

ANALOGS OF CUNTZ ALGEBRAS ON L^p SPACES

N. CHRISTOPHER PHILLIPS

ABSTRACT. For $d = 2, 3, \dots$ and $p \in [1, \infty)$, we define a class of representations ρ of the Leavitt algebra L_d on spaces of the form $L^p(X, \mu)$, which we call the spatial representations. We prove that for fixed d and p , the Banach algebra $\mathcal{O}_d^p = \overline{\rho(L_d)}$ is the same for all spatial representations ρ . When $p = 2$, we recover the usual Cuntz algebra \mathcal{O}_d . We give a number of equivalent conditions for a representation to be spatial. We show that for distinct $p_1, p_2 \in [1, \infty)$ and arbitrary $d_1, d_2 \in \{2, 3, \dots\}$, there is no nonzero continuous homomorphism from $\mathcal{O}_{d_1}^{p_1}$ to $\mathcal{O}_{d_2}^{p_2}$.

The algebras that we call the Leavitt algebras L_d (see Definition 1.1) are a special case of algebras introduced in characteristic 2 by Leavitt [20], and generalized to arbitrary ground fields in [21]. The Cuntz algebra \mathcal{O}_d , introduced in [9], can be defined as the norm closure of the range of a $*$ -representation of L_d on a Hilbert space. (Cuntz did not define \mathcal{O}_d this way.) The Cuntz algebras have turned out to be one of the most fundamental families of examples of C*-algebras. Leavitt algebras were long obscure, but they have recently attracted renewed attention.

In this paper we study the analogs of Cuntz algebras on L^p spaces. That is, we consider the norm closure of the range of a representation of L_d on a space of the form $L^p(X, \mu)$. It turns out that there is a rich theory of such algebras, of which we exhibit the beginning in this paper.

Our main results are as follows. There are many possible L^p analogs of Cuntz algebras (although we mostly do not know for sure that they really are essentially different), but there is a natural class of such algebras, namely those that come from what we call spatial representations (Definition 7.4(2)). Spatial representations are those for which the standard generators form an L^p analog of what is called a row isometry in multivariable operator theory on Hilbert space. (The survey article [11] emphasizes the more general row contractions on Hilbert spaces, but row isometries also play a significant role. See especially Section 6.2 of [11].) We give a number of rather different equivalent conditions for a representation to be spatial—further evidence that this is a natural class. For fixed $p \in [1, \infty) \setminus \{2\}$, we prove a uniqueness theorem: the Banach algebras coming from any two spatial representations are isometrically isomorphic via an isomorphism which sends

2000 *Mathematics Subject Classification*. Primary 46H05, 47L10; Secondary 46H35.

This material is based upon work supported by the US National Science Foundation under Grants DMS-0701076 and DMS-1101742. It was also partially supported by the Centre de Recerca Matemàtica (Barcelona) through a research visit conducted during 2011.

the standard generators to the standard generators. We call the Banach algebra obtained this way \mathcal{O}_d^p . The usual Cuntz algebra is \mathcal{O}_d^2 . We further obtain a strong dependence on p . Specifically, for $p_1 \neq p_2$ and any $d_1, d_2 \in \{2, 3, \dots\}$, there is no nonzero continuous homomorphism from $\mathcal{O}_{d_1}^{p_1}$ to $\mathcal{O}_{d_2}^{p_2}$.

Some of our results are valid, with the same proofs, for the infinite Leavitt algebra L_∞ (which gives L^p analogs of \mathcal{O}_∞) and for the Cohn algebras (which give L^p analogs of Cuntz's algebras E_d). In such cases, we include the corresponding results. However, in many cases, the results, or at least the proofs, must be modified. We do not go in that direction in this paper.

The methods here have little in common with C^* -algebra methods. Indeed, the results on spatial representations have no analog for C^* -algebras, and the result on nonexistence of nonzero continuous homomorphisms does not make sense if one only considers C^* -algebras. Uniqueness of \mathcal{O}_d^p is of course true when $p = 2$, but, as far as we can tell, our proof for $p \neq 2$ does not work when $p = 2$.

This paper is organized as follows. In Section 1, we define Leavitt algebras and Cohn algebras, and give some basic facts about them which will be needed in the rest of the paper. In Section 2, we discuss representations on Banach spaces. We define several natural conditions on representations (weaker than being spatial). We describe ways to get new representations from old ones, some of which work in general and some of which are special to Leavitt and Cohn algebras. Section 3 contains a large collection of examples of representations on L^p spaces.

In Section 6 we develop the theory of (semi)spatial partial isometries on L^p spaces associated to σ -finite measure spaces. The results here are the basic technical tools needed to prove our main results. Roughly speaking, a semispacial partial isometry from $L^p(X, \mu)$ to $L^p(Y, \nu)$ comes from a map from a subset of Y to a subset of X . Lamperti's Theorem [18], which plays a key role, asserts that for $p \in (0, \infty) \setminus \{2\}$, every isometry from $L^p(X, \mu)$ to $L^p(Y, \nu)$ is semispacial.

It is unfortunately not really true that semispacial partial isometries from $L^p(X, \mu)$ to $L^p(Y, \nu)$ come from point maps. Instead, they come from suitable homomorphisms of σ -algebras. For our theory of spatial partial isometries, we need a much more extensive theory of these than we have been able to find in the literature. In Section 4 we recall some standard facts about Boolean σ -algebras, and in Section 5 we discuss the maps on functions and measures induced by a suitable homomorphism of the Boolean σ -algebras of measurable sets mod null sets.

Sections 7, 8, and 9 contain our main results. In Section 7, we give equivalent conditions for representations of Leavitt algebras on L^p spaces to be spatial. Along the way, we define spatial representations of the algebra M_d of $d \times d$ matrices, and we give a number of equivalent conditions for a representation of M_d to be spatial. In Section 8, we prove that and two spatial representations of the Leavitt algebra L_d on L^p spaces give the same norm on L_d , and thus lead to isometrically isomorphic Banach algebras. In Section 9, we prove the nonexistence of nonzero continuous homomorphisms between the resulting algebras for different values of p .

In [22], we will show that \mathcal{O}_d^p is an amenable purely infinite simple Banach algebra, and in [23], we will show that its topological K-theory is the same as for the ordinary Cuntz algebra \mathcal{O}_d . The methods in these papers are much closer to C*-algebra methods.

Scalars will always be \mathbb{C} . Much of what we do also works for real scalars. We use complex scalars for our proof of the equivalence of several of the conditions in Theorem 7.2 for a representation of M_d to be spatial. We do not know whether complex scalars are really necessary.

We have tried to make this paper accessible to operator algebraists who are not familiar with operators on spaces of the form $L^p(X, \mu)$.

We are grateful to Joe Diestel, Ilijas Farah, Coenraad Labuschagne, and Volker Runde for useful email discussions and for providing references. We are especially grateful to Bill Johnson for extensive discussions about Banach spaces, and to Guillermo Cortiñas and María Eugenia Rodríguez, who carefully read an early draft and whose comments led to numerous corrections and improvements. Some of this work was carried out during a visit to the Instituto Superior Técnico, Universidade Técnica de Lisboa, and during an extended research visit to the Centre de Recerca Matemàtica (Barcelona). I also thank the Research Institute for Mathematical Sciences of Kyoto University for its support through a visiting professorship. I am grateful to all these institutions for their hospitality.

1. LEAVITT AND COHN ALGEBRAS

In this section, we define Leavitt algebras and some of their relatives. We describe a grading, a linear involution, and a conjugate linear involution. We give some useful computational lemmas.

Definition 1.1. Let $d \in \{2, 3, 4, \dots\}$. We define the *Leavitt algebra* L_d to be the universal complex associative algebra on generators

$$s_1, s_2, \dots, s_d, t_1, t_2, \dots, t_d$$

satisfying the relations

$$(1.1) \quad t_j s_j = 1 \quad \text{for } j \in \{1, 2, \dots, d\},$$

$$(1.2) \quad t_j s_k = 0 \quad \text{for } j, k \in \{1, 2, \dots, d\} \text{ with } j \neq k,$$

and

$$(1.3) \quad \sum_{j=1}^d s_j t_j = 1.$$

These algebras were introduced in Section 3 of [20] (except that the base field there is $\mathbb{Z}/2\mathbb{Z}$), and they are simple (Theorem 2 of [21], with an arbitrary choice of the field).

Definition 1.2. Let $d \in \{2, 3, 4, \dots\}$. We define the *Cohn algebra* C_d to be the universal complex associative algebra on generators

$$s_1, s_2, \dots, s_d, t_1, t_2, \dots, t_d$$

satisfying the relations (1.1) and (1.2) (but not (1.3)).

These algebras are a special case of algebras introduced at the beginning of Section 5 of [8]. What we have called C_d is called $U_{1,d}$ in [8], and also in [8] the field is allowed to be arbitrary. Our notation, and the name ‘‘Cohn algebra’’, follow Definition 1.1 of [4], except that we specifically take the field to be \mathbb{C} and suppress it in the notation.

Definition 1.3. Let $d \in \{2, 3, 4, \dots\}$. We define the (*infinite*) *Leavitt algebra* L_∞ to be the universal complex associative algebra on generators $s_1, s_2, \dots, t_1, t_2, \dots$ satisfying the relations

$$(1.4) \quad t_j s_j = 1 \quad \text{for } j \in \mathbb{Z}_{>0}$$

and

$$(1.5) \quad t_j s_k = 0 \quad \text{for } j, k \in \mathbb{Z}_{>0} \text{ with } j \neq k.$$

When it is necessary to distinguish the generators of L_∞ from those of L_d and C_d , we denote them by $s_1^{(\infty)}, s_2^{(\infty)}, \dots, t_1^{(\infty)}, t_2^{(\infty)}, \dots$.

This algebra is simple, by Example 3.1(ii) of [3].

