HEREDITARY COMPLETENESS FOR SYSTEMS OF EXPONENTIALS AND REPRODUCING KERNELS

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Abstract. We prove that any complete and minimal system of exponentials $\{e^{i\lambda_n t}\}$ in $L^2(-a,a)$ is hereditarily complete up to a one-dimensional defect. This means that there is at most one (up to a constant factor) function f which is orthogonal to all the summands in its formal Fourier series $\sum_n (f,\tilde{e}_n)e^{i\lambda_n t}$, where $\{\tilde{e}_n\}$ is the system biorthogonal to $\{e^{i\lambda_n t}\}$. An analogous result is obtained for systems of reproducing kernels in some de Branges spaces. On the other hand, we show that in general there exist non-hereditarily complete systems of reproducing kernels in de Branges spaces, thus answering a question posed by N. Nikolski.

1. Introduction and main results

1.1. **Hereditary completeness in general setting.** A system of vectors $\{x_n\}_{n\in N}$ in a separable Hilbert space H is said to be *exact* if it is both *complete* (i.e., $\overline{Span}\{x_n\} = H$) and *minimal* (i.e., $\overline{Span}\{x_n\}_{n\neq n_0} \neq H$ for any n_0). Given an exact system we consider its (unique) *biorthogonal* system $\{x'_n\}_{n\in N}$ which satisfies $(x_m, x'_n) = \delta_{mn}$. Then to every element $x \in H$ we associate its formal Fourier series

$$x \sim \sum_{n\in N} (x, x'_n) x_n.$$

A natural condition is that this correspondence is one-to-one: no nonzero vector generates zero series, in other words the biorthogonal system $\{x'_n\}$ is also complete. Another important property is the possibility of reconstructing the vector x from its Fourier series

$$x \in \overline{Span}\,\{(x, x'_n)x_n\}.$$

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If this holds, we say that the system $\{x_n\}_{n\in N}$ is *hereditarily complete*. We will use an equivalent description: for any partition $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$, the system

$$\{x_n\}_{n\in N_1} \cup \{x'_n\}_{n\in N_2}$$

is complete in H. In the opposite situation (i.e., when $\{x_n\}$ and $\{x'_n\}$ are complete, but $\{x_n\}$ is not hereditarily complete) we say that system is non-hereditarily complete.

The importance of this notion is related to the spectral synthesis problem for linear operators. If $\{x_n\}$ is the sequence of eigenfunctions and root functions of some compact operator, then the hereditary completeness of $\{x_n\}$ is equivalent to the possibility of the so-called strong spectral synthesis for this operator, i.e., its restriction to any invariant subspace is complete (see [13] or [11, Chapter 4]).

The condition that the biorthogonal system $\{x'_n\}$ is also complete in H is by no means automatic and corresponding examples can be easily constructed. It is less trivial to give examples of the situations where both $\{x_n\}$ and $\{x'_n\}$ are complete, but the system $\{x_n\}$ fails to be hereditarily complete. In fact, first examples go back to Hamburger [10] who constructed a compact operator with a complete set of eigenvectors, whose restriction to an invariant subspace is a nonzero Volterra operator (and, hence, is not complete). Further examples of non-hereditarily complete systems were found by Markus [13] and Nikolski [14], while a general approach to constructing non-hereditarily complete systems was developed by Dovbysh, Nikolski and Sudakov [6, 7]. Any non-hereditarily complete system gives an example of an exact system which is not a summation basis. On the other hand, uniform minimality and closeness to an orthonormal system may be combined with nonhereditary completeness [7].

1.2. **Hereditary completeness for exponential systems.** It is natural to study the problem of hereditary completeness for special systems in functional spaces, e.g. those which appear as families of eigenvectors and root vectors of a certain operator. Exponential systems form an important class in this respect. Let $\Lambda = \{\lambda_n\} \subset \mathbb{C}$ and let $e_\lambda(t) = \exp(i\lambda t)$. We consider the exponential system $\{e_\lambda\}_{\lambda\in\Lambda}$ in $L^2(-a, a)$, $a > 0$. It was shown by Young [16] that, in contrast to the general situation, for any exact system of exponentials its biorthogonal system is complete. Another approach to this problem was suggested in [9], where it is shown that any exact system of exponentials is the system of eigenfunctions of the differentiation operator $i\frac{d}{dx}$ in $L^2(-a,a)$ with a certain generalized boundary condition.

The problem of hereditary completeness for exponential systems remains open. We will show that the hereditary completeness holds up to a possible one-dimensional defect.

Applying the Fourier transform $\mathcal{F}$ one reduces the problem for exponential systems in $L^2(-\pi,\pi)$ to the same problem for systems of reproducing kernels in the Paley–Wiener space $\mathcal{P}W_\pi = \mathcal{F}L^2(-\pi,\pi)$. Recall that the reproducing kernel of $\mathcal{P}W_\pi$ corresponding to a point $\lambda \in \mathbb{C}$ is of the form

$$K_\lambda(z) = \frac{\sin \pi(z-\overline{\lambda})}{\pi(z-\overline{\lambda})}, \qquad f(\lambda) = (f, K_\lambda)_{\mathcal{P}W_\pi}.$$

Let $\Lambda \subset \mathbb{C}$ be such that the system of reproducing kernels $\{K_\lambda\}_{\lambda\in\Lambda}$ is exact in the Paley–Wiener space $\mathcal{P}W_\pi$. Then the biorthogonal system $\{g_\lambda\}$ is given by

$$g_\lambda(z) = \frac{G(z)}{G'(\lambda)(z-\lambda)},$$

where G is the so-called generating function of the set Λ. By the above-mentioned result of Young, $\{g_\lambda\}$ is also an exact system. It is well known that G is a function of exponential type π and has only simple zeros at the points of Λ.

Theorem 1.1. *If $\{K_\lambda\}_{\lambda\in\Lambda}$ is exact in the Paley–Wiener space $\mathcal{P}W_\pi$, then for any partition $\Lambda = \Lambda_1 \cup \Lambda_2$, the orthogonal complement in $\mathcal{P}W_\pi$ to the system*

$$\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2} \tag{1.1}$$

is at most one-dimensional.

Moreover, this exceptional one-dimensional complement may happen only in the case when the sequence Λ_1 has zero density. Given a sequence Λ set

$$D_+(\Lambda) = \limsup_{r\to\infty} \frac{n_r(\Lambda)}{2r},$$

where $n_r(\Lambda)$ is the usual counting function of the sequence Λ, $n_r(\Lambda) = \#\{\lambda \in \Lambda, |\lambda| \le r\}$.

Theorem 1.2. *Let $\Lambda \subset \mathbb{R}$, let the system $\{K_\lambda\}_{\lambda\in\Lambda}$ be exact in $\mathcal{P}W_\pi$, and let the partition $\Lambda = \Lambda_1 \cup \Lambda_2$ satisfy $D_+(\Lambda_1) > 0$. Then the system (1.1) is complete in $\mathcal{P}W_\pi$.*

1.3. **Reproducing kernels of the de Branges spaces.** The above results may be extended to the de Branges spaces. Let E be an entire function in the Hermite–Biehler class, that is E has no zeros in $\overline{\mathbb{C}_+}$, and

$$|E(z)| > |E^*(z)|, \qquad z \in \mathbb{C}_+,$$

where $E^*(z) = \overline{E(\overline{z})}$. With any such function we associate the *de Branges space* $\mathcal{H}(E)$ which consists of all entire functions F such that F/E and F^*/E restricted to $\mathbb{C}_+$ belong to the Hardy space $H^2 = H^2(\mathbb{C}_+)$. The inner product in $\mathcal{H}(E)$ is given by

$$(F, G)_E = \int_{\mathbb{R}} \frac{F(t)\overline{G(t)}}{|E(t)|^2}\, dt.$$

The Hilbert spaces of entire functions $\mathcal{H}(E)$ were introduced by L. de Branges [5] in connection with inverse spectral problems for differential operators. These spaces are also of a great interest from the function theory point of view. The Paley–Wiener space $\mathcal{P}W_a$ is the de Branges space corresponding to $E(z) = \exp(-iaz)$.

An important characteristics of the de Branges space $\mathcal{H}(E)$ is its phase function, that is, an increasing C^∞-function φ such that $E(t)\exp i\varphi(t) \in \mathbb{R}$, $t \in \mathbb{R}$ (thus, essentially, $\varphi = -\arg E$ on $\mathbb{R}$). Clearly, for $\mathcal{P}W_a$, $\varphi(t) = at$. If $\varphi' \in L^\infty(\mathbb{R})$ (in which case we say that φ has sublinear growth), the space $\mathcal{H}(E)$ shares certain properties with the Paley–Wiener spaces.

