SPECTRUM OF SOME TRIANGULATED CATEGORIES

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Abstract. In this paper, we compute the triangular spectrum (as defined by P. Balmer) of two classes of tensor triangulated categories which are quite common in algebraic geometry. One of them is the derived category of G-equivariant sheaves on a smooth scheme X, for a finite group G. The other class is the derived category of split superschemes.

1. Introduction

This paper studies the prime spectrum of two tensor triangulated categories. Triangulated categories have been one of the most influential objects in mathematics. Introduced by Grothendeick and Verdier to study Serre duality in a relative setting, this idea was soon developed by Verdier and Illusie who studied the derived category of the abelian category of coherent sheaves, and the triangulated category of perfect complexes respectively. Slowly the abstract homological construction of triangulated categories permeated into other subjects like topology, modular representation theory and even Kasparov's KK theory. Balmer's paper [3] gives a nice summary of the elegant history.

In algebraic geometry, triangulated categories mostly appear as the derived category of the abelian category of coherent sheaves on a variety and as the category of perfect complexes on a variety. The later category, as was observed by Neeman [22], are just the compact objects of the derived category of the abelian category of quasi-coherent sheaves (in case the scheme is quasi compact and separated). From now on we shall call the derived category of the category of coherent sheaves to be the derived category of the variety. Gabriel [10] and Rosenberg [24] proved that the category of quasi coherent sheaves completely determine the underlying variety. Bondal and Orlov [6] proved that a smooth variety can be reconstructed from the derived category of coherent sheaves provided that either the canonical bundle or the anti-canonical bundle is ample. But the ampleness condition here is crucial, as Mukai [19] gave an example of two nonisomorphic varieties whose derived categories are equivalent.

Balmer [3] proved that in addition to the triangulated structure on a derived category, if we also consider the tensor structure induced by the tensor structure in the category of coherent sheaves, we have enough information to reconstruct the variety. He gave a method to reconstruct, by constructing "the Spec" of the tensor triangulated category. The definition of Spec is quite general and applies

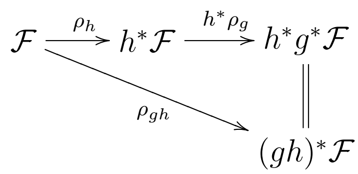
CRM Preprint Series number 1085

to any tensor triangulated category. Spectrum has been computed for few other triangulated categories, for example [4]. In his ICM talk [4, section 4.1], Balmer stressed the importance of computing Spec for more examples. We demonstrate two such examples. In both these examples, the Spec turns out to be a scheme. This reconfirms the already known fact that the Spec is not a good invariant of the tensor triangulated category. This raises the question of whether one can define a finer geometric invariant. Some possible answers are discussed in the first author's thesis with HBNI.

In section 2, we recall the definition of Spec. We also recall some facts about G sheaves and prove some lemmas which shall be useful in the next section.

In section 3 we compute the Spec of the derived category of the abelian category of coherent G-equivariant sheaves on some smooth quasi-projective scheme X. Since the scheme is quasi projective there exists an orbit space, see [20], which we denote as X/G. As G is a finite group and hence we get a finite map $\pi\colon X \to X/G$ which is also a perfect morphism. Recall that a G equivariant sheaf is defined as follows

Definition 1.1. A G-sheaf (or G-equivariant sheaf or an equivariant sheaf with respect to the group G) on X is a sheaf $\mathcal{F}$ together with isomorphisms $\rho_g\colon \mathcal{F} \to g^*\mathcal{F}$ for all $g \in G$ such that following diagram



is commutative for any pair $g, h \in G$. A G-sheaf is a pair $(\mathcal{F}, \rho)$.

The category of coherent G-sheaves is denoted as $\mathfrak{Coh}^G(X)$ and for simplicity we denote by $\mathcal{D}^G(X)$, the bounded derived category of coherent G-sheaves. Consider the affine map $\pi\colon X \to X/G$. Then $\mathcal{D}^G(X)$ admits a functor from the category of perfect complexes (see [27]) $\mathcal{D}^{per}(X/G)$,

$$\pi^*\colon \mathcal{D}^{per}(X/G) \longrightarrow \mathcal{D}^G(X).$$

Since we consider only quasi projective varieties therefore the perfect complexes are nothing but bounded complexes of vector bundles [27].

We prove the following theorem.

Theorem 1.2. *Assume that the scheme X is a smooth quasi projective variety over a field k of characteristic p with an action of a finite group G. If $p > 0$, assume that the order of G is coprime to p. The induced map*

$$\operatorname{Spec}(\pi^*)\colon \operatorname{Spec}(\mathcal{D}^G(X)) \longrightarrow \operatorname{Spec}(\mathcal{D}^{per}(X/G))$$

is an isomorphism of locally ringed spaces.

The proof involves some computation using results from representation theory.

Finally, in section 4, we compute the Spec of the tensor triangulated category of perfect complexes over a split superscheme.

Superschemes, studied by Manin and Deligne (see for example [18]), are also an important object of study in modern algebraic geometry, specially due to applications in physics. The following definition of split superscheme is given in [Pg. 84-85, Manin [17]].

Definition 1.3. (1) A ringed space $(X, \mathcal{O}_X)$ is called *superspace* if the ring $\mathcal{O}_X(U)$ associated to any open subset U is supercommutative and each stalk is local ring. A *superspace* is called *superscheme* if in addition the ringed space $(X, \mathcal{O}_{X,0})$ is a scheme and $\mathcal{O}_{X,1}$ is a coherent sheaf over $\mathcal{O}_{X,0}$ (where the subscript 0 denotes the even part and the subscript 1 denotes the odd part). We shall denote by J_X the ideal sheaf generated by $\mathcal{O}_{X,1}$ inside $\mathcal{O}_X$.

(2) A superscheme $(X, \mathcal{O}_X)$ is called *split* if the graded sheaf $Gr\mathcal{O}_X$ with mod 2 grading is isomorphic as a locally superringed sheaf to $\mathcal{O}_X$. Here the graded sheaf

$$Gr\mathcal{O}_X := \oplus_{i\geq 0} J_X^i / J_X^{i+1} \text{ where } J_X^0 := \mathcal{O}_X.$$

Manin has also given example of superschemes which are not split superschemes. An important example of a split superscheme is super projective space $\mathbb{P}^{n|m}$. We consider the triangulated category $\mathcal{D}^{per}(X)$ of "perfect complexes" (the definition being modified appropriately in the super setting) on this superscheme.

Theorem 1.4. *Let X be a split superscheme. Let $X_0 = (X, \mathcal{O}_{X,0})$ be the 0-th part of this superscheme. Here X_0 is by definition a scheme. Then we have an isomorphism of locally ringed spaces*

$$f\colon X_0 \longrightarrow \operatorname{Spec}(\mathcal{D}^{per}(X)).$$

The proof of homeomorphism adapts the classification of thick tensor ideals due to Thomason [26] as demonstrated by Balmer [3]. Again, following Balmer [3] we use the generalized localization theorem of [Theorem 2.1, Neeman [22]] to finish the proof.

Finally, we would like to mention that recently we came across a paper [16] which proves a version of theorem 1.2 for stacks. But we would like to mention that our proof is different and is completely scheme theoretic.

This article contains proofs of the results announced in [9].

Acknowledgement: We would like to take this opportunity to thank Prof. Kapil Paranjape and Prof. V. Srinivas for their encouragement and insightful comments.

We also would like to thank the referee for their suggestions.

2. Preliminaries

In this section we shall recall various basic definitions and facts which are used explicitly or implicitly later.

2.1. **Some definitions from category theory.** As we are borrowing many definitions and results from Balmer's papers [2] [3] so we shall work only with an essentially small categories i.e. categories equivalent to a small category. We recall first some basic definitions,

Definition 2.1 (Semi simple abelian category)**.** An abelian category is called *semisimple* if every short exact sequence splits.

Definition 2.2 (Triangulated category)**.** An additive category $\mathcal{D}$ with a functorial isomorphism T, (called *translation* or *shift*,) and a collection of sextuple (a, b, c, f, g, h) with objects a, b, c and morphisms f, g, h, called *distinguished triangles*, satisfying certain axioms, (cf. [28] [14],) is called *triangulated category.* Traditionally the image of any object, say a, via functor T^i is denoted as $a[i]$ and a distinguished triangle is denoted in a similar way as short exact sequences: $a \to b \to c \to a[1]$.

An additive functor $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ between two triangulated categories $\mathcal{D}_1$ and $\mathcal{D}_2$ is called a *triangulated functor* if it commutes with the translation functor i.e. $F \circ T = T \circ F$ and takes distinguished triangles to distinguished triangles, i.e. $F(a) \to F(b) \to F(c) \to F(a)[1]$ is distinguished for every distinguished triangles $a \to b \to c \to a[1]$.

Example 2.3. Let $\mathcal{A}$ be an abelian category and $K^*(\mathcal{A})$ (resp. $\mathcal{D}^*(\mathcal{A})$), for $(* = -, +$ or $b)$, be the homotopy (resp. derived) category of an abelian category $\mathcal{A}$. Then both additive categories are triangulated categories, see [11] for proof. In particular we are interested in the cases when $\mathcal{A} = \mathcal{C}\mathfrak{oh}^G(X)$ for some variety X with an action of some finite group G; see subsection 2.3 for more details. When group G is trivial then $\mathcal{A}$ is an abelian category of coherent sheaves on variety X. Another class of examples which we shall consider later comes from an abelian categories $\mathcal{A} = \mathcal{C}\mathfrak{oh}(\mathcal{O}_X)$ for some split superscheme X.

Example 2.4. The category $\mathcal{D}^{per}$ of perfect complexes on a scheme is a triangulated category. See [27] for definitions.

2.2. **Triangular spectrum.** In this section we shall recall some definitions and results from Balmer's papers [2] and [3]. Suppose $\mathcal{D}$ is an essentially small triangulated category.

Definition 2.5. A *tensor triangulated category* is a triple $(\mathcal{D}, \otimes, 1)$ consisting of a triangulated category with symmetric monoidal bifunctor which is exact in each variable. The unit is denoted by 1 (or Id).

Definition 2.6. A *thick tensor ideal* $\mathcal{A}$ of $\mathcal{D}$ is a full subcategory containing 0 and satisfying the following conditions:

(a): $\mathcal{A}$ is *triangulated*: if any two terms of a distinguished triangle are in $\mathcal{A}$ then third term is also in $\mathcal{A}$. In particular direct sum of any two objects of $\mathcal{A}$ is again in $\mathcal{A}$ and this we refer as an **additivity**.
(b): $\mathcal{A}$ is *thick*: If $a \oplus b \in \mathcal{A}$ then $a \in \mathcal{A}$.
(c): $\mathcal{A}$ is *tensor ideal*: if a or $b \in \mathcal{A}$ then $a \otimes b \in \mathcal{A}$.

If $\mathcal{E}$ is any collection of $\mathcal{D}$ then we shall denote by $\langle \mathcal{E} \rangle$ the smallest thick tensor ideal generated by this subset in $\mathcal{D}$.

Now we shall give an explicit description of a thick tensor ideal generated by some collection $\mathcal{E}$ in a tensor triangulated category. We first use some definitions from Bondal [5] here. Recall $add(\mathcal{E})$ was defined as an additive category generated by $\mathcal{E}$ and closed under taking shifts inside $\mathcal{D}$. Similarly define $ideal(\mathcal{E})$ as a full subcategory generated by objects of the form $a \otimes x$ for each $a \in \mathcal{D}$ and $x \in \mathcal{E}$. Since $(\mathbb{1} \oplus \cdots \oplus \mathbb{1}) \otimes X$ and $\mathbb{1}[k] \otimes X$ is contained in $ideal(\mathcal{E})$, it is closed under taking finite direct sum, shifts and tensoring with any object of $\mathcal{D}$. Recall the operation defined on sub categories i.e. $\mathcal{A} \star \mathcal{B}$ is the full subcategory generated by objects x which fits in a distinguished triangle of the form

$$a \longrightarrow x \longrightarrow b \longrightarrow a[1] \text{ with } a \in \mathcal{A} \text{ and } b \in \mathcal{B}.$$

This was observed by Bondal [5] et. al. that if $\mathcal{A}$ and $\mathcal{B}$ are closed under shifts and direct sums then $\mathcal{A} \star \mathcal{B}$ is also closed under shifts and direct sums. Similarly we can see that if $\mathcal{A}$ and $\mathcal{B}$ are tensor ideal then $\mathcal{A} \star \mathcal{B}$ is also tensor ideal. Take $smd(\mathcal{A})$ to be the full subcatgory generated by all direct summands of objects of $\mathcal{A}$. Now combining these two operations we can define a new operation on collections of subcategories as follows,

$$\mathcal{A} \diamond \mathcal{B} := smd(\mathcal{A} \star \mathcal{B}).$$

Using this operation we can define the full subcategories $\langle \mathcal{E} \rangle^n$ for each non-negative integer as

$$\langle \mathcal{E} \rangle^n := \langle \mathcal{E} \rangle^{n-1} \diamond \langle \mathcal{E} \rangle^0 \text{ where } \langle \mathcal{E} \rangle^0 := smd(ideal(\mathcal{E})).$$

Now we can see following description of ideal generated by a collection $\mathcal{E}$,

Lemma 2.7. $\langle \mathcal{E} \rangle = \cup_{n \geq 0} \langle \mathcal{E} \rangle^n$.

Proof of the above lemma follows from the fact that right hand side subcategory is a thick tensor ideal and contains every thick tensor ideal containing the collection $\mathcal{E}$.

Definition 2.8. (a) An additive functor, $F \colon \mathcal{D}_1 \to \mathcal{D}_2$, is called an *exact (or triangulated)* if it commutes with translation functor and takes distinguished triangle to a distingushed triangle.
(b) An exact functor, $F \colon \mathcal{D}_1 \to \mathcal{D}_2$, is called a *tensor functor* if there exists a natural isomorphism $\eta(a, b) \colon F(a) \otimes F(b) \to F(a \otimes b)$ for objects a and b of $\mathcal{D}_1$.
(c) A tensor functor, $F \colon \mathcal{D}_1 \to \mathcal{D}_2$, is called *dominant* if $\langle F(\mathcal{D}_1) \rangle = \mathcal{D}_2$.

Note that every unital tensor functors is a dominant tensor functor.

Definition 2.9. A *prime ideal* of $\mathcal{D}$ is a proper thick tensor ideal $\mathcal{P} \subsetneq \mathcal{D}$ such that $a \otimes b \in \mathcal{P} \implies a \in \mathcal{P}$ or $b \in \mathcal{P}$. And *triangular spectrum* of $\mathcal{D}$ is defined as set of all prime ideals, i.e.

$$\mathrm{Spc}(\mathcal{D}) = \{\mathcal{P} \mid \mathcal{P} \text{ is a prime ideal of } \mathcal{D}\}.$$

The Zariski topology on this set is defined as follows: closed sets are of the form

$$Z(\mathcal{S}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{D}) \mid \mathcal{S} \cap \mathcal{P} = \varnothing\},$$

where $\mathcal{S}$ is a family of objects of $\mathcal{D}$; or equivalently we can define the open subsets to be of the form

$$U(\mathcal{S}) := \mathrm{Spc}(\mathcal{D}) \backslash Z(\mathcal{S}).$$

In particular, we shall denote by

$$\mathrm{supp}(a) := Z(\{a\}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{D}) \mid a \notin \mathcal{P}\},$$

the basic closed sets and hence $U(\{a\})$ are the basic open sets.

A collection of objects $\mathcal{S} \subset \mathcal{D}$ is called a *tensor multiplicative family* of objects if $1 \in \mathcal{S}$ and if $a, b \in \mathcal{S} \implies a \otimes b \in \mathcal{S}$.

We shall recall here the following lemma(Lemma 2.2 in Balmer's paper [3]) which we shall need later,

Lemma 2.10. *Let $\mathcal{D}$ be a nontrivial tensor triangulated category and $\mathcal{I} \subset \mathcal{D}$ be a thick tensor ideal. Suppose $\mathcal{S} \subset \mathcal{D}$ is a tensor multiplicative family of objects such that $\mathcal{S} \cap \mathcal{I} = \varnothing$ Then there exists a prime ideal $\mathcal{P} \in \mathrm{Spc}(\mathcal{D})$ such that $\mathcal{I} \subset \mathcal{P}$ and $\mathcal{P} \cap \mathcal{S} = \varnothing$.*

Balmer [3] had also proved the functoriality of Spc on all essentially small tensor triangulated category with a morphism given by an unital tensor functors but it is not difficult to see that it is also true for an essentially small tensor triangulated categories with morphism given by a dominant tensor functor i.e. we have following result,

Proposition 2.11. *Given $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ a dominant tensor functor, the map $\mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{D}_2) \to \mathrm{Spc}(\mathcal{D}_1)$ defined as $\mathcal{P} \mapsto F^{-1}(\mathcal{P})$ is well defined, continuous and for all objects $a \in \mathcal{D}_1$, we have $\mathrm{Spc}(F)^{-1}(\mathrm{supp}(a)) = \mathrm{supp}(F(a))$ in $\mathrm{Spc}(\mathcal{D}_2)$.*

This defines a contravariant functor $\mathrm{Spc}(-)$ from the category of essentially small tensor triangulated categories with dominant tensor functors as morphisms to the category of topological spaces. So if F, G are two dominant tensor functors then $\mathrm{Spc}(G \circ F) = \mathrm{Spc}(F) \circ \mathrm{Spc}(G)$.

