\sum_{i=1}^{l} |F_i|^2_-)^{1/2}.$$

Here and in the sequel both for the case $l = 1$ and for any arbitrary $l \in N$ we use the same notations: $|\ |_-$, $<,>$.

Conversely for any antilinear continuous functional $g(v)\,(v \in \mathcal{H}^l_+)$ there is a unique element $F \in \mathcal{H}^l_-$ suchthat $g(v) =< F, v >, \forall v \in \mathcal{H}^l_+$. At the same time the norm of the functional g is equal to $|F|_-$.

In what follows, antilinear continuous functionals on $\mathcal{H}^l_+$ are identified with the corresponding elements of the space $\mathcal{H}^l_-$.

4. By the aid of the Hardy inequality under the condition (1.3) we have

$$|\mathcal{A}[u, v]| \leq M|u|_+|v|_+ \quad (\forall u, v \in \mathcal{H}^l_+).$$

Thus it is possible to introduce an operator $\mathcal{A} : \mathcal{H}^l_+ \to \mathcal{H}^l_-$, acting by the equality

$$< \mathcal{A}u, v >= \mathcal{A}[u, v] \qquad (\forall u, v \in \mathcal{H}^l_+).$$

Let A be the same operator as in theorems 1.1, 1.2. The following theorem is valid:

Theorem 1.3. *For sufficiently large by modules $\lambda \in S$ there are continuous inverses*

$$(\mathcal{A} - \lambda E)^{-1} : \mathcal{H}^l_- \to \mathcal{H}^l_-, \quad (A - \lambda E)^{-1} : \mathcal{H}^l \to \mathcal{H}^l,$$

and an equality

$$(\mathcal{A} - \lambda E)^{-1} u = (A - \lambda E)^{-1} u \qquad (\forall u \in \mathcal{H}^l)$$

holds. At the same time $Au = \mathcal{A}u\,(\forall u \in D(A))$*, and*

$$D(A) = \{u \in \mathcal{H}^l_+ \,:\, \mathcal{A}u \in \mathcal{H}^l\}.$$

A similar result for partial differential operators with scalar coefficients has been received in article [19]. Note that the first part of the assertion of the theorem 1.3 can be proven by the scheme of the article [19] only in a case of additional assumption that

$$|\,arg\,\{\gamma(t) < a(t)h, h >_{\mathbf{C}^l}\}| < \frac{\pi - \varepsilon}{2} \quad (\forall t \in \bar{J},\, 0 \neq h \in \mathbf{C}^l), \tag{1.4}$$

where $\varepsilon > 0, \gamma(t) \in C(\bar{J}), \gamma(t) \neq 0\,(\forall t \in \bar{J})$. Here and in the sequel it is considered that the function $arg\,z$ takes values in on the interval $(-\pi, \pi]$.

In particular, it followes from (1.4)that

$$|\,arg\,\gamma(t)\mu_j(t)| \leq \frac{\pi - \varepsilon}{2} \quad (\forall t \in \bar{J},\, j = \overline{1, l}).$$

5. The proof of the theorem 1.2 is produced in §2. Note that following bound of the resolvent of the operator A in the sector S is obtained in §2:

$$\|(A - \lambda E)^{-1}\| \leq M(|\lambda|)^{-1}, (\lambda \in S,\, |\lambda| \geq c(S)),$$

where $c(S) > 0$. Summability of the Fourier series of elements $f \in \mathcal{H}^l$ with respect to the system of root vector-functions of the operator A by the Abel method with brackets was established in [1]. In this work completeness of the system of root vector-functions of the operator A in $\mathcal{H}^l$ is proved.

In §3 we describe the domain of the operator A. In §4 we investigate asymptotic behavior of eigenvalues of the operator A.

2. Resolvent estimate of the operator A

1. Let P be a selfadjoint operator in $\mathcal{H}$ associated with the bilinear form

$$P'[u, v] = (\rho^\theta u^{(m)}, \rho^\theta v^{(m)}), D[P'] = \mathcal{H}_+.$$

Below we need the following (see [1, p.36]).

Lemma 2.1. *There exists continuous inverse operator* $T_\omega : \mathcal{H}_- \to \mathcal{H}, \omega \geq 1$*, such that* $T_\omega u = (P + \omega E)^{-\frac{1}{2}}u$*,* $\forall u \in \mathcal{H}$, moreover

$$|T_\omega F| \leq M|F|_{-\nu}\ (\forall \omega \geq 1, \nu \in [1, 2\omega), \forall F \in \mathcal{H}_{-\nu}),$$

where a number $M > 0$ dose not depend on ω, ν.