A crucial property of the de Branges spaces is the existence of orthogonal bases of reproducing kernels corresponding to real points [5]. For $\alpha \in [0, \pi)$ we consider the set of points $t_n \in \mathbb{R}$ such that

$$\varphi(t_n) = \alpha + \pi n, \qquad n \in \mathbb{Z}. \tag{1.2}$$

It should be mentioned that the points t_n may exist not for all $n \in \mathbb{Z}$ (e.g., the sequence $\{t_n\}$ may be one-sided, that is, t_n may exist only for $n \ge n_0$). If the points t_n are defined by (1.2), then the system of reproducing kernels $\{K_{t_n}\}$ is an orthogonal basis for $\mathcal{H}(E)$ for each $\alpha \in [0, \pi)$ except, may be, one (α is an exceptional value if and only if $e^{i\alpha}E - e^{-i\alpha}E^* \in \mathcal{H}(E)$).

The completeness of a system biorthogonal to an exact system of reproducing kernels was studied in [2, 8]. In particular, it was shown in [8] that such biorthogonal systems are always complete when $\varphi' \in L^\infty(\mathbb{R})$. The following extension of this result is obtained in [2]: if, for some $N > 0$, $\varphi'(t) = O(|t|^N)$, $|t| \to \infty$, then either $e^{i\alpha}E - e^{-i\alpha}E^* \in \mathcal{H}(E)$ for some $\alpha \in [0,\pi)$, or any system biorthogonal to an exact system of reproducing kernels is complete in $\mathcal{H}(E)$.

The method of the proof of Theorem 1.1 extends to the case of the de Branges spaces with sublinear growth of the phase.

Theorem 1.3. *Let $\mathcal{H}(E)$ be a de Branges space such that $\varphi' \in L^\infty(\mathbb{R})$. If the system of reproducing kernels $\{K_\lambda\}_{\lambda\in\Lambda}$ is exact in $\mathcal{H}(E)$, then for any partition $\Lambda = \Lambda_1 \cup \Lambda_2$, the orthogonal complement in $\mathcal{H}(E)$ to the system*

$$\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2} \tag{1.3}$$

is at most one-dimensional.

A crucial step in the proofs of Theorems 1.1 and 1.3 is the use of expansions of functions in $\mathcal{P}W_\pi$ or in $\mathcal{H}(E)$ with respect to *two different* orthogonal bases of reproducing kernels. At first glance it may look like an artificial trick; however it should be noted that the existence of two orthogonal bases of reproducing kernels is a property which characterizes de Branges spaces among all Hilbert spaces of entire functions (see [3, 4]). Therefore, we believe this method to be intrinsically connected with the deep and complicated geometry of de Branges spaces.

We do not know if there really exists one-dimensional defect in the conditions of Theorem 1.3 (and even in the Paley–Wiener space $\mathcal{P}W_\pi$), but we are able to show that in general de Branges spaces nonhereditary completeness is possible.

Theorem 1.4. *There exists a de Branges space $\mathcal{H}(E)$ and an exact system of reproducing kernels $\{K_\lambda\}$ such that its biorthogonal system is complete, but the original system $\{K_\lambda\}$ is non-hereditarily complete.*

This theorem answers a question about hereditary completeness of systems of reproducing kernels in the model subspaces of the Hardy space (de Branges spaces being a special case) posed by N. Nikolski.

Throughout this paper the notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \le CV(z)$ holds for all z in the set in question, which may be a Hilbert space, a set of complex numbers, or a suitable index set. We write $U(z) \asymp V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

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2. Preliminaries

Note that if $\Lambda = \Lambda_1 \cup \Lambda_2$, and one of the sets Λ_1 or Λ_2 is finite, then the corresponding system (1.1) is complete by a simple Hilbert space argument. Therefore, from now on we exclude the case when one of the sets Λ_1, Λ_2 is finite.

Let $h \in \mathcal{P}W_\pi$ be a function orthogonal to the system (1.1). Assume that $\Lambda \cap \mathbb{Z} = \emptyset$ and write the expansion of the vector h with respect to the Shannon–Kotelnikov–Whittaker orthonormal basis $K_n(z) = \frac{\sin \pi(z-n)}{\pi(z-n)}$,

$$h(z) = \sum_n \overline{a_n} K_n(z) = \frac{1}{\pi} \sum_n \overline{a_n} (-1)^n \frac{\sin \pi z}{z-n},$$

where $a_n = \overline{h(n)}$ and $\|h\|^2 = \sum_n |a_n|^2 < \infty$. The fact that h is orthogonal to $\big\{\frac{G(z)}{z-\lambda}\big\}_{\lambda \in \Lambda_1}$ is equivalent to

$$\left(\frac{G(z)}{z-\lambda}, h\right) = \frac{1}{\pi} \sum_n \frac{a_n G(n)}{n-\lambda} = 0, \qquad \lambda \in \Lambda_1, \tag{2.1}$$

while $(h, K_\lambda) = 0$, $\lambda \in \Lambda_2$, implies that

$$\sum_n \frac{\overline{a_n}(-1)^n}{\lambda - n} = 0, \qquad \lambda \in \Lambda_2. \tag{2.2}$$

Without loss of generality we may assume that h does not vanish at integers, that is, $a_n \neq 0$, $n \in \mathbb{Z}$. Otherwise we can expand h with respect to some other basis $\{K_{n+\alpha}\}$, $\alpha \in (0,1)$.

Now let G_2 be a canonical product of genus 1 constructed by the zero set Λ_2, and let $G_1 = G/G_2$. The function G_2 is defined uniquely up to an exponential factor $e^{\gamma z}$. Note that the zeros of G satisfy the Blaschke condition in $\mathbb{C}_+$ and in $\mathbb{C}_-$. Therefore, we may choose γ such that $G_2^*/G_2 = B_1/B_2$ for some Blaschke products B_1 and B_2. Hence G_1^*/G_1 is the ratio of two Blaschke products as well, since G^*/G is of this form for any generating function G of a complete minimal system of reproducing kernels.

We can rewrite conditions (2.1)–(2.2) as

$$\sum_n \frac{a_n G(n)}{z-n} = \frac{G_1(z) S_1(z)}{\sin \pi z}, \tag{2.3}$$

$$\sum_n \frac{\overline{a_n}(-1)^n}{z-n} = \frac{G_2(z)S_2(z)}{\sin \pi z}, \tag{2.4}$$

where S_1 and S_2 are some entire functions.

The pairs (S_1, S_2) of entire functions satisfying (2.3)–(2.4) parametrize all functions orthogonal to (1.1). We will denote the set of such pairs by $\Sigma(\Lambda_1, \Lambda_2)$. Note that *the function $S_2 = h/G_2$ does not depend on the choice of the orthogonal basis* $\{k_{n+\alpha}\}$ up to a nonzero scalar factor (we will use this fact repeatedly), while S_1 will depend on the choice of the basis.

Comparing the residues at n we get

$$S_1(n) = (-1)^n a_n G_2(n), \qquad G_2(n)S_2(n) = \overline{a_n}. \tag{2.5}$$

Put $S = S_1 S_2$. Then

$$S(n) = S_1(n)S_2(n) = (-1)^n |a_n|^2. \tag{2.6}$$

Lemma 2.1. *The function G_1S_1 is in $\mathcal{P}W_\pi + z\mathcal{P}W_\pi$.*

Proof. If w is a zero of G_1S_1, then it follows from (2.3) and the inclusion $G(n)(1+|n|)^{-1} \in \ell^2$ that

$$\frac{G_1(z)S_1(z)}{z-w} = \sin \pi z \sum_n \frac{a_n G(n)}{(n-w)(z-n)} \in \mathcal{P}W_\pi. \tag{2.7}$$ □

Lemma 2.2. *Let $h \in \mathcal{P}W_\pi$ be orthogonal to some system of the form (1.1) and let $(S_1, S_2) \in \Sigma(\Lambda_1, \Lambda_2)$. Then $S \in \mathcal{P}W_\pi + c \sin \pi z$ for some $c \in \mathbb{C}$.*

Proof. Consider the function $Q \in \mathcal{P}W_\pi$ which solves the interpolation problem $Q(n) = (-1)^n |a_n|^2$, $n \in \mathbb{Z}$ (where a_n are the coefficients in the expansion $h = \sum_n \overline{a}_n K_n$) and put $\tilde{S} = S - Q$. Then $\tilde{S}$ vanishes on $\mathbb{Z}$ and so $\tilde{S}(z) = H(z)\sin \pi z$. It remains to show that H is a constant. Note that $G_2S_2 = h \in \mathcal{P}W_\pi$ and, by Lemma 2.1, $G_1S_1 \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi$. Hence,

$$G(z)S(z) = \sin^2 \pi z \cdot \sum_n \frac{a_n G(n)}{z-n} \cdot \sum_n \frac{\overline{a_n}(-1)^n}{z-n} \in \mathcal{P}W_{2\pi} + z\mathcal{P}W_{2\pi}, \tag{2.8}$$

and, since $G \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi$ and $GQ \in \mathcal{P}W_{2\pi} + z\mathcal{P}W_{2\pi}$, also

$$G(z)\tilde{S}(z) = G(z)H(z)\sin \pi z \in \mathcal{P}W_{2\pi} + z\mathcal{P}W_{2\pi}.$$