Proof. (Similar to Balmer [2]). □

Corollary 2.12. *If a tensor functor $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ is an equivalence then every quasi-inverse functor of F is a dominant tensor functor. And also $\mathrm{Spc}(F)$ is a homeomorphism.*

Proof. First observe that the continuous map $\mathrm{Spc}(F)$ given by a dominant tensor functor is independent of natural isomorphism defining the tensor functor (recall definition 2.8). Now using functoriality of above proposition we have an homeomorphism whenever a quasi-inverse of F is an dominant tensor functor. Suppose G is a quasi-inverse of F. Since $G \circ F \simeq Id$, the exact functor G is dominant. Suppose $\eta\colon F \circ G \to Id$ and $\mu\colon G \circ F \to Id$ are natural isomorphisms. Now we get a required natural isomorphism by composing as follows,

$$G(a) \otimes G(b) \xrightarrow{\mu^{-1}} GF(G(a) \otimes G(b)) = G(FG(a) \otimes FG(b)) \xrightarrow{G(\eta_a \otimes \eta_b)} G(a \otimes b).$$

Here we used a fact that $G(\eta_a \otimes \eta_b)$ gives a natural transformation. □

Now we shall recall the definition of a structure sheaf defined on $\mathrm{Spc}(\mathcal{D})$ as in Balmer [3].

Definition 2.13. For any open set $U \subset \mathrm{Spc}(\mathcal{D})$, let $Z := \mathrm{Spc}(\mathcal{D}) \setminus U$ be a closed complement and let $\mathcal{D}_Z$ be the thick tensor ideal of $\mathcal{D}$ supported on Z. We denote by $\mathcal{O}_{\mathcal{D}}$ the sheafification of following presheaf of rings: $U \mapsto \mathrm{End}(1_U)$ where $1_U \in \frac{\mathcal{D}}{\mathcal{D}_Z}$ is the image of the unit 1 of $\mathcal{D}$ via the localisation map. And the restriction maps are defined using localisation maps in the obvious way. The sheaf of commutative ring $\mathcal{O}_{\mathcal{D}}$ makes the topological space $\mathrm{Spc}(\mathcal{D})$ a ringed space, which we shall denote by $\mathrm{Spec}(\mathcal{D}) := (\mathrm{Spc}(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$.

The following theorem was proved in Balmer [3] which computes the spectrum for certain tensor triangulated categories.

Theorem 2.14 (Balmer)**.** *For X a topologically noetherian scheme,*

$$\mathrm{Spec}(\mathcal{D}^{per}(X)) \simeq X.$$

2.3. ***G*-sheaves.** Throughout this section, k is a field and G be a finite group whose order is coprime to the characteristic of k. Let X be a smooth quasi-projective variety over k, with an action of a finite group G i.e. there is a group homomorphism from G to the automorphism group of algebraic variety X. We say G acts freely on X if $gx \neq x$ for any $x \in X$ and any $g \in G$ with $g \neq e$. Recall following general result proved in Mumford's book [20] for the existence of group quotient,

Theorem 2.15. *Let X be an algebraic variety and G a finite group of automorphisms of X. Suppose that for any $x \in X$, the orbit Gx of x is contained in an affine open subset of X. Then there is a pair (Y, π) where Y is a variety and $\pi\colon X \to Y$ a morphism, satisfying:*

1. *as a topological space, (Y, π) is the quotient of X for the G-action; and*
2. *if $\pi_*(\mathcal{O}_X)^G$ denotes the subsheaf of G-invariants of $\pi_*(\mathcal{O}_X)$ for the action of G on $\pi_*(\mathcal{O}_X)$ deduced from 1, the natural homomorphism $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ is an isomorphism.*

The pair (Y, π) is determined up to an isomorphism by these conditions. The morphism π is finite, surjective and separable. Y is affine if X is affine.

If further G acts freely on X, π is an étale morphism.

In the remark after the proof, Mumford further showed that quasi-projective varieties always satisfies the hypothesis of above theorem. We denote this quotient space (if it exists) as X/G. For a variety X with a G action, and $H \subset G$ a subgroup, let X^H be the subvariety of fixed points of H.

Proposition 2.16. *With the notation in the above paragraph,*

(1) *X^H is a closed subvariety.*
(2) *If $H_1 \subseteq H_2$ are subgroups then we have a reverse inclusion $X^{H_2} \subseteq X^{H_1}$*
(3) *If Z is any G-invariant component of X then there exists an open subset of Z with free action of G/H for unique subgroup H. A G-invariant component is defined to be a minimal G-invariant subset of X with dimension equal to* $\dim X$. *Here dimension of an algebraic set is the maximum of the dimensions of its irreducible subsets.*
(4) *If Z is any G-invariant algebraic subset of X then there exists the set of subgroups H_i for $i = 1, \dots, r$ and open subsets $W_i, i = 1, \dots, r$ of Z such that G/H_i acts freely on W_i for $i = 1, \dots, r$. Here r is the number of G-invariant components of Z. Also note that the open subsets W_i are pairwise disjoint, and* $\dim(Z \setminus \cup_i W_i) < \dim Z$.

Proof of 1. Since $X^H = \cap_{h \in H} X^h$ where X^h is a fixed points of automorphism corresponding to h under the action. It is enough to prove that the invariant of any automorphism of a variety is a closed subset. Since X is a quasi-projective noetherian variety and hence separated. Therefore the diagonal and the graph of any automorphism will be closed subset of $X \times X$. The intersection of graph of automorphism with the diagonal will be closed subset of the diagonal. Hence the invariant of the automorphism h will be closed in X.

Proof of 2. It clearly follows from the formulae $X^{H_i} = \cap_{h \in H_i} X^h$.

Proof of 3. Since for any algebraic subset there exists the subgroup H such that G/H acts faithfully (or effectively), we can assume that G acts faithfully on Z. Since for a faithful action, Z^H is a proper subset of Z for any nontrivial normal subgroup H of G, the open subset of Z defined as

$$W = Z - (\cup_{H \trianglelefteq G} Z^H),$$

where union on right side is over all nontrivial normal subgroups, is non-empty and it is easy to see that G acts freely on W.

Proof of 4. Using *3.*, it is enough to prove that any algebraic subset can be uniquely written as union of G-invariant components of Z, and an algebraic subset of dimension strictly less than $\dim Z$. Since Z is noetherian, it will be finite union of irreducible closed subsets. Take finite set S of generic points of irreducible subsets of Z, which have the same dimension as Z. Now the action of G on Z induces an action on the finite set S; since an automorphism of Z will take any irreducible subset to another irreducible subset of same dimension. Thus S can be uniquely written as a disjoint union of G-invariant subsets. By taking union of closure of these generic points in each invariant subset, we get the G-invariant components of Z. Clearly, any nonempty intersection of W_i and W_j for $i \neq j$ will give a proper G-invariant component, and this will contradict the minimality. □

We shall now look at some properties of G-sheaves (definition 1.1). The G-sheaves form a category $\mathcal{QC}\mathfrak{oh}^G(X)$ as follows. Given two G-sheaves $(\mathcal{F}, \rho)$ and $(\mathcal{G}, \psi)$, the group of morphisms of $\mathcal{O}_X$-modules $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$ gets a G action, where $g \in G$ acts on θ to give $\psi_g^{-1} \circ g^*\theta \circ \rho_g$. $\mathrm{Hom}_{\mathcal{QC}\mathfrak{oh}^G(X)}((\mathcal{F}, \rho), (\mathcal{G}, \psi))$ is defined to be the group of G-invariant morphisms in $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$.

$\mathcal{QC}\mathfrak{oh}^G(X)$ is an abelian category. Define $\mathcal{C}\mathfrak{oh}^G(X)$ to be the abelian subcategory of $\mathcal{QC}\mathfrak{oh}^G(X)$ consisting of objects $(\mathcal{F}, \rho)$, for which $\mathcal{F}$ is coherent. In Tohoku paper of Grothendieck [13] it was proved that $\mathcal{QC}\mathfrak{oh}^G(X)$ has enough injectives. Also, for finite G and quasi-projective X, there is an ample invertible G-sheaf, allowing G-equivariant locally free resolutions (see [13], [7]). Therefore derived functors of various functors like π_*, π^* and $\otimes$ will always exist similar to non-equivariant case and for simplicity we shall use same notation.

Definition 2.17. Let $\mathcal{D}^G(X)$ be the bounded derived category of $\mathcal{C}\mathfrak{oh}^G(X)$.

Remark 2.18. $\mathcal{C}\mathfrak{oh}^G(X)$ can be given a tensor structure $\otimes\colon \mathcal{C}\mathfrak{oh}^G(X) \times \mathcal{C}\mathfrak{oh}^G(X) \to \mathcal{C}\mathfrak{oh}^G(X)$ in the obvious way. The tensor structure on the derived category $\mathcal{D}^G(X)$ is given by the derived functor of $\otimes$. For more details on this tensor structure, one can refer to [7]

Also $\mathcal{D}^G(X)$ has a natural structure of a k-linear category. We shall use this fact later.

Remark 2.19. Note that here and elsewhere (for example theorem 1.2 and its special cases mentioned later), we assume X to be smooth to make the definition of $\mathcal{D}^G(X)$ meaningful. It might be possible that with a proper but more general definition of $\mathcal{D}^G(X)$ we can remove the assumption that X is smooth. But we will not consider that question in this paper.

Given an algebraic variety X with an action of a finite group G we have a natural morphism $\pi\colon X \to Y := X/G$ which further gives a functor $\pi_*\colon \mathcal{C}\mathfrak{oh}^G(X) \to \mathcal{C}\mathfrak{oh}^G(Y)$ and by taking G-invariant part of image we can define a functor $\pi_*^G\colon \mathcal{C}\mathfrak{oh}^G(X) \to \mathcal{C}\mathfrak{oh}(Y)$ i.e. $\pi_*^G(\mathcal{F}, \rho) = (\pi_*(\mathcal{F}, \rho))^G$ for all $(\mathcal{F}, \rho) \in \mathcal{C}\mathfrak{oh}^G(X)$. We have following result when G acts freely on X (see Mumford's book [20] for proof).

Proposition 2.20. *Let $\pi\colon X \to Y$ be a natural quotient morphism given by free action of the finite group G on X. The map $\pi^*\colon \mathcal{C}\mathfrak{oh}(Y) \to \mathcal{C}\mathfrak{oh}^G(X)$ is an* *equivalence of abelian categories with the quasi-inverse* π_*^G*. Further locally free sheaves corresponds to locally free sheaves of the same rank.*

We can extend above equivalence to get a tensor equivalence π^* between the categories $\mathcal{D}^b(Y)$ and $\mathcal{D}^G(X)$.

Next we prove that there exists a canonical (or isotypic) decomposition, similar to finite dimensional representation of finite groups. Suppose X is a smooth quasi projective variety over a field k, with the structure morphism $\eta\colon X \to \mathrm{Spec}(k)$. The category of all coherent sheaves on affine variety $\mathrm{Spec}(k)$ can be identified with category of all finite dimensional vector spaces and the category of all G-equivariant sheaves can be identified with finite dimensional k-linear G representations. By using properties of the pullback functor η^* we can prove the following basic results; see [7] for details and some similar results.

Lemma 2.21. (1) $\eta^*(k) = \mathcal{O}_X$.
(2) $\eta^*(V_1 \otimes V_2) = \eta^*(V_1) \otimes \eta^*(V_2)$.
(3) $\eta^*(V^*) = (\eta^*(V))^*$.
(4) *If* $X = \mathrm{Spec}(R)$ *is an affine variety then we have follwoing relation between functors:*
  (a) $\eta^*(V) \otimes \widetilde{M} = \widetilde{V \otimes M}$.
  (b) $\pi_\lambda(\widetilde{M}) = \widetilde{\pi_\lambda(M)}$

Let $(\mathcal{G}, \lambda)$ be an object in $\mathfrak{Coh}^G(X)$. We shall denote $\eta^*(V) \otimes \mathcal{G}$ by $V \otimes \mathcal{G}$ for simplicity. If we take the G-invariant projector defined by $\pi_\lambda := \frac{1}{|G|}\sum_{g\in G}\lambda_g^*$, the invariance functor $(\ )^G$ is the same as $\mathcal{G} \mapsto \pi_\lambda(\mathcal{G})$. Using this, we get following functor.

Lemma 2.22. *Let* $(\mathcal{G}, \lambda)$ *be an object of* $\mathfrak{Coh}^G(X)$*. The association* $\mathcal{G}$ *going to* $\mathrm{Hom}_G(V, \mathcal{G}) := \pi_\lambda(\eta^*(V^*) \otimes \mathcal{G})$ *is functorial. Moreover, the functor* $\mathrm{Hom}_G(V, \_)\colon \mathfrak{Coh}^G(X) \to \mathfrak{Coh}(X)$ *is an exact functor, and hence so is its right derived functor.*

Notice that each object contained in the image of the functor $\mathrm{Hom}_G(V, \_)$ are trivial G-sheaves. Hence the image is contained in $\mathfrak{Coh}(X)$. Let V_λ be an irreducible representation of the group G. We have the evaluation map from $V_\lambda \otimes V_\lambda^*$ to k. We shall denote $\eta^*(V) \otimes \mathcal{G}$ by $V \otimes \mathcal{G}$ for simplicity. We can pullback usual evaluation map from representation category to bounded derived category of G-equivariant sheaves. Using above lemma 2.21 we get following morphism,

$$\eta^*(ev) \otimes id\colon V_\lambda \otimes V_\lambda^* \otimes \mathcal{F} \longrightarrow \mathcal{F}.$$

Now by using the fact that the G-invariant part of a G-module V is a direct summand of $V^* \otimes V$, and the map $\eta^*(ev) \otimes id$ we get following map, which also we denote by ev,

$$ev\colon \oplus_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, \mathcal{F}^\cdot) \longrightarrow \mathcal{F}^\cdot.$$

We have following lemma which is used later to prove canonical decomposition.

Lemma 2.23. *The association sending $\mathcal{F}^{\cdot}$ to $\oplus_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, \mathcal{F}^{\cdot})$ gives an exact functor from $\mathcal{D}^b(\mathcal{C}\mathfrak{oh}^G(X))$ to itself. Further, the objectwise morphism ev induces a natural transformation between this functor and the identity functor.*

Proof. Since the association $\mathrm{Hom}_G(V, \_)$ is a functor using lemma 2.22. Hence it is easy to see thet the association taking $\mathcal{F}^{\cdot}$ to $\oplus_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, \mathcal{F}^{\cdot})$ is functorial. Consider a morphism $f\colon \mathcal{F}_1^{\cdot} \to \mathcal{F}_2^{\cdot}$ in $\mathcal{D}^b(\mathcal{C}\mathfrak{oh}^G(X))$. Now the naturality of morphism *ev* follows from the commutativity of following diagrams,

$$\begin{array}{ccccc}
\bigoplus_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, \mathcal{F}_1^{\cdot}) & \longrightarrow & \bigoplus_\lambda V_\lambda \otimes V_\lambda^* \otimes \mathcal{F}_1^{\cdot} & \longrightarrow & \mathcal{F}_1^{\cdot} \\
\downarrow & & \downarrow & & \downarrow f \\
\bigoplus_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, \mathcal{F}_2^{\cdot}) & \longrightarrow & \bigoplus_\lambda V_\lambda \otimes V_\lambda^* \otimes \mathcal{F}_2^{\cdot} & \longrightarrow & \mathcal{F}_2^{\cdot}
\end{array}$$

Here $f\colon \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism compatible with action of finite group G and therefore gives commutativity of left square. □

We prove the canonical decomposition of any object using pullback and reduction to affine case.

Proposition 2.24. *Suppose X is an algebraic set (need not be smooth variety) over a field K with trivial action of a finite group G whose order is coprime to* $\mathrm{char}(K)$*. Let $\mathcal{F}^{\cdot}$ be a bounded complex of G-equivariant coherent sheaves i.e. ampl$(\mathcal{F}^{\cdot}) < \infty$. There exists a direct sum decomposition of $\mathcal{F}^{\cdot}$ as follows,*

$$\mathcal{F}^{\cdot} = \oplus_\lambda V_\lambda \otimes \mathcal{F}_\lambda$$

where V_λ are finite dimensional irreducible representations of G and $\mathcal{F}_\lambda = (V_\lambda^* \otimes \mathcal{F}^{\cdot})^G =: \mathrm{Hom}_G(V_\lambda, \mathcal{F}^{\cdot})$. Here the complexes $\mathcal{F}_\lambda$ are trivial G-equivariant sheaves or usual sheaves.

Proof. We shall divide proof into two steps. In the first step we prove the case of coherent sheaf concentrated in degree zero (which we refer as a pure sheaf). In the second step we prove isomorphism of the map *ev* using the first step.

Step 1. Let $\mathcal{F}^{\cdot}$ be a complex with a coherent sheaf concentrated at zero, say $\mathcal{F}$. We can assume that the variety X is affine as it is enough to prove isomorphism on any affine cover. Hence we can assume that $\mathcal{F} = \widetilde{M}$. Using again lemma 2.21 we can reduce to proving bijection of following map

$$ev\colon\ \oplus_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, M) \longrightarrow M.$$

This map is an equivariant morphism, see [Page 184, [12]] for more discussions on this. It is enough to prove that the map *ev* is bijection as a k-linear morphism but this follows from the lemma 2.25.