2. Let T_ω be the same operator as in lemma 2.1, $\mathcal{T}_\omega : \mathcal{H} \to \mathcal{H}_-$ – inverse operator with respect to the operator $T_\omega : \mathcal{H}_- \to \mathcal{H}$. As in lemma 2.1 it is proved that

$$|\mathcal{T}_\omega u|_{-\nu} \leq M|u|\ (\forall \omega \geq 1, \nu \in [1, 2\omega), \forall u \in \mathcal{H}),$$

where $M > 0$ is independent of ω, ν. At the same time if $u \in \mathcal{H}_\nu$, then

$$\mathcal{T}_\omega u = (P + \omega E)^{\frac{1}{2}}u.$$

Introduce operators $T^l_\omega : \mathcal{H}^l_- \to \mathcal{H}^l, \mathcal{T}^l_\omega : \mathcal{H}^l \to \mathcal{H}^l_-$ by the formulas:

$$T^l_\omega = diag\{T_\omega, ..., T_\omega\},\ \mathcal{T}^l_\omega = diag\{\mathcal{T}_\omega, ..., \mathcal{T}_\omega\}.$$

We put $P_l = diag\{P, ..., P\}$.

The following theorem is valid

Theorem 2.1. *For $\lambda \in S$, $|\lambda| \geq \sigma$, where $\sigma > 0$ is sufficiently large, the following representation hold*

$$(\mathcal{A} - \lambda E)^{-1} = (P_l + |\lambda|E)^{-1}\Phi(\lambda)T_\lambda \tag{2.1}$$

$$(A - \lambda E)^{-1} = (P_l + |\lambda|E)^{-\frac{1}{2}}\Phi(\lambda)(P_l + |\lambda|E)^{-\frac{1}{2}}, \tag{2.1$'$}$$

where $\Phi(\lambda) : \mathcal{H}^l \to \mathcal{H}^l$ – bounded operator and

$$\sup_{\lambda\in S,\ |\lambda|\geq\sigma} ||\Phi(\lambda)|| < +\infty. \tag{2.2}$$

Proof. We prove that if $\nu = |\lambda|$ then for sufficiently large in magnitude $\lambda \in S$,

$$|(P_l + |\lambda|E)^{\frac{1}{2}}(\mathcal{A}_\nu - \lambda E)^{-1}\mathcal{T}^l_{|\lambda|}u| \leq M|u|\ (\forall u \in \mathcal{H}^l)$$

we use the equality (see [1], §4, (4.6), (4.7))

$$(\mathcal{A}_\nu - \lambda E)^{-1} = X_\nu(\lambda)\Gamma'_\nu(\lambda).$$

It is clear that

$$|T^l_{|\lambda|}\Gamma^{(\lambda)}_{|\lambda|}\mathcal{T}^l_{|\lambda|}u| \leq |\Gamma^{(\lambda)}_{|\lambda|}\mathcal{T}^l_{|\lambda|}u|_{-|\lambda|} \leq M_1|\mathcal{T}^l_{|\lambda|}u|_{-|\lambda|} \leq M_2|u|,\ (\lambda \in S, |\lambda| \geq \sigma_1).$$

It remains to show that (see [1], §4, (4.6), (4.7))

$$|(P_l + |\lambda|E)^{\frac{1}{2}}X_{|\lambda|}(\lambda)\mathcal{T}^l_{|\lambda|}u| \leq M_3|u|,\ (u \in \mathcal{H}^l). \tag{2.3}$$

Using (4.3), (3.13) from [1] we bring a proof of the estimate (2.3), as above, to the proof of the following inequality:

$$|(P_l + |\lambda|E)^{\frac{1}{2}}R_{k,|\lambda|}(\lambda)\mathcal{T}_{|\lambda|}v| \leq M_4|v|,\ (v \in \mathcal{H}).$$

This inequality for numbers $\lambda \in S$ sufficiently large in magnitude follows from representation (3.12) from [1]. Thus we have

$$(\mathcal{A}_\nu - \lambda E)^{-1} = (P_l + |\lambda|E)^{-\frac{1}{2}}\Phi(\lambda)T^l_{|\lambda|},\ (\lambda \in S, |\lambda| \geq \sigma), \tag{2.4}$$

where $\Phi(\lambda) : \mathcal{H}^l \to \mathcal{H}^l$ – a bounded operator satisfying the estimate (2.2).