Remark 1.4. The algebras L_d , C_d , and L_∞ are all examples of Leavitt path algebras. For L_d see Example 1.4(iii) of [2], for C_d see Section 1.5 of [1], and for L_∞ see Example 3.1(ii) of [3]. (Warning: There are two possible conventions for the choice of the direction of the arrows in the graph, and both are in common use.)

Lemma 1.5. Let C_d be as in Definition 1.2 and let L_d be as in Definition 1.1, with the generators named as there (using the same names in both kinds of algebras). For $d_1, d_2 \in \{2, 3, 4, \dots, \infty\}$ with $d_1 < d_2$, there is a unique homomorphism $\iota_{d_1, d_2}: C_{d_1} \rightarrow L_{d_2}$ such that $\iota_{d_1, d_2}(s_j) = s_j$ and $\iota_{d_1, d_2}(t_j) = t_j$ for $j \in \{1, 2, \dots, d_1\}$. Moreover, ι_{d_1, d_2} is injective.

Proof. Existence and uniqueness of ι_{d_1, d_2} are immediate from the definitions of the algebras as universal algebras on generators and relations.

We prove injectivity. We presume that there is a purely algebraic proof, but one can easily see this by comparing with the C*-algebras, following Remark 2.9 below. Let E_{d_1} be the extended Cuntz algebra, as in Remark 2.9. There is a commutative diagram

$$\begin{array}{ccc} C_{d_1} & \xrightarrow{\iota_{d_1, d_2}} & L_{d_2} \\ \downarrow & & \downarrow \\ E_{d_1} & \longrightarrow & \mathcal{O}_{d_2}. \end{array}$$

The left vertical map is injective by Theorem 7.3 of [25] and Remark 1.4, and the bottom horizontal map is well known to be injective. Therefore ι_{d_1, d_2} is injective. \square

Lemma 1.6. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3).

- (1) There exists a unique conjugate linear antimultiplicative involution $a \mapsto a^*$ on A such that $s_j^* = t_j$ and $t_j^* = s_j$ for all j .
- (2) There exists a unique linear antimultiplicative involution $a \mapsto a'$ on A such that $s'_j = t_j$ and $t'_j = s_j$ for all j .

The properties of $a \mapsto a^*$ are just the algebraic properties of the adjoint of a complex matrix:

$$(a + b)^* = a^* + b^*, \quad (\lambda a)^* = \bar{\lambda} a^*, \quad (ab)^* = b^* a^*, \quad \text{and} \quad (a^*)^* = a$$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$. The properties of $a \mapsto a'$ are the same, except that it is linear: $(\lambda a)' = \lambda a'$ for all $a \in A$ and $\lambda \in \mathbb{C}$.

Proof of Lemma 1.6. See Remark 3.4 of [25], where explicit formulas, valid for any graph algebra, are given, and Remark 1.4. Both parts may also be easily obtained using the universal properties of algebras on generators and relations: $a \mapsto \bar{a}$ is a homomorphism from A to its opposite algebra, and $a \mapsto a^*$ is the composition of $a \mapsto \bar{a}$ with a homomorphism from A to its complex conjugate algebra. \square

One can get Lemma 1.6(1) by using the fact (Theorem 7.3 of [25] and Remark 1.4) that there are injective maps from L_d , C_d , and L_∞ to C^* -algebras which preserve the intended involution. (For L_d and L_∞ , injectivity is automatic because the algebras are simple. See Remark 2.9 for definitions of $*$ -representations on Hilbert spaces.)

Proposition 1.7. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3). Then there is a unique \mathbb{Z} -grading on A determined by

$$\deg(s_j) = 1 \quad \text{and} \quad \deg(t_j) = -1$$

for all j .

Proof. The proof is easy. (See after Definition 3.12 in [25].) \square

We will need some of the finer algebraic structure of L_d , and associated notation. We roughly follow the beginning of Section 1 of [9], starting with 1.1 of [9].

Notation 1.8. Let $d \in \{2, 3, 4, \dots, \infty\}$, and let $n \in \mathbb{Z}_{\geq 0}$. For $d < \infty$, we define $W_n^d = \{1, 2, \dots, d\}^n$, and we define $W_n^\infty = (\mathbb{Z}_{>0})^n$. Thus, W_n^d is the set of all

sequences $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(n))$ with $\alpha(l) \in \{1, 2, \dots, d\}$ (or $\alpha(l) \in \mathbb{Z}_{>0}$ if $d = \infty$) for $l = 1, 2, \dots, n$. We set

$$W_\infty^d = \prod_{n=0}^{\infty} W_n^d.$$

We call the elements of W_∞^d *words* (on $\{1, 2, \dots, d\}$ or $\mathbb{Z}_{>0}$ as appropriate). If $\alpha \in W_\infty^d$, the *length* of α , written $l(\alpha)$, is the unique number $n \in \mathbb{Z}_{\geq 0}$ such that $\alpha \in W_n^d$. Note that there is a unique word of length zero, namely the empty word, which we write as \emptyset . For $\alpha \in W_m^d$ and $\beta \in W_n^d$, we denote by $\alpha\beta$ the concatenation, a word in W_{m+n}^d .

Notation 1.9. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3). Let $n \in \mathbb{Z}_{\geq 0}$, and let

$$\alpha = (\alpha(1), \alpha(2), \dots, \alpha(n)) \in W_n^d.$$

If $n \geq 1$, we define $s_\alpha, t_\alpha \in A$ by

$$s_\alpha = s_{\alpha(1)}s_{\alpha(2)} \cdots s_{\alpha(n-1)}s_{\alpha(n)} \quad \text{and} \quad t_\alpha = t_{\alpha(n)}t_{\alpha(n-1)} \cdots t_{\alpha(2)}t_{\alpha(1)}.$$

We take $s_\emptyset = t_\emptyset = 1$.

For emphasis: in the definition of t_α , we take the generators $t_{\alpha(l)}$ *in reverse order*. We do this to get convenient formulas in Lemma 1.10(3). In particular, when working with Cuntz algebras, one simply uses s_j^* in place of t_j , and we want to have $s_\alpha^* = t_\alpha$.

Lemma 1.10. Let the notation be as in Notation 1.8 and Notation 1.9.

- (1) Let $\alpha, \beta \in W_\infty^d$. Then $s_{\alpha\beta} = s_\alpha s_\beta$ and $t_{\alpha\beta} = t_\beta t_\alpha$.
- (2) In the \mathbb{Z} -grading on A of Proposition 1.7, we have $\deg(s_\alpha) = l(\alpha)$ and $\deg(t_\alpha) = -l(\alpha)$ for all $\alpha \in W_\infty^d$.
- (3) Let $\alpha \in W_\infty^d$. Then the involutions of Lemma 1.6 satisfy $s'_\alpha = s_\alpha^* = t_\alpha$ and $t'_\alpha = t_\alpha^* = s_\alpha$.
- (4) Let

$$a_1, a_2, \dots, a_n \in \{s_1, s_2, \dots\} \cup \{t_1, t_2, \dots\}.$$

Suppose $a_1 a_2 \cdots a_n \neq 0$. Then there exist unique $\alpha, \beta \in W_\infty^d$ such that $a_1 a_2 \cdots a_n = s_\alpha t_\beta$.

- (5) Let $\alpha, \beta \in W_\infty^d$ satisfy $l(\alpha) = l(\beta)$. Then $t_\beta s_\alpha = 1$ if $\alpha = \beta$, and $t_\beta s_\alpha = 0$ otherwise.

Proof. Parts (1), (2), (3), and (5) are obvious. (Part (5) is also in Lemma 1.2(b) of [9].)

Using Part (3), we see that Part (4) is Lemma 1.3 of [9]. \square

Lemma 1.11. Let $d \in \{2, 3, 4, \dots\}$, let L_d be as in Definition 1.1, and let $m \in \mathbb{Z}_{\geq 0}$. Then the collection $(s_\alpha t_\beta)_{\alpha, \beta \in W_m^d}$ is a system of matrix units for a unital subalgebra of L_d isomorphic to M_{d^m} . That is, identifying M_{d^m} with the linear

maps on a vector space with basis W_m^d , with matrix units $e_{\alpha,\beta}$ for $\alpha, \beta \in W_m^d$, there is a unique homomorphism $\varphi_m: M_{d^m} \rightarrow L_d$ such that $\varphi_m(e_{\alpha,\beta}) = s_\alpha t_\beta$.

Proof. We prove that $\sum_{\alpha \in W_m^d} s_\alpha t_\alpha = 1$, by induction on m . The case $m = 1$ is relation (1.3) in Definition 1.1. Assuming the result holds for m , use this and the case $m = 1$ at the last step to get

$$\sum_{\alpha \in W_{m+1}^d} s_\alpha t_\alpha = \sum_{\beta \in W_m^d} \sum_{j=1}^d s_\beta s_j t_j t_\beta = \sum_{\beta \in W_m^d} s_\beta \left(\sum_{j=1}^d s_j t_j \right) t_\beta = 1.$$

The statement of the lemma now follows from Lemma 1.10(5), or is Proposition 1.4 of [9]. \square

Lemma 1.12. Let $d \in \{2, 3, 4, \dots\}$, let $m \in \mathbb{Z}_{>0}$, and let $a_1, a_2, \dots, a_m \in L_d$. Then there exist $n \in \mathbb{Z}_{\geq 0}$, a finite set $F \subset W_\infty^d$, and numbers $\lambda_{k,\alpha,\beta} \in \mathbb{C}$ for $k = 1, 2, \dots, m$, $\alpha \in F$, and $\beta \in W_n^d$, such that

$$(1.6) \quad a_k = \sum_{\alpha \in F} \sum_{\beta \in W_n^d} \lambda_{k,\alpha,\beta} s_\alpha t_\beta$$

for $k = 1, 2, \dots, m$.