We may divide by $\sin \pi z$ in the space $\mathcal{P}W_\pi$, and so

$$G(z)H(z) \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi.$$

Since G is an entire function of exponential type π, we conclude that H is of zero exponential type. Now if H has at least one zero z_1, we conclude that $\frac{H(z)G(z)}{z-z_1} \in \mathcal{P}W_\pi$ which contradicts the fact that Λ is a uniqueness set for the Paley–Wiener space. Thus, H is a constant. □

Lemma 2.3. *Let $h \in \mathcal{P}W_\pi$ be orthogonal to some system of the form (1.1) and let $(S_1, S_2) \in \Sigma(\Lambda_1, \Lambda_2)$. Then both functions S_1/S_1^* and S_2/S_2^* are ratios of two Blaschke products.*

Proof. The zero sets of S_1 and S_2 satisfy the Blaschke condition in $\mathbb{C}_+$ and in $\mathbb{C}_-$ since $G_1S_1 \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi$ and $h = G_2S_2 \in \mathcal{P}W_\pi$. Thus, it remains to show that S_1/S_1^* and S_2/S_2^* have no exponential factors. By Lemma 2.2 we know that S satisfies this property. Indeed, if $c \neq 0$ this is obvious, whereas if $c = 0$, then the function S coincides with the function $Q \in \mathcal{P}W_\pi$ which is real on $\mathbb{R}$ and has at least one zero in each interval $(n, n+1)$. So the size of the conjugate indicator diagram of the function GS equals 4π. Hence, the size of the conjugate indicator diagram both for G_1S_1 and for G_2S_2 equals 2π. Since $G_2S_2 \in \mathcal{P}W_\pi$, we obtain that $G_2S_2/(G_2^*S_2^*)$ is a ratio of two Blaschke products. By the construction of G_2, the same is true for S_2 and, hence, for S_1. □

Lemma 2.4. *If $(S_1, S_2) \in \Sigma(\Lambda_1, \Lambda_2)$, then also $(S_1^*, S_2^*) \in \Sigma(\Lambda_1, \Lambda_2)$.*

Proof. By Lemma 2.3, S_1^*/S_1 is of the form B_1/B_2 for some Blaschke products B_1 and B_2. We consider the following representation

$$\frac{G_1(z)S_1(z)}{\sin \pi z} \cdot \frac{S_1^*(z)}{S_1(z)} = \sum_n \frac{a_n G(n)}{z-n} \cdot \frac{S_1^*(n)}{S_1(n)} + H(z), \tag{2.9}$$

where H is an entire function (which holds since the residues at integers coincide). On the other hand, $G_1S_1^* \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi$, whence $|H(z)| \lesssim 1 + |z|$ and so H is a polynomial of degree at most 1. Finally, (2.3) implies that $e^{-\pi|y|}|G_1(iy)S_1(iy)| \to 0$, $|y| \to \infty$. Since the function S_1^*/S_1 is reciprocal to itself at conjugate points, we conclude that $\min(|H(iy)|, |H(-iy)|) \to 0$, $|y| \to \infty$, and so $H \equiv 0$.

Set $b_n = a_n S_1^*(n)/S_1(n)$. We can use an analogous argument to show that

$$\frac{G_2(z)S_2^*(z)}{\sin \pi z} = \sum_n \frac{\overline{a_n}(-1)^n}{z-n} \cdot \frac{S_2^*(n)}{S_2(n)}. \tag{2.10}$$

Comparing the residues we get

$$\overline{a_n}(-1)^n S_2^*(n)/S_2(n) = \overline{b_n}(-1)^n.$$

Thus, the pair (S_1^*, S_2^*) corresponds to the sequence $\{b_n\}$ in equations (2.3) and (2.4). This means that $(S_1^*, S_2^*) \in \Sigma(\Lambda_1, \Lambda_2)$. □

By Lemma 2.4, if $(S_1, S_2) \in \Sigma(\Lambda_1, \Lambda_2)$, then $(S_1 + S_1^*, S_2 + S_2^*) \in \Sigma(\Lambda_1, \Lambda_2)$ and $(iS_1 - iS_1^*, -iS_2 + iS_2^*) \in \Sigma(\Lambda_1, \Lambda_2)$. Thus, in what follows we may assume that the functions S_1 and S_2 are real on $\mathbb{R}$. In this case we have an immediate corollary from (2.6).

Corollary 2.5. *If S_1 and S_2 are real on $\mathbb{R}$, then each open interval $(n, n+1)$, $n \in \mathbb{Z}$, contains exactly one zero of S, and S has no other zeros.*

Proof. Since S is real on $\mathbb{R}$ and changes the sign at $n \in \mathbb{Z}$, it has at least one zero in every interval $(n, n+1)$. Choosing a zero in each interval we construct the (principal value) canonical product S_0. Then $S = S_0 H$ for some entire function H of zero exponential type which is real on $\mathbb{R}$. Clearly, $|S_0(iy)| \gtrsim |y|^{-1} e^{\pi|y|}$, $|y| \to \infty$. By Lemma 2.2 we have $S \in \mathcal{P}W_\pi + c \sin \pi z$. Hence, $|H(iy)| \lesssim |y|$, $|y| \to \infty$, which implies that H is a polynomial of degree at most 1. Since the signs of $S(n)$ interchange, S cannot have two zeros in any of the intervals $(n, n+1)$. Thus, H is a constant. □

3. Proofs of Theorems 1.1 and 1.2

We are now ready to prove the main results on hereditary completeness for exponential systems.

3.1. **Completeness up to a one-dimensional defect.** *Proof of Theorem 1.1.* Without loss of generality assume that $\Lambda \cap \mathbb{Z} = \emptyset$. Let $f = \sum_{n \in \mathbb{Z}} \overline{a_n} K_n$ and $h = \sum_{n \in \mathbb{Z}} \overline{b_n} K_n$ be two linearly independent vectors orthogonal to (1.1) and let (S_1, S_2) and (T_1, T_2) be the corresponding pairs of entire functions from $\Sigma(\Lambda_1, \Lambda_2)$. Since, by Lemma 2.4, the pairs $(S_1 + S_1^*, S_2 + S_2^*)$, $(iS_1 - iS_1^*, -iS_2 + iS_2^*)$, $(T_1 + T_1^*, T_2 + T_2^*)$, and $(iT_1 - iT_1^*, -iT_2 + iT_2^*)$ also belong to $\Sigma(\Lambda_1, \Lambda_2)$, we may assume from the very beginning that the pairs (S_1, S_2) and (T_1, T_2) are linearly independent and all functions S_1, S_2, T_1, and T_2 are real on $\mathbb{R}$.

Using equations (2.5) for S and T we get

$$S_1(n) T_2(n) \overline{G_2(n)} = T_1(n) S_2(n) \overline{G_2(n)} = (-1)^n G_2(n) a_n b_n,$$

and hence,

$$S_1(n) T_2(n) = S_2(n) T_1(n) = \beta_n a_n b_n,$$

with $|\beta_n| = 1$.

Denote by Q the function in PW_π which solves the interpolation problem $Q(n) = \beta_n a_n b_n$. Then

$$T_1(z)S_2(z) = Q(z) + a(z)\sin \pi z, \qquad S_1(z)T_2(z) = Q(z) + b(z)\sin \pi z,$$

for some entire functions a and b. We show now that a and b are constants.

In what follows we denote by $\mathcal{P}W_\pi + \mathbb{C}\sin \pi z$ the class of functions of the form $f + c\sin \pi z$, where $f \in \mathcal{P}W_\pi$, $c \in \mathbb{C}$. Note that the functions $S = S_1S_2$ and $T = T_1T_2$ are in $\mathcal{P}W_\pi + \mathbb{C}\sin \pi z$ by Lemma 2.2. Moreover, note that the pair (S_1+T_1, S_2+T_2) corresponds to the vector $f+h$ while the pair (S_1+iT_1, S_2-iT_2) corresponds to the vector $f + ih$. Applying again Lemma 2.2 we get $U = (S_1 + T_1)(S_2 + T_2)$ and $V = (S_1 + iT_1)(S_2 - iT_2)$ are in $\mathcal{P}W_\pi + \mathbb{C}\sin \pi z$. Hence the functions

$$S_1T_2 + S_2T_1 = U - S - T, \qquad i(S_2T_1 - S_1T_2) = V - S - T$$

belong to $\mathcal{P}W_\pi + \mathbb{C}\sin \pi z$. Thus, S_1T_2, $S_2T_1 \in \mathcal{P}W_\pi + \mathbb{C}\sin \pi z$, and we conclude that a and b are constants.