Note that

$$(\mathcal{A}_\nu - \lambda E)^{-1}F = (\mathcal{A} - \lambda E)^{-1}F.\ (\forall\ \nu \geq 1, F \in \mathcal{H}^l_-). \tag{2.5}$$

For $u \in \mathcal{H}^l$ we have

$$T^l_{|\lambda|}u = (P_l + |\lambda|E)^{-\frac{1}{2}}u,\ (\mathcal{A} - \lambda E)^{-1}u = (A - \lambda E)^{-1}u,$$

which together with (2.4), (2.5) proves (2.1), (2.1$'$). The theorem is proven.

3. From representation (2.1′) it follows that

$$||(A-\lambda E)^{-1}|| \le M|\lambda|^{-1}. \ (\lambda \in S, |\lambda| \ge \sigma). \tag{2.6}$$

Since the order of the resolvent of the operator P_l equals to $\frac{1}{2m}$, it follows from (2.1′) that the order of the resolvent of the operator A is at most $\frac{1}{2m}$. From here and from (2.6), applying theorem 6.4.1 from [20], we establish that the system of root vector-functions of operator A is complete in $\mathcal{H}^l$.

Note that summability of the Fourier series of elements $f \in \mathcal{H}^l$ with respect to the system of root vector-functions of the operator A by the Abel method with brackets was established in [1].

4. Let H be a Hilbert space. Denote by $\sigma_\tau(H)$, $\tau > 0$ the space of operators $L \in \sigma_\infty(H)$, which series of s-numbers in order τ is convergent ([21]):

$$||L||_\tau = \left(\sum_{j=1}^{\infty} s_j^\tau(L)\right)^{\frac{1}{\tau}} < +\infty.$$

(The lower bound of numbers τ such that $L \in \sigma_\tau(H)$ is called the order of the operator L.)

Denote by $\mu_1(t), \mu_2(t), ...$ the sequence of e.v. of operator $L \in \sigma_\infty(H)$ numbered according nonincreasing their magnitudes and taken with their algebraic multiplicities. Note that

$$s_j\left((L^*L)^{\frac{1}{2}}\right) = \mu_j(L), \ j = 1, 2, ... \ .$$

In what follows, we need the following well known inequalities (for example, see [21])

$$\sum_{j=1}^{+\infty} |\mu_j(t)| \le ||L||_1, \ (\forall L \in \sigma_1(H)) \tag{2.7}$$

$$||LL'||_p \le ||L||_p||L'||, \ ||L'L||_p \le ||L'||||L||_p, \tag{2.8}$$

if $L \in \sigma_p(H), p \ge 1, L'$ is a bounded operator;

$$||L_1...L_r||_p \le ||L_1||_{\kappa_1}...||L_r||_{\kappa_r}, \tag{2.9}$$

if $L_j \in \sigma_{\kappa_j}(H)$, $1 \le p \le \kappa_j$ $(j = \overline{1,r})$, $\sum\limits_{j=1}^{r} \kappa_j^{-1} = \frac{1}{p}$. From (2.9) for $L_1 = ... = L_r = L \in \sigma_1(H), \kappa_j = r$ $(j = \overline{1,r})$ it follows that

$$||L^r||_1 \le ||L||_r^r. \tag{2.10}$$

From (2.7) follows convergent of the series:

$$SpL \overset{def}{=} \sum_{j=1}^{+\infty} mu_j(t), \ \forall L \in \sigma_1(H).$$

5. In the last part of current section we prove the assertion of the theorem 1.2 on spectral estimation of operator A.

Denote by $\lambda_1, \lambda_2, ...$ the sequence of e.v. of operator A numbered according nonincreasing their magnitudes and taken with their algebraic multiplicities.