Proof. Since a_1, a_2, \dots, a_m are linear combinations of products of the s_j and t_j , it suffices to prove the statement when a_1, a_2, \dots, a_m are products of the s_j and t_j . By Lemma 1.10(4), we may assume $a_k = s_{\alpha_k} t_{\beta_k}$ with $\alpha_k, \beta_k \in W_\infty^d$. Set

$$n = \max(l(\beta_1), l(\beta_2), \dots, l(\beta_m)).$$

For $k = 1, 2, \dots, m$, set $l_k = n - l(\beta_k)$. Take

$$F = \bigcup_{k=1}^m \{\alpha_k \alpha : \alpha \in W_{l_k}^d\}.$$

Lemma 1.11 and Lemma 1.10(1) imply

$$a_k = s_{\alpha_k} t_{\beta_k} = \sum_{\alpha \in W_{l_k}^d} s_{\alpha_k} s_\alpha t_\alpha t_{\beta_k} = \sum_{\alpha \in W_{l_k}^d} s_{\alpha_k \alpha} t_{\beta_k \alpha}.$$

This expression has the form in (1.6). \square

Definition 1.13. Let A be any of L_d, C_d , or L_∞ . Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d) \in \mathbb{C}^d$. (For $A = L_\infty$, take $d = \infty$, take $\mathbb{C}^d = \bigoplus_{j=1}^\infty \mathbb{C}$, and take $\lambda = (\lambda_1, \lambda_2, \dots)$.) Define $s_\lambda, t_\lambda \in A$ by

$$s_\lambda = \sum_{j=1}^d \lambda_j s_j \quad \text{and} \quad t_\lambda = \sum_{j=1}^d \lambda_j t_j.$$

In principle, this notation conflicts with Notation 1.9, but no confusion should arise.

Lemma 1.14. Let the notation be as in Definition 1.13. Let $\lambda, \gamma \in \mathbb{C}^d$. Then

$$t_\lambda s_\gamma = \left(\sum_j \lambda_j \gamma_j \right) \cdot 1.$$

Proof. This is immediate from the relations $t_j s_k = 1$ for $j = k$ and $t_j s_k = 0$ for $j \neq k$. \square

2. REPRESENTATIONS ON BANACH SPACES

In this section, we discuss representations of Leavitt and Cohn algebras on Banach spaces. Much of what we say makes sense for representations on general Banach spaces, but some only works for representations on spaces of the form $L^p(X, \mu)$. Some of the constructions work for general algebras, but some are special to representations of Leavitt and Cohn algebras. Some of what we do is intended primarily to establish notation and conventions. (For example, representations are required to be unital, and isomorphisms are required to be surjective.)

All Banach spaces in this article will be over \mathbb{C} .

Notation 2.1. Let E and F be Banach spaces. We denote by $L(E, F)$ the Banach space of all bounded linear operators from E to F , and by $K(E, F) \subset L(E, F)$ the closed subspace of all compact linear operators from E to F . When $E = F$, we get the Banach algebra $L(E)$ and the closed ideal $K(E) \subset L(E)$.

The following definition summarizes terminology for Banach spaces that we use. We will need both isometries and isomorphisms of Banach spaces.

Definition 2.2. If E and F are Banach spaces, we say that $a \in L(E, F)$ is an *isomorphism* if a is bijective. (By the Open Mapping Theorem, a^{-1} is also bounded.) If an isomorphism exists, we say E and F are *isomorphic*.

We say that $a \in L(E, F)$ is an *isometry* if $\|a\xi\| = \|\xi\|$ for every $\xi \in E$. (We do not require that a be surjective.) If there is a surjective isometry from E to F , we say that E and F are *isometrically isomorphic*.

If A and B are Banach algebras, we say that a homomorphism $\varphi: A \rightarrow B$ is an *isomorphism* if it is continuous and bijective. (By the Open Mapping Theorem, φ^{-1} is also continuous.) If an isomorphism exists, we say A and B are *isomorphic*. If in addition φ is isometric, we call it an *isometric isomorphism*. If such a map exists, we say A and B are *isometrically isomorphic*.

For emphasis (because some authors do not use this convention): *isomorphisms are required to be surjective*.

The following notation for duals is intended to avoid conflict with the notation for adjoints.

Notation 2.3. Let E be a Banach space. We denote by E' its dual Banach space $L(E, \mathbb{C})$, consisting of all bounded linear functionals on E . If F is another

Banach space and $a \in L(E, F)$, we denote by a' the element of $L(F', E')$ defined by $a'(\omega)(\xi) = \omega(a\xi)$ for $\xi \in E$ and $\omega \in F'$.

We will also need some notation for specific spaces.

Notation 2.4. For any set S , we give $l^p(S)$ the usual meaning (using counting measure on S), and we take (as usual) $l^p = l^p(\mathbb{Z}_{>0})$. For $d \in \mathbb{Z}_{>0}$ and $p \in [1, \infty]$, we let $l^p_d = l^p(\{1, 2, \dots, d\})$. We further let $M^p_d = L(l^p_d)$ with the usual operator norm, and we algebraically identify M^p_d with the algebra M_d of $d \times d$ complex matrices in the standard way.

We warn of a notational conflict. Many articles on Banach spaces use $L_p(X, \mu)$ rather than $L^p(X, \mu)$, and use l^d_p for what we call l^p_d . Our convention is chosen to avoid conflict with the standard notation for the Leavitt algebra L_d of Definition 1.1.

For fixed d , the norms on the various M^p_d are of course equivalent, but they are not equal. The following example illustrates this.

Example 2.5. Let $d \in \mathbb{Z}_{>0}$. Let $\eta, \mu \in \mathbb{C}^d$, and let $\omega_\eta: \mathbb{C}^d \rightarrow \mathbb{C}$ be the linear functional $\omega_\eta(\xi) = \sum_{j=1}^d \eta_j \xi_j$. Let $a \in M_d$ be the rank one operator given by $a\xi = \omega_\eta(\xi)\mu$. Let $p, q \in [1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, and regard a as an element of M^p_d . Then one can calculate that $\|a\| = \|\mu\|_p \|\eta\|_q$.

The following terminology and related observation will be used many times.

Definition 2.6. Let (X, \mathcal{B}, μ) be a measure space and let $p \in [1, \infty]$. For a function $\xi \in L^p(X, \mu)$ (or, more generally, any measurable function on X) and a subset $E \subset X$, we will say that ξ is *supported* in E if $\xi(x) = 0$ for almost all $x \in X \setminus E$. If I is a countable set, and $(\xi_i)_{i \in I}$ is a family of elements of $L^p(X, \mu)$ or measurable functions on X , we say that the ξ_i have *disjoint supports* if there are disjoint subsets $E_i \subset X$ such that ξ_i is supported in E_i for all $i \in I$.

Remark 2.7. Let (X, \mathcal{B}, μ) be a measure space, let $p \in [1, \infty)$, let I be a countable set, and let $\xi_i \in L^p(X, \mu)$, for $i \in I$, have disjoint supports. Then

$$\left\| \sum_{i \in I} \xi_i \right\|_p^p = \sum_{i \in I} \|\xi_i\|_p^p.$$

Definition 2.8. Let A be any unital complex algebra. Let E be a nonzero Banach space. A *representation* of A on E is a unital algebra homomorphism from A to $L(E)$.

We do not say anything about continuity. We will mostly be interested in representations of L_d, C_d , and L_∞ , for which we do not use a topology on the algebra, or of M_d , for which all representations are continuous by finite dimensionality.

Remark 2.9. The well known representations of L_d, C_d , and L_∞ are those on a Hilbert space H . Choose any d isometries $w_1, w_2, \dots, w_d \in L(H)$ (or, for the case

of L_∞ , isometries $w_1, w_2, \dots \in L(H)$ with orthogonal ranges. Then we obtain a representation $\rho: C_d \rightarrow L(H)$ or $\rho: L_\infty \rightarrow L(H)$ by setting $\rho(s_j) = w_j$ and $\rho(t_j) = w_j^*$ for all j . If $d < \infty$ and $\sum_{j=1}^d w_j w_j^* = 1$, we get a representation of L_d . These representations are even *-representations: making A a *-algebra as in Lemma 1.6(1), we have $\rho(a^*) = \rho(a)^*$ for all $a \in A$.

The closures $\overline{\rho(A)}$ do not depend on the choice of ρ (in case $A = C_d$, provided $\rho\left(1 - \sum_{j=1}^d s_j s_j^*\right) \neq 0$), and are the usual Cuntz algebra \mathcal{O}_d when $A = L_d$ (including the case $d = \infty$), and the extended Cuntz algebras E_d when $A = C_d$. See Theorem 1.12 of [9] for L_d and L_∞ , and see Lemma 3.1 of [10] for C_d .

Further examples of representations of L_d , C_d , and L_∞ will be given in Section 3.

Representations of L_d , C_d , and L_∞ have a kind of rigidity property. It is stronger for L_d than for the others: a representation is determined by the images of the s_j or by the images of the t_j .

Lemma 2.10. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3). Let B be a unital algebra over \mathbb{C} , and let $\varphi, \psi: L_d \rightarrow B$ be unital homomorphisms such that for all j we have $\varphi(s_j) = \psi(s_j)$ and $\varphi(s_j t_j) = \psi(s_j t_j)$. Then $\varphi = \psi$. The same conclusion holds if we replace $\varphi(s_j) = \psi(s_j)$ with $\varphi(t_j) = \psi(t_j)$.