Step 2. Recall that for any object $\mathcal{F}^{\cdot}$ of $\mathcal{D}^b(\mathcal{C}\mathfrak{oh}^G(X)$ we have a integral function

$$\mathrm{ampl}\colon \mathcal{D}^b(\mathcal{C}\mathfrak{oh}^G(X)) \longrightarrow \mathbb{Z}; \quad \mathcal{F}^{\cdot} \longmapsto |\{i \in \mathbb{Z}/\mathcal{H}^i(\mathcal{F}^{\cdot}) \neq 0\}|$$

i.e. cardinality of nonzero cohomologies of bounded complex. Take induction on $\mathrm{ampl}(\mathcal{F}^\cdot)$. If $\mathrm{ampl}(\mathcal{F}^\cdot) = 1$ then it is isomorphic to shift of some coherent sheaf. Using lemma 2.23 we know that the morphism ev commutes with shifts and hence proof of first step of induction follows from **Step 1**. Now we prove that the induction proceeds. For that, suppose that $\mathrm{ampl}(\mathcal{F}^\cdot) = m$. Note that $\mathcal{F}^\cdot$ can be represented by a complex, whose objects have an action of G, and the action of G on the objects are compatible with the differential maps of the complex. Then, we have the usual distinguished triangle, defining a truncation:

$$\tau^{\leq n-1}\mathcal{F}^\cdot \longrightarrow \mathcal{F}^\cdot \longrightarrow \mathcal{H}^n(\mathcal{F}^\cdot)[-n] \xrightarrow{+} .$$

Here n is the maximal degree with nontrivial cohomology of complex $\mathcal{F}^\cdot$. Note that, $\tau^{\leq n-1}\mathcal{F}^\cdot$ and $\mathcal{H}^n(\mathcal{F}^\cdot)[-n]$ both have G-actions compatible with that on $\mathcal{F}$. By induction hypothesis, we also have following isomorphisms

$$ev\colon \bigoplus_\lambda V_\lambda \otimes (\tau^{\leq n-1}\mathcal{F}^\cdot)_\lambda \longrightarrow \tau^{\leq n-1}\mathcal{F}^\cdot$$
$$ev\colon \bigoplus_\lambda V_\lambda \otimes (\mathcal{H}^n(\mathcal{F}^\cdot)[-n])_\lambda \longrightarrow \mathcal{H}^n(\mathcal{F}^\cdot)[-n].$$

Also, $(\tau^{\leq n-1}\mathcal{F})_\lambda = \mathrm{Hom}_G(V_\lambda, \tau^{\leq n-1}\mathcal{F}^\cdot)$, $(\mathcal{H}^n(\mathcal{F})[-n])_\lambda = \mathrm{Hom}_G(V_\lambda, \mathcal{H}^n(\mathcal{F})[-n])$. Now using lemma 2.23 we have following commutative diagram,

$$\begin{array}{ccccccc}
\bigoplus_\lambda V_\lambda \otimes (\tau^{\leq n-1}\mathcal{F}^\cdot)_\lambda & \longrightarrow & \bigoplus_\lambda V_\lambda \otimes \mathcal{F}^\cdot_\lambda & \longrightarrow & \bigoplus_\lambda V_\lambda \otimes (\mathcal{H}^n(\mathcal{F}^\cdot)[-n])_\lambda & \xrightarrow{+1} & . \\
\downarrow & & \downarrow & & \downarrow & & \\
\tau^{\leq n-1}\mathcal{F}^\cdot & \longrightarrow & \mathcal{F}^\cdot & \longrightarrow & \mathcal{H}^n(\mathcal{F}^\cdot)[-n] & \xrightarrow{+1} &
\end{array}$$

i.e. we have morphism of two distinguished triangles. The extreme vertical arrows are isomorphisms by induction. Hence so is the middle. This completes the induction step.

Hence using these two steps we have the canonical decomposition as stated and further it is easy to observe that $\mathcal{F}^\cdot_\lambda$ are trivial as G-sheaves i.e. all ρ_g are identity, see definition 1.1. □

Lemma 2.25. *Suppose M is a k-linear G-representation (need not be finite dimensional) for finite group G. The following canonical evaluation map is an isomorphism*

$$ev\colon \oplus_\lambda V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, M) \longrightarrow M.$$

Proof. See [Proposition 4.1.15, [12]]. □

We shall use proposition 2.24 in the following form.

Corollary 2.26. *Let X be a smooth algebraic variety defined over k with a G action. Let $U \subset X$ be a (possibly singular) G-invariant, locally closed subset, with $i_U\colon U \to X$ being the inclusion. Suppose H is a subgroup of G with the* *property that it acts trivially on U. Then for any object $(\mathcal{G}, \rho) \in \mathcal{D}^G(X)$ we have the canonical decomposition,*

$$(\mathcal{F}, \rho) = \oplus_\lambda W_\lambda \otimes (\mathcal{F}, \rho)_\lambda$$

where $\mathcal{F} = i_U^* \mathcal{G}$ and $(\mathcal{F}, \rho)_\lambda = (W_\lambda^* \otimes (\mathcal{F}, \rho))^H$ and W_λ is a finite dimensional irreducible representation of the subgroup H, and sum is over all finite dimensional irreducible representation of H. The subgroup H acts trivially on $(\mathcal{F}, \rho)_\lambda$ and this will induce the natural action of the group G/H on $(\mathcal{F}, \rho)_\lambda$.

Proof. Note that $\mathcal{G}$ has finite amplitude length, and hence so does $\mathcal{F}$. Thus the above proposition applies to U. □

Note that if G acts trivially on X then we can take $H = G$ and as a particular case we shall get the canonical decomposition,

$$(\mathcal{F}, \rho) = \oplus_\lambda V_\lambda \otimes (\mathcal{F}, \rho)_\lambda$$

where $(\mathcal{F}, \rho)_\lambda = (V_\lambda^* \otimes (\mathcal{F}, \rho))^G$ and V_λ is a finite dimensional irreducible representation of the group G. Now we shall give a distinguished triangle for any complex of G-equivariant coherent sheaf $\mathcal{F}$ over X. We have following result,

Proposition 2.27. *Let G, k and X be as above.*

1. *Suppose U is any G-invariant open subset of X with induced action. If π_ρ denotes the projector $\frac{1}{|G|} \sum_{g \in G} \rho_g^*$ on X where ρ_g is automorphism of X coming from the action of G then $i_U^* \circ \pi_\rho = \pi_\rho \circ i_U^*$. Here, π_ρ is also used to denote its restriction on open set U.*
2. *Suppose G acts faithfully on X. If $\mathcal{F} \in \mathcal{D}^G(X)$ with $\mathrm{supph}(\mathcal{F}) = X$ then we have a distinguished triangle*

$$\pi^* \pi_*^G(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_1$$

with $\mathrm{supph}(\mathcal{F}_1) \subsetneq \mathrm{supph}(\mathcal{F})$. Same is true if we have faithful action of G on $\mathrm{supph}(\mathcal{F}) \subsetneq X$.

Proof of 1. Since U is a G-invariant subset, each automorphism ρ_g of X induces an automorphism. For simplicity we use the same notation ρ_g. Now assertion immediately follows from the following commutative square,

$$\begin{array}{ccc} U & \xhookrightarrow{i_U} & X \\ \downarrow{\scriptstyle \rho_g} & & \downarrow{\scriptstyle \rho_g} \\ U & \xhookrightarrow{i_U} & X \end{array}$$

flat base change and additivity.

Proof of 2. Since G acts faithfully on X we can use proposition 2.16 to get an open subset $U \subseteq X$ with free action of group G . We shall use induction on amplitude length, $\mathrm{ampl}(\mathcal{F})$. When $\mathrm{ampl}(\mathcal{F}) = 1$ then $\mathcal{F}$ is a shift of a coherent sheaf so enough to prove for coherent sheaf. Now using the fact that $\mathrm{supph}(\mathcal{F}) = X$

we have $i_U^*(\mathcal{F}) \neq 0$. There is a natural morphism coming from adjunction and inclusion of G-invariant part, say $\eta\colon \pi^*\pi_*^G(\mathcal{F}) \to \mathcal{F}$. Using flat base change and part 1 of 2.27 we get an isomorphism $i_U^*\pi^*\pi_*^G(\mathcal{F}) \simeq \pi^*\pi_*^G(i_U^*\mathcal{F})$. Now this will give an isomorphism, as G act freely on U, i.e. $i_U^*(\eta)\colon i_U^*\pi^*\pi_*^G(\mathcal{F}) \to i_U^*\mathcal{F}$ is an isomorphism. Hence cone of the map η will have support outside an open set U. This completes the first step of induction.

Now assume the for all $\mathcal{F}$ with $\operatorname{ampl}(\mathcal{G}) \leq (n-1)$ we have such a distinguished triangle. Now consider $\mathcal{F}$ with $\operatorname{ampl}(\mathcal{F}) = n$ with highest cohomology in degree n. We have usual truncation distinguished triangle $\tau^{\leq(n-1)}(\mathcal{F}) \to \mathcal{F} \to \mathcal{H}^n(\mathcal{F})[-n]$. Using exactness of i_U^* and argument similar to first step of induction we have a following commutative diagram (we have used same notation η for different sheaves),

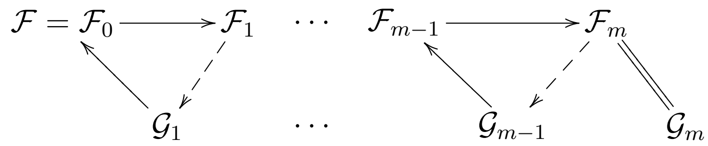
$$\begin{array}{ccccc} i_U^*\pi^*\pi_*^G\tau^{\leq(n-1)}(\mathcal{F}) & \longrightarrow & i_U^*\pi^*\pi_*^G\mathcal{F} & \longrightarrow & i_U^*\pi^*\pi_*^G\mathcal{H}^n(\mathcal{F})[-n] \\ \downarrow{\scriptstyle i_U^*(\eta)} & & \downarrow{\scriptstyle i_U^*(\eta)} & & \downarrow{\scriptstyle i_U^*(\eta)} \\ i_U^*\tau^{\leq(n-1)}(\mathcal{F}) & \longrightarrow & i_U^*\mathcal{F} & \longrightarrow & i_U^*\mathcal{H}^n(\mathcal{F})[-n] \end{array}$$

Since both the extreme vertical arrows are isomorphism using induction hypothesis, we have isomorphism of the middle $i_U^*(\eta)$. Therefore cone of the map η will have proper support. ☐

Lemma 2.28. *Let $\pi\colon X \to Y$ be the quotient map as before.*

(1) *Given $\mathcal{F} \in \mathcal{D}^G(X)$ we have* $\operatorname{supph}(\pi_*\mathcal{F}) = \pi(\operatorname{supph}\mathcal{F})$.

(2) *There exists a tower of distinguished triangles for each object $\mathcal{F}$ in $\mathcal{D}^G(X)$,*



where $\mathcal{G}_i = \bigoplus_{\lambda_i} W_{\lambda_i} \otimes \pi^*\pi_*^{G/H_i}(\mathcal{F}_{\lambda_i})$ with the sum being over the irreducible representations of the corresponding H_i's, $\operatorname{supph}(\mathcal{F}_m) \subsetneq \ldots \subsetneq \operatorname{supph}(\mathcal{F})$. *Here, for a representation R and a sheaf $\mathcal{H}$, $R \otimes \mathcal{H}$ means $i_{Z,*}(\eta_Z^* R \otimes \mathcal{H})$ where $Z = \operatorname{supph}\mathcal{H}$ and $\eta_Z\colon Z \to \operatorname{Spec} k$ is the corresponding structure morphism.*

Furthermore

$$\operatorname{supph}(\pi_*^{G/H_j}(\mathcal{F}_{\lambda_j})) \subseteq \operatorname{supph}(\pi_*(\mathcal{F}_{\lambda_j})) = \pi(\operatorname{supph}(\mathcal{F}_{\lambda_j})).$$

Proof of 1. Consider $\mathcal{F} \in \mathcal{D}^G(X)$ a complex of G-sheaves. We have the special case of the Grothendieck-Leray spectral sequence [15, Pg. 74 (3.4)] as follows,

$$E_2^{p,q} = R^p\pi_*(\mathcal{H}^q(\mathcal{F})) \Longrightarrow R^{p+q}\pi_*(\mathcal{F}).$$

Since $R^p\pi_* = 0$ for each $p > 0$ the above spectral sequence will degenerate and we get that $\pi_*(\mathcal{H}^i(\mathcal{F})) = \mathcal{H}^i(\pi_*(\mathcal{F}))$. Here as before $\mathcal{H}^i(\mathcal{F})$ represents the i-th cohomology sheaf of the complex $\mathcal{F}$. Now this will give following equality.

$$\mathrm{supph}(\pi_*(\mathcal{F})) = \cup_i \mathrm{supp}(\mathcal{H}^i(\pi_*\mathcal{F})) = \cup_i \mathrm{supp}(\pi_*\mathcal{H}^i(\mathcal{F}))$$

suppose we prove the assertion for pure sheaves, i.e. complexes of sheaves concentrated on degree 0, then following observation will complete the proof.

$$\mathrm{supph}(\pi_*\mathcal{F}) = \cup_i \mathrm{supp}(\pi_*\mathcal{H}^i(\mathcal{F})) = \cup_i \pi(\mathrm{supp}(\mathcal{H}^i(\mathcal{F}))) = \pi(\mathrm{supph}(\mathcal{F})).$$

Now it remains to prove the assertion for pure sheaves. We shall denote by $\mathcal{F}_U$ the restriction of the sheaf $\mathcal{F}$ on the open set U of X. Suppose V_j is an open affine cover of Y and $U_j := \pi^{-1}(V_j)$ is the affine open cover of X. We shall denote the restriction of the map π on U_j with same notation π. Now using the flat base change we have $\pi_*(\mathcal{F}_{U_j}) = (\pi_*\mathcal{F})_{V_j}$ for any sheaf $\mathcal{F}$ on X. Suppose the above assertion is true for affine case then following observations will complete the proof.

$$\begin{aligned}\pi(\mathrm{supp}(\mathcal{F})) &= \pi(\cup_j(\mathrm{supp}(\mathcal{F}) \cap U_j)) = \cup_j \pi(\mathrm{supp}\,\mathcal{F}_{U_j}) = \cup_j \mathrm{supp}(\pi_*\mathcal{F}_{U_j}) \\ &= \cup_j \mathrm{supp}((\pi_*\mathcal{F})_{V_j}) = \cup_j \mathrm{supp}(\pi_*\mathcal{F}) \cap V_j = \mathrm{supp}(\pi_*\mathcal{F}).\end{aligned}$$

It remains to prove the assertion for pure sheaves on affine varieties. Suppose $\pi\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is a map, and $\tilde{N}$ is a pure sheaf on $\mathrm{Spec}(B)$, corresponding to the B-module N. Since A and B are noetherian ring therefore this reduces to following fact.

$$V(\mathrm{ann}(_AN)) = \pi(V(\mathrm{ann}(N)).$$

Here $\mathrm{ann}(N)$ denotes the annihilator ideal and $V(\mathrm{ann}(N))$ denotes the closed set given by all prime ideal containing the ideal $\mathrm{ann}(N)$. Let $\bar{\pi}\colon A \to B$ be the algebra map corresponding to π.

Now to show $V(\mathrm{ann}(_AN)) = \pi(V(\mathrm{ann}(N))$ it is enough to prove that

$$\bar{\pi}^{-1}(\mathrm{ann}(N)) = \mathrm{ann}(_AN).$$

This follows as $x \in \bar{\pi}^{-1}(\mathrm{ann}(N))$ iff $\bar{\pi}(x)N = 0$. This is equivalent to $x(_AN) = 0$ which in turn holds iff $x \in \mathrm{ann}(_AN)$. This concludes the proof of 1.

Proof of 2. To prove the first part we use induction on the dimension of the homological support of $\mathcal{F}$. Note that the homological support is invariant under the action of G. If dimension is zero then it will be set of G-invariant points and we shall get the direct sums of skyscrapers on these points. If we have free action of G/H for some subgroup H then we shall have the canonical decomposition by 2.26. This will prove that the induction starts.