Using (2.1′), (2.8)-(2.10) we obtain

$$||(A-\lambda E)^{-r}||_1 \le ||(P_l+|\lambda|E)^{-\frac{1}{2}}\Phi(\lambda)(P_l+|\lambda|E)^{-\frac{1}{2}}||_r^r \le M||(P_l+|\lambda|)^{-\frac{1}{2}}||_{2r}^{2r},$$

$$(\lambda \in S, |\lambda| \ge \sigma) \tag{2.11}$$

where $r = 4m, \sigma > 0$ is a sufficient large number. It is known that

$$N_0(t) \overset{def}{=} \sum_{\omega_j \le t} 1 \sim const \cdot t^{\frac{1}{2m}} \ (t \to +\infty),$$

where $\omega_1, \omega_2, ...$ denotes the sequence of e.v. of the operator L. Therefore

$$||(P_l+|\lambda|E)^{-\frac{1}{2}}||_{8m}^{8m} = \sum_{j=1}^{+\infty}(\omega_j+|\lambda|)^{-4m} = \int_0^{+\infty}\frac{dN_0(t)}{(t+|\lambda|)^{4m}} \le M|\lambda|^{\frac{1}{2m}-4m} \ (|\lambda| \ge 1).$$

From here and from (2.7), (2.11) we conclude that

$$\sum_{j=1}^{+\infty}|(\lambda_j-\lambda)^{-4m}| \le M|\lambda|^{\frac{1}{2m}-4m} \ (\lambda \in S, |\lambda| \ge \sigma).$$

Let us choose a number $\varphi \in (-\pi; \pi]$ such that the ray $\Gamma = \{\lambda = te^{i\varphi} : t \ge 0\}$ be the mean line of the angle S. Then

$$|z| + |\lambda| \le c'|z-\lambda| \ (\forall z \notin S, \lambda \in \Gamma), \tag{2.13}$$

where $c' > 0$ depends only on the spread of the angle S. For sufficiently large $j \ge j_0$ we have $\lambda_j \notin S$. It is clear that

$$N(t) = \int_0^t dN(\tau) \le (2t)^{4m}\int_0^t \frac{dN(\tau)}{(t+\tau)^{4m}} \le (2t)^{4m}\int_0^{+\infty}\frac{dN(\tau)}{(t+\tau)^{4m}} =$$

$$= (2t)^{4m}\sum_{j=1}^{+\infty}(|\lambda_j|+t)^{-4m},$$

where $N(t) = card\{j : |\lambda_j| \le t\}$. Therefore (see (2.12), (2.13))

$$N(t) \le M_1 + M_2 t^{4m}\sum_{j=j_0}^{+\infty}|\lambda_j - te^{i\varphi}|^{-4m} \le M_2 t^{\frac{1}{2m}} \ (t \ge 1).$$

Thus proof of the theorem 1.2 is completely finished.

3. Description of the domain of operator A

1. Let A be the same operator as in the theorem 1.1 and coefficients

$$a_{ij}(t) \in C^j(J;\, End\, \mathbf{C}^l)\ (i,j = \overline{0,m}). \tag{3.1}$$

Theorem 3.1. *The domain $D(A)$ of the operator A is described as a class of vector-functions $u \in W^{2m}_{2,loc}(J)^l \cap \mathcal{H}^l_+$ such that*

$$f = \sum_{i,j=0}^{m} (-1)^j (p_i(t)p_j(t)a_{ij}(t)u^{(i)}(t))^{(j)} \in \mathcal{H}^l.$$

At the same time $f = Au$.

Proof. Let $u \in W^{2m}_{2,loc}(J)^l \cap \mathcal{H}^l_+$ and a vector function $f(t) \in \mathcal{H}^l$. Then for arbitrary vector-function $v(t) \in C_0^\infty(J)^l$ by integrating by parts we obtain

$$(f,v) = \sum_{i,j=0}^{m} (p_i(t)a_{ij}(t)u^{(i)}(t), p_j(t)v^{(j)}(t)) = \mathcal{A}[u,v].$$

These equalities are extended by continuity for any $v \in \mathcal{H}^l_+$. Therefore according the theorem 1.1 $u \in D(A)$, $f = Au$.

Conversely, let $u \in D(A)$, $f_1 = Au$. Then

$$(f_1, v) = \sum_{i,j=0}^{m} (p_i a_{ij} u^{(i)}, p_j v^{(j)}),\ \forall v \in C_0^\infty(J)^l,$$

so that an element

$$f_2 = \sum_{i,j=0}^{m} (-1)^j (p_i(t)p_j(t)a_{ij}(t)u^{(j)}(t))^{(j)},$$

considered in the sense of distributions belongs to the space $\mathcal{H}^l$. $f_1 = f_2$. Furthermore it follows from general theory of elliptic equations that $u \in W^{2m}_{2,loc}(J)^l$.