Proof. Assume $\varphi(s_j) = \psi(s_j)$ for all j . Using the relations $t_j s_j t_j = t_j$ at the first step and $\varphi(t_j)\varphi(s_j) = 1$ at the last step, we calculate:

$$\begin{aligned} \varphi(t_j) &= \varphi(t_j)\varphi(s_j t_j) \\ &= \varphi(t_j)\psi(s_j t_j) = \varphi(t_j)\psi(s_j)\psi(t_j) = \varphi(t_j)\varphi(s_j)\psi(t_j) = \psi(t_j). \end{aligned}$$

The first statement follows. If instead $\varphi(t_j) = \psi(t_j)$ for all j , similar reasoning (using $s_j t_j s_j = s_j$) gives

$$\begin{aligned} \varphi(s_j) &= \varphi(s_j t_j)\varphi(s_j) \\ &= \psi(s_j t_j)\varphi(s_j) = \psi(s_j)\psi(t_j)\varphi(s_j) = \psi(s_j)\varphi(t_j)\varphi(s_j) = \psi(s_j). \end{aligned}$$

This completes the proof. \square

Lemma 2.11. Let $d \in \{2, 3, 4, \dots\}$, let B be a unital algebra over \mathbb{C} , and let $\varphi, \psi: L_d \rightarrow B$ be unital homomorphisms such that $\varphi(s_j) = \psi(s_j)$ for $j \in \{1, 2, \dots, d\}$. Then $\varphi = \psi$. The same conclusion holds if we replace $\varphi(s_j) = \psi(s_j)$ with $\varphi(t_j) = \psi(t_j)$.

Proof. For $j \in \{1, 2, \dots, d\}$, define idempotents $e_j, f_j \in B$ by $e_j = \varphi(s_j t_j)$ and $f_j = \psi(s_j t_j)$. By Lemma 2.10, it suffices to show that $e_j = f_j$ for all j .

First assume that $\varphi(s_j) = \psi(s_j)$ for all j . Using this statement at the first and third steps, $t_j s_k = 0$ for $j \neq k$ at the second step, and $\sum_{k=1}^d e_k = 1$ at the last

step, we have

$$(2.1) \quad f_j e_j = \psi(s_j t_j s_j) \varphi(t_j) = \sum_{k=1}^d \psi(s_j t_j s_k) \varphi(t_k) = \sum_{k=1}^d f_j e_k = f_j.$$

If now $j \neq k$, then

$$f_k e_j = f_k e_k e_j = 0.$$

This equation, together with $\sum_{k=1}^d f_k = 1$, gives

$$(2.2) \quad e_j = \sum_{k=1}^d f_k e_j = f_j e_j.$$

The proof is completed by combining (2.1) and (2.2).

Now assume that $\varphi(t_j) = \psi(t_j)$ for all j . With similar justifications, we get

$$f_j e_j = \psi(s_j) \varphi(t_j s_j t_j) = \sum_{k=1}^d \psi(s_k) \varphi(t_k s_j t_j) = \sum_{k=1}^d f_k e_j = e_j.$$

So for $j \neq k$ we have $f_j e_k = f_j f_k e_k = 0$. Combining these results gives $f_j = \sum_{k=1}^d f_j e_k = f_j e_j$, whence $e_j = f_j$. \square

The analog of Lemma 2.11 for L_∞ and C_d is false. See Example 3.5.

The following definition gives several natural conditions to ask of a representation of L_d , C_d , or L_∞ on a Banach space E . The condition in (3) is motivated by the following property of a *-representation ρ of L_d or C_d on a Hilbert space H (as in Remark 2.9): for $\lambda_1, \lambda_2, \dots, \lambda_d \in \mathbb{C}$ and $\xi \in H$, we have

$$(2.3) \quad \left\| \rho \left(\sum_{j=1}^d \lambda_j s_j \right) \xi \right\| = \|(\lambda_1, \lambda_2, \dots, \lambda_d)\|_2 \|\xi\|.$$

In Definition 7.4 and Definition 7.6, we will see further conditions on representations which are natural when $E = L^p(X, \mu)$.

Definition 2.12. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2) or L_∞ (Definition 1.3). Let E be a nonzero Banach space, and let $\rho: A \rightarrow L(E)$ be a representation.

- (1) We say that ρ is *contractive on generators* if for every j , we have $\|\rho(s_j)\| \leq 1$ and $\|\rho(t_j)\| \leq 1$.
- (2) We say that ρ is *forward isometric* if $\rho(s_j)$ is an isometry for every j .
- (3) We say that ρ is *strongly forward isometric* if ρ is forward isometric and (following Definition 1.13) for every $\lambda \in \mathbb{C}^d$, the element $\rho(s_\lambda)$ is a scalar multiple of an isometry.

Remark 2.13. A representation which is contractive on generators is clearly forward isometric.

A representation of L_d which is contractive on generators need not be strongly forward isometric. See Example 3.11 below. We will see in Example 3.5 below that a strongly forward isometric representation of L_∞ need not be contractive on generators. We do not know whether this can happen for L_d with d finite.

We now describe several ways to make new representations from old ones. The first two (direct sums and tensoring with the identity on some other Banach space) work for representations of general algebras. They also work for more general choices of norms on the direct sum and tensor product than we consider here. For simplicity, we restrict to specific choices which are suitable for representations on spaces of the form $L^p(X, \mu)$.

Lemma 2.14. Let A be a unital complex algebra, and let $p \in [1, \infty]$. Let $n \in \mathbb{Z}_{>0}$, and for $l = 1, 2, \dots, n$ let $(X_l, \mathcal{B}_l, \mu_l)$ be a σ -finite measure space and let $\rho_l: A \rightarrow L(L^p(X_l, \mu_l))$ be a representation. Equip $E = \bigoplus_{l=1}^n L^p(X_l, \mu_l)$ with the norm

$$\|(\xi_1, \xi_2, \dots, \xi_n)\| = \left(\sum_{l=1}^n \|\xi_l\|_p^p \right)^{1/p}.$$

Then there is a unique representation $\rho: A \rightarrow L(E)$ such that

$$\rho(a)(\xi_1, \xi_2, \dots, \xi_n) = (\rho_1(a)\xi_1, \rho_2(a)\xi_2, \dots, \rho_n(a)\xi_n)$$

for $a \in L_d$ and $\xi_l \in L^p(X_l, \mu_l)$ for $l = 1, 2, \dots, n$. If A is any of L_d, C_d , or L_∞ , and each ρ_l is contractive on generators or forward isometric, then so is ρ .

Proof. This is immediate. □

Remark 2.15. The norm used in Lemma 2.14 identifies E with $L^p(\coprod_{l=1}^n X_l)$, using the obvious measure. We write this space as

$$L^p(X_1, \mu_1) \oplus_p L^p(X_2, \mu_2) \oplus_p \cdots \oplus_p L^p(X_n, \mu_n).$$

We write the representation ρ as

$$\rho = \rho_1 \oplus_p \rho_2 \oplus_p \cdots \oplus_p \rho_n,$$

and call it the p -direct sum of $\rho_1, \rho_2, \dots, \rho_n$.

Example 3.11 below shows that if $\rho_1, \rho_2, \dots, \rho_n$ are strongly forward isometric, it does not follow that ρ is strongly forward isometric.

One can form a p -direct sum $\bigoplus_{i \in I} \rho_i$ over an infinite index set I provided $\sup_{i \in I} \|\rho_i(a)\| < \infty$ for all $a \in A$.

We now consider tensoring with the identity on some other Banach space. This requires the theory of tensor products of Banach spaces and of operators on them. We will consider only a very special case.

Fix $p \in [1, \infty)$. We need a tensor product, defined on pairs of Banach spaces both of the form $L^p(X, \mu)$ for σ -finite measures μ , for which one has a canonical isometric identification

$$L^p(X, \mu) \otimes L^p(Y, \nu) = L^p(X \times Y, \mu \times \nu).$$

(One can't reasonably expect something like this for $p = \infty$.) The tensor product

$$L^p(X, \mu) \widetilde{\otimes}_{\Delta_p} L^p(Y, \nu)$$

described in Chapter 7 of [12] will serve our purpose. To simplify the notation, we simply write $L^p(X, \mu) \otimes_p L^p(Y, \nu)$.

We note that there is a more general construction, the M -norm of [7], defined before (4) on page 3 of [7]. This norm is defined for the tensor product of a Banach lattice (this includes all spaces $L^p(X, \mu)$ for all p) and a Banach space. Theorem 3.2(1) of [7] shows that whenever $p \in [1, \infty)$ and (X, \mathcal{B}, μ) is a finite measure space, then the completion of $E \otimes_{\text{alg}} L^p(X, \mu)$ in this norm is isometrically isomorphic to the space of L^p functions on X with values in E . In particular, regardless of the value of p , this norm gives the properties in Theorem 2.16.

Theorem 2.16. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces. Let $p \in [1, \infty)$. Write $L^p(X, \mu) \otimes_p L^p(Y, \nu)$ for the Banach space completed tensor product $L^p(X, \mu) \widetilde{\otimes}_{\Delta_p} L^p(Y, \nu)$ defined in 7.1 of [12]. Then there is a unique isometric isomorphism

$$L^p(X, \mu) \otimes_p L^p(Y, \nu) \cong L^p(X \times Y, \mu \times \nu)$$

which identifies, for every $\xi \in L^p(X, \mu)$ and $\eta \in L^p(Y, \nu)$, the element $\xi \otimes \eta$ with the function $(x, y) \mapsto \xi(x)\eta(y)$ on $X \times Y$. Moreover:

- (1) Under the identification above, the linear span of all $\xi \otimes \eta$, for $\xi \in L^p(X, \mu)$ and $\eta \in L^p(Y, \nu)$, is dense in $L^p(X \times Y, \mu \times \nu)$.
- (2) $\|\xi \otimes \eta\|_p = \|\xi\|_p \|\eta\|_p$ for all $\xi \in L^p(X, \mu)$ and $\eta \in L^p(Y, \nu)$.
- (3) The tensor product \otimes_p is commutative and associative.
- (4) Let

$$(X_1, \mathcal{B}_1, \mu_1), \quad (X_2, \mathcal{B}_2, \mu_2), \quad (Y_1, \mathcal{C}_1, \nu_1), \quad \text{and} \quad (Y_2, \mathcal{C}_2, \nu_2)$$

be σ -finite measure spaces. Let

$$a \in L(L^p(X_1, \mu_1), L^p(X_2, \mu_2)) \quad \text{and} \quad b \in L(L^p(Y_1, \nu_1), L^p(Y_2, \nu_2)).$$

Then there exists a unique

$$c \in L(L^p(X_1 \times Y_1, \mu_1 \times \nu_1), L^p(X_2 \times Y_2, \mu_2 \times \nu_2))$$

such that, making the identification above, $c(\xi \otimes \eta) = a\xi \otimes b\eta$ for all $\xi \in L^p(X_1, \mu_1)$ and $\eta \in L^p(Y_1, \nu_1)$. We call this operator $a \otimes b$.