Assume that $a \neq 0$. Let us denote by s_m the zero of S_2 in the interval $[m - 1/2, m + 1/2]$ for those m for which such a zero exists. Then

$$Q(s_m) + a(-1)^m \sin \pi(s_m - m) = 0,$$

whence

$$\sum |s_m - m|^2 \asymp \sum \sin^2 \pi(s_m - m) \asymp \sum |Q(s_m)|^2 < \infty.$$

On the other hand, the zeros of S_2 do not depend on the choice of the basis, they are the zeros of h/G_2. Expanding with respect to another basis (say, $\{n + \delta\}$ with small δ) we conclude that $\sum |s_m - m - \delta|^2 < \infty$ for the zeros s_m of S_2 with $s_m \in [m + \delta - 1/2, m + \delta + 1/2]$. This is obviously wrong.

Thus, we have proved that $a = b = 0$, and so $S_1T_2 = T_1S_2 = Q$. Since S_1 has no common zeros with S_2 (we choose the basis so that all a_n are nonzero) we conclude that the zero sets of S_2 and T_2 coincide, and, thus, $g = ch$ for some constant c, a contradiction. □

3.2. **Proof of Theorem 1.2.** The following proposition plays the key role in the proof of Theorem 1.2. In Section 4 we prove a slightly stronger result which applies to general de Branges spaces (see Proposition 4.4). We prefer, however, to include an elementary proof to make the exposition concerning exponential systems more self-contained.

Proposition 3.1. *Let $S \in \mathcal{PW}_\pi + \mathbb{C}\sin\pi z$ be a real entire function with real zeros interlacing with $\mathbb{Z}$. If $\sum_{n\in\mathbb{Z}} |S(n)| < \infty$, then for any $\delta > 0$ we have*

$$L_\delta := \lim_{N\to\infty} \frac{1}{N} \text{card}\,\big\{|k| \le N : \text{dist}\,(\mathcal{Z}_S \cap [k, k+1], \mathbb{Z}) > \delta\big\} = 0.$$

Proof. Let $S(n) = (-1)^n c_n$. Without loss of generality we may assume that $c_n > 0$ and $\sum_{n\in\mathbb{Z}} c_n = 1$. Then $S(z)/\sin\pi z$ is a Herglotz function in $\mathbb{C}_+$ and

$$\frac{S(z)}{\sin\pi z} = b + \sum_{n\in\mathbb{Z}} \frac{c_n}{z-n}$$

for some $b \in \mathbb{R}$. Set $s(x) = \sum_{n\in\mathbb{Z}} \frac{c_n}{x-n}$.

Case 1. If $b \neq 0$, then

$$\lim_{x\in\mathcal{Z}_S, |x|\to\infty} \text{dist}\,(x, \mathbb{Z}) = 0.$$

This follows from the fact that for any $\delta > 0$ we have $s(x) \to 0$ as $|x| \to \infty$ and $\text{dist}\,(x, \mathbb{Z}) \ge \delta$.

Case 2. Suppose that $b = 0$. Fix two positive numbers $\delta < 1/4$ and $\eta < \delta^3$ and choose M so that $\sum_{|n|\le M} c_n > 1 - \eta$.

Now let the integer N be so large that $\delta N > M$. Put

$$E_N = \left\{x \in \mathbb{R} : \left|\sum_{n\in\mathbb{Z}} \frac{c_n}{x-n}\right| \ge \frac{1}{N}\right\}.$$

By Boole's lemma, $|E_N| = 2N$ (by $|E|$ we denote the Lebesgue measure of the set E).

Next, set

$$F_N = \left\{x \in \mathbb{R} : \left|\sum_{|n|>M} \frac{c_n}{x-n}\right| \ge \frac{\delta}{2N}\right\}.$$

Then

$$|F_N| \le \frac{4N\eta}{\delta}.$$

Let $J_N = [-N - \delta N - M, N + \delta N + M]$. Since

$$\left|\sum_{|n|\le M} \frac{c_n}{x-n}\right| \le \frac{1}{(1+\delta)N}, \qquad x \notin J_N,$$

we have, for $x \in E_N \setminus J_N$,

$$\left|\sum_{|n|>M} \frac{c_n}{x-n}\right| \ge \frac{1}{N} - \frac{1}{(1+\delta)N} = \frac{\delta}{(1+\delta)N},$$

and so $x \in F_N$. We conclude that $E_N \setminus J_N \subset F_N$.

Consider the family $\mathcal{I}_N$ of the intervals of the form $I_k = [k, k+1] \subset J_N$ with $|k| \ge M + \delta N$ and with the following two properties:

$$(I_k^* \cap E_N) \setminus F_N \neq \emptyset, \qquad I_k^* = [k+\delta, k+1-\delta]; \tag{3.1}$$

$$|I_k \cap F_N| < \delta. \tag{3.2}$$

We will show that, for sufficiently large N, we have

$$\operatorname{card} \mathcal{I}_N \ge (1 - A_1\delta)|J_N|, \tag{3.3}$$

where A_1 is some absolute (numeric) constant. In what follows, symbols A_1, A_2, etc. will denote different absolute constants.

If $(I_k^* \cap E_N) \setminus F_N = \emptyset$ (i.e., the interval I_k^* does not satisfy (3.1)), then $I_k^* \subset (J_N \setminus E_N) \cup F_N$ and

$$\begin{aligned} |(J_N \setminus E_N) \cup F_N| &\le |J_N| - |J_N \cap E_N| + |F_N| \\ &= |J_N| - |E_N| + |E_N \setminus J_N| + |F_N| \\ &\le 2N + 2\delta N + 2M - 2N + \frac{8N\eta}{\delta} \le A_2 \delta N. \end{aligned}$$

Hence, for the number N_1 of those intervals I_k^* which do not satisfy (3.1), we have the estimate

$$N_1(1 - 2\delta) \le A_3 \delta N.$$

On the other hand, for the number N_2 of those intervals I_k which do not satisfy (3.2), we get $N_2\delta \le \frac{4N\eta}{\delta}$, and so $N_2 \le \frac{4N\eta}{\delta^2} \le A_4\delta N$, since $\eta < \delta^3$. Thus, for sufficiently large N,

$$\operatorname{card} \mathcal{I}_N \ge 2N - N_1 - N_2 \ge 2N - A_5 \delta N.$$

The latter inequality implies (3.3).

Now, if $I_k \in \mathcal{I}_N$, then there exists a point $y \in (I_k^* \cap E_N) \setminus F_N$ and so we have

$$\Big| \sum_{|n| \le M} \frac{c_n}{y - n} \Big| \ge \frac{1}{N} - \frac{\delta}{2N} > \frac{1}{2N}.$$

For any $x \in I_k^*$ using the fact that $|k| \ge M + \delta N$ we get

$$\Big| \sum_{|n| \le M} \frac{c_n}{x - n} - \sum_{|n| \le M} \frac{c_n}{y - n} \Big| \le \sum_{|n| \le M} \frac{c_n |x - y|}{|(x-n)(y-n)|} \le \frac{1}{\delta^2 N^2} \le \frac{1}{4N} \tag{3.4}$$

for sufficiently large N, and hence,

$$\left|\sum_{|n|\le M}\frac{c_n}{x-n}\right|\ge\frac{1}{4N},\qquad x\in I_k^*.$$

Suppose that for some $w\in I_k^*$ we have $s(w)=\sum_{n\in\mathbb{Z}}\frac{c_n}{w-n}=0$. Then

$$\left|\sum_{|n|>M}\frac{c_n}{w-n}\right|\ge\frac{1}{4N}>\frac{\delta}{N}.$$

So $w\in F_N$ and, moreover, since the function under the modulus sign is monotone on I_k we obtain that either $[k,w]\subset F_N$ or $[w,k+1]\subset F_N$, which is impossible due to (3.2).

Thus, the zeros of s (and hence of S) on $I_k\in\mathcal{I}_N$ are in $I_k\setminus I_k^*$. It follows from (3.3) that

$$L_\delta=\limsup_{N\to\infty}\frac{1}{N}\text{card}\left\{|k|\le N:\text{dist}\left(\mathcal{Z}_S\cap[k,k+1],\mathbb{Z}\right)>\delta\right\}\le A\delta$$

for some absolute constant A. Since L_δ is a non-increasing nonnegative function of δ on $(0,1/4)$, it follows that $L_\delta\equiv 0$. □

Proof of Theorem 1.2. Assume that there is a nontrivial function h orthogonal to the system (1.1) such that $D_+(\Lambda_1)>0$. Denote by $\mathcal{Z}_1$ and $\mathcal{Z}_2$ the zero sets of S_1 and S_2, respectively.

Since $G_1S_1\in\mathcal{P}W_\pi+z\mathcal{P}W_\pi$, we have

$$D(\Lambda_1\cup\mathcal{Z}_1)=\lim_{r\to\infty}\frac{n_r(\Lambda_1\cup\mathcal{Z}_1)}{2r}\le\pi,$$

and so

$$D_-(\mathcal{Z}_1)=\liminf_{r\to\infty}\frac{n_r(\mathcal{Z}_1)}{2r}<\pi.$$

Since S is of exponential type π, we have $D_+(\mathcal{Z}_2)>0$.