For the induction step, assume that for all $\mathcal{G}$ with $\dim \mathrm{supph}(\mathcal{G}) \leq n-1$, we have a tower as in the statement of the lemma. Now consider $\mathcal{F}$ with $\dim \mathrm{supph}(\mathcal{F}) = n$. Here $\mathrm{supph}(\mathcal{F})$ is a union of G-invariant components and using the proposition 2.16 we get subsets U_i, open in $\mathrm{supph}(\mathcal{F})$ for $i = 1, \ldots, r$ and subgroups H_i for $i = 1, \ldots, r$. As observed before, these U_i are mutually disjoint and there is a free action of group G/H_i on U_i for $i = 1, \ldots, r$. Consider the open subset $U_1 \subset \mathrm{supph}(\mathcal{F})$. Let i_{U_1} be the inclusion of U_1 in X. By 2.26, we can decompose $i^*_{U_1}(\mathcal{F})$ as

$$i^*_{U_1}(\mathcal{F}) = \bigoplus_{\lambda} W_\lambda \otimes \mathcal{F}_\lambda$$

where each W_λ is an irreducible representation of subgroup H_1, and the $\mathcal{F}_\lambda$'s are G/H_1-sheaves over the open subset U_1. Using adjunction and 2.20, we get a canonical isomorphism, $\eta_\lambda \colon \pi^*\pi_*^{G/H_1}(\mathcal{F}_\lambda) \to \mathcal{F}_\lambda$ in $\mathcal{D}^G(U_1)$. Putting these together, we get an isomorphism

$$\bigoplus_{\lambda} W_\lambda \otimes \pi^*\pi_*^{G/H_1}(\mathcal{F}_\lambda) \xrightarrow{\sim} i^*_{U_1}(\mathcal{F}) = \bigoplus_{\lambda} W_\lambda \otimes \mathcal{F}_\lambda. \tag{1}$$

Let $\mathcal{F}_{\lambda_1} = (i_{U_1})_*\mathcal{F}_\lambda$. Then, $\mathcal{F}_\lambda \cong i^*_{U_1}\mathcal{F}_{\lambda_1}$, since the adjunction map $i^*_{U_1} i_{U_1 *}\mathcal{F}_{\lambda_1} \to \mathcal{F}_{\lambda_1}$ induces an isomorphism on stalks, as U_1 is open in $\mathrm{supph}(\mathcal{F}_{\lambda_1})$. Also, since U_1 is open in $\mathrm{supph}(\mathcal{F})$, there exists an open subset $\tilde{U_1} \subset X$ such that $\tilde{U_1} \cap \mathrm{supph}(\mathcal{F}) = U_1$. Let $\bar{U_1} = \tilde{U_1} \cup (X \setminus \mathrm{supph}(\mathcal{F})$. Now we shall prove that

$$\pi^*\pi_*^{G/H_1} i^*_{\bar{U}_1}(\mathcal{F}_{\lambda_1}) \cong i^*_{\bar{U}_1}\pi^*\pi_*^{G/H_1}(\mathcal{F}_{\lambda_1}).$$

This follows from flat base change and some functorial properties, by considering the diagram,

$$\begin{array}{ccc} \bar{U}_1 & \xrightarrow{i_{\bar{U}_1}} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ V_1 & \xrightarrow{i_{V_1}} & Y \end{array}$$

and from the following sequence of canonical isomorphisms,

$$\begin{aligned} i^*_{\bar{U}_1}(\pi^*\pi_*^{G/H_1}(\mathcal{F}_{\lambda_1})) &\cong \pi^* i^*_{V_1}(\pi_*(\mathcal{F}_{\lambda_1}))^{G/H_1} \cong \pi^*(i^*_{V_1}\pi_*(\mathcal{F}_{\lambda_1}))^{G/H_1} \\ &\cong \pi^*(\pi_* i^*_{\bar{U}_1}(\mathcal{F}_{\lambda_1}))^{G/H_1} = \pi^*\pi_*^{G/H_1} i^*_{\bar{U}_1}(\mathcal{F}_{\lambda_1}). \end{aligned}$$

The isomorphism, proved in the previous paragraph, and equation (1) implies that the map $i^*_{U_1}(\tilde{\eta})$ is an isomorphism by looking at stalks, where $\tilde{\eta}$ is the map $\oplus_\lambda W_\lambda \otimes \pi^*\pi_*^{G/H_1}(\mathcal{F}_{\lambda_1}) \to \mathcal{F}$ coming from the appropriate adjunction maps. We shall denote $\oplus_\lambda W_\lambda \otimes \pi^*\pi_*^{G/H_1}(\mathcal{F}_{\lambda_1})$ by $\mathcal{G}_1$. Now using 1. of 2.28 we get that $\mathrm{supph}(\pi_*^{G/H_1}(\mathcal{F}_{\lambda_1})) \subseteq \mathrm{supph}(\pi_*(\mathcal{F}_{\lambda_1})) = \pi(\mathrm{supph}(\mathcal{F}_{\lambda_1}))$.

From the above discussion, the cone of the map $\tilde{\eta}$, say $\mathcal{F}_1$, will have the property that $i^*_{U_1}(\mathcal{F}_1) = 0$ and hence $\mathrm{supph}(\mathcal{F}_1) \subseteq (\mathrm{supph}(\mathcal{F}) \setminus U_1) \subsetneq \mathrm{supph}(\mathcal{F})$. Now we can proceed similarly with $\mathcal{F}_1$ whose support has less number of G-invariant components than $\mathcal{F}$ and hence in finitely many steps (in less than r steps) the dimension of homological support will drop. Hence we shall get $\mathcal{F}_i$ and $\mathcal{G}_i$ for $i = 1, \ldots, s$ with the stated restrictions on supports. The dimension of $\mathrm{supph}(\mathcal{F}_s) \leq n-1$ and that concludes the induction step. ☐

3. Example: Derived category of equivariant sheaves

In this section we shall compute Balmer's triangular spectrum for some particular examples. This computation of triangular spectrum also motivates the need for some finer geometric structures attached to a given tensor triangulated category.

Throughout this section, G is a finite group and k is a field whose characteristic is coprime to the order of G. The varieties we consider will be defined over this field k.

3.1. **Equivariant sheaves.** In this example we shall compute Balmer's triangular spectrum for equivariant sheaves over some quasi-projective varieties with G-action. We shall first do some particular cases before going to general case. The general case will be done in the next subsection 3.2.

Note that the next subsection 3.2 does not depend on this one. This subsection is only there to provide a motivation for the result and to demonstrate some easy proofs in simple cases.

Case 1: X is a point. Let G and k be as above. As usual $\mathcal{R}\mathfrak{ep}(G)$ is the category of all finite dimensional k linear representation of a group G. We can define a strict symmetric monoidal structure on this category using the usual tensor product of representations i.e. if V_1 and V_2 are two representations of G then $V_1 \otimes V_2$ is the tensor product as k vector spaces with diagonal action. We shall denote the bounded derived category of abelian category $\mathcal{R}\mathfrak{ep}(G)$ (resp. $\mathcal{R}\mathfrak{ep}(\{0\})$, for the trivial group $\{0\}$) as $\mathcal{D}_{k[G]}$ (resp. $\mathcal{D}_k$). We can extend the above tensor product of representations to get a symmetric tensor triangulated structure on $\mathcal{D}_{k[G]}$.

Proposition 3.1. $\mathrm{Spec}(\mathcal{D}_{k[G]}) \cong \mathrm{Spec}(\mathcal{D}_k) \cong \mathrm{Spec}(k)$.

We give two proofs of this proposition. The second proof generalizes to the final general case. The reason for including the first proof is purely to demonstrate another method of seeing the above statement, and has no other implication in the paper.

First proof. Since $\mathcal{R}\mathfrak{ep}(\{0\})$ is a semisimple abelian category with k as its unit it is easy to see that $\mathrm{Spec}(\mathcal{D}_k) \cong \mathrm{Spec}(k)$ as a variety. Therefore it is enough to prove the first isomorphism. The unit object of $\mathcal{D}_{k[G]}$ is k with endomorphism ring isomorphic to k so it remains to say that the trivial ideal, i.e. ideal with only zero object, is the only prime ideal. To prove this observe that for any nonzero finite dimensional representation of G, say W, we have the representation $W^* \otimes W \simeq \mathrm{End}_k(W, W)$ containing G invariant element given by identity endomorphism. Now this G invariant element will give trivial representation as an summand of $W^* \otimes W$. So if any primes ideal contains any any non-zero representation then using thickness we will get unit object inside prime ideal which is absurd. $\square$

As mentioned earlier, for the sake of generalisation we shall give another proof of proposition 3.1.

Second proof of Proposition 3.1. Consider the two exact tensor functors $F\colon \mathcal{D}_{k[G]} \to \mathcal{D}_k$ and $G\colon \mathcal{D}_k \to \mathcal{D}_{k[G]}$ where F is the forgetful functor and G comes from the augmentation map of the group algebra $k[G]$ i.e. sending each complex of vector space to a complex of $k[G]$ module with the trivial action of a group G. Note that $F \circ G = Id$ and hence $\mathrm{Spec}(G) \circ \mathrm{Spec}(F) = \mathrm{Id}$. Hence the following lemma will complete the proof of the proposition. □

Lemma 3.2. $\mathrm{Spec}(F) \circ \mathrm{Spec}(G) = \mathrm{Id}$.

Proof. Let $\mathcal{P} \in \mathrm{Spec}(\mathcal{D}_{k[G]})$ be a prime ideal. We want to prove that $(G \circ F)^{-1}(\mathcal{P}) = \mathcal{P}$. If $V \in \mathcal{M}\mathfrak{od}(k[G])$ is any $k[G]$-module, then we have the canonical decomposition,

$$V = \bigoplus_{\lambda} V_\lambda \otimes (V_\lambda^* \otimes V)^G$$

where V_λ is an irreducible representation of a group G. Further $(V_\lambda^* \otimes V)^G$ is a direct summand of $(V_\lambda^* \otimes V)$ as is seen using the projector $\frac{1}{|G|}\sum_{g\in G} \rho_g$ where ρ_g comes from the action of a group G on $(V_\lambda^* \otimes V)$. Since any complex in $\mathcal{D}_{k[G]}$ is isomorphic to the direct sum of translates of the cohomology of that complex, to prove above assertion its enough to prove that $(G \circ F)^{-1}(\mathcal{P} \cap \mathcal{M}\mathfrak{od}(k[G])) = \mathcal{P} \cap \mathcal{M}\mathfrak{od}(k[G])$. Observe that,

$$\begin{aligned} V \in (\mathcal{P} \cap \mathcal{M}\mathfrak{od}(k[G])) &\iff (V_\lambda \otimes V)^G \in (\mathcal{P} \cap \mathcal{M}\mathfrak{od}(k[G]) \\ &\qquad \text{using thickness and additivity} \\ &\iff (V_\lambda \otimes V)^G \in (G \circ F)^{-1}(\mathcal{P} \cap \mathcal{M}\mathfrak{od}(k[G])) \\ &\qquad \text{Since } (G \circ F)(W) = W \text{ if } G \text{ acts trivially on } W \\ &\iff V \in (G \circ F)^{-1}(\mathcal{P} \cap \mathcal{M}\mathfrak{od}(k[G])) \\ &\qquad \text{using thickness and additivity.} \end{aligned}$$

The above observation completes the proof of the lemma, and hence of proposition 3.1. □

Case 2: X smooth variety with a trivial G action. In this case, we shall extend the above example. Let X be a smooth variety considered as a space with the trivial action of a finite group G. Recall the definitions and some properties of a G-sheaves from the preliminary section 2. Let $\mathcal{C}\mathfrak{oh}(X)$ (resp. $\mathcal{C}\mathfrak{oh}^G(X)$) be the abelian category of all coherent sheaves (resp. coherent G-sheaves) over X. We have two functors F and G similar to the previous example defined as follows,

$$\begin{aligned} F\colon \mathcal{C}\mathfrak{oh}^G(X) &\longrightarrow \mathcal{C}\mathfrak{oh}(X) \qquad \& \qquad & G\colon \mathcal{C}\mathfrak{oh}(X) &\longrightarrow \mathcal{C}\mathfrak{oh}^G(X) \\ (\mathcal{F}, \rho) &\longmapsto \mathcal{F} & \mathcal{F} &\longmapsto (\mathcal{F}, id) \end{aligned}$$

Note that the functor F (respectively G) is a faithful (respectively fully faithful) exact functor. Thus we get two exact derived functors of the above two functors, $F\colon \mathcal{D}^G(X) \to \mathcal{D}^b(X)$ and $G\colon \mathcal{D}^b(X) \to \mathcal{D}^G(X)$ which by abuse of notation are denoted by the same symbols.

Recall that $\mathcal{D}^G(X)$ and $\mathcal{D}^b(X)$ are a tensor triangulated categories which makes the functors F and G unital tensor functors and hence using the functorial property of "Spec" we shall get two morphisms $\mathrm{Spec}(F)\colon \mathrm{Spec}(\mathcal{D}^b(X)) \to \mathrm{Spec}(\mathcal{D}^G(X))$ and $\mathrm{Spec}(G)\colon \mathrm{Spec}(\mathcal{D}^G(X)) \to \mathrm{Spec}(\mathcal{D}^b(X))$.

Proposition 3.3. $\mathrm{Spec}(\mathcal{D}^G(X)) \cong \mathrm{Spec}(\mathcal{D}^b(X)) \cong X$.

Proof. Here, the second isomorphism was proved by Balmer [3] which enables him to reconstruct the variety from its associated tensor triangulated category of coherent sheaves. We shall use the idea of previous example to prove the first isomorphism.

Since $F \circ G = Id$, functoriality of the "Spec" will give $\mathrm{Spec}(G) \circ \mathrm{Spec}(F) = \mathrm{Id}$. Now it remains to prove that $\mathrm{Spec}(F) \circ \mathrm{Spec}(G) = \mathrm{Id}$. Note that every object $(\mathcal{F}, \rho) \in \mathcal{D}^G(X)$ has the canonical decomposition as follows,

$$(\mathcal{F}, \rho) = \bigoplus_{\lambda} V_\lambda \otimes (\mathcal{F}, \rho)_\lambda$$

where $(\mathcal{F}, \rho)_\lambda = (V_\lambda^* \otimes (\mathcal{F}, \rho))^G$ and V_λ is a finite dimensional irreducible representation of the group G, see section 2 for proof. Also note that $(\mathcal{F}, \rho)_\lambda$ is an ordinary sheaf with the trivial action of a group G and also using similar projector as above, i.e. $\frac{1}{|G|}\sum_{g \in G} \rho_g$, we can prove that $(\mathcal{F}, \rho)_\lambda$ is an direct summand of the sheaf $(V_\lambda^* \otimes (\mathcal{F}, \rho))$. Now we use the following lemma.

Lemma 3.4. $\mathrm{Spec}(F) \circ \mathrm{Spec}(G) = \mathrm{Id}$.

Proof. Let $\mathcal{P} \in \mathrm{Spec}(\mathcal{D}^G(X))$ be a prime ideal. We want to prove that $(G \circ F)^{-1}(\mathcal{P}) = \mathcal{P}$. Now using the canonical decomposition of each objects of the triangulated category $\mathcal{D}^G(X)$. we have,

$$\begin{aligned}(\mathcal{F}, \rho) \in \mathcal{P} &\iff (\mathcal{F}, \rho)_\lambda \in \mathcal{P} \text{ using thickness, additivity and projector} \\ &\iff (\mathcal{F}, \rho)_\lambda \in (G \circ F)^{-1}(\mathcal{P}) \\ &\qquad \text{Since } (G \circ F)(\mathcal{F}, id) = (\mathcal{F}, id) \text{ if } G \text{ acts trivially i.e. } \rho = id \\ &\iff (\mathcal{F}, \rho) \in (G \circ F)^{-1}(\mathcal{P}) \\ &\qquad \text{using thickness, additivity and projector.}\end{aligned}$$

Hence the above observation completes the proof of lemma. □

Now, using the above lemma, it follows that $\mathrm{Spec}(F)$ is an isomorphism between $\mathrm{Spec}(\mathcal{D}^G(X))$ and $\mathrm{Spec}(\mathcal{D}^b(X))$. □

Remark 3.5. The proof for the case of trivial action on smooth varieties does not need the assumption of quasi-projectivity on the variety X. However, the condition of quasi-projectivity is necessary for the general case, to ensure the existence of the quotient variety.

Case 3: G acts freely on a smooth variety X. Now we shall consider the case where a finite group G acts freely on X. We refer to section 2 for the definition. Recall that we have a canonical map $\pi\colon X \to Y := X/G$ which is a G-equivariant map with the trivial action of G on Y. Now we can also define two functors associated with π: $\pi^*\colon \mathfrak{Coh}(Y) \to \mathfrak{Coh}^G(X)$ and $\pi_*^G\colon \mathfrak{Coh}^G(X) \to \mathfrak{Coh}(Y)$ where $\pi_*^G = G$-equivariant part of π_*. We had also seen in 2 that π^* is a tensor functor in general; and when G acts freely it is also an equivalence of categories with π_*^G as its quasi-inverse. Hence we shall get an equivalence of the tensor triangulated categories $\mathcal{D}^b(Y)$ and $\mathcal{D}^G(X)$. Since an equivalence gives an isomorphism of "Spec", (cf. section 2), therefore we get an isomorphism $\mathrm{Spec}(\pi^*)\colon \mathrm{Spec}(\mathcal{D}^G(X)) \to \mathrm{Spec}(\mathcal{D}^b(Y))$ with its inverse given by $\mathrm{Spec}(\pi_*^G)$. In fact using case 2 and this argument, we can give slightly more general statement as follows.

Corollary 3.6. *Suppose finite group G acts freely on a quasi-projective variety X modulo some normal subgroup H. In other words, the subgroup H acts trivially, and the induced action of the quotient group G/H is free. Then*

$$\mathrm{Spec}(\mathcal{D}^G(X)) \cong \mathrm{Spec}(\mathcal{D}^b(Y)) \cong Y$$

where $Y := X/G$ as before.