2. In connection with the theorem 3.1 we note that the space $\mathcal{H}^l_+$ when $-\frac{1}{2} < \theta < m - \frac{1}{2}$ is described (see [20]) as a class $u(t) \in \mathcal{H}^l$ with fined norm

$$|u|_+ = \left(\int_J |\rho^{2\theta}(t)u(t)|^2 dt + \int_J |u(t)|^2 dt \right)^{\frac{1}{2}} < +\infty, \tag{3.2}$$

which have zero traces

$$u^{(j)}(0) = u^{(j)}(1) = 0,\ j = 0, 1, ..., s_0 - 1;$$

here s_0 is an integer such that $m - \theta - \frac{1}{2} \le s_0 < m - \theta + \frac{1}{2}$. If $\theta \le -\frac{1}{2}$ or $m - \frac{1}{2} \le \theta < m$, then the space $\mathcal{H}^l_+$ consists of vector-functions $u(t) \in \mathcal{H}^l$ ([20]) with finite norms $|u|_+$ (3.2).

3. Together with theorem 3.1 the following theorem is valid

Theorem 3.2. *Let condition (3.1) is satisfied and*

$$|a_{ij}^{(k)}(t)| \le M\{t(1-t)\}^{-k}, \ (k=0,1,...,j).$$

Let furthermore $\theta + \frac{1}{2} \notin \{1,2,...,m\}$. Then the domain of the operator A is described as a class of vector-functions $u \in W_{2,loc}^{2m}(J)^l \cap \mathcal{H}_+^l$ such that

$$p_0(t)u(t), \ \sum_{i,j=0}^{m} (-1)^j (p_i(t)p_j(t)a_{ij}(t)u^{(j)}(t))^{(j)} \in \mathcal{H}^l.$$

4. Asymptotic distribution of eigenvalues of operator A

1. Let A be thesame operator as in the theorem 1.1. Suppose that an eigenvalues $\mu_1(t),...,\mu_l(t)$ of matrix $a(t)$ are located on the complex plane in the following way:

$$\mu_1(t),...,\mu_n(t) \in R_+ \stackrel{def}{=} \{z \in \mathbb{C} : Rez > 0, Imz = 0\}, \ \mu_{n+1}(t),...,\mu_l(t) \notin \overline{\Phi},$$

where $1 \le n \le l$. $\Phi = \{z \in \mathbb{C} : |argz| < \varphi\}, \varphi \in (0,\pi)$. Then according the theorem 1.3 in any closed sector $S \subset \overline{\Phi} \backslash R_+$, with vertex the origin includes a finite number of e.v. of operator A. It follows from here that

$$\lim_{j\to+\infty} arg\lambda_j = 0,$$

where $\lambda_1, \lambda_2, ...$ denotes a sequence of e.v. of operator A, located in the angle Φ and numerated in order to nondecreasing their magnitude taking into account root multiplicities.

The following theorem is valid

Theorem 4.1. *For the function*

$$N(t) = card\{j : |\lambda_j| \le t\},$$

the asymptotic formula

$$N(t) \sim ct^{\frac{1}{2m}}, \ c = \frac{1}{\pi} \sum_{j=1}^{n} \int_0^1 \rho^{-\frac{\theta}{m}}(t) \mu_j^{-\frac{1}{2m}}(t) dt$$

holds as $t \to +\infty$.

Analogous result for the second order differential operators was established in [4, 13]. However we note that the scheme of works [4, 13] is not applicable for the case $m > 1$ even if condition (1.4) is satisfied. An significant moment of methods of current work is that we “single out” in an explicit form a principle part of “generalized resolvent” as operator acting from $\mathcal{H}_{-\nu}^l$ to $\mathcal{H}_{\nu}^l$.

In combination with application of some another analytic techniques it allows us to compute a principle part of asymptotic of the function $Sp(A - zE)^{-1}$ as $z \to +\infty$ by some half-line $\Gamma \subset \overline{\Phi} \backslash R_+$ with beginning from zero. Obtained asymptotic formulas here even for the case $m = 1$ related to more general class of operators then those in works [4, 13].

2. To prove theorem 4.1 we use (4.6), (4.7) (see [1], §4) for $\nu = |\lambda|$. Let P_l, T^l_ω, $\mathcal{T}^l_\omega$ are the same operators as in subsection 1 of section 2.