The function $S_2=h/G_2$ does not depend on the choice of the basis, and replacing if necessary the basis $\{K_n\}$ by the basis $\{K_{n+\alpha}\}$ we may find α such that for a subsequence $\tilde{\mathcal{Z}}_2$ of $\mathcal{Z}_2$ with positive upper density we have $\text{dist}\,(\tilde{\mathcal{Z}}_2,\mathbb{Z}+\alpha)\ge 1/4$. Without loss of generality assume that this holds for $\alpha=0$. Construct the function S_1 corresponding to this basis by formula (2.3). Then for $S=S_1S_2$ we have $\sum_{n\in\mathbb{Z}}|S(n)|<\infty$. Note that by Corollary 2.5 the zeros of S interlace with $\mathbb{Z}$. By Proposition 3.1 all zeros of S except the set of zero density are close to $\mathbb{Z}$, and we come to a contradiction. □

4. Extensions to the de Branges spaces

4.1. **Preliminary remarks.** We start with a general construction of functions biorthogonal to a system of reproducing kernels. Let $\mathcal{H}(E)$ be a de Branges space, let φ be the corresponding phase function. As usual, we write $E = A - iB$. To avoid inessential difficulties we will always assume that $A \notin \mathcal{H}(E)$. Let $\Lambda \subset \mathbb{C}$ be such that the system of reproducing kernels $\{K_\lambda\}_{\lambda\in\Lambda}$ of the space $\mathcal{H}(E)$ is exact. Then there exists the generating function, that is, an entire function $G \in \mathcal{H}(E) + z\mathcal{H}(E)$, such that $GH \notin \mathcal{H}(E)$ for any nontrivial entire function H, and vanishing exactly on the set Λ. The biorthogonal system to $\{K_\lambda\}_{\lambda\in\Lambda}$ is given by

$$g_\lambda(z) := \frac{G(z)}{G'(\lambda)(z-\lambda)}.$$

We will assume that $\{g_\lambda\}_{\lambda\in\Lambda}$ is also an exact system in $\mathcal{H}(E)$ (recall that this is the case, e.g., when $\varphi' \in L^\infty(\mathbb{R})$ [8] or when φ' has at most power growth and $\Theta = E^*/E$ has no finite derivative at ∞ [2]).

Let $h \in \mathcal{H}(E)$ be orthogonal to $\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2}$. Denote by $T = \{t_n\}$ the zero set of A (assume that $T \cap \Lambda = \emptyset$) and recall that the functions

$$\frac{A(z)}{z-t_n} = \pi i \frac{K_{t_n}(z)}{E(t_n)}$$

form an orthogonal basis in $\mathcal{H}(E)$ [5, Theorem 22] and $\left\|\frac{A(z)}{z-t_n}\right\|^2 = \pi\varphi'(t_n)$. Expanding

$$h(z) = A(z)\sum_n \frac{\overline{a}_n \mu_n^{1/2}}{z-t_n}, \qquad \{a_n\} \in \ell^2,$$

where $\mu_n = 1/\varphi'(t_n)$, we conclude that

$$\sum_n \frac{\overline{a}_n \mu_n^{1/2}}{z-t_n} = \frac{G_2(z)S_2(z)}{A(z)} \tag{4.1}$$

for some entire function S_2. As in the Paley–Wiener case we assume that G_2 is an entire function which vanishes exactly on Λ_2 and $G_2^*/G_2 = B_1/B_2$ for some Blaschke products B_1 and B_2. On the other hand, since $h \perp g_\lambda$, $\lambda \in \Lambda_1$, we obtain

$$\sum_n \frac{G(t_n)}{E(t_n)} \frac{a_n \mu_n^{1/2}}{z-t_n} = i\frac{G_1(z)S_1(z)}{A(z)} \tag{4.2}$$

for some entire function S_1 (argue as in the Paley–Wiener case). Comparing the residues we get

$$S_1(t_n)G_1(t_n) = -i\frac{a_n\mu_n^{1/2}A'(t_n)G(t_n)}{E(t_n)}, \tag{4.3}$$

and

$$S_2(t_n)G_2(t_n) = \overline{a}_n\mu_n^{1/2}A'(t_n). \tag{4.4}$$

Hence, for $S = S_1S_2$, we have

$$S(t_n) = -i|a_n|^2\mu_n(A'(t_n))^2/E(t_n).$$

Since $A'(t_n) = (-1)^n|E(t_n)|\varphi'(t_n)$ (the phase φ is chosen so that $\varphi(t_n) = \pi n$), we get

$$S(t_n) = |a_n|^2A'(t_n). \tag{4.5}$$

From now on we assume that φ is of *tempered growth*, that is,

$$\varphi'(t) = O(|t|^N), \qquad |t| \to \infty, \tag{4.6}$$

for some N. It follows from (4.6) that, for any $F \in \mathcal{H}(E)$,

$$\frac{|F(x)|}{|E(x)|} \le \frac{\|K_x\|_E\|F\|_E}{|E(x)|} = \Big(\frac{\varphi'(x)}{\pi}\Big)^{1/2}\|F\|_E \lesssim (|x|+1)^{N/2}, \quad x \in \mathbb{R}.$$

Using the same arguments as in the proof of Lemma 2.1 we get $G_1S_1 \in \mathcal{H}(E) + z\mathcal{H}(E)$. Hence,

$$G(z)S(z) = A^2(z)\sum_n \frac{\overline{a}_n\mu_n^{1/2}}{z-t_n}\cdot\sum_n\frac{G(t_n)}{E(t_n)}\frac{a_n\mu_n^{1/2}}{z-t_n} \in \mathcal{P}_{\frac{N}{2}+1}\cdot\mathcal{H}(E^2), \tag{4.7}$$

where $\mathcal{P}_M$ is the set of polynomials of degree at most M.

Arguing analogously to the proof of Lemma 2.2 we obtain the following growth restriction.

Lemma 4.1. *Assume that φ is of tempered growth. Let $h \in \mathcal{H}(E)$ be orthogonal to some system $\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2}$ and let (S_1, S_2) be the corresponding pair. Then $S \in \mathcal{P}_M \cdot \mathcal{H}(E)$ for some M.*

Proof. We give a sketch of the proof. Let $M \ge \frac{N}{2}+1$. By (4.5), we have

$$\frac{|S(t_n)|}{|E(t_n)|(\varphi'(t_n))^{1/2}} = |a_n|^2(\varphi'(t_n))^{1/2} \lesssim |a_n|^2|t_n|^{N/2},$$

and, dividing out sufficiently many zeros $s_1, \dots, s_M$ of S we obtain that

$$\sum_n \frac{|\tilde{S}(t_n)|^2}{|E(t_n)|^2 \varphi'(t_n)} < \infty, \qquad \tilde{S}(z) = \frac{S(z)}{(z-s_1)\cdots(z-s_M)}.$$

Now let Q be the (unique) function in $\mathcal{H}(E)$ which solves the interpolation problem $Q(t_n) = \tilde{S}(t_n)$. Using (4.7) and an analogous estimate for GQ, we obtain that $G(\tilde{S}-Q) \in \mathcal{P}_M \cdot \mathcal{H}(E^2)$. Since $\tilde{S}-Q$ vanishes on $\{t_n\}$, we have $G(\tilde{S}-Q) = GAH \in \mathcal{P}_M \cdot \mathcal{H}(E^2)$ for some entire function H. We want to show that H is a polynomial of degree at most $M+1$, whence $\tilde{S} = Q + AH \in \mathcal{P}_{M+1} \cdot \mathcal{H}(E)$. Clearly, H is of zero exponential type. If H has at least $M+2$ zeros, then dividing them out we obtain an entire function $\tilde{H}$ such that $GA\tilde{H} \in \mathcal{H}(E^2)$ and $|G(iy)\tilde{H}(iy)|/|E(iy)| = o(y^{-1})$, $|y| \to \infty$ (we use the fact that $|A(iy)|/|E(iy)| \gtrsim y^{-1}$, $y \to +\infty$). Let v_n be such that $\varphi(v_n) = \pi/2 + \pi n$ (thus, $\{v_n\}$ is the support of another orthogonal family of reproducing kernels). Since $|A(v_n)| = |E(v_n)|$, we conclude that

$$G(v_n)\tilde{H}(v_n)/E(v_n) \in L^2(\nu), \qquad \nu = \sum_n (\varphi'(v_n))^{-1} \delta_{v_n}.$$

Now it remains to apply [5, Theorem 26] to conclude that $G\tilde{H} \in \mathcal{H}(E)$, a contradiction to the fact that G is the generating function of a complete system of kernels. $\square$

By (the proof of) Lemma 4.1, $S = QH_1 + AH_2$ for some polynomials H_1, H_2, and, hence, S/S^* is a Blaschke product. Suppose that $G_2S_2/(G_2^*S_2^*)$ is not a Blaschke product. Then for small ε, we have $G_2S_2e^{i\varepsilon z} \in \mathcal{H}(E)$; analogously, $G_1S_1e^{-i\varepsilon z} \in \mathcal{H}(E)$, and, hence, $(S_1e^{i\varepsilon z}, S_2e^{-i\varepsilon z}) \in \Sigma(\Lambda_1, \Lambda_2)$ and $((1+e^{i\varepsilon z})S_1, (1+e^{-i\varepsilon z})S_2) \in \Sigma(\Lambda_1, \Lambda_2)$. Applying (4.5) and Lemma 4.1 to the function $z \mapsto S(z)(1+e^{i\varepsilon z})(1+e^{-i\varepsilon z})$, we obtain a contradiction. Thus, S_2/S_2^* and S_1/S_1^* are Blaschke products.