Proof. As mentioned above, the proof goes in similar lines as in case 2, using a more general canonical decomposition of objects of $\mathcal{D}^G(X)$:

$$(\mathcal{F},\rho) = \bigoplus_{\lambda} W_\lambda \otimes (\mathcal{F},\rho)_\lambda$$

where $(\mathcal{F},\rho)_\lambda = (W_\lambda^* \otimes (\mathcal{F},\rho))^H$, W_λ is a finite dimensional irreducible representation of the group H, and the group G/H acts naturally on $(\mathcal{F},\rho)_\lambda$. See corollary 2.26 for the proof. ☐

Finally we tackle the general case. Since the proof is a bit long, we devote a full subsection to it.

3.2. **Case 4: The general case.** In this case we shall consider the more general situation of a finite group G acting on a smooth quasi-projective variety X and we further assume that the group G acts faithfully. As is assumed throughout the section, the order of G and the characteristic of the base field k are coprime to each other. Define $\pi\colon X \to Y := X/G$ as above an G-equivariant map. Here the action of G on Y is trivial. Note that for a finite group, the quotient space always exists 2. We shall now prove the following.

Proposition 3.7. $\mathrm{Spec}(\mathcal{D}^G(X)) \cong \mathrm{Spec}(\mathcal{D}^{per}(Y) \cong Y$.

Here again as before the second isomorphism is a particular case of the more general reconstruction result of Balmer [2] [3]. Hence we shall just prove the first isomorphism. We know there are two exact functors $\pi^*\colon \mathcal{D}^{per}(Y) \to \mathcal{D}^G(X)$ and $\pi_*\colon \mathcal{D}^G(X) \to \mathcal{D}^{per}(Y)$. We also know that the map π^* is an unital tensor functor and hence it will give the map $\mathrm{Spec}(\pi^*)\colon \mathrm{Spec}(\mathcal{D}^G(X)) \to \mathrm{Spec}(\mathcal{D}^{per}(Y))$. Note that π_* need not be a tensor functor. We shall prove that $\mathrm{Spec}(\pi^*)$ is a closed bijection and induces an isomorphism for the structure sheaves.

To simplify the proof we will break it in several steps. The first two steps will prove that $\mathrm{Spec}(\pi^*)$ gives a bijection of sets on the underlying topological spaces of the two Specs in question. The next step will show that the underlying topological spaces are homeomorphic. Then finally in step 4 we prove that the Specs of the tensor triangulated categories under consideration, are isomorphic as ringed spaces.

Step 1: $\mathrm{Spec}(\pi^*)$ *is onto.* Suppose $\mathfrak{q} \in \mathrm{Spec}(\mathcal{D}^{per}(Y))$ is a prime ideal then we want to construct an prime ideal $\mathfrak{p}$ in $\mathrm{Spec}(\mathcal{D}^G(X))$ such that $\mathfrak{q} = (\pi^*)^{-1}(\mathfrak{p})$. Recall that $\langle\pi^*(\mathfrak{q})\rangle$ denotes the thick tensor ideal generated by the image of $\mathfrak{q}$ via functor π^* in a tensor triangulated category $\mathcal{D}^G(X)$. We have a following lemma which uses the explicit description of thick tensor ideal $\langle\pi^*(\mathfrak{q})\rangle$.

Lemma 3.8. $\pi_*(\langle\pi^*(\mathfrak{q})\rangle) \subseteq \mathfrak{q}$.

Proof. To prove this lemma, we use lemma 2.7 i.e.

$$\langle\pi^*(\mathfrak{q})\rangle = \cup_{n\geq 0}\langle\pi^*(\mathfrak{q})\rangle^n$$

where $\langle\pi^*(\mathfrak{q})\rangle^n$ constructed inductively by taking $\langle\pi^*(\mathfrak{q})\rangle^0$ as the summands of tensor ideal generated by $\pi^*(\mathfrak{q})$ and $\langle\pi^*(\mathfrak{q})\rangle^n$ to be the thick tensor ideal containing cone of morphism between any two objects of $\langle\pi^*(\mathfrak{q})\rangle^{(n-1)}$ and$\langle\pi^*(\mathfrak{q})\rangle^0$. Here cone of a morphism refers to the third object of any distinguised triangle having this morphism as a base or equivalently we can use $\diamond$ operation. The above equality follows from the lemma 2.7 proved earlier.

We shall use induction on n in the above explicit description. For $n = 0$, given $\mathcal{F} \in \mathfrak{q}$,

$$\pi_*(\pi^*(\mathcal{F}) \otimes \mathcal{G}) = \mathcal{F} \otimes \pi_*(\mathcal{G}) \in \mathfrak{q},$$

and hence $\pi_*(\langle\pi^*(\mathfrak{q})\rangle^0) \subseteq \mathfrak{q}$ using thickness of $\mathfrak{q}$.

Using induction suppose we know that $\pi_*(\langle\pi^*(\mathfrak{q})\rangle^{(n-1)}) \subseteq \mathfrak{q}$. Since π_* is an exact functor, it follows that the image under π_* of a cone of any morphism is a cone of π_* of the morphism. Hence using the triangulated ideal property and thickness of $\mathfrak{q}$ it follows that $\pi_*(\langle\pi^*(\mathfrak{q})\rangle^n) \subseteq \mathfrak{q}$. Therefore we have $\pi_*(\langle\pi^*(\mathfrak{q})\rangle) = \pi_*(\cup_{n\geq 0}\langle\pi^*(\mathfrak{q})\rangle^n) \subseteq \mathfrak{q}$. □

Lemma 3.9. $\pi^*(\mathcal{D}^{per}(Y) \setminus \mathfrak{q}) \cap \langle\pi^*(\mathfrak{q})\rangle = \varnothing$.

Proof. To prove this by contradiction, suppose that there exists an object $\mathcal{G} \in (\mathcal{D}^{per}(Y)\backslash\mathfrak{q})$ such that $\pi^*(\mathcal{G}) \in \langle\pi^*(\mathfrak{q})\rangle$. Then using the above lemma $\pi_*(\pi^*\mathcal{G}) \in \mathfrak{q}$. On the other hand, the projection formula implies $\pi_*(\pi^*\mathcal{G}) = \mathcal{G} \otimes \pi_*(\mathcal{O}_X)$, which we saw is in $\mathfrak{q}$.

Using the primality of $\mathfrak{q}$ it follows that $\pi_*(\mathcal{O}_X) \in \mathfrak{q}$. Now $(\pi_*(\mathcal{O}_X))^G = \mathcal{O}_Y$ is a direct summand of $\pi_*(\mathcal{O}_X)$ by the canonical decomposition of a G-sheaves on Y. Hence $\mathcal{O}_Y$ is an object of $\mathfrak{q}$; which is absurd. □

To complete Step 1, we apply Balmer's result 2.10 to get an prime ideal $\mathfrak{p}$, such that $\pi^*(\mathcal{D}^{per}(Y) \setminus \mathfrak{q}) \cap \mathfrak{p} = \varnothing$ and $\langle \pi^*(\mathfrak{q}) \rangle \subseteq \mathfrak{p}$. Hence we shall get $\mathfrak{q} = (\pi^*)^{-1}(\mathfrak{p})$ which proves the surjectivity of the map $\mathrm{Spec}(\pi^*)$.

Step 2: Injectivity of $\mathrm{Spec}(\pi^*)$. First we prove a technical lemma.

Lemma 3.10. *Let $\pi\colon X \to Y$ be the quotient map as before. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{D}^G(X)$ and suppose that $(\pi^*)^{-1}(\mathfrak{p}) = \mathfrak{q}_y$. Here, y is the point in Y corresponding to $\mathfrak{q}_y$ in* $\mathrm{Spec}(\mathcal{D}^{per}(Y)) \cong Y$.

1. *Let $\mathcal{F} \in \mathcal{D}^G(X)$ be such that its homological support is contained in $(X \setminus \pi^{-1}(y))$. Then, it is an object of $\mathfrak{p}$.*
2. *Let $\mathcal{F}$ be an object of $\mathfrak{p}$. Then* $\mathrm{supph}(\mathcal{F}) \subseteq (X \setminus \pi^{-1}(y))$.

Proof of 1. Using 2. of 2.28, there is a tower whose lower terms $\mathcal{G}_i := \bigoplus_{\lambda_i} W_{\lambda_i} \otimes \pi^*\pi_*^{G/H_i}(\mathcal{F}_{\lambda_i})$ have support contained in the subset $X - \pi^{-1}(y)$. Since $\mathrm{supph}(W_{\lambda_i} \otimes \mathcal{O}_X) = X$, we have $\mathrm{supph}(\pi^*\pi_*^{G/H_i}(\mathcal{F}_{\lambda_i})) \subseteq X - \pi^{-1}(y)$. Using 1. of 2.28, the support of $\pi_*^{G/H_i}(\mathcal{F}_{\lambda_i})$ will be in $Y - y$ and hence $\pi_*^{G/H_i}(\mathcal{F}_{\lambda_i}) \in \mathfrak{q}_y$. We know $\pi^*(\mathfrak{q}_y) \subseteq \mathfrak{p}$ where $\mathrm{Spec}(\pi^*)(\mathfrak{p}) = \mathfrak{q}_y$ is given. This will prove $\pi^*\pi_*^{G/H_i}(\mathcal{F}_{\lambda_i}) \in \mathfrak{p}$ and hence $\mathcal{G}_i \in \mathfrak{p}$. Now using the tower and the definition of a triangulated ideal, $\mathcal{F}$ is contained in $\mathfrak{p}$.

Proof of 2. Suppose $\mathrm{supph}(\mathcal{F}) \cap \pi^{-1}(y) \neq \varnothing$ and hence we get $\mathcal{F}' = \mathcal{F} \otimes \mathcal{O}_{\pi^{-1}(\bar{y})} \in \mathfrak{p}$. Observe that $\mathrm{supph}(\mathcal{F}') = \pi^{-1}(\bar{y}) = \overline{\pi^{-1}(y)}$. Now applying the same procedure as in 2. of 2.28, we shall get a distinguished triangle

$$\bigoplus_{\lambda} W_\lambda \otimes \pi^*\pi_*^{G/H}(\mathcal{F}'_\lambda) \to \mathcal{F}' \to \mathcal{F}'' \to$$

with $\mathrm{supph}(\mathcal{F}'') \subsetneq \mathrm{supph}(\mathcal{F}')$. Since the G-invariant subset $\mathrm{supph}(\mathcal{F}'')$ is a proper subset of $\overline{\pi^{-1}(y)}$ therefore $\mathrm{supph}(\mathcal{F}'') \cap \pi^{-1}(y) = \varnothing$. Using 1. above, we get that $\mathcal{F}'' \in \mathfrak{p}$. Hence using triangulated ideal property the third object of distinguished triangle will be in $\mathfrak{p}$ i.e. $\bigoplus_{\lambda} W_\lambda \otimes \pi^*\pi_*^{G/H}(\mathcal{F}'_\lambda) \in \mathfrak{p}$. But this gives $\pi^*\pi_*^{G/H}(\mathcal{F}'_\lambda) \in \mathfrak{p}$ with $\mathrm{supph}(\pi_*^{G/H}(\mathcal{F}'_\lambda)) \subseteq \bar{y}$ as $W_\lambda \otimes \mathcal{O}_X$ is not in any $\mathfrak{p}$ because $W_\lambda^* \otimes W_\lambda \otimes \mathcal{O}_X$ contains the $\mathcal{O}_X$ as direct summand, see Proposition 10.30 of [8]. And, at least for one λ, say λ_0, we have $\mathrm{supph}(\pi_*^{G/H}(\mathcal{F}'_\lambda)) = \bar{y}$ which gives $\pi_*^{G/H}(\mathcal{F}'_{\lambda_0}) \notin \mathfrak{q}_y$. This is a contradiction as $\pi^*(\mathcal{D}^{per}(Y)) \cap \mathfrak{p} = \pi^*(\mathfrak{q}_y)$. □

Proposition 3.11. *The map* $\mathrm{Spec}(\pi^*)\colon \mathrm{Spec}(\mathcal{D}^G(X)) \to \mathrm{Spec}(\mathcal{D}^{per}(Y))$ *is an injective map where X is a smooth quasi-projective varieties of dimension n.*

Proof. We prove this proposition by contradiction. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be two distinct points of $\mathrm{Spec}(D^G(X))$ which maps to the same point $\mathfrak{q}_y$ i.e. $(\pi^*)^{-1}(\mathfrak{p}_1) = (\pi^*)^{-1}(\mathfrak{p}_2) = \mathfrak{q}_y$. Let $\mathcal{F} \in \mathfrak{p}_1$ be an complex of G-equivariant sheaves. Now we use the above lemma.

Using 2., we have $\mathrm{supph}(\mathcal{F}) \subseteq (X - \pi^{-1}(y))$. Therefore using 1., and the fact that $(\pi^*)^{-1}(\mathfrak{p}_2) = \mathfrak{q}_y$, we get that $\mathcal{F} \in \mathfrak{p}_1 \cap \mathfrak{p}_2$. Hence $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$. Similarly, $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$

implying that $\mathfrak{p}_1 = \mathfrak{p}_2$. This contradicts the assumption that $\mathfrak{p}_1 \neq \mathfrak{p}_2$, and hence proves the proposition. □

Step 3: Spec(π^*) *is closed and hence is a homeomorphism.* Here we need bijection of the above step to prove closedness of the map Spec(π^*). We shall use the fact that $W \otimes \mathcal{O}_X \notin \mathfrak{p}$ for any finite dimensional representation and any prime ideal $\mathfrak{p}$. This follows from the fact that the representation on $W^* \otimes W \otimes \mathcal{O}_X$, coming from $W \otimes \mathcal{O}_X$, has the trivial representation as a direct summand, see Proposition 10.30 of [8]. Since supp(a), $a \in \mathcal{D}^G(X)$, are the basic closed sets therefore it is enough to prove that image under the map Spec(π^*) are closed. Now to prove this we shall use the description given in lemma 2.28 for any object of $\mathcal{D}^G(X)$. Letting $b_{\lambda_j} = \pi_*^{G/H_j}(a_{\lambda_j})$ for simplicity, we have the following lemma.

Lemma 3.12. Spec(π^*)(supp$(a)) = \bigcup_j \bigcup_{\lambda_j}$ supp(b_{λ_j}).

Proof. Given $a \in \mathfrak{p}$ we have b_{λ_j}'s as in lemma 2.28. Now,

$$\begin{aligned} a \in \mathfrak{p} &\iff W_{\lambda_j} \otimes \pi^*(b_{\lambda_j}) \in \mathfrak{p}\ \forall j, \lambda_j \\ &\iff \pi^*(b_{\lambda_j}) \in \mathfrak{p}, \text{ since } \ W_{\lambda_j} \otimes \mathcal{O}_X \notin \mathfrak{p}. \\ \text{Therefore } a \notin \mathfrak{p} &\Leftrightarrow \exists\ \lambda_j \text{ such that } \pi^*(b_{\lambda_j}) \notin \mathfrak{p}. \end{aligned}$$

Let $\mathfrak{p} \in$ supp(a) and hence by the definition $a \notin \mathfrak{p}$. Now using the above observation there exists a λ_j such that $\pi^*(b_{\lambda_j}) \notin \mathfrak{p}$ i.e. $b_{\lambda_j} \notin (\pi^*)^{-1}(\mathfrak{p}) =$ Spec$(\pi^*)(\mathfrak{p})$ and hence Spec$(\pi^*)(\mathfrak{p}) \in$ supp(b_{λ_j}). Therefore Spec(π^*)(supp$(a)) \subseteq \cup_j \cup_{\lambda_j}$ supp(b_{λ_j}).

Conversely suppose $\mathfrak{q} \in \cup_j \cup_{\lambda_j}$ supp(b_{λ_j}) and hence $\mathfrak{q} \in$ supp(b_{λ_j}) for some λ_j. Therefore by definition $b_{\lambda_j} \notin \mathfrak{q}$ but using the bijection of the map Spec(π^*) we have $b_{\lambda_j} \notin (\pi^*)^{-1}(\mathfrak{p}) = \mathfrak{q}$ for some $\mathfrak{p}$. Now it follows that $\pi^*(b_{\lambda_j}) \notin \mathfrak{p}$ and once again using the above observation we have $a \notin \mathfrak{p}$ i.e. $\mathfrak{p} \in$ supp(a). Hence we have $\cup_j \cup_{\lambda_j}$ supp$(b_{\lambda_j}) \subseteq$ Spec(π^*)(supp(a)). □

Since union in right hand side of above lemma is finite it follows that the image of supp(a) under the map Spec(π^*) is closed for all $a \in \mathcal{D}^G(X)$. Hence the map Spec(π^*) is a closed map and therefore it is a homeomorphism.

Step 4: Spec(π^*) *is an isomorphism.* In this step we shall prove that the above homeomorphism spec(π^*) is, in fact, an isomorphism. We begin by proving the following lemma which we shall use later.