Denote by $u_1, u_2, ...$ orthonormal sequence of eigenfunctions of the operator the P_l. Let $P_l u_j = \omega_j u_j$, $\omega_1 \le \omega_2 \le \ldots$. Since $u_1, u_2, ...$ – orthonormal basis in $\mathcal{H}^l$ $(A - \lambda E)^{-1} u_j = (\mathcal{A}_\nu - \lambda E)^{-1} u_j$ $\forall \nu \ge 1$, it follows that

$$sp(A - \lambda E)^{-1} = \sum_{j=1}^{+\infty} ((A - \lambda E)^{-1} u_j, u_j) = \sum_{j=1}^{+\infty} ((\mathcal{A}_\nu - \lambda E)^{-1} u_j, u_j) =$$

$$= \sum_{j=1}^{+\infty} (X_\nu(\lambda) u_j, u_j) + \sum_{j=1}^{+\infty} (X_\nu(\lambda)\Gamma_\nu(\lambda) u_j, u_j), \ (\lambda \in S, |\lambda| \ge \sigma = \sigma(S)), \quad (4.1)$$

where $S \subset \overline{\Phi} \backslash R_+$ – arbitrary closed angle with vertex the origin. Taking into account

$$(P_l + |\lambda| E)^{\pm\frac{1}{2}} u_j = (\omega_j + |\lambda|)^{\pm\frac{1}{2}} u_j,$$

we obtain

$$\sum_{j=1}^{+\infty} (X_\nu(\lambda)\Gamma_\nu(\lambda) u_j, u_j) = \sum_{j=1}^{+\infty} (X_\nu(\lambda)\Gamma_\nu(\lambda)(P_l + |\lambda|E)^{\frac{1}{2}} u_j, (P_l + |\lambda|E)^{-\frac{1}{2}} u_j) =$$

$$= \sum_{j=1}^{+\infty} ((P_l + |\lambda|E)^{-\frac{1}{2}} X_\nu(\lambda) \mathcal{T}_{|\lambda|} T_{|\lambda|} \Gamma_\nu(\lambda) \mathcal{T}_{|\lambda|} u_j, u_j). \quad (4.2)$$

According (4.6) (see [1], §4) for $\nu = |\lambda|, u \in \mathcal{H}^l$ we have

$$|\mathcal{T}_{|\lambda|} \Gamma_\nu(\lambda) T_{|\lambda|} u| \le M |\Gamma_\nu(\lambda) T_{|\lambda|} u|_{-|\lambda|} \le M_1 |\lambda|^{-\varepsilon'} |T_{|\lambda|} u|_{-|\lambda|} = M_2 |\lambda|^{-\varepsilon'} |u|.$$

Thus operator $\mathcal{T}_{|\lambda|} \Gamma_\nu(\lambda) T_{|\lambda|}$ induces a bounded operator in $\mathcal{H}^l$, which norm dose not exceed $M_2 |\lambda|^{-\varepsilon'}$. From here by (4.1), (4.2) we find

$$Z(\lambda) \stackrel{def}{=} |sp(A - \lambda E)^{-1} - \sum_{j=1}^{+\infty} (X_\nu(\lambda) u_j, u_j)| \le M |\lambda|^{-\varepsilon'} ||(P_l + |\lambda|E)^{-\frac{1}{2}} X_\nu(\lambda) \mathcal{T}_{|\lambda|}||_1.$$

Here regardless of the fact that $\mathcal{T}_{|\lambda|}$ –unbounded operator in $\mathcal{H}^l$, operator $X_\nu(\lambda)\mathcal{T}_{|\lambda|}$ induces in $\mathcal{H}^l$ a bounded operator. Applying (2.3) we obtain

$$Z(\lambda) \le M |\lambda|^{-\varepsilon'} |(P_l + |\lambda|E)^{-1}|_1 \le M_1 |\lambda|^{\frac{1}{2m} - 1 - \varepsilon'}.$$