- (5) The operator $a \otimes b$ of (4) satisfies $\|a \otimes b\| = \|a\| \cdot \|b\|$.
- (6) The tensor product of operators defined in (4) is associative, bilinear, and satisfies (when the domains are appropriate) $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.

Proof. The identification of $L^p(X, \mu) \widetilde{\otimes}_{\Delta_p} L^p(Y, \nu)$ is in 7.2 of [12]. Part (1) is part of the definition of a tensor product of Banach spaces. Part (2) is in 7.1 of [12]. Part (3) follows from the corresponding properties of products of measure spaces. Parts (4) and (5) are a special case of 7.9 of [12], or of Theorem 1.1 of [14]. Part (6) follows from part (4) and part (1) by examining what happens on elements of the form $\xi \otimes \eta$. \square

In fact, the statements about tensor products of operators in Theorem 2.16(4) and Theorem 2.16(5) are valid in considerably greater generality; see, for example, Theorem 1.1 of [14]. We need a slightly more general statement in the proof of the following lemma.

Lemma 2.17. Let A be a unital complex algebra, let $p \in [1, \infty)$, let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces, and let $\rho: A \rightarrow L(L^p(X, \mu))$ be a representation. Then there is a unique representation $\rho \otimes_p 1: A \rightarrow L^p(X \times Y, \mu \times \nu)$ such that, following Theorem 2.16(4), we have $(\rho \otimes_p 1)(a) = \rho(a) \otimes 1$ for all $a \in A$. This representation satisfies $\|(\rho \otimes_p 1)(a)\| = \|\rho(a)\|$ for all $a \in A$. If A is any of L_d , C_d , or L_∞ , and ρ is any of contractive on generators, forward isometric, or strongly forward isometric, then so is $\rho \otimes_p 1$.

Proof. Existence of $\rho \otimes_p 1$ and $\|(\rho \otimes_p 1)(a)\| = \|\rho(a)\|$ follow from parts (4), (5), and (6) of Theorem 2.16. If A is one of L_d , C_d , or L_∞ and ρ is contractive on generators, then the norm equation implies that $\rho \otimes_p 1$ is also contractive on generators.

To prove that $\rho \otimes_p 1$ is forward isometric or strongly forward isometric when ρ is, it suffices to prove that if $s \in L(L^p(X, \mu))$ is isometric (not necessarily surjective), then so is $s \otimes 1 \in L(L^p(X \times Y, \mu \times \nu))$. Let $E \subset L^p(X, \mu)$ be the range of s . Let $t: E \rightarrow L^p(X, \mu)$ be the inverse of the corestriction of s . Then $\|t\| = 1$. Let $F \subset L^p(X \times Y, \mu \times \nu)$ be the closed linear span of all $\xi \otimes \eta$ with $\xi \in E$ and $\eta \in L^p(Y, \nu)$. Then Theorem 1.1 of [14] implies that $t \otimes 1: F \rightarrow L^p(X \times Y, \mu \times \nu)$ is defined and satisfies $\|t \otimes 1\| = 1$. Since $\|s \otimes 1\| = 1$ and $(t \otimes 1)(s \otimes 1) = 1$, this implies that $s \otimes 1$ is isometric. \square

In the proof of Lemma 2.17, we could also have used Lamperti's Theorem (Theorem 6.9 below) instead of Theorem 1.1 of [14] to show that $\|t \otimes 1\| = 1$.

Finally, we present constructions of new representations that are special to the kinds of algebras we consider. They will play important technical roles.

Lemma 2.18. Let $d \in \{2, 3, 4, \dots\}$, let E be a nonzero Banach space, and let $\rho: L_d \rightarrow L(E)$ be a representation. Let $u \in L(E)$ be invertible. Then there is a unique representation $\rho^u: L^d \rightarrow L(E)$ such that for $j = 1, 2, \dots, d$ we have

$$\rho^u(s_j) = u\rho(s_j) \quad \text{and} \quad \rho^u(t_j) = \rho(t_j)u^{-1}.$$

Assume further that u is isometric. If ρ is contractive on generators, forward isometric, or strongly forward isometric (Definition 2.12), then so is ρ^u .

Proof. For the first part, we check the relations (1.1), (1.2), and (1.3) in Definition 1.1. For $j \in \{1, 2, \dots, d\}$ we have

$$[\rho(t_j)u^{-1}][u\rho(s_j)] = \rho(t_j)\rho(s_j) = 1,$$

for distinct $j, k \in \{1, 2, \dots, d\}$ we have

$$[\rho(t_j)u^{-1}][u\rho(s_k)] = \rho(t_j)\rho(s_k) = 0,$$

and we have

$$\sum_{j=1}^d [u\rho(s_j)][\rho(t_j)u^{-1}] = u \left(\sum_{j=1}^d s_j t_j \right) u^{-1} = u \cdot 1 \cdot u^{-1} = 1.$$

The second part follows directly from the definitions of the conditions on the representation. \square

For representations of C_d and L_∞ , we only need one sided invertibility.

Lemma 2.19. Let $d \in \{2, 3, 4, \dots\}$, let E be a nonzero Banach space, and let $\rho: C_d \rightarrow L(E)$ be a representation. Let $u, v \in L(E)$ satisfy $vu = 1$. Then there is a unique representation $\rho^{u,v}: C^d \rightarrow L(E)$ such that for $j = 1, 2, \dots, d$ we have

$$\rho^{u,v}(s_j) = u\rho(s_j) \quad \text{and} \quad \rho^{u,v}(t_j) = \rho(t_j)v.$$

If u is an isometry and ρ is forward isometric or strongly forward isometric, then so is $\rho^{u,v}$. If $\|u\|, \|v\| \leq 1$ and ρ is contractive on generators, then so is $\rho^{u,v}$.

Proof. The proof is essentially the same as that of Lemma 2.18, using the relations (1.1) and (1.2) in Definition 1.1 (see Definition 1.2). Since no relation involves $s_j t_j$, we do not need to have $uv = 1$. \square

Lemma 2.20. Let $\rho: L_\infty \rightarrow L(E)$ be a representation. Let $u, v \in L(E)$ satisfy $vu = 1$. Then there is a unique representation $\rho^{u,v}: L_\infty \rightarrow L(E)$ such that for $j = 1, 2, \dots$ we have

$$\rho^{u,v}(s_j) = u\rho(s_j) \quad \text{and} \quad \rho^{u,v}(t_j) = \rho(t_j)v.$$

If u is an isometry and ρ is forward isometric or strongly forward isometric, then so is $\rho^{u,v}$. If $\|u\|, \|v\| \leq 1$ and ρ is contractive on generators, then so is $\rho^{u,v}$.

Proof. The proof is the same as that of Lemma 2.19, using the relations (1.4) and (1.5) in Definition 1.3. \square

Examples 3.4 and 3.5 illustrate this construction.

Lemma 2.21. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3). Let E be a nonzero Banach space, and let $\rho: A \rightarrow L(E)$ be a representation. Using Notation 2.3 and the notation of Lemma 1.6(2), define a function $\rho': A \rightarrow L(E')$ by $\rho'(a) = \rho(a)'$. Then ρ' is a representation of A , called the dual of ρ .

Proof. This follows from Lemma 1.6(2). \square

3. EXAMPLES OF REPRESENTATIONS

In this section, we give a number of examples of representations of Leavitt algebras on Banach spaces. These examples illustrate some of the possible behavior of representations. We begin with basic examples of representations on l^p .

Example 3.1. Fix $p \in [1, \infty]$. We take $l^p = l^p(\mathbb{Z}_{>0})$. Let $d \in \{2, 3, 4, \dots\}$. Define functions $f_1, f_2, \dots, f_d: \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ by

$$f_{d,j}(n) = d(n-1) + j$$

for $n \in \mathbb{Z}_{>0}$. The functions f_j are injective and have disjoint ranges whose union is $\mathbb{Z}_{>0}$. For $j = 1, 2, \dots, d$, define $v_{d,j}, w_{d,j} \in L(l^p)$ by, for $\xi = (\xi(1), \xi(2), \dots) \in l^p$ and $n \in \mathbb{Z}_{>0}$,

$$(v_{d,j}\xi)(n) = \begin{cases} \xi(f_{d,j}^{-1}(n)) & n \in \text{ran}(f_{d,j}) \\ 0 & n \notin \text{ran}(f_{d,j}) \end{cases}$$

and

$$(w_{d,j}\xi)(n) = \xi(f_{d,j}(n)).$$

Then $v_{d,j}$ is isometric, $w_{d,j}$ is contractive, and there is a unique representation $\rho: L_d \rightarrow L(l^p)$ such that

$$\rho(s_j) = v_{d,j} \quad \text{and} \quad \rho(t_j) = w_{d,j}$$

for $j = 1, 2, \dots, d$.