Now by an argument, analogous to Lemma 2.4, the pair (S_1^*, S_2^*) also corresponds to some function orthogonal to $\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2}$. Thus, we may always find functions S_1, S_2 which are real on $\mathbb{R}$. By (4.5) the function S changes its sign at adjacent points t_n (as usual we assume that the basis is chosen so that all coefficients a_n are nonzero), and thus, there is a zero of S in each of the intervals (t_n, t_{n+1}). We have an analog of Corollary 2.5.

Lemma 4.2. *Assume that φ is of tempered growth. If a pair (S_1, S_2) corresponds to a function $h \in \mathcal{H}(E)$ orthogonal to some system $\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2}$ and S_1*

and S_2 are real on $\mathbb{R}$, then $S = S_0H$, where S_0 has exactly one zero in any interval (t_n, t_{n+1}) and H is a polynomial.

4.2. **Proof of Theorem 1.3.** Without loss of generality assume that $\Lambda\cap\{t_n\} = \emptyset$, where $\varphi(t_n) = \pi n$, $n \in \mathbb{Z}$. Let f and h be orthogonal to the system (1.3),

$$f(z) = A(z)\sum_n \frac{\overline{a}_n\mu_n^{1/2}}{z-t_n}, \qquad h(z) = A(z)\sum_n \frac{\overline{b}_n\mu_n^{1/2}}{z-t_n}, \qquad \{a_n\},\ \{b_n\} \in \ell^2.$$

Let (S_1, S_2) and (T_1, T_2) be the corresponding pairs of entire functions such that S_1, S_2, T_1 and T_2 are real on $\mathbb{R}$. Using the equations (4.3)–(4.4) in the same way as in the proof of Theorem 1.1, we obtain

$$S_1(t_n)T_2(t_n) = T_1(t_n)S_2(t_n) = a_nb_n|E(t_n)|\varphi'(t_n)\beta_n,$$

where $|\beta_n| = 1$. The hypothesis $\sup_n|\varphi'(t_n)| < \infty$ implies that

$$\sum_n \frac{|S_1(t_n)T_2(t_n)|^2}{|E(t_n)|^2\varphi'(t_n)} = \sum_n \frac{|S_1(t_n)T_2(t_n)|^2\varphi'(t_n)}{(A'(t_n))^2} < \infty.$$

Since $\{K_{t_n}\}$ is an orthogonal basis in $\mathcal{H}(E)$ and $\|K_{t_n}\|_E^2 = |E(t_n)|^2\varphi'(t_n)/\pi$, we conclude that there exists a unique function $Q \in \mathcal{H}(E)$ which solves the interpolation problem $Q(t_n) = a_nb_n|E(t_n)|\varphi'(t_n)\beta_n$. Then

$$T_1(z)S_2(z) = Q(z) + a(z)A(z), \qquad S_1(z)T_2(z) = Q(z) + b(z)A(z),$$

for some entire functions a and b. We show now that a and b are polynomials.

Note that by Lemma 4.1 the functions $S = S_1S_2$ and $T = T_1T_2$ as well as $(S_1+T_1)(S_2+T_2)$ and $(S_1+iT_1)(S_2-iT_2)$ are in $\mathcal{P}_M\cdot\mathcal{H}(E)$. Hence, the functions S_1T_2 and S_2T_1, and, consequently, the functions $S_1T_2 - Q$ and $S_2T_1 - Q$ are in $\mathcal{P}_M\cdot\mathcal{H}(E)$. Now assume that $F = AH$ for some entire function H of zero exponential type and $F \in \mathcal{P}_M\cdot\mathcal{H}(E)$. We claim that H must be a polynomial. Indeed, if H has at least M zeros z_j, then dividing F by $\prod_{j=1}^M(z-z_j)$ we obtain a function in $\mathcal{H}(E)$ which vanishes on $\{t_n\}$ and, thus, is identically zero. Applying this argument to S_1T_2-Q and S_2T_1-Q we conclude that a and b are polynomials.

Now assume that $a \neq 0$. Let us denote by s_m the zero of S_2 such that $|\varphi(s_m)-\varphi(t_m)| \le \pi/2$ whenever such a zero exists. Then

$$Q(s_m) + a(s_m)A(s_m) = 0.$$

Note that $\{t_m\}$ is separated sequence (i.e., $\inf_{n\neq m}|t_n - t_m| > 0$) and so s_m is at most a union of two separated sequences. By a simple variant of Carleson embedding theorem for the de Branges spaces with $\varphi' \in L^\infty(\mathbb{R})$ (an explicit statement may be found in [1, Theorem 5.1], though the proof may be recovered already from [15, Theorem 2]) we have

$$\sum_m \frac{|Q(s_m)|^2}{|E(s_m)|^2} < \infty$$

for any $Q \in \mathcal{H}(E)$, whence

$$\sum_m \frac{|A(s_m)|^2}{|E(s_m)|^2} < \infty.$$

By the definition of the phase function, $|A(s_m)| = |E(s_m)\sin(\varphi(s_m) - \varphi(t_m))|$. Thus, we obtain that

$$\sum_m \sin^2(\varphi(s_m) - \varphi(t_m)) \asymp \sum_m (\varphi(s_m) - \varphi(t_m))^2 < \infty.$$

To complete the proof we apply once again the argument with the shift of the basis. The zeros of S_2 do not depend on the choice of the basis. Expanding with respect to another basis, say $\{K_{\tilde{t}_n}\}$, with $\varphi(\tilde{t}_n) = \delta + \pi n$ for some small δ, we get that $\sum_m (\varphi(s_m) - \varphi(\tilde{t}_m))^2 < \infty$. However, $|\varphi(t_m) - \varphi(\tilde{t}_m)| = \delta$ and we come to a contradiction.

Thus, we have proved that $a = b = 0$, and so $S_1 T_2 = T_1 S_2 = Q$. Since S_1 has no common zeros with S_2 (we choose the basis so that all a_n are nonzero) we conclude that the zero sets of S_2 and T_2 coincide, and, thus, f is proportional to h.

Remark 4.3. It is easy to show that if φ is of tempered growth, then the orthogonal complement to the system (1.3) is always finite dimensional, with a bound on the dimension depending only on N. Indeed, by Lemma 4.2, there exists $M = M(N)$ such that for any pair (S_1, S_2) which corresponds to a function f in the orthogonal complement to (1.3) and is real on $\mathbb{R}$, we have $S = S_0 H$, $H \in \mathcal{P}_M$. In particular, any interval (t_n, t_{n+1}) contains at most $M + 1$ zeros of S.

Now assume that the orthogonal complement to (1.3) contains at least $M + 3$ linearly independent vectors $f_{j,0}$, $j = 1, \dots M + 3$, such that the corresponding functions $S_{1,j,0}$, $S_{2,j,0}$ are real on $\mathbb{R}$. Considering linear combinations (with real coefficients) $f_{j,1} = f_{j,0} - \alpha_j f_{M+3,0}$, $j = 1, \dots, M + 2$, we may achieve that the functions $S_{1,j,1}$ corresponding to $f_{j,1}$ have a common zero at $x_1 \in (t_0, t_1)$. Repeating this procedure we obtain a nonzero function $f_{M+2,1}$ in the orthogonal complement to (1.3) such that the corresponding function $S_{1,M+2,1}$ vanishes at $M + 2$ distinct points $x_1, \dots x_{M+2} \in (t_0, t_1)$ which gives a contradiction.

4.3. **Density results.** Let a pair (S_1, S_2) correspond to a function $h \in \mathcal{H}(E)$ orthogonal to some system $\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2}$ and let S_1 and S_2 be real on $\mathbb{R}$. We will show that most of the zeros of S are in a certain sense close to the set $\{t_n\}$ (the support of a Clark basis). Thus, the zeros of S_2 which do not depend on the choice of the basis form a small proportion of the zeros of S (see Corollary 4.5 below).