Future we have (see [1], §4, (4.3)),

$$\sum_{j=1}^{+\infty} (X_\nu(\lambda) u_j, u_j) = \sum_{j=1}^{+\infty} (U(\mathcal{B}_\nu - \lambda E)^{-1} U^{-1} u_j, u_j) = \sum_{j=1}^{+\infty} ((\mathcal{B}_\nu - \lambda E)^{-1} u_j, u_j) =$$

$$= \sum_{k=1}^{l} Sp(\widetilde{Q}_k - \lambda E)^{-1}.$$

Here operators $\widetilde{Q}_k,\ k = \overline{1, l}$ are defined in the space $\mathcal{H}$ by following way:

$$D(\widetilde{Q}_k) = \{v \in \mathcal{H}_+ : Q_{\nu,k} v \in \mathcal{H}\}, \forall \nu \geq 1,$$

$$\widetilde{Q}_k v = Q_{\nu,k} v,\ \forall v \in D(\widetilde{Q}_k).$$

Operators $Q_{\nu,k}$ were introduced in [1, §4, subsec. 1]. Note that

$$(\mathcal{B}_\nu - \lambda E)^{-1} = diag\{(Q_{\nu,1} - \lambda E)^{-1}, ..., (Q_{\nu,\ l} - \lambda E)^{-1}\}.$$

Operators $\widetilde{Q}_1, ..., \widetilde{Q}_l$ are defined above does not depend on $\nu \geq 1$.

Applying theorem 1.3 to current case when $l = 1, A = \widetilde{Q}_j, j = \overline{1, l}$ we obtain that operator $\widetilde{Q}_j, j = \overline{n+1, l}$ has a finite e.v. in angle Φ. Since $\mu_j(t) \in R_+ (j = \overline{1, n})$ then $\widetilde{Q}_j = \widetilde{Q}_j^* \geq 0, (j = \overline{1, n})$. Thus we have

$$Sp(A - \lambda E)^{-1} = \sum_{i=1}^{+\infty} \sum_{k=1}^{l} (\lambda_{i,k} - \lambda)^{-1} + O(|\lambda|^{\frac{1}{2m} - 1 - \varepsilon'}),\ (\lambda \in S, |\lambda| \geq \sigma(S)),\ (4.3)$$

where $\varepsilon' > 0, S \subset \overline{\Phi} \backslash R_+$ – a closed angle with vertex the origin and $\lambda_{1,k}, \lambda_{2,k}, ...$ denotes the sequence of e.v. of operator $\widetilde{Q}_k$ are numbered according nonincreasing their magnitudes.

Let $\psi \in (0, \varphi)$,

$$\mathcal{L} = \{z \in \mathbb{C} : arg z = \pm\psi\} \cup \{0\}$$

be a contour turning round R_+ from the left. Choose a number $a, \delta > 0$ such that the following conditions be satisfied:

(i) $|(arg\lambda_j') \pm \varphi| \geq \delta, |(arg\lambda_{j,k}) \pm \varphi| \geq \delta$, if $|\lambda_j'| \geq a$ or $|\lambda_{j,k}| \geq a,\ (j = 1, 2, ..., k = \overline{1, l})$ accordingly;

(ii) $\lambda_{j,k} \notin \Phi,\ (k = \overline{n+1, l})$, if $|\lambda_{j,k}| \geq a$.

Here $\lambda_1', \lambda_2', ...$ denotes a sequence of e.v. of operator A numbered according nonincreasing their magnitudes.

Then in a case when $|\lambda_j'| \geq a, |\lambda_{j,k}| \geq a,\ \ \lambda \in \mathcal{L}$ we have $|\lambda - \lambda_j'|^{-1} \leq M|\lambda_j'|^{-\tau}|\lambda|^{\tau-1}, |\lambda - \lambda_{j,k}|^{-1} \leq M|\lambda_{j,k}|^{-\tau}|\lambda|^{\tau-1},\ \ \tau \in \left(\frac{1}{2m}, 1\right)$. Therefore

$$\sum_{q \leq |\lambda_j'|} |\lambda_j' - \lambda|^{-1} \leq M_1 r(q) |\lambda|^{\tau-1}, \qquad (4.4)$$

$$r(q) \overset{def}{=} \sum_{q \leq |\lambda_j'|} |\lambda_j'|^{-\tau} \to 0,\ (q \to +\infty). \qquad (4.5)$$

Here we use assertion of the theorem 1.2 on spectral estimate of operator A.

Future we have

$$\frac{1}{2\pi i}\int\limits_{\mathcal{L}}\left(\sum_{a<|\lambda_j'|\le q}(t+\lambda)^{-1}(\lambda-\lambda_j')^{-1}\right)d\lambda=\sum_{a<|\lambda_j'|\le q}{}^{'}(t+\lambda_j')^{-1}, \tag{4.6}$$

where the summation $\sum'$ being taken over those j for which $|arg\lambda_j'|<\psi$.