If we let $\delta_n \in l^p$ be the sequence given by $\delta_n(n) = 1$ and $\delta_n(m) = 0$ for $m \neq n$, then

$$v_{d,j}\delta_n = \delta_{f_{d,j}(n)} \quad \text{and} \quad w_{d,j}\delta_n = \begin{cases} \delta_{f_{d,j}^{-1}(n)} & n \in \text{ran}(f_{d,j}) \\ 0 & n \notin \text{ran}(f_{d,j}) \end{cases}$$

for all $n \in \mathbb{Z}_{>0}$. For $p \in [1, \infty)$, these formulas determine $v_{d,j}, w_{d,j} \in L(l^p)$ uniquely.

The representation ρ is clearly contractive on generators (Definition 2.12(1)). One can check directly that ρ is strongly forward isometric (Definition 2.12(3)), with

$$\left\| \rho \left(\sum_{j=1}^d \lambda_j s_j \right) \xi \right\| = \|(\lambda_1, \lambda_2, \dots, \lambda_d)\|_p \|\xi\|.$$

(Or use Lemma 7.7 below.)

Example 3.2. Let the notation be as in Example 3.1. Then there is a unique representation $\pi: C_d \rightarrow L(l^p)$ such that

$$\pi(s_j) = v_{d+1,j} \quad \text{and} \quad \pi(t_j) = w_{d+1,j}$$

for $j = 1, 2, \dots, d$. Since $\sum_{j=1}^d w_{d+1,j} v_{d+1,j} \neq 1$, this representation does not descend to a representation of L_d .

Example 3.3. Let $p \in [1, \infty]$. We obtain a representation ρ of L_∞ on l^p by the same method as in Example 3.1, taking $v_{\infty,j}$ and $w_{\infty,j}$ to come from the functions

$$f_{\infty,j}(n) = 2^j n - 2^{j-1}$$

for $j, n \in \mathbb{Z}_{>0}$. Again, this representation is contractive on generators and strongly forward isometric.

In Example 3.3, the ranges of the $\rho(s_j)$ span a dense subspace of l^p , except when $p = \infty$. The following example shows that this need not be the case.

Example 3.4. Let $p \in [1, \infty]$. Let $\rho: L_\infty \rightarrow L(l^p)$ be as in Example 3.3. Then there exists a unique representation $\pi: L_\infty \rightarrow L(l^p)$ such that for $\pi(s_j) = \rho(s_{j+1})$ and $\pi(t_j) = \rho(t_{j+1})$ for all $j \in \mathbb{Z}_{>0}$. Like ρ , this representation is contractive on generators and strongly forward isometric.

This example can be obtained from ρ by the method of Lemma 2.20. Define $u, v \in L(l^p)$ by

$$(u\xi)(n) = \begin{cases} \xi(n/2) & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases} \quad \text{and} \quad (v\xi)(n) = \xi(2n)$$

for $\xi \in l^p$ and $n \in \mathbb{Z}_{>0}$. Using the notation of Example 3.1, for $p \neq \infty$ these are determined by

$$u\delta_n = \delta_{2n} \quad \text{and} \quad v\delta_n = \begin{cases} \delta_{n/2} & n \text{ is even} \\ 0 & n \text{ is odd.} \end{cases}$$

Then $\pi = \rho^{u,v}$.

Let ρ be as in Example 3.3, and let π be as in Example 3.4. We do not know whether the Banach algebras $\overline{\rho(L_\infty)}$ and $\overline{\pi(L_\infty)}$ are isometrically isomorphic, or even whether they are isomorphic.

The representations of Example 3.3 and Example 3.4 are both very strongly tied to the structure of l^p as a space of functions. (They are spatial in the sense of Definition 7.4(2) below.) The following example is less regular.

Example 3.5. Let $p \in [1, \infty]$. Let the notation be as in Example 3.3 and Example 3.4. Define $y \in L(l^p)$ by

$$(y\xi)(n) = \begin{cases} \xi(2n) & n \text{ is even} \\ \xi(2n) + \xi(n) & n \text{ is odd.} \end{cases}$$

Thus, in the notation of Example 3.1, we have

$$y\delta_n = \begin{cases} \delta_{n/2} & n \text{ is even} \\ \delta_n & n \text{ is odd.} \end{cases}$$

We have $yu = 1$, so, using Lemma 2.20, we obtain a representation $\sigma = \rho^{u,y}$ of L_∞ .

We have $\sigma(s_j) = \pi(s_j)$ for all j , but $\sigma(t_1) \neq \pi(t_1)$. Indeed, $\pi(t_1)(\delta_1 + \delta_2) = \delta_1$, but $y(\delta_1 + \delta_2) = 2\delta_1$, so $\sigma(t_1)(\delta_1 + \delta_2) = 2\delta_1$. Thus the analog of Lemma 2.11 for L_∞ is false. By restriction, it also fails for C_d .

The representation σ is strongly forward isometric, since π is. For $p = 1$, it is easy to check that $\|y\| \leq 1$, from which it follows that σ is contractive on

generators. Now suppose $p \neq 1$. We saw above that $\sigma(t_1)(\delta_1 + \delta_2) = 2\delta_1$. Taking $\frac{1}{p} = 0$ when $p = \infty$, we have

$$\|\sigma(t_1)(\delta_1 + \delta_2)\| = 2 > 2^{1/p} = \|\delta_1 + \delta_2\|.$$

So σ is not contractive on generators. In particular, a strongly forward isometric representation of L_∞ need not be contractive on generators. We do not know whether this can happen for representations of L_d when d is finite.

Example 3.6. The restrictions of the representations of Examples 3.3, 3.4, and 3.5 to the standard copy of C_d in L_∞ (see Lemma 1.5) are representations of C_d on l^p .

In particular, strongly forward isometric does not imply contractive on generators for representations of C_d .

Example 3.7. Let $\rho: L_d \rightarrow L(l^p)$ be as in Example 3.1. One immediately checks that there is a representation $\sigma: L_d \rightarrow L(l^p)$ such that for $j = 1, 2, \dots, d$,

$$\sigma(s_j) = \frac{1}{2}\rho(s_j) \quad \text{and} \quad \sigma(t_j) = 2\rho(t_j).$$

Then σ has the property (part of the definition of being strongly forward isometric, Definition 2.12(3)) that for every $\lambda \in \mathbb{C}^d$, the element $\sigma(s_\lambda)$ is a scalar multiple of an isometry. Moreover, $\|\sigma(s_j)\| \leq 1$ for $j = 1, 2, \dots, d$. Also, the values of σ on $s_j t_j \in L_d$ are the same as for the strongly forward isometric representation ρ . However, σ is not strongly forward isometric.

The Banach algebras $\overline{\rho(L_d)}$ and $\overline{\sigma(L_d)}$ are isometrically isomorphic—in fact, they are equal.

Taking $u = 1\frac{1}{2}$, and following Lemma 2.18, we can write $\sigma = \rho^u$.

Example 3.8. Let $\rho: L_d \rightarrow L(l^p)$ be as in Example 3.1, and let $\sigma: L_d \rightarrow L(l^p)$ be as in Example 3.7. Let $\pi = \rho \oplus_p \sigma$ be as in Remark 2.15. Then $\|\pi(s_j)\| = 1$ for $j = 1, 2, \dots, d$, but π is not forward isometric and not contractive on generators.

It seems unlikely that the Banach algebras $\overline{\rho(L_d)}$ and $\overline{\pi(L_d)}$ are isomorphic.

Taking $v = \text{diag}(1, \frac{1}{2})$ and using the notation of Lemma 2.18, we can write $\pi = (\rho \oplus_p \rho)^v$.

Example 3.8 was obtained using Lemma 2.18 with a diagonal scalar matrix. In the following example, we instead use a nondiagonalizable scalar matrix. We do not know how the Banach algebras obtained as the closures of the ranges differ.

Example 3.9. Let $\rho: L_d \rightarrow L(l^p)$ be as in Example 3.1. Let $\rho \oplus_p \rho$ be as in Remark 2.15. Set

$$w = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in L(l^p \oplus_p l^p).$$

Let τ be the representation $\tau = (\rho \oplus_p \rho)^w$ of Lemma 2.18. We have

$$\tau(s_j) = \begin{pmatrix} \rho(s_j) & \rho(s_j) \\ 0 & \rho(s_j) \end{pmatrix} \quad \text{and} \quad \tau(t_j) = \begin{pmatrix} \rho(t_j) & -\rho(t_j) \\ 0 & \rho(t_j) \end{pmatrix}$$

for $j = 1, 2, \dots, d$.

We now turn to a different kind of modification of our basic example, using automorphisms of L_d rather than Lemma 2.18.

Example 3.10. Let $d \in \{2, 3, 4, \dots\}$ and let $p \in [1, \infty]$. Set $\omega = \exp(2\pi i/d)$. Let q be the conjugate exponent, that is, $\frac{1}{p} + \frac{1}{q} = 1$. For $k = 1, 2, \dots, d$, define elements of L_d by

$$v_k = d^{-1/p} \sum_{j=1}^d \omega^{jk} s_j \quad \text{and} \quad w_k = d^{-1/q} \sum_{j=1}^d \omega^{-jk} t_j.$$

(We take $d^{-1/p} = 1$ when $p = \infty$ and $d^{-1/q} = 1$ when $p = 1$.) It follows from Lemma 1.14 that $w_j v_k = 1$ for $j = k$ and $w_j v_k = 0$ for $j \neq k$. A computation shows that $\sum_{k=1}^d v_k w_k = 1$. Therefore there is an endomorphism $\varphi: L_d \rightarrow L_d$ such that $\varphi(s_k) = v_k$ and $\varphi(t_k) = w_k$ for $k = 1, 2, \dots, d$. A computation shows that

$$s_k = d^{-1/q} \sum_{j=1}^d \omega^{-jk} v_j \quad \text{and} \quad t_k = d^{-1/p} \sum_{j=1}^d \omega^{jk} w_j.$$

Therefore φ is bijective.

It follows that whenever ρ is a representation of L_d on a Banach space E , then $\rho \circ \varphi$ is also a representation. Moreover, $\overline{(\rho \circ \varphi)(L_d)} = \overline{\rho(L_d)}$.