Lemma 3.13. *There exist a natural transformation $\eta \colon \pi^*\pi_*^G \to Id$ (resp. $\mu \colon Id \to \pi_*^G\pi^*$) such that $\eta(\mathcal{O}_X) = id$ (resp. $\mu(\mathcal{O}_Y) = id$) where $\pi^*\pi_*^G(\mathcal{O}_X) = \mathcal{O}_X$ (resp. $\pi_*^G\pi^*(\mathcal{O}_Y) = \mathcal{O}_Y$).*

Proof. We shall prove the existence of η, as μ can be found using similar arguments. Since the functor π^* is a left adjoint of the functor π_* we have a natural transformation $\eta' \colon \pi^*\pi_* \to$ Id given by the adjunction property. We also have a natural transformation given by inclusion of G-invariant part of sheaves on Y, say I. Now composing with the functors π^* and π_* we get another natural transformation which composed with η' gives the η i.e. $\eta := \eta' \circ (\pi^* \cdot I \cdot \pi_*)$. Now to prove $\eta(\mathcal{O}_X) = \mathrm{Id}$ we can assume that X is an affine variety. Suppose $\tilde{A}$ is a structure sheaf of X. As A is flat over A^G we can reduce to computing a map from $\pi^*\pi^G_*(\tilde{A}) \to \tilde{A}$, in place of its derived functors. Now clearly the multiplication map $A \otimes ({}_BA)^G \to A$ is just inverse of the natural identification map of A with $A \otimes ({}_BA)^G$. Hence the map $\eta(\mathcal{O}_X)\colon \tilde{A} \to \tilde{A}$ is an identity map. Similarly we can prove that $\mu(\mathcal{O}_Y) = \mathrm{Id}$. □

Recall the definitions of structure sheaves and associated map of the sheaves given by the unital tensor functor of underlying tensor triangulated categories 2.2 i.e. given an unital functor $\pi^*\colon \mathcal{D}^{per}(Y) \to \mathcal{D}^G(X)$ the morphism $\mathrm{Spec}(\pi^*)$ induces a map of the structure sheaves, $\mathrm{Spec}(\pi^*)^\#\colon \mathcal{O}_Y \to \mathcal{O}_X$. We shall prove that this map is an isomorphism by observing that $\mathrm{Spec}(\pi^*)^\#(V)$ is an isomorphism for every open set $V \subseteq \mathrm{Spec}(\mathcal{D}^{per}(Y))$. If we take $U = \pi^{-1}(V)$, $Z = Y \setminus V$ and $Z' = X \setminus U$ then we have a functor $\pi^*_V\colon \frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)} \to \frac{\mathcal{D}^G(X)}{\mathcal{D}^G_{Z'}(X)}$ which will induce a map $\mathrm{Spec}(\pi^*)^\#(V) := \pi^*_V\colon \mathrm{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y) \to \mathrm{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}^G_{Z'}(X)}}(\mathcal{O}_X)$.

Lemma 3.14. *The map* $\pi^*_V\colon \mathrm{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y) \to \mathrm{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}^G_{Z'}(X)}}(\mathcal{O}_X)$ *is surjective.*

Proof. Suppose $[\mathcal{O}_Y \xleftarrow{s} \mathcal{G} \xrightarrow{a} \mathcal{O}_Y]$ is an element of $\mathrm{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y)$ then the map π^* will send it to an element $[\mathcal{O}_X \xleftarrow{\pi^*(s)} \pi^*(\mathcal{G}) \xrightarrow{\pi^*(a)} \mathcal{O}_X]$ of $\mathrm{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}^G_{Z'}(X)}}(\mathcal{O}_X)$. It is now enough to prove that this map is a bijection.

Let $[\mathcal{O}_X \xleftarrow{t} \mathcal{F} \xrightarrow{b} \mathcal{O}_X] \in \mathrm{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}^G_{Z'}(X)}}(\mathcal{O}_X)$ be a given element then using the functor π^G_* we shall get an element $[\mathcal{O}_Y \xleftarrow{\pi^G_*(t)} \pi^G_*(\mathcal{F}) \xrightarrow{\pi^G_*(b)} \mathcal{O}_Y] \in \mathrm{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y)$ as $\mathrm{supph}(C(\pi^G_*(t))) \subseteq Z$ using the flat base change and the canonical isomorphism, $i^*_V\pi^G_*(\mathcal{F}) \simeq (i^*_V\pi_*(\mathcal{F}))^G \simeq \pi^G_*(i^*_U\mathcal{F}) \xrightarrow{i^*_U(t)} \pi^G_*(i^*_U\mathcal{O}_X) \simeq \mathcal{O}_V$. Now we want to prove that

$$[\mathcal{O}_X \xleftarrow{t} \mathcal{F} \xrightarrow{b} \mathcal{O}_X] = [\mathcal{O}_X \xleftarrow{\pi^*\pi^G_*(t)} \pi^*\pi^G_*(\mathcal{F}) \xrightarrow{\pi^*\pi^G_*(b)} \mathcal{O}_X].$$

Using the lemma 3.13, we have a natural map $\eta(\mathcal{F})\colon \pi^*\pi^G_*(\mathcal{F}) \to \mathcal{F}$, so to prove the assertion it is now enough to check that $t \circ \eta(\mathcal{F}) = \pi^*\pi^G_*(t)$, $b \circ \eta(\mathcal{F}) = \pi^*\pi^G_*(b)$ and the cone of $\eta(\mathcal{F})$ is supported on Z' that is $C(\eta(\mathcal{F})) \in \mathcal{D}^G_{Z'}(X)$. Here the first two assertions follows from the following commutative diagrams which are a consequence of lemma 3.13.

$$\begin{array}{ccc} \pi^*\pi^G_*(\mathcal{F}) & \xrightarrow{\eta(\mathcal{F})} & \mathcal{F} \\ {\scriptstyle \pi^*\pi^G_*(t)}\downarrow & & \downarrow{\scriptstyle t} \\ \mathcal{O}_X & \overset{\eta(\mathcal{O}_X)}{=\!=\!=} & \mathcal{O}_X \end{array} \qquad \begin{array}{ccc} \pi^*\pi^G_*(\mathcal{F}) & \xrightarrow{\eta(\mathcal{F})} & \mathcal{F} \\ {\scriptstyle \pi^*\pi^G_*(b)}\downarrow & & \downarrow{\scriptstyle b} \\ \mathcal{O}_X & \overset{\eta(\mathcal{O}_X)}{=\!=\!=} & \mathcal{O}_X \end{array}$$

Now the last assertion $C(\eta(\mathcal{F})) \in \mathcal{D}^G_{Z'}(X)$ is equivalent to $i^*_U C(\eta(\mathcal{F})) \simeq 0$ in $\mathcal{D}^G(U)$ but as the functor i^*_U is exact this assertion is same as $C(i^*_U\eta(\mathcal{F})) \simeq 0$. Since a cone of an isomorphism is zero it is enough to check that the map $i^*_U\eta(\mathcal{F})$ is an isomorphism. And this follows from the following commutative diagram.

$$\begin{array}{ccc} i^*_U\pi^*\pi^G_*(\mathcal{F}) & \xrightarrow{i^*_U\eta(\mathcal{F})} & i^*_U\mathcal{F} \\ \downarrow\wr & & \| \\ \pi^*\pi^G_*(i^*_U\mathcal{F}) & \xrightarrow{\eta(i^*_U\mathcal{F})} & i^*_U\mathcal{F} \\ {\scriptstyle \pi^*\pi^G_* i^*_U(t)}\downarrow\wr & & {\scriptstyle i^*_U(t)}\downarrow\wr \\ \pi^*\pi^G_*(\mathcal{O}_U) & \overset{\eta(\mathcal{O}_U)}{=\!=\!=\!=} & \mathcal{O}_U \end{array}$$

In above diagram we had used the same notations π and η for its restriction on open subsets. Here the top left vertical isomorphism comes from the flat base change formula and using the following canonical isomorphism.

$$i^*_U\pi^*\pi^G_*(\mathcal{F}) \simeq \pi^* i^*_V(\pi_*(\mathcal{F}))^G \simeq \pi^*(i^*_V\pi_*(\mathcal{F}))^G \simeq \pi^*(\pi_* i^*_U(\mathcal{F}))^G = \pi^*\pi^G_*(i^*_U\mathcal{F}).$$

This proves that π^*_V is surjective. □

Lemma 3.15. *π^*_V is injective.*

Proof. Let $[\mathcal{O}_Y \xleftarrow{s} \mathcal{G} \xrightarrow{a} \mathcal{O}_Y] \in \mathrm{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y)$ maps to zero in $\mathrm{End}_{\frac{\mathcal{D}^G(X)}{\mathcal{D}^G_{Z'}(X)}}(\mathcal{O}_X)$ i.e. $[\mathcal{O}_X \xleftarrow{\pi^*(s)} \pi^*(\mathcal{G}) \xrightarrow{\pi^*(a)} \mathcal{O}_X] = 0$ which is equivalent to the existence of $\mathcal{F}$ and a map $t\colon \mathcal{F} \to \pi^*\mathcal{G}$ with $\mathrm{supph}(C(t)) \subseteq Z'$ such that $\pi^*(a) \circ t = 0$. Now the map $\pi^G_*(t)\colon \pi^G_*(\mathcal{F}) \to \pi^G_*\pi^*(\mathcal{G})$ gives $\pi^G_*\pi^*(a) \circ \pi^G_*(t) = 0$ and as proved earlier we know that $\mathrm{supph}(C(\pi^G_*(t))) \subseteq Z$ whenever $\mathrm{supph}(C(t)) \subseteq Z'$. Hence the element $[\mathcal{O}_Y \xleftarrow{\pi^G_*\pi^*(s)} \pi^G_*\pi^*(\mathcal{G}) \xrightarrow{\pi^G_*\pi^*(a)} \mathcal{O}_Y] = 0$ in $End_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y)$. We shall prove that $[\mathcal{O}_Y \xleftarrow{s} \mathcal{G} \xrightarrow{a} \mathcal{O}_Y] = [\mathcal{O}_Y \xleftarrow{\pi^G_*\pi^*(s)} \pi^G_*\pi^*(\mathcal{G}) \xrightarrow{\pi^G_*\pi^*(a)} \mathcal{O}_Y]$ as an elements of $\mathrm{End}_{\frac{\mathcal{D}^{per}(Y)}{\mathcal{D}^{per}_Z(Y)}}(\mathcal{O}_Y)$. Now using lemma 3.13 we have a map $\mu(\mathcal{G})\colon \mathcal{G} \to \pi^G_*\pi^*(\mathcal{G})$

which gives the following commutative diagrams as before using lemma 3.13,

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\mu(\mathcal{G})} & \pi_*^G\pi^*(\mathcal{G}) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \pi_*^G\pi^*(s)} \\ \mathcal{O}_Y & \overset{\mu(\mathcal{O}_Y)}{=\!=\!=} & \mathcal{O}_Y \end{array} \qquad \begin{array}{ccc} \mathcal{G} & \xrightarrow{\mu(\mathcal{G})} & \pi_*^G\pi^*(\mathcal{G}) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \pi_*^G\pi^*(a)} \\ \mathcal{O}_Y & \overset{\mu(\mathcal{O}_Y)}{=\!=\!=} & \mathcal{O}_Y \end{array}$$

Therefore it remains to prove that $i_V^*C(\mu(\mathcal{G})) = 0$ but as before this is equivalent to proving $C(i_V^*\mu(\mathcal{G})) = 0$ since the functor i_V^* is an exact functor. Again using the fact that a cone of an isomorphism is zero it is enough to prove that $i_V^*\mu(\mathcal{G})$ is an isomorphism. This clearly follows from the following commutative diagrams,

$$\begin{array}{ccc} i_V^*\mathcal{G} & \xrightarrow{i_V^*\mu(\mathcal{G})} & i_V^*\pi_*^G\pi^*(\mathcal{G}) \\ \| & & \wr\downarrow \\ i_V^*\mathcal{G} & \xrightarrow{\mu(i_V^*\mathcal{G})} & \pi_*^G\pi^*(i_V^*\mathcal{G}) \\ {\scriptstyle i_V^*(s)}\downarrow\wr & & \wr\downarrow{\scriptstyle \pi_*^G\pi^*i_V^*(s)} \\ \mathcal{O}_V & \overset{\mu(\mathcal{O}_V)}{=\!=\!=} & \pi_*^G\pi^*(\mathcal{O}_V). \end{array}$$

Here again as earlier the top right vertical isomorphism comes from the flat base change and the following sequence of natural isomorphisms.

$$i_V^*\pi_*^G\pi^*(\mathcal{G}) \simeq i_V^*(\pi_*\pi^*\mathcal{G})^G \simeq (i_V^*\pi_*\pi^*\mathcal{G})^G \simeq \pi_*^G i_U^*\pi^*\mathcal{G} \simeq \pi_*^G\pi^*(i_V^*\mathcal{G}).$$

This proves injectivity of the map π_V^*. □

From the above two lemmas it follows that π_V^* is an isomorphism and hence $\mathrm{Spec}(\pi^*)$ is an isomorphism of the varieties $\mathrm{Spec}(\mathcal{D}^{per}(Y))$ and $\mathrm{Spec}(\mathcal{D}^G(X))$.

4. Example: Superschemes

In this section, we shall recall the basic definition of superscheme and some properties of it. Then, we shall relate various notions for superschemes with usual schemes.

4.0.1. *Superalgebra.* An associative $\mathbb{Z}/2\mathbb{Z}$-grading ring is an associative ring R with direct sum decomposition $R = R^0 \oplus R^1$ as an additive group so that multiplication preserves the grading i.e. $R^iR^j \subseteq R^{i+j}$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$. There exists a parity function which takes values in ring $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ for every homogeneous element of R i.e. if $r \in R^i$ then parity denoted $\bar{r} = i$. Now we restrict to following important class of rings.

Definition 4.1. An associative $\mathbb{Z}/2\mathbb{Z}$ graded ring with unity, $R = R^0 \oplus R^1$ is called supercommutative if the supercommutator of a ring R is zero i.e. $[r_1, r_2] := r_1r_2 - (-1)^{\bar{r_1}\bar{r_2}}r_2r_1 = 0$ for all $r_1, r_2 \in R$. Further ring is called k-superalgebra if R is supercommutative k-algebra with $k \subseteq R^0$.

As usual we can define an abelian category of left modules over any k-superalgebra R, say $\mathcal{M}\mathfrak{od}(R)$. An object of this category is a $\mathbb{Z}/2\mathbb{Z}$- graded abelian group with a left R-module structure which is compatible with the grading i.e. $R^iM^j \subseteq M^{i+j}$ for all $i, j = 0, 1$. Morphism between these objects is a graded morphism compatible with the action of R. Similarly there exists a parity function defined for each homogeneous element of a module M denoted as above. We can define the parity change functor $\Pi\colon \mathcal{M}\mathfrak{od}(R) \to \mathcal{M}\mathfrak{od}(R); M \mapsto \Pi M$ with $\mathbb{Z}/2\mathbb{Z}$ grading given by $(\Pi M)^0 = M^1$ and $(\Pi M)^1 = M^0$. There exist an exact faithful functor from $\mathcal{M}\mathfrak{od}(R)$ as follows,

$$ff\colon \mathcal{M}\mathfrak{od}(R) \longrightarrow \mathcal{M}\mathfrak{od}(R^0) \times \mathcal{M}\mathfrak{od}(R^0).$$

A canonical right module structure on left R modules is given by $mr := (-1)^{\bar{m}\bar{r}}rm$. Now using this structure we can define tensor product of two left R- modules M_1 and M_2 as quotient of $M_1 \otimes_{R^0} M_2$ with submodule generated by homogeneous elements $\{r_1m_1 \otimes m_2 - (-1)^{\bar{m}_1}m_1 \otimes r_1m_2 | r_1 \in R^1, m_i \in M^i\}$. Here $M_1 \otimes_{R^0} M_2$ is defined as a tensor product of two $\mathbb{Z}/2\mathbb{Z}$ graded modules over a commutative ring R^0. The tensor product $M_1 \otimes_R M_2$ is then a $\mathbb{Z}/2\mathbb{Z}$ graded module with $\overline{m \otimes n} = \bar{m}\bar{n}$. The commutativity constraint is similar to the case of tensor product of supervector spaces. Note that in general above two forgetful functors are not tensor functors. Another important notion in commutative algebra is localization. It is easy to define localization of rings and modules if multiplicative set is contained in center of a ring. For super commutative ring we can define localization at any homogeneous prime ideal. It is easy to observe that given a R module M with a prime ideal $\mathfrak{p}$, the localization $M_{\mathfrak{p}} = 0$ iff $({}_{R^0}M)_{\mathfrak{p}} = 0$ (or $({}_{(R/J)}M)_{\mathfrak{p}} = 0$ where $J := R{\cdot}R^1$. We can also prove followng version of Nakayama lemma for superrings.

Proposition 4.2 (Nakayama's lemma)**.** *Suppose a finitely generated R module M satisfies $IM = M$ for the homogenous ideal I given by the intersection of all maximal homogenous ideals then $M = 0$.*

Proof. It is similar to the case of rings. First observe that every element of R_1 is nilpotent, and hence belongs to I. Hence for any element $a := a_0 + a_1 \in I$ with $a_i \in R_i$, the element $1 - a$ is a unit in R iff $1 - a_0$ is unit in R_0. But $R_0 \cap I$ is the Jacobson radical of R_0, and $a_0 = a - a_1 \in I \cap R_0$ and therefore $1 - a_0$ is unit in R_0. Thus $1 - a$ is a unit in R. Rest of the proof is similar to the commutative case, [see Prop. 2.6, [1]]. □

Now using Nakayama's lemma we get following result whose proof is similar to the commutative case.