Taking into account that (see (4.4), (4.5))

$$\lim_{q\to+\infty} r(q)\int\limits_{\mathcal{L}}|\lambda|^{\tau-1}|t+\lambda|^{-1}d\lambda=0,$$

and passing to the limit in (4.6) as $q\to+\infty$, we find

$$\frac{1}{2\pi i}\int\limits_{\mathcal{L}}(t+\lambda)^{-1}\left(\sum_{a<|\lambda_j'|}(\lambda-\lambda_j')^{-1}\right)d\lambda=\sum_{a<|\lambda_j'|}{}^{'}(t+\lambda_j')^{-1}, \tag{4.7}$$

Analogously

$$\frac{1}{2\pi i}\int\limits_{\mathcal{L}}(t+\lambda)^{-1}\left(\sum_{a<|\lambda_{j,k}|}(\lambda-\lambda_{j,k})^{-1}\right)d\lambda=\sum_{a<|\lambda_{j,k}|}{}^{''}(t+\lambda_{j,k})^{-1},\ k=\overline{1,l}, \tag{4.8}$$

where the summation $\sum''$ being taken over those j for which $|arg\lambda_{j,k}|\le\psi$.

Operators $\widetilde{Q}_{n+1},...,\widetilde{Q}_l$ have a finite number of e.v. in angle Φ. From here and from (4.3), (4.7), (4.8) we conclude that

$$\sum_{j=1}^{+\infty}(t+\lambda_j)^{-1}=\sum_{j=1}^{+\infty}\sum_{k=0}^{n}(t+\lambda_{j,k})^{-1}+O(t^{\frac{1}{2m}-1-\varepsilon'}),\ t\to+\infty.$$

Since $arg\lambda_j\to 0(j\to+\infty)$, then $\lambda_j|\lambda_j|^{-1}\to 1\ (j\to+\infty)$. Thus for $q=1,2,...$ we have

$$\sum_{j=q}^{+\infty}|(t+\lambda_j)^{-1}-(t+|\lambda_j|)^{-1}|\le 2\sum_{j=q}^{+\infty}\left\{\frac{|\lambda_j-|\lambda_j||}{(t+|\lambda_j|)^2}\right\}\le$$

$$\le c_q\sum_{j=q}^{+\infty}\frac{|\lambda_j|}{(t+|\lambda_j|)^2}\le c_q'\sum_{j=q}^{+\infty}(t+|\lambda_j|)^{-1},$$

where $c_q,c_q'\to 0\ (q\to+\infty)$. It is easy to drive from here that

$$\sum_{j=1}^{+\infty}(t+\lambda_j)^{-1}\sim\sum_{j=1}^{+\infty}(t+|\lambda_j|)^{-1}\ (t\to+\infty).$$

For e.v. of operator $Q_k,\ k=\overline{1,n}$ it is known that

$$\sum_{j=1}^{+\infty}(t+\lambda_{j,k})^{-1}\sim\int\limits_0^{+\infty}\frac{dN_j(\tau)}{\tau+t},\ (t\to+\infty),$$

where

$$N_j(\tau)=\frac{1}{\pi}\tau^{\frac{1}{2m}}\int\limits_0^{+\infty}\rho^{-\frac{\theta}{m}}(t)\mu_j^{-\frac{1}{2m}}(t)dt.$$

Thus we have

$$\int\limits_0^{+\infty}\frac{dN(\tau)}{\tau+t}\sim\int\limits_0^{+\infty}\frac{d\widetilde{N}(\tau)}{\tau+t},\ (t\to+\infty),$$

$$\widetilde{N}(\tau)=\sum_{j=1}^{n}N_j(\tau).$$

Applying appropriate tauberian theorem we obtain the formula

$$N(t)\sim\sum_{j=1}^{n}N_j(t),\ (t\to+\infty),$$

which proves theorem 4.1.

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Makhmadrakhim G. Gadoev
Head of Department of Mathematics
Mirny Polytechnic Institute (branch) of North-Eastern Federal University named after M. K. Ammosov
ul. Tikhonova, 5/1
Mirny, Republic of Sakha (Yakutia)
678170 Russia
E-mail address: gadoev@rambler.ru

Sulaimon A. Iskhokov
Department of Mathematics
Mirny Polytechnic Institute (branch) of North-Eastern Federal University named after M. K. Ammosov
ul. Tikhonova, 5/1
Mirny, Republic of Sakha (Yakutia)
678170 Russia
E-mail address: sulaimon@mail.ru