Now take ρ to be as in Example 3.1. We claim that $\rho \circ \varphi$ is contractive on generators and strongly forward isometric. To prove this, it is convenient to introduce the matrix

$$u = (u_{j,k})_{j,k=1}^d = d^{-1/p} \begin{pmatrix} \omega & \omega^2 & \dots & \omega^{d-1} & 1 \\ \omega^2 & \omega^4 & \dots & \omega^{d-2} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega^{d-1} & \omega^{d-2} & \dots & \omega & 1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

We then have

$$\varphi(s_k) = \sum_{j=1}^d u_{j,k} s_j.$$

Therefore, for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d) \in \mathbb{C}^d$, and following Definition 1.13,

$$\varphi(s_\lambda) = \sum_{k=1}^d \sum_{j=1}^d \lambda_k u_{j,k} s_j = \sum_{j=1}^d (u\lambda)_j s_j = s_{u\lambda}.$$

So, for $\lambda \in \mathbb{C}^d$ and $\xi \in l^p$, Example 3.1 implies that

$$\|(\rho \circ \varphi)(s_\lambda)\xi\|_p = \|u\lambda\|_p \|\xi\|_p.$$

In particular, taking $\lambda = \delta_j$, for $p \neq \infty$ we get

$$\|(\rho \circ \varphi)(s_j)\xi\|_p = \|u\delta_j\|_p \|\xi\|_p = \left(d^{-1} \sum_{k=1}^d |\omega^{jk}|^p \right)^{1/p} \|\xi\|_p = \|\xi\|_p.$$

One also checks that $\|u\delta_j\|_p = 1$ when $p = \infty$. We conclude that $\rho \circ \varphi$ is strongly forward isometric.

It remains to prove that $\rho \circ \varphi$ is contractive on generators. Assume $p \neq 1, \infty$. Let $\xi = (\xi(1), \xi(2), \dots) \in l^p$. Let $w_{d,j}$ be as in Example 3.1. Then

$$\sum_{j=1}^d \|w_{d,j}\xi\|_p^p = \sum_{j=1}^d \sum_{n=1}^{\infty} |\xi(d(n-1) + j)|^p = \sum_{m=1}^{\infty} |\xi(m)|^p = \|\xi\|_p^p.$$

Now, for $k \in \{1, 2, \dots, d\}$, we get, using Hölder's inequality at the second step and $|\omega^{-jk}| = 1$ at the third step,

$$\begin{aligned} \|(\rho \circ \varphi)(t_k)\xi\|_p^p &= d^{-p/q} \sum_{m=1}^{\infty} \left| \sum_{j=1}^d \omega^{-jk} (w_{d,j}\xi)(m) \right|^p \\ &\leq d^{-p/q} \sum_{m=1}^{\infty} \left(\sum_{j=1}^d |\omega^{-jk}|^q \right)^{p/q} \left(\sum_{j=1}^d |(w_{d,j}\xi)(m)|^p \right) \\ &= d^{-p/q} d^{p/q} \sum_{j=1}^d \|w_{d,j}\xi\|_p^p = \|\xi\|_p^p. \end{aligned}$$

Thus ρ is contractive on generators. Easier calculations show that ρ is contractive on generators when $p = 1$ and $p = \infty$ as well.

For $p \in [1, \infty) \setminus \{2\}$, we claim that, unlike the representation ρ of Example 3.18, we have in general $\|(\rho \circ \varphi)(s_\lambda)\xi\|_p \neq \|\lambda\|_p \|\xi\|_p$. Since we have already seen that $\|(\rho \circ \varphi)(s_\lambda)\xi\|_p = \|u\lambda\|_p \|\xi\|_p$, it suffices to find some $\lambda \in \mathbb{C}^d$ such that $\|u\lambda\|_p \neq \|\lambda\|_p$.

For an explicit easily checked example, take $d = 2$, $p = 3$, and $\lambda = (1, 2)$. Then

$$\|\lambda\|_p^p = 1 + 2^p = 9, \quad u\lambda = 2^{-1/3}(1, 3), \quad \text{and} \quad \|u\lambda\|_p^p = \frac{1}{2}(1 + 3^p) = 14.$$

For arbitrary $p \in [1, \infty) \setminus \{2\}$ and arbitrary d , define $\sigma: M_d^p \rightarrow L(l^p)$ by $\sigma(a) = uau^{-1}$ for $a \in M_d^p$. Let $e_{j,k} \in M_d^p$ be the usual matrix unit. Then one checks that $\sigma(e_{1,1})$ is not multiplication by any characteristic function. This violates condition (5) in Theorem 7.2 below. Therefore σ is not isometric. So u is not isometric.

In the following examples derived from Example 3.10, we exclude $p = 2$ and $p = \infty$. When $p = 2$, we do not get new behavior for the closure of the image of L_d . We have not checked what happens when $p = \infty$.

Example 3.11. Let $d \in \{2, 3, 4, \dots\}$ and let $p \in [1, \infty) \setminus \{2\}$. Let ρ be as in Example 3.1, and let φ and $\rho \circ \varphi$ be as in Example 3.10. Let $\pi = \rho \oplus_p (\rho \circ \varphi)$, as in Remark 2.15. Then π is forward isometric and contractive on generators because ρ and $\rho \circ \varphi$ are. However, for $p \neq 2$, Example 3.8 and Example 3.1 show that there is $\lambda \in \mathbb{C}^d$ and $\xi \in l^p$ such that $\|\rho(s_\lambda)\xi\| \neq \|(\rho \circ \varphi)(s_\lambda)\xi\|$. Therefore $\pi(s_\lambda)$ is not a scalar multiple of an isometry, so π is not strongly forward isometric.

We do not know whether the Banach algebras $\overline{\rho(L_d)}$ and $\overline{\pi(L_d)}$ are isomorphic.

There are more complicated versions of Example 3.10 which also give representations which are strongly forward isometric and contractive on generators. For example, one might split the generators into families and treat each family separately in the manner of Example 3.10. We give three special cases which are easy to write down and which we want for specific purposes.

Example 3.12. Let $d = 3$, let $p \in [1, \infty) \setminus \{2\}$, and let ρ be as in Example 3.1 with this choice of d . There is a unique automorphism ψ of L_3 such that

$$\psi(s_1) = s_1, \quad \psi(s_2) = 2^{-1/p}(s_2 + s_3), \quad \psi(s_3) = 2^{-1/p}(s_2 - s_3),$$

and

$$\psi(t_1) = t_1, \quad \psi(t_2) = 2^{-1/q}(t_2 + t_3), \quad \psi(t_3) = 2^{-1/q}(t_2 - t_3).$$

Then $\rho \circ \psi$ is strongly forward isometric, with

$$\|(\rho \circ \psi)(s_\lambda)\xi\|_p = (|\lambda_1|^p + \frac{1}{2}|\lambda_2 + \lambda_3|^p + \frac{1}{2}|\lambda_2 - \lambda_3|^p)^{1/p} \|\xi\|_p,$$

and contractive on generators. Moreover, $\sigma = \rho \oplus_p (\rho \circ \psi)$, as in Remark 2.15, is forward isometric and contractive on generators, but not strongly forward isometric.

Letting π be as in Example 3.11 with $d = 3$, we do not know whether the Banach algebras $\overline{\sigma(L_3)}$ and $\overline{\pi(L_3)}$ are isomorphic. We do note that $\overline{\pi(L_3)}$ has an isometric automorphism which cyclically permutes the elements $\pi(s_j)$, but there is no isometric automorphism of $\overline{\sigma(L_3)}$ which does this. Possibly there isn't even any continuous automorphism of $\overline{\sigma(L_3)}$ which does this.

Example 3.13. Let $p \in [1, \infty) \setminus \{2\}$. Let $d_0, n \in \{2, 3, \dots\}$. Set $d = nd_0$. In L_d , call the standard generators $s_{j,m}$ and $t_{j,m}$ for $j = 1, 2, \dots, d_0$ and $m = 1, 2, \dots, n$. By reasoning similar to that of Example 3.10, there is an automorphism α of L_d such that, for $j = 1, 2, \dots, d_0$ and $l = 1, 2, \dots, n$, we have

$$(3.1) \quad \alpha(s_{k,m}) = d^{-1/p} \sum_{j=1}^d \omega^{jk} s_{j,m} \quad \text{and} \quad \alpha(t_{k,m}) = d^{-1/q} \sum_{j=1}^d \omega^{-jk} t_{j,m}.$$

Take ρ to be as in Example 3.1. Then $\rho \circ \alpha$ is a representation of L_d which is strongly forward isometric and contractive on generators, and for which one has $\overline{(\rho \circ \alpha)(L_d)} = \overline{\rho(L_d)}$ but, in general, $\|(\rho \circ \alpha)(s_\lambda)\xi\|_p \neq \|\lambda\|_p \|\xi\|_p$.

We presume that $\rho \oplus_p (\rho \circ \alpha)$ is essentially different from both ρ and the representation $\rho \oplus_p (\rho \circ \varphi)$ of Example 3.11. However, we do not have a proof of anything like this.

Example 3.14. Let the notation be as in Example 3.13. Define a homomorphism $\beta: C_{d_0} \rightarrow L_d$ by $\beta(s_j) = s_{j,1}$ and $\beta(t_j) = t_{j,1}$ for $j = 1, 2, \dots, d_0$. Then $\rho \circ \beta$ is a representation of C_{d_0} which is strongly forward isometric and contractive on generators, and for which one has $\|(\rho \circ \beta)(s_\lambda)\xi\|_p = \|\lambda\|_p \|\xi\|_p$ for $\lambda \in \mathbb{C}^{d_0}$ and $\xi \in L^p$.

The next example is the analog of Example 3.13 for L_∞ .