Corollary 4.3. *Suppose $(R, \mathfrak{m})$ is a local superring. Let M, M_1 and M_2 be finitely generated R modules.*

(1) *A finitely generated module $M = 0$ if and only if $M \otimes R/\mathfrak{m} = 0$.*
(2) *$M_1 \otimes M_2 = 0$ if and only if $M_1 = 0$ or $M_2 = 0$.*

4.0.2. *Split Superscheme.* Given any topological space X we can define a super ringed space by attaching a sheaf of superrings on X. We shall denote a sheaf of superrings with $\mathbb{Z}/2\mathbb{Z}$ grading as $\mathcal{O}_X = \mathcal{O}_{X,0} \oplus \mathcal{O}_{X,1}$. Similarly we can define sheaf of modules and parity change functor Π over such a ringed space as before. We have following definition,

Definition 4.4. A ringed space $(X, \mathcal{O}_X)$ is called a *superspace* if ring $\mathcal{O}_X(U)$ associated to any open subset U is supercommutative and each stalk is local ring. A *superspace* is called *superscheme* if in addition ringed space $(X, \mathcal{O}_{X,0})$ is a scheme and $\mathcal{O}_{X,1}$ is a coherent sheaf over $\mathcal{O}_{X,0}$.

A superscheme X is called quasi compact and quasi separated if $(X, \mathcal{O}_{X,0})$ is quasi compact and quasi separated. Similarily a superscheme is (topologically) Noetherian if $(X, \mathcal{O}_{X,0})$ is (topologically) Noetherian. We shall use these notion later to borrow results developed by Grothendieck. We say that a superscheme is *affine* if the even part of structure sheaf $(X, \mathcal{O}_{X,0})$ is affine. It is easy to see that any affine superscheme gives a super commutative ring. Equivalently an affine superscheme associated to any super commutative ring can be defined in a manner similar to usual affine schemes. Note that in the definition of superscheme the odd part is a coherent sheaf of modules over the even part. Therefore if even part of a superscheme is noetherian then we shall get the left (or two sided) noetherian superscheme. Given a superscheme $(X, \mathcal{O}_X)$ we can define sheaf of ideal [17] $J_X := \mathcal{O}_X \cdot \mathcal{O}_{X,1}$. Define $GrX := \oplus_{i \geq 0} J_X^i / J_X^{i+1}$ where $J_X^0 := \mathcal{O}_X$ and we denote the first term of GrX as $Gr_0 X = \mathcal{O}_X / J_X$. Now using these notation we can define structure sheaves of *even* scheme and *reduced* scheme associated to superscheme X as follows,

$$\mathcal{O}_{X_{rd}} := Gr_0 X \text{ and } \mathcal{O}_{X_{red}} := \mathcal{O}_X / \sqrt{J_X}.$$

Here J_X / J_X^2 is a locally free sheaf of finite rank $0|d$ for some d over $\mathcal{O}_{X_{rd}}$. And GrX is a Grassmann algebra over $\mathcal{O}_{X_{rd}}$ of locally free sheaf J_X / J_X^2. Following particular class of superschemes are defined in paper of Manin [17].

Definition 4.5. A superscheme $(X, \mathcal{O}_X)$ is called *split* if the graded sheaf GrX with mod 2 grading is isomorphic as a locally superringed sheaf to the structure sheaf $\mathcal{O}_X$.

Manin [17] had also given a way to construct such a *split* superscheme. If we take purely even scheme $(X, \mathcal{O}_X)$ and a locally free sheaf $\mathcal{V}$ over $\mathcal{O}_X$ then we can define symmetric algebra of odd locally free sheaf $\Pi\mathcal{V}$, which is denoted $S(\Pi\mathcal{V})$ (see Manin [17]), then $(X, S(\Pi\mathcal{V}))$ is a split superscheme. An important example is given by projective superscheme $\mathbb{P}^{m|n}$ where the locally free sheaf $\mathcal{V}$ is $\mathcal{O}(-1)^n$. An example of a nonsplit superscheme given in Manin [17] is Grassmann superscheme $G(1|1, \mathbb{C}^{2|2})$ which is also an example of non superprojective scheme. We can define an abelian category of sheaf of left modules over $\mathcal{O}_X$, denoted $\mathcal{M}\mathfrak{od}^s(X)$ or $\mathcal{M}\mathfrak{od}(\mathcal{O}_X)$. As above we have a natural right module structure given by the Koszul sign rule. When $(X, \mathcal{O}_X)$ is affine superscheme given by super ring R then we can define the sheaf of module associated to any R-module M similar to commutative case. Hence we can define quasi-coherent and coherent sheaves over any superscheme. Therefore we shall get two abelian subcategories namely category of all quasi-coherent sheaves and coherent sheaves. We denote them by $\mathcal{QC}\mathfrak{oh}(\mathcal{O}_X)$ and $\mathcal{C}\mathfrak{oh}(\mathcal{O}_X)$ respectively. Now similar to affine case we have forgetful functor as follows,

$$ff\colon \mathcal{M}\mathfrak{od}(\mathcal{O}_X) \longrightarrow \mathcal{M}\mathfrak{od}(\mathcal{O}_{X,0}) \times \mathcal{M}\mathfrak{od}(\mathcal{O}_{X,0}).$$

It is an exact faithful functor. we can easily see that

$$\mathcal{QC}\mathfrak{oh}(\mathcal{O}_X) = ff^{-1}(\mathcal{QC}\mathfrak{oh}(\mathcal{O}_{X,0}) \times \mathcal{QC}\mathfrak{oh}(\mathcal{O}_{X,0}))$$
$$\mathcal{C}\mathfrak{oh}(\mathcal{O}_X) = ff^{-1}(\mathcal{C}\mathfrak{oh}(\mathcal{O}_{X,0}) \times \mathcal{C}\mathfrak{oh}(\mathcal{O}_{X,0})).$$

We can define the tensor product of two sheaves of modules over superscheme similar to usual scheme. We shall use the canonical identification of sheaf of left and right modules by Koszul sign rule. Define tensor product of two sheaves of modules $\mathcal{F}_1$ and $\mathcal{F}_2$ as the sheaf associated to pre sheaf given by

$$U \mapsto (\mathcal{F}_1 \otimes \mathcal{F}_2)(U) := \mathcal{F}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}_2(U).$$

Note that with this definition of tensor structure the commutative constraint is given by sign rule i.e.

$$\begin{aligned} &\mathcal{F} \otimes \mathcal{G} \;\cong\; \mathcal{G} \otimes \mathcal{F} \text{ where the isomorphism is given by,} \\ &f \otimes g \longmapsto -g \otimes f \text{ if both } \mathcal{F} \text{ and } \mathcal{G} \text{ are odd,} \\ &f \otimes g \longmapsto g \otimes f \text{ otherwise.} \end{aligned}$$

Now we can prove some easy properties of this tensor product by just reducing to affine case,

Lemma 4.6. *Suppose $(X, \mathcal{O}_X)$ is a split superscheme and $\mathcal{F}$ and $\mathcal{G}$ are quasi coherent sheaves. Then we have*

(1) $(\Pi\mathcal{F}) \otimes \mathcal{G} = \mathcal{F} \otimes (\Pi\mathcal{G}) = \Pi(\mathcal{F} \otimes \mathcal{G})$
(2) *$\mathcal{F} \otimes \mathcal{O}_{X_{rd}}$ has trivial action of J_X and hence it is a quasi coherent $\mathcal{O}_{X_{rd}}$ sheaf.*

Given a split superscheme $(X, \mathcal{O}_X = S^{\cdot}(\Pi\mathcal{V}) = \Pi\Lambda^{\cdot}(\mathcal{V}))$ there is one more forgetful functor as follows,

$$ff\colon \mathcal{M}\mathfrak{od}(\mathcal{O}_X) \longrightarrow \mathcal{M}\mathfrak{od}(\mathcal{O}_{X_{rd}}) \times \mathcal{M}\mathfrak{od}(\mathcal{O}_{X_{rd}}).$$

This functor is defined using the obvious inclusion of $\mathcal{O}_{X_{rd}}$ inside $\mathcal{O}_X$ which comes from the definition of split superscheme. Note also that the Grassmann algebra constructed from locally free sheaf $\mathcal{V}$ gives a locally free sheaf of $\mathcal{O}_{X_{rd}}$ module. Therefore structure sheaf $\mathcal{O}_X$ is locally free sheaf as a $\mathcal{O}_{X_{rd}}$ module.

Similar to usual scheme we can take $\mathcal{D}(X) := \mathcal{D}(\mathcal{M}\mathfrak{od}(X))$ the derived category of abelian category $\mathcal{M}\mathfrak{od}(X)$. There are various triangulated subcategories like $\mathcal{D}^{\sharp}_{qc}(X) := \mathcal{D}^{\sharp}(\mathcal{QC}\mathfrak{oh}(X))$ and $\mathcal{D}^{\sharp}_{coh}(X) := \mathcal{D}^{\sharp}(\mathcal{C}\mathfrak{oh}(X))$ where $\sharp = +, -, b$ or $\emptyset$. For convenience we shall denote by $\mathcal{D}^{\sharp}(X^0) := \mathcal{D}^{\sharp}(\mathcal{M}\mathfrak{od}(\mathcal{O}^0_X))$ (resp. $\mathcal{D}^{\sharp}(X_{rd}) := \mathcal{D}^{\sharp}(\mathcal{M}\mathfrak{od}(\mathcal{O}_{X_{rd}}))$) the derived category of modules over purely even scheme $(X, \mathcal{O}^0_X)$ (resp. $X_{rd} = (X, Gr_0X)$). Similar notation we can have for the other subcategories. Following criterion using Nakayama's lemma will be used later.

Proposition 4.7. *Suppose $(R, \mathfrak{m})$ is a local superring. Suppose $M^{\cdot}, M^{\cdot}_1$ and $M^{\cdot}_2$ are bounded complexes of finitely generated R-modules.*

1. *$M^{\cdot}$ is acyclic iff $M^{\cdot} \otimes R/\mathfrak{m}$ is acyclic.*
2. *$M^{\cdot}_1 \otimes M^{\cdot}_2$ is acyclic iff $M^{\cdot}_1$ or $M^{\cdot}_2$ is acyclic.*

Proof. The proof of (i) is similar to the proof of Thomason [lemma 3.3 (a), [26]]. Indeed, using spectral sequence mentioned in the proof of Thomason [lemma 3.3 (a), [26]] the proof reduces to the case of finitely generated modules which follows from the above reuslt 4.3.
The proof of (ii) follows from the proof of (i) using following natural isomorphism

$$(M^{\cdot}_1 \otimes M^{\cdot}_2) \otimes R/\mathfrak{m} \simeq (M^{\cdot}_1 \otimes R/\mathfrak{m}) \otimes_{R/\mathfrak{m}} (M^{\cdot}_2 \otimes R/\mathfrak{m}). \qquad \square$$

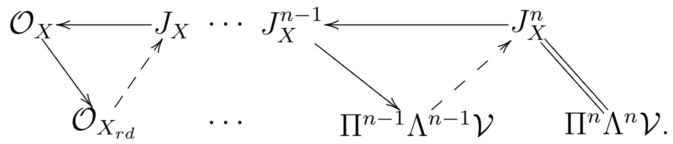
Next we give some results which we need in computation of spectrum.

Proposition 4.8. *Suppose $(X, \mathcal{O}_X)$ is a split superscheme and $\mathcal{F}$ and $\mathcal{G}$ are quasi coherent sheaves.*

1. *Any $\mathcal{O}_{X_{rd}}$ quasi coherent sheaf $\mathcal{F}^0$ is also a $\mathcal{O}_X$ quasi coherent sheaf via canonical projection $\mathcal{O}_X \to \mathcal{O}_{X_{rd}}$. Hence we get a functor $\mathbf{i_{rd}} \colon \mathcal{D}_{qc}(X_{rd}) \to \mathcal{D}_{qc}(X)$.*
2. *The functor $\mathbf{i_{rd}}$ is a tensor functor and the images of this functor is tensor ideal in $\mathcal{D}_{qc}(X)$. The functor $\mathbf{i_{rd}}$ is in fact a dense tensor functor.*

Proof. The proof of *1* is clear from the definition. Hence we just indicate the proof of *2*.

Proof of 2. Given any quasi coherent sheaf $\mathcal{F}$, observe that $\mathbf{i_{rd}}(\mathcal{F})$ has the trivial action of ideal sheaf J_X. Therefore by definition of tensor product it follows that $\mathbf{i_{rd}}$ is a tensor functor. Also observe that given a sheaf of $\mathcal{O}_X$ module, $\mathcal{F}$, the tensor $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{rd}}$ has the trivial action of the ideal sheaf J_X and hence it will be in the image of the functor $\mathbf{i_{rd}}$. Since $(X, \mathcal{O}_X)$ is a split superscheme, we have identification of $\mathcal{O}_X$ with GrX. The sheaf GrX is an exterior algebra over purely odd locally free sheaf $\Pi\mathcal{V} := J_X/J^2_X$ and each subquotientsJ^i_X/J^{i+1}_X can be identified with $\Pi^i\Lambda^i\mathcal{V}$. Hence each subquotients are purely odd or purely even locally free sheaves. The $\mathbb{Z}$-grading on sheaf GrX gives a filtration for structure sheaf $\mathcal{O}_X$ and hence we have following tower for structure sheaf $\mathcal{O}_X$,



In above tower, each of the terms in the lower row is complex of either purely odd or purely even sheaves. And using property of tensor proved above, we have $\Pi^i\Lambda^i\mathcal{V} = (\Pi^i\mathcal{O}_{X_{rd}}) \otimes \Lambda^i\mathcal{V}$. Therefore the ideal generated by the image of the functor $\mathbf{i_{rd}}$ contains the all the terms in the lower row of the above tower and hence $\mathbf{i_{rd}}$ is a dense tensor functor. □

We now define the another important triangulated subcategory of $\mathcal{D}_{qc}(X)$.

Definition 4.9. A complex $\mathcal{F}^{\cdot}$ of quasi coherent sheaves of modules over the superscheme $(X, \mathcal{O}_X)$ is called *strictly perfect* if it is quasi isomorphic to a bounded complex of locally free coherent sheaves of $\mathcal{O}_X$ modules. A complex $\mathcal{F}^{\cdot}$ is called *perfect* if it is locally quasi isomorphic to a bounded complex of locally free coherent sheaves.

We shall denote the triangulated subcategory of all perfect complexes by $\mathcal{D}^{per}(X)$. As in the scheme case, we can extend various functors defined on sheaves of modules over the superscheme to these triangulated categories. Hence we can prove $\mathcal{D}^{per}(X)$ is a tensor triangulated category with the tensor structure given by the derived functor of the usual tensor product defined as above. We can extend the forgetful functor defined earlier using exactness,

$$ff\colon \mathcal{D}^{\sharp}(X) \longrightarrow \mathcal{D}^{\sharp}(X^0) \times \mathcal{D}^{\sharp}(X^0) \quad \text{and} \quad ff\colon \mathcal{D}^{\sharp}_{qc}(X) \longrightarrow \mathcal{D}^{\sharp}_{qc}(X^0) \times \mathcal{D}^{\sharp}_{qc}(X^0).$$

Here $\sharp \in \{+, -, b, \varnothing\}$. We can have similar forgetful functors in the case of coherent sheaves. If we restrict to split superschemes, we can also define forgetful functors in the case of locally free sheaves (or vector bundles). Hence for a split superscheme, we have the following forgetful functor for the triangulated subcategory of perfect complexes,

$$ff\colon \mathcal{D}^{per}(X) \longrightarrow \mathcal{D}^{per}(X_{rd}) \times \mathcal{D}^{per}(X_{rd})$$

Note that this functor may not be a tensor functor.

4.1. **Main Results.** Now we have following result which gives a way to get back quasi coherent complexes over superscheme with a pair of quasi coherent complexes over the *purely even superscheme* (or *usual scheme*). As in the case of schemes we can define the *support* of a quasicoherent sheaf as a subset of X containing all super prime ideals where the stalk of the sheaf is nonzero. Since nontriviality of the stalk at any point $\mathfrak{p}$ is a local property we can check it in an affine open set containing $\mathfrak{p}$. Now from the earlier observation $\mathcal{F}_{\mathfrak{p}} = 0$ iff $\mathcal{F}^0_{\mathfrak{p}} = 0 = \mathcal{F}^1_{\mathfrak{p}}$ as stalks of a sheaves of $\mathcal{O}_{X_{rd}}$ modules $\mathcal{F}^0$ and $\mathcal{F}^1$. Therefore for a quasi coherent sheaf $\mathcal{F}$ we have $\text{supp}(\mathcal{F}) = \text{supp}(ff(\mathcal{F})) = \text{supp}(\mathcal{F}^0) \cup \text{supp}(\mathcal{F}^1)$. Now the assignment of support can be extended to the derived category as follows,

$$\text{supph}(\mathcal{F}^\cdot) := \cup_{i\in\mathbb{Z}} \text{supp}(\mathcal{H}^i(\mathcal{F}^\cdot)).$$

This association can be restricted to the thick subcategory $\mathcal{D}^{per}(X)$ for quasi compact and quasi separated scheme X. As the forgetful functor is an exact functor we have the following relation between supports as in the case of sheaves,

$$\text{supph}(\mathcal{F}^\cdot) = \text{supph}(ff(\mathcal{F}^\cdot)) = \text{supph}(\mathcal{F}^0) \cup \text{supph}(\mathcal{F}^1)$$

Above observation gives following result similar to the result of Thomason [lemma 3.3 (c), [26]].