By Lemma 4.2, $S = S_0H$, where S_0 has exactly one zero in each of the intervals (t_n, t_{n+1}) and H is a polynomial. Moreover, by (4.5) we have $\{S(t_n)/A'(t_n)\} \in \ell^1$, whence $\{S_0(t_n)/A'(t_n)\} \in \ell^1$. By Lemma 4.1 we have $S \in \mathcal{P}_M \cdot \mathcal{H}(E)$ for some M, whence S_0/A grows at most polynomially along $i\mathbb{R}_+$. Since the zeros of A and S_0 interlace, the function S_0/A is a Herglotz function and thus has a representation

$$\frac{S_0(z)}{A(z)} = az + b + \sum_n \frac{c_n}{z - t_n}, \qquad \{c_n\} \in \ell^1. \tag{4.8}$$

We will show that in this case the zeros of S_0 (and S) must be necessarily close (in some sense) to the points t_n. The case when $a \neq 0$ or $b \neq 0$ should be treated exactly as in Proposition 3.1. The remaining case follows from the following proposition (apparently, known to experts).

Proposition 4.4. *Let $t_n \in \mathbb{R}$, $n \in \mathbb{Z}$, $t_n \to \pm\infty$, $n \to \pm\infty$, and let $\mu_n > 0$, $\sum_n \mu_n = M < \infty$. Let A be an entire function which is real on $\mathbb{R}$ and has only simple real zeros coinciding with the points $\{t_n\}$. Define an entire function B by the Herglotz representation*

$$\frac{B(z)}{A(z)} = \sum_n \frac{\mu_n}{z - t_n}.$$

If we denote by s_n the zero of B in (t_n, t_{n+1}), then

$$\sum_{s_n>0} \frac{t_{n+1} - s_n}{s_n} < \infty, \qquad \sum_{s_n<0} \frac{s_n - t_n}{|s_n|} < \infty. \tag{4.9}$$

Proof. The zeros of B are simple and interlace with the zeros of A. Since $\operatorname{Im} \frac{B}{A} > 0$ in $\mathbb{C}_+$, the function $E = A - iB$ is in the Hermite–Biehler class and so we can define the de Branges space $\mathcal{H}(E)$. The measure $\mu = \sum_n \mu_n \delta_{t_n}$ is a corresponding Clark measure for which the embedding operator $E^{-1}\mathcal{H}(E) \to L^2(\mu)$ is unitary.

Consider the inner function $\Theta = E^*/E$. Since $2A/E = 1 + \Theta$ and $2B/E = i(\Theta - 1)$, we have

$$i\frac{1 - \Theta(z)}{1 + \Theta(z)} = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z} \sim i\frac{M}{y}, \qquad z = iy, \ y \to \infty.$$

Hence,

$$\frac{1+\Theta(iy)}{1-\Theta(iy)} \sim \frac{y}{M}, \qquad y\to\infty. \tag{4.10}$$

It is well known and easy to see that the function Θ may be reconstructed from the sets $\{t_n\}=\{\Theta=1\}$ and $\{s_n\}=\{\Theta=-1\}$ by the formula

$$\log\frac{\Theta+1}{\Theta-1} = c+\int_{\mathbb{R}}\Big(\frac{1}{t-z}-\frac{t}{t^2+1}\Big)f(t)dt,$$

where

$$f(t)=\begin{cases}-1/2, & t\in(t_n,s_n),\\ 1/2, & t\in(s_n,t_{n+1}),\end{cases}$$

and $c\in\mathbb{R}$ (essentially, this is a very special case of the Krein spectral shift formula [12]). Then, by (4.10), we have

$$\mathrm{Re}\int_{\mathbb{R}}\Big(\frac{1}{t-iy}-\frac{t}{t^2+1}\Big)f(t)dt=\int_{\mathbb{R}}\frac{(1-y^2)t}{(t^2+y^2)(t^2+1)}f(t)dt=\log y+O(1), \quad y\to\infty.$$

A direct computation shows, however, that

$$\int_{\mathbb{R}}\frac{(y^2-1)|t|}{(t^2+y^2)(t^2+1)}|f(t)|dt=\log y+O(1), \qquad y\to\infty,$$

whence

$$\int_{\{t:\ tf(t)>0\}}\frac{(y^2-1)\,tf(t)}{(t^2+y^2)(t^2+1)}dt=O(1), \qquad y\to\infty,$$

and therefore

$$\int_{\{t:\ tf(t)>0\}}\frac{|tf(t)|}{t^2+1}dt<\infty.$$

Since $tf(t)>0$ for $t\in(s_n,t_{n+1})$, $s_n>0$, or $t\in(t_n,s_n)$, $s_n<0$, we have

$$\sum_{s_n>0}\int_{s_n}^{t_{n+1}}\frac{dt}{t}=\sum_{s_n>0}\ln\frac{t_{n+1}}{s_n}<\infty, \qquad \sum_{s_n<0}\int_{t_n}^{s_n}\frac{dt}{|t|}=\sum_{s_n<0}\ln\frac{|t_n|}{|s_n|}<\infty.$$

The latter convergences are obviously equivalent to (4.9). $\square$

As a corollary we immediately obtain a slightly refined version of Proposition 3.1. Moreover, if $t_n=n$, $n\in\mathbb{Z}$, $A(z)=\sin\pi z$, and $S=S_1S_2$ is the function arising from the possible one-dimensional defect in the Paley–Wiener space, then

$$\sum_{s\in\mathcal{Z}_2}\frac{1}{|s|}<\infty.$$

Indeed, the zero set $\mathcal{Z}_2$ of the function S_2 does not depend on the choice of the basis, therefore applying Proposition 4.4 to $t_n = n$ and $t_n = n + \delta$ (e.g., $\delta = \frac{1}{2}$), $n \in \mathbb{Z}$, we obtain

$$\sum_{s\in\mathcal{Z}_2,\, s>0} \frac{[s]+1-s}{s} < \infty, \qquad \sum_{s\in\mathcal{Z}_2,\, s>0} \frac{[s-\delta]+1+\delta-s}{s} < \infty.$$

Under natural regularity conditions, Proposition 4.4 implies the following closeness of the sequences $\{t_n\}$ and $\{s_n\}$.

Corollary 4.5. *Let A, B, $\{t_n\}$ and $\{s_n\}$ be as in Proposition 4.4. Put $I_n = [t_n, t_{n+1}]$. Assume that $|I_k| \asymp |I_n|$, $n \le k \le 2n$, with the constants independent on k, n, and also that $|t_{2n}| \ge \rho|t_n|$ with some $\rho > 1$. Then for any $\delta > 0$ the set $\mathcal{N}$ of indices n such that $t_n > 0$ and $t_{n+1} - s_n \ge \delta|I_n|$ (respectively, $t_n < 0$ and $s_n - t_n \ge \delta|I_n|$) has zero density.*

Proof. Note that $|t_k| \asymp |t_n|$, $n \le k \le 2n$. If the upper density of $\mathcal{N}$ is positive, then there exists a sequence $M_j \to \infty$ such that

$$\sum_{n\in[M_j,2M_j]\cap\mathcal{N}} \frac{t_{n+1}-s_n}{s_n} \gtrsim \sum_{n\in[M_j,2M_j]\cap\mathcal{N}} \frac{t_{n+1}-t_n}{t_n} \gtrsim \sum_{n\in[M_j,2M_j]} \frac{t_{n+1}-t_n}{t_n}$$
$$\gtrsim \log\frac{t_{2M_j}}{t_{M_j}} \ge \log\rho,$$

and the first series in (4.9) diverges, a contradiction. □

5. An example of a nonhereditarily complete system of reproducing kernels

In this section we prove Theorem 1.4, i.e., we construct a de Branges space $\mathcal{H}(E)$ and a complete and minimal system of reproducing kernels $\{K_\lambda\}_{\lambda\in\Lambda}$ such that its biorthogonal system is also complete, but the system $\{K_\lambda\}_{\lambda\in\Lambda}$ is not hereditarily complete.

We have seen in the previous section that the existence of a non-hereditarily complete system of reproducing kernels generated by some function G in the de Branges space $\mathcal{H}(E)$ is equivalent to the solvability of the equations

$$\sum_n \frac{\overline{a}_n \mu_n^{1/2}}{z-t_n} = \frac{G_2(z)S_2(z)}{A(z)},$$

$$\sum_n \frac{G(t_n)}{E(t_n)} \cdot \frac{a_n\mu_n^{1/2}}{z-t_n} = i\frac{G_1(z)S_1(z)}{A(z)} \tag{5.1}$$

for some nonzero $\{a_n\} \in \ell^2$ and some entire functions S_1 and S_2. If all the above objects are found, then $h = G_2S_2$ is orthogonal to the corresponding system. The corresponding equations will be constructed as small perturbations of an orthogonal expansion in a de Branges space with respect to a reproducing kernels basis.

We construct the space $\mathcal{H}(E)$ and the functions G_1, G_2, S_1 and S_2 in the reverse order. Namely, we start with the construction of the function S. Let $a_n \in \mathbb{R}$ be such that

$$|a_n| = |n|^{-3/4}, \qquad n \neq 0$$

(the signs will be specified later on, the value of $|a_0| > 0$ is not important). Put

$$\frac{S(z)}{\sin \pi z} = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{a_n^2}{z-n}.$$

Then S has exactly one zero z_n in each interval $(n, n+1)$ and, by Proposition 3.1, we have $|z_n - (n+1)| \to 0$, $n \to +\infty$, with a possible exception of a set of zero density (in fact, using the explicit form of the masses a_n^2 one can show that the limit over all indices n is zero). We represent S as the product

$$S = S_1S_2 = T_0T_1S_2,$$

where T_0 is a canonical product with the zeros $s_n = z_{4^n}$ in intervals $(4^n, 4^n+1)$ and S_2 is a canonical product with the zeros $z_{4^n+2^n}$ in $(4^n+2^n, 4^n+2^n+1)$, $n \geq 2$. Next we construct h. We will construct it as $h = \tilde{T}_0T_1S_2$ where $\tilde{T}$ is a perturbation of the function T_0 such that

$$\frac{h(z)}{\sin \pi z} = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{c_n|a_n|}{z-n}, \tag{5.2}$$

$$\sum_{n\in\mathbb{Z}} c_n^2 = \infty, \qquad \sum_{n\neq 0} \frac{c_n^2}{n^2} < \infty. \tag{5.3}$$

Condition (5.2) means that

$$\frac{S(z)}{\sin \pi z} \cdot \frac{\tilde{T}_0(z)}{T_0(z)} = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{\tilde{T}_0(n)}{T_0(n)} \cdot \frac{a_n^2}{z-n},$$

and $c_n = |a_n|\tilde{T}_0(n)/T_0(n)$. Let us show that all these conditions may be satisfied. Without loss of generality assume that $|s_n - 4^n| > |s_n - (4^n+1)|$. Then we shift each zero s_n of T_0 from $(4^n, 4^n+1)$ in the following way:

$$\tilde{s}_n = 4^n + 1 - (4^n)^{3/4}\ \mathrm{dist}(s_n, \mathbb{Z})\rho_n, \qquad n \ge 2.$$

We may choose $\rho_n \in (1,2)$ such that $\mathrm{dist}(\tilde{s}_n, \mathbb{Z}\setminus\{4^n+1\}) > \frac{1}{10}$ and zero sets of $\tilde{T}_0$ and T_1S_2 do not intersect. It is easy to see that with such choice of zeros for $\tilde{T}_0$ for $x \in (4^n - 4^{n-1}, 4^n + 4^{n-1})$ we have

$$\left|\frac{\tilde{T}_0(x)}{T_0(x)}\right| \asymp \left|\frac{x-\tilde{s}_n}{x-s_n}\right|.$$

Then we obtain

$$|c_{4^n+1}| \asymp \left|\frac{\tilde{T}_0(4^n+1)}{T(4^n+1)}\right| \cdot |a_{4^n+1}| \asymp 1.$$

Moreover, it is easy to see that for any $n \in \mathbb{Z}$, $n \ge 0$, we have

$$|n|^{-1} \lesssim \left|\frac{\tilde{T}_0(n)}{T_0(n)}\right| \lesssim |n|^{3/4}. \tag{5.4}$$

Hence, $|c_n| \lesssim 1$ and, thus, (5.3) is satisfied. Moreover, $|\tilde{T}_0(iy)/T_0(iy)| \asymp 1$, and so both terms in (5.2) tend to zero along $i\mathbb{R}$. We conclude that the interpolation formula holds.

Next we introduce a de Branges space $\mathcal{H}(E)$. Put $\mu_n = c_n^2$ and $\mu = \sum_{n\in\mathbb{Z}} \mu_n\delta_n$. By (5.3), $\int (1+t^2)^{-1} d\mu(t) < \infty$, and we can define a meromorphic inner function Θ by the formula

$$\frac{1+\Theta(z)}{1-\Theta(z)} = \frac{1}{\pi i}\int \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) d\mu(t), \qquad z \in \mathbb{C}_+.$$

Then $\Theta = E^*/E$ for some entire function E in the Hermite–Biehler class. We may assume that E is of finite exponential type and does not vanish on $\overline{\mathbb{C}_+}$. Moreover, since the zero set of $2A = E + E^*$ coincides with $\mathbb{Z}$, we may choose E so that $A(z) = \sin \pi z$. Now, if we choose the signs of a_n so that $\mathrm{sign}\, a_n = \mathrm{sign}\, c_n$, formula (5.2) becomes

$$\frac{h(z)}{\sin \pi z} = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{a_n|c_n|}{z-n} = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{a_n\mu_n^{1/2}}{z-n}.$$

Hence, $h \in \mathcal{H}(E)$.

We have $h = \tilde{T}_0T_1S_2$. Put $G_2 = \tilde{T}_0T_1$. Then $h = G_2S_2$ and it remains to construct G_1 so that G is the generating function of a complete and minimal system and (5.1) is satisfied.

We will construct G_1 as a small perturbation of S_2 as we did above. We need to satisfy $G \notin \mathcal{H}(E)$, $G \in \mathcal{H}(E) + z\mathcal{H}(E)$ and (5.1) which is rewritten as

$$\frac{S(z)}{\sin \pi z} \cdot \frac{G_1(z)}{S_2(z)} = -i \sum_{n\in\mathbb{Z}} \frac{G_1(n)}{S_2(n)} \cdot \frac{h(n)}{E(n)} \cdot \frac{a_n \mu_n^{1/2}}{z-n}. \tag{5.5}$$

Note that in any de Branges space we have $iA'(t_n) = -E(t_n)\varphi'(t_n)$. Since in our case $A(z) = \sin \pi z$, we have $E(n) = -\pi i(-1)^n(\varphi'(n))^{-1} = -\pi i(-1)^n \mu_n$. Then (5.5) simplifies to

$$\frac{S(z)}{\sin \pi z} \cdot \frac{G_1(z)}{S_2(z)} = \frac{1}{\pi} \sum_{n\in\mathbb{Z}} \frac{G_1(n)}{S_2(n)} \cdot \frac{h(n)}{(-1)^n |c_n|} \cdot \frac{a_n}{z-n}.$$

The residues, obviously, coincide.

Applying the above construction to S_2 in place of T_0 we construct G_1 (again we may assume that G_1 has no common zeros with $\tilde{T}_0 T_1$) so that

$$|n|^{-1} \lesssim \left|\frac{G_1(n)}{S_2(n)}\right| \lesssim |n|^{3/4}$$

and

$$\left|\frac{G_1(4^n+2^n+1)}{S_2(4^n+2^n+1)}\right| \cdot |a_{4^n+2^n+1}| \asymp 1.$$

Hence,

$$|G(4^n+2^n+1)| = \left|\frac{G_1(4^n+2^n+1)}{S_2(4^n+2^n+1)}\right| \cdot |a_{4^n+2^n+1}| \cdot |c_{4^n+2^n+1}| \asymp |c_{4^n+2^n+1}|.$$

Hence, $\sum_{n\in\mathbb{Z}} |G(n)|^2 |c_n|^{-2} = \infty$. Thus, $G \notin \mathcal{H}(E)$. However,

$$\sum_{n\neq 0} |G(n)|^2 |n|^{-2} |c_n|^{-2} < \infty,$$

whence $\frac{G(z)}{(z-\lambda)E(z)} \in L^2(\mu)$ for the zeros λ of G. Also $|G_1(iy)/S_2(iy)| \asymp 1$, so

$$\left|\frac{G(iy)}{\sin \pi iy}\right| \asymp \left|\frac{S(iy)}{\sin \pi iy}\right| \asymp |y|^{-1}, \qquad |y| \to \infty, \tag{5.6}$$

and by [5, Theorem 26], $G \in \mathcal{H}(E) + z\mathcal{H}(E)$. Estimate (5.6) also implies the interpolation formula (5.5).

It remains to show that G is the generating function of a complete and minimal system of kernels such that its biorthogonal is also complete. The first follows from (5.6) since if $GH \in \mathcal{H}(E)$ for some entire function H of zero exponential type, then H should be at most a polynomial, which contradicts to the fact that $G \notin \mathcal{H}(E)$. Note also that, by (5.4) $|c_n| \gtrsim |n|^{-1}|a_n|$, thus $\mu_n \gtrsim |n|^{-4}$ and also $\sum_{n\in\mathbb{Z}} \mu_n = \infty$. Then, by [2, Theorem 1.2], the system biorthogonal to

$\{k_\lambda : G(\lambda) = 0\}$ is automatically complete. This completes the construction of the example (and, thus, the proof of Theorem 1.4).

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