Lemma 4.10. *Suppose X is a quasi compact and quasi separated superscheme and $\mathcal{F}^\cdot \in \mathcal{D}^{per}(X)$. Then the subset* $\text{supph}(\mathcal{F}^\cdot)$ *is closed and* $X - \text{supph}(\mathcal{F}^\cdot)$ *is quasi compact subset of X.*

Using this property of supports we can prove the following result,

Lemma 4.11. *The pair* (X, supph) *defined as above gives a support data on the triangulated category $\mathcal{D}^{per}(X)$.*

Proof. Since the forgetful functor is an exact functor and we have the equality $(\mathcal{F}^\cdot) = \text{supph}(ff(\mathcal{F}^\cdot))$ therefore the support data properties (SD 1)-(SD 4) [3] are easy to prove. We shall just prove (SD 5) here. Since checking nontriviality of the stalk is a local question, we can assume that X is an affine superscheme. First we observe that any perfect complex $\mathcal{F}^\cdot$ is a strict perfect complex and hence a bounded complex of fintely generated projective modules. Hence by taking local superring $R = \mathcal{O}_{X,x}$ the proof follows from the above result 4.7. □

We shall now prove that above support data is in fact classifying support data as defined in Balmer [3]. We need following classification (see [3]) of thick tensor subcategories of $\mathcal{D}^{per}(X)$ which we prove by relating it with the case of schemes.

Proposition 4.12. *Given a quasi compact and quasi separated superscheme $(X, \mathcal{O}_X)$ we have a bijection,*

$$\theta \to \{Y \subset X | Y \textit{ specialisation closed}\} \xrightarrow{\sim} \{\mathcal{I} \subset \mathcal{D}^{per}(X) \,|\, \mathcal{I} \textit{ radical thick } \otimes\textit{-ideal}\}$$

defined by $Y \mapsto \{\mathcal{F}^\cdot \in \mathcal{D}^{per}(X) | \text{supph}(\mathcal{F}^\cdot) \subset Y\}$, *with inverse, say η,* $\mathcal{I} \mapsto \text{supph}(\mathcal{I}) := \cup_{\mathcal{F}^\cdot \in \mathcal{I}} \text{supph}(\mathcal{F}^\cdot)$.

Proof. Using support data properties (SD 1) - (SD 5) we can prove that $\theta(Y)$ is a radical thick tensor ideal and hence the map θ is well defined. To prove that $\eta(\mathcal{I})$ is a specialisation closed subset it is enough to prove that for any $y \in \eta(\mathcal{I})$ there is a closed set containing this point. By definition y is in the homological support of some object $\mathcal{F}^\cdot \in \mathcal{I}$. Hence $y \in \text{supph}(ff(\mathcal{F}^\cdot))$ which is a closed subset.
It is easy to check that $\eta \circ \theta(Y) \subseteq Y$ and $\mathcal{I} \subseteq \theta \circ \eta(\mathcal{I})$. To prove that $Y \subseteq \eta \circ \theta(Y)$ it is enough to show that for any closed subset Z there exists an object with

support Z. But there exists an $\mathcal{O}_{X_{rd}}$ perfect sheaf with support Z and hence via the natural map $\mathcal{O}_X \to \mathcal{O}_{X_{rd}}$ we get a perfect sheaf with support Z.
Finally to prove that $\theta \circ \eta(\mathcal{I}) \subseteq \mathcal{I}$ it is enough to prove that for any $\mathcal{F}^\cdot \in \theta \circ \eta(\mathcal{I})$ the object $\mathcal{F}^\cdot \in \mathcal{I}$. Now following the proof of theorem 3.15 of Thomason [26] we reduce to proving that if $\operatorname{supph}(\mathcal{F}^\cdot) \subseteq \operatorname{supph}(\mathcal{G}^\cdot)$ for some object of $\mathcal{G}^\cdot \in \mathcal{I}$ then $\mathcal{F}^\cdot \in \mathcal{I}$. But $\mathcal{F}^\cdot \otimes \mathcal{O}_{X_{rd}}$ will be in thick tensor ideal generated by $\mathcal{G}^\cdot \otimes \mathcal{O}_{X_{rd}}$ as there is a dense tensor inclusion of $\mathcal{D}^{per}(X_{rd})$ in $\mathcal{D}^{per}(X)$ (see 4.8). This will prove $\mathcal{F}^\cdot \otimes \mathcal{O}_{X_{rd}} \in \mathcal{I}$. $\mathcal{I}$ is the intersection of all the prime ideals containing $\mathcal{I}$ and $\mathcal{O}_{X_{rd}}$ is not in any prime ideal. Hence $\mathcal{F}^\cdot \in \mathcal{I}$ □

With this result it follows that $(X, \operatorname{supph})$ is a classifying support data on the tensor triangulated category $\mathcal{D}^{per}(X)$ for a topologically Noetherian superscheme X, see Balmer [def. 5.1, [3]] for definitions. Using Theorem 5.2 of Balmer [3] we get following corollary,

Corollary 4.13. *The canonical map* $f\colon X \to \operatorname{Spc}(\mathcal{D}^{per}(X))$ *given by* $x \mapsto \{\mathcal{F}^\cdot \in \mathcal{D}^{per}(X) \mid x \notin \operatorname{supph}(\mathcal{F}^\cdot)\}$ *is a homeomorphism.*

5. Localization theorem and spectrum for a split superscheme

We shall prove a localization theorem (similar to that proved by Thomason) for split superschemes by using the generalisation of Thomason's result proved by Neeman [21]. First we recall some notation. Given a closed subset Z of X we can define the full triangulated subcategory $\mathcal{D}_{qc,Z}(X) \subseteq \mathcal{D}_{qc}(X)$ consisting of all objects with homological support contained in Z. Suppose U is the open complement of closed subset Z. There is a canonical restriction functor $j^*\colon \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(U)$ and clearly it will be the trivial functor on the thick subategory $\mathcal{D}_{qc,Z}(X)$. Here we are using the unbounded complex of quasi coherent sheaves over superschemes. To extend various functors to unbounded complexes we need notion of K-injective (K-projective) resolutions, see [25]. Following definition was given in [25].

Definition 5.1. An unbounded complex $A^\cdot$ of an abelian category is called K-injective (resp. K-projective) if for every acyclic complex $S^\cdot$, the complex $\operatorname{Hom}^\cdot(S^\cdot, A^\cdot)$ (resp. $\operatorname{Hom}^\cdot(A^\cdot, S^\cdot)$) is acyclic.

It is proved in the same paper, that an abelian category for which inverse (resp. direct) limit exists, and which has enough injectives (resp. projectives) admits a K-injective (resp. K-projective) resolution for any unbounded complex , see [cor 3.9 (resp. cor. 3.5), [25]]. Similar to the scheme case the abelian category $\mathcal{QC}\mathfrak{oh}(X)$ of all quasi coherent shaves over superscheme has arbitrary small coproducts. Therefore we can extnd various functors to unbounded derived category as demonstrated by Spaltenstein, [see sec. 6, [25]]. Moreover the abelian category $\mathcal{QC}\mathfrak{oh}(X)$ will have K-flat resolution for every unbounded complex and hence derived functor of tensor product functor can be extended to unbounded derived category and various relation among these functors can be extended from bounded derived category case to unbounded derived category, see [25] for more details. We have following result which is proved similar to the case of schemes,

Proposition 5.2. *The canonical functor induced from the functor j^*, which by abuse of notation we call $j^*\colon \mathcal{D}_{qc}(X)/\mathcal{D}_{qc,Z}(X) \xrightarrow{\sim} \mathcal{D}_{qc}(U)$ is an equivalence.*

Proof. Using K-injective resolution we can derive j_* to unbounded derived category and we can prove similar to scheme case that it gives inverse to above functor j^*. □

Another notion which we need is that of a compact object in a triangulated category and compactly generated triangulated category.

Definition 5.3. (a) An object t in a triangulated category, which is closed under the formation of arbitrary small coproducts, is said to be *compact* if $Hom(t, \_)$ respects coproducts. In a triangulated category $\mathcal{T}$, the full subcategory of all compact objects is denoted as $\mathcal{T}^c$.

(b) A triangulated category $\mathcal{T}$, which is closed under formation of arbitrary small coproducts, is said to be *compactly generated* if there exists a set T of compact objects s.t. $\mathcal{T}$ is a smallest triangulated subcategory containing T which is closed under coproducts and distinguished triangles . Equivalently, $\mathcal{T}$ is called *compactly generated* iff $T^{\perp} := \{x \in \mathcal{T} | Hom_{\mathcal{T}}(t, x) = 0 \text{ for all } t \in T\} = 0$. The set of compact objects T is called *generating set* if further T is closed under suspension or translation.

An example of such triangulated category can be gven using derived category of left R-modules and category of quasi coherent sheaves over superschemes. Hence the abelian category $\mathcal{QC}\mathfrak{oh}(X)$ is cocomplete for every superschemes. A result [remark 1.2.2, [23]] of Neeman says that distinguished triangles are preserved under coproducts i.e. in a cocomplete triangulated category coproduct of distinguished triangle is distinguished. Now we shall recall the theorem 2.1 of Neeman [22] which is proved in great generality and is a slight strengthening of theorem 2.1 of Neeman [21].

Theorem 5.4 (Neeman [21] [22])**.** *Let $\mathcal{S}$ be a compactly generated triangulated category. Let R be a set of compact objects of $\mathcal{S}$ closed under suspension. Let $\mathcal{R}$ be the smallest full subcategory of $\mathcal{S}$ containing R and closed with respect to coproducts and triangles. Let $\mathcal{T}$ be the Verdier quotient $\mathcal{S}/\mathcal{R}$. Then we know:*

1. *The category $\mathcal{R}$ is compactly generated, with R as a generating set.*
2. *If R happens to be a generating set for all of $\mathcal{S}$, then $\mathcal{R} = \mathcal{S}$.*
3. *If $R \subset \mathcal{R}$ is closed under the formation of triangles and direct summands, then it is all of $\mathcal{R}^c$. In any case $\mathcal{R}^c = \mathcal{R} \cap \mathcal{S}^c$.*
4. *The induced functor $F\colon \mathcal{S}^c/\mathcal{R}^c \to \mathcal{T}^c$ is fully faithful and every object of $\mathcal{T}^c$ is isomorphic to direct summand of image of the functor F. In particular, if $\mathcal{T}^c$ is an idempotent complete then we get an equivalence from idempotent completion $\widetilde{\mathcal{S}^c/\mathcal{R}^c}$ to the triangulated category $\mathcal{T}^c$.*

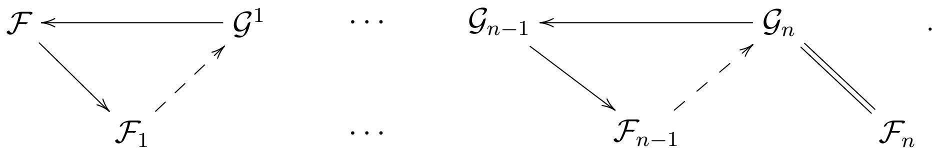
In our particular situation we take $\mathcal{S} := \mathcal{D}_{qc}(X), \mathcal{R} := \mathcal{D}_{qc,Z}(X)$ and as we proved above in 5.2 the quotient will be $\mathcal{T} := \mathcal{D}_{qc}(U)$. We shall now prove following result which will provide all hypothesis required for the application of Neeman's theorem.

Proposition 5.5. *Following statements are true for any split superscheme* $(X, \mathcal{O}_X)$

(1) *The triangulated category* $\mathcal{D}_{qc}(X)$ *is closed under the formation of arbitrary small coproducts.*
(2) *The triangulated category* $\mathcal{D}_{qc}(X)$ *is a compactly generated category.*
(3) $\mathcal{D}_{qc,Z}(X)^c \simeq \mathcal{D}_Z^{per}(X)$ *for any closed subset* Z *of* X.

Proof. Proof of 1. This is similar to the scheme case, as in example 1.3 of Neeman [22].

Proof of 2. Suppose $T \in \mathcal{D}_{qc}(X)$ denotes the set of objects obtained by taking the image of all perfect complexes of $\mathcal{O}_{X_{rd}}$ under the functors i_{rd} and Π applied in that order. Let $\mathcal{F} \in \mathcal{D}_{qc}(X)$. Since every unbounded complex of quasi coherent sheaves over superscheme X has K-flat resolution, we can assume that $\mathcal{F}$ is a K-flat. Now using the tower 4.8 of structure sheaf $\mathcal{O}_X$ we have following tower for $\mathcal{F} \in \mathcal{D}_{qc}(X)$,



The base of above tower,$\mathcal{F}_i := \mathcal{F} \otimes_{\mathcal{O}_X} \Pi^i \Lambda^i(\mathcal{V}) \in Im(\mathbf{i_{rd}})$, is generated by objects of the set T. Hence every object $\mathcal{F} \in \mathcal{D}_{qc}(X)$ is generated by the set T. It is now enough to prove that all objects of set T is compact in $\mathcal{D}_{qc}(X)$. Since Π computes with coproducts it is enough to prove compactness of the image of the functor $\mathbf{i_{rd}}$ restricted to compact objects. Let $\mathcal{S}$ be image of a $\mathcal{O}_{X_{rd}}$ perfect complex. We want to prove that $\mathrm{Hom}(\mathcal{S}, \_)$ commutes with small coproducts i.e.

$$\mathrm{Hom}(\mathcal{S}, \bigoplus_{\alpha\in\Lambda} \mathcal{F}_\alpha) \simeq \bigoplus_{\alpha\in\Lambda} \mathrm{Hom}(\mathcal{S}, \mathcal{F}_\alpha).$$

Considering above tower for each $\mathcal{F}_\alpha$ we get coproduct of tower as above. Using remark that small coproducts preserve distinguished triangles, [remark 1.2.2, [23]], we get tower of distinguished triangles for $\bigoplus_{\alpha\in\Lambda} \mathcal{F}_\alpha$. If we denote by $\mathcal{F}_{\alpha,i}$ the lower terms of the corresponding towers then we have following isomorphism using functor $\mathbf{i_{rd}}$

$$\mathrm{Hom}(\mathcal{S}, \bigoplus_{\alpha\in\Lambda} \mathcal{F}_{\alpha,i}) \simeq \bigoplus_{\alpha\in\Lambda} \mathrm{Hom}(\mathcal{S}, \mathcal{F}_{\alpha,i}).$$

Using *dévissage* the proof follows from long exact sequence associated to $\mathrm{Hom}(\mathcal{S}, \_)$ and five lemma.

Proof of 3. It is enough to prove that all perfect complexes are compact objects. Indeed, the full subcategory of perfect complexes is closed under triangles and direct summands as in the case of schemes. Hence by taking R to be all perfect complexes the above result of Neeman proves that all compact objects are perfect complexes. Now to prove that every perfect complex is a compact object we have to first observe the following,

$$(H^0(\mathcal{RH}om(\mathcal{F},\mathcal{G})))^0 = Hom_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}).$$

Here $\mathcal{RH}om(\mathcal{F},\mathcal{G})$ is (internal) homomorphisms between $\mathcal{F}$ and $\mathcal{G}$. Now the rest of the proof is similar to the proof given in example 1.3 of Neeman [22]. □

Using the above result it is easy to deduce following corollary,

Corollary 5.6. *Given a split superscheme* $(X, \mathcal{O}_X)$ *we have an equivalence of tensor triangulated categories,* $F\colon \widetilde{\mathcal{D}^{per}(X)/\mathcal{D}^{per}_Z}(X) \xrightarrow{\sim} \mathcal{D}^{per}(U)$.

Proof. It is enough to observe that j^* induces a tensor functor. □

As in Balmer [3] we shall use this localisation result to give a relation between structure sheaves. Balmer [3] has defined structure sheaf of $\mathrm{Spc}(\mathcal{K})$ for any tensor triangulated category CK as a sheaf associated to the presheaf given by $U \mapsto End_{\mathcal{K}/\mathcal{K}_Z}(1_U)$ where U is an open set and $1_U \in (\mathcal{K}/\mathcal{K}_Z)$ is the image of tensor unit $1 \in \mathcal{K}$. Define $\mathrm{Spec}(\mathcal{D}^{per}(X)) := (\mathrm{Spc}(\mathcal{D}^{per}(X)), \mathcal{O}_{\mathcal{D}^{per}(X)})$ the locally ringed space associated to tensor triangulated category $\mathcal{D}^{per}(X)$. Now the homeomorphism f defined in 4.13 above for a split superscheme gives a map of locally ringed spaces, $f\colon (X \simeq X^0, \mathcal{O}_{X^0}) \to \mathrm{Spec}(\mathcal{D}^{per}(X))$. Here the map of structure sheaves comes from the identification given in corollary 5.6. We have the following result similar to Theorem 6.3 of Balmer [3],

Theorem 5.7. *Suppose X is a topologically noetherian (that is, all open subsets are quasi compact) split superscheme. The map f defined as above gives an isomorphism of locally ringed spaces i.e.* $X^0 \simeq \mathrm{Spec}(\mathcal{D}^{per}(X))$.

Proof. Using the homeomorphism f it is enough to prove isomorphism of structure sheaves. Hence we can assume that the superscheme is affine. Now using the remark 8.2 of Balmer [2] and localisation theorem 5.6 we can prove that the induced map of sheaves is an isomorphism. □

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