Example 3.15. Let $p \in [1, \infty) \setminus \{2\}$ and let $d_0 \in \{2, 3, \dots\}$. In L_∞ , call the standard generators $s_{j,m}$ and $t_{j,m}$ for $j = 1, 2, \dots, d_0$ and $m \in \mathbb{Z}_{>0}$. Then there is an automorphism α of L_∞ defined by the formula (3.1), but now for $j = 1, 2, \dots, d_0$ and $m \in \mathbb{Z}_{>0}$.

Take ρ to be as in Example 3.3. Then $\rho \circ \alpha$ is a representation of L_∞ which is strongly forward isometric and contractive on generators, and for which one has $\overline{(\rho \circ \alpha)(L_\infty)} = \overline{\rho(L_\infty)}$ but, in general, $\|(\rho \circ \alpha)(s_\lambda)\xi\|_p \neq \|\lambda\|_p \|\xi\|_p$.

We can then form the direct sum representation $\pi = \rho \oplus_p (\rho \circ \alpha)$ as in Remark 2.15. We do not know whether $\overline{\pi(L_\infty)}$ is isomorphic to $\overline{\rho(L_\infty)}$ as a Banach algebra.

Example 3.16. Let the notation be as in Example 3.15. Define a homomorphism β from L_∞ (with conventionally named generators) to L_∞ (with generators named as in Example 3.15) by $\beta(s_j) = s_{j,1}$ and $\beta(t_j) = t_{j,1}$ for $j = 1, 2, \dots, d_0$. Then $\rho \circ \beta$ is a representation of L_∞ which is strongly forward isometric and contractive on generators, and for which one has $\overline{(\rho \circ \beta)(L_\infty)} = \overline{\rho(L_\infty)}$ for $\lambda \in \mathbb{C}^\infty$ and $\xi \in L^p$. We do not know whether $\overline{(\rho \circ \beta)(L_\infty)}$ is isomorphic to $\overline{\rho(L_\infty)}$ as a Banach algebra.

Remark 3.17. Example 3.10 is based on the Fourier transform from functions on \mathbb{Z}_d to functions on \mathbb{Z}_d . We have not investigated the possibility of using other finite abelian groups.

We finish this section with basic examples of representations on $L^p([0, 1])$.

Example 3.18. Let $d \in \{2, 3, 4, \dots\}$ and let $p \in [1, \infty]$. For $j = 1, 2, \dots, d$, define a function

$$g_j: [0, 1] \rightarrow \left[\frac{j-1}{d}, \frac{j}{d} \right]$$

by $g_j(x) = d^{-1}(j + x - 1)$. Then we claim that there is a unique representation $\rho: L_d \rightarrow L(L^p([0, 1]))$ such that for $j = 1, 2, \dots, d$, $x \in [0, 1]$, and $\xi \in L^p([0, 1])$ we have

$$(\rho(s_j)\xi)(x) = \begin{cases} d^{1/p}\xi(g_j^{-1}(x)) & x \in \left[\frac{j-1}{d}, \frac{j}{d} \right] \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\rho(t_j)\xi)(x) = d^{-1/p}\xi(g_j(x)).$$

The proof of the claim is a straightforward verification that the proposed operators $\rho(s_j)$ and $\rho(t_j)$ satisfy the defining relations for L_d in Definition 1.1.

It is easy to check that ρ is contractive on generators and is forward isometric. In fact, ρ is strongly forward isometric. This follows from general theory. (See Lemma 7.9 below.) But it is also easily checked, using Remark 2.7, that for $\lambda \in \mathbb{C}^d$ and $\xi \in L^p([0, 1])$, we have $\|\rho(s_\lambda)\xi\|_p = \|\lambda\|_p \|\xi\|_p$.

Example 3.19. Let $p \in [1, \infty]$. We give an example of a representation of L_∞ on $L^p([0, 1])$. In the following, if $p = \infty$ then expressions with p in the denominator are taken to be zero.

For $j \in \mathbb{Z}_{>0}$, define a function

$$f_j: [0, 1] \rightarrow \left[\frac{1}{2^j}, \frac{1}{2^{j-1}} \right]$$

by $f_j(x) = 2^{-j}(1 + x)$. Then we claim that there is a unique representation $\rho: L_\infty \rightarrow L(L^p([0, 1]))$ such that for $j \in \mathbb{Z}_{>0}$, $x \in [0, 1]$, and $\xi \in L^p([0, 1])$ we have

$$(\rho(s_j)\xi)(x) = \begin{cases} 2^{j/p}\xi(f_j^{-1}(x)) & x \in \left[\frac{1}{2^j}, \frac{1}{2^{j-1}} \right] \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\rho(t_j)\xi)(x) = 2^{-j/p}\xi(f_j(x)).$$

The proof of the claim is a straightforward verification that the proposed operators $\rho(s_j)$ and $\rho(t_j)$ satisfy the defining relations for L_∞ in Definition 1.3.

One can check that ρ is contractive on generators and forward isometric. (We will see in Lemma 7.9 below that it is in fact strongly forward isometric.)

4. BOOLEAN σ -ALGEBRAS

In this section and the next, we describe the background for the characterization, due to Lamperti, of isometries on $L^p(X, \mu)$. Parts of this material can be found in Section 1 of Chapter X of [13] and in Lamperti's paper [18], but these references contain only enough to state and prove the characterization theorem, not enough to make serious use of it. A somewhat more systematic presentation can be found in Chapter 15 of [24], especially Section 2, and we use the terminology from there, but we need more than is there, and the form in which it is presented there makes citation of specific results difficult.

For this section, on abstract Boolean σ -algebras, we follow [16] and just state the basic results. (However, we use notation more suggestive of unions and intersections than the notation of [16]: our $E \vee F$ is $E + F$ there, our $E \wedge F$ is $E \cdot F$ there, and our E' is $-E$ there.)

Definition 4.1 (Definition 1.1 of [16]). A *Boolean algebra* is a set \mathcal{B} with two commutative associative binary operations

$$(E, F) \mapsto E \vee F \quad \text{and} \quad (E, F) \mapsto E \wedge F,$$

a unary operation $E \mapsto E'$, and distinguished elements 0 and 1, satisfying the following for all $E, F, G \in \mathcal{B}$:

- (1) $E \vee (E \wedge F) = E$ and $E \wedge (E \vee F) = E$.
- (2) $E \wedge (F \vee G) = (E \wedge F) \vee (E \wedge G)$ and $E \vee (F \wedge G) = (E \vee F) \wedge (E \vee G)$.
- (3) $E \vee E' = 1$ and $E \wedge E' = 0$.

Subalgebras and homomorphisms of Boolean algebras have the obvious meanings. (See Definitions 1.7 and 1.3 of [16].)

Example 4.2. The standard example is the power set $\mathcal{P}(X)$ of a set X , with

$$E \vee F = E \cup F, \quad E \wedge F = E \cap F, \quad E' = X \setminus E, \quad 0 = \emptyset. \quad \text{and} \quad 1 = X.$$

We therefore refer to the operations in a Boolean algebra as union, intersection, and complementation.

Proofs that the axioms imply the other expected properties can be found in Section 1.5 of [16]. However, for the purposes of finite algebraic manipulations, it is easier to rely on the following theorem, which implies that all finite identities which hold among sets also hold in any Boolean algebra.

Theorem 4.3 (Theorem 2.1 of [16]). Let \mathcal{B} be a Boolean algebra. Then there exists a set X and an isomorphism of \mathcal{B} with a Boolean subalgebra of $\mathcal{P}(X)$.

Definition 4.4. Let \mathcal{B} be a Boolean algebra, and let $E, F \in \mathcal{B}$. We define $E \leq F$ to mean $E \wedge F = E$. We say that E and F are *disjoint* if $E \wedge F = 0$. We define the *symmetric difference* of E and F to be

$$E \triangle F = (E \wedge F') \vee (E' \wedge F).$$

In $\mathcal{P}(X)$, disjointness means that the intersection is empty, $E \leq F$ means $E \subset F$, and symmetric difference has its usual meaning. Thus, by Theorem 4.3, the relation of Definition 4.4 is a partial order on \mathcal{B} , in which 0 is the least element and 1 is the greatest element. In particular, if $E \leq F$ and $F \leq E$, then $E = F$.

Definition 4.5 (Section 15.2 of [24]). A *Boolean σ -algebra* is a Boolean algebra \mathcal{B} in which whenever $E_1, E_2, \dots \in \mathcal{B}$, then there is a least element $E \in \mathcal{B}$ such that $E_n \leq E$ for all $n \in \mathbb{Z}_{>0}$.

In Definition 1.28 of [16], a Boolean σ -algebra is called a σ -complete Boolean algebra. (It is also required that greatest lower bounds of countable collections exist, but this follows from Definition 4.5 by complementation.)

Definition 4.6. Let \mathcal{B} be a Boolean σ -algebra. The element E in Definition 4.6 is denoted $\bigvee_{n=1}^{\infty} E_n$. We call it the *union* of the E_n . We further define $\bigwedge_{n=1}^{\infty} E_n = (\bigvee_{n=1}^{\infty} E_n)'$, and call it the *intersection* of the E_n .

The operations $\bigvee_{n=1}^{\infty} E_n$ and $\bigwedge_{n=1}^{\infty} E_n$ behave as expected. (This is not proved in [24], but is proved in [16].)

Lemma 4.7. Let \mathcal{B} be a Boolean σ -algebra.

(1) Let $E_1, E_2, \dots \in \mathcal{B}$. Then

$$\left(\bigvee_{n=1}^{\infty} E_n \right)' = \bigwedge_{n=1}^{\infty} E_n' \quad \text{and} \quad \left(\bigwedge_{n=1}^{\infty} E_n \right)' = \bigvee_{n=1}^{\infty} E_n'.$$

(2) Let $E_1, E_2, \dots, F \in \mathcal{B}$. Then

$$F \vee \bigvee_{n=1}^{\infty} E_n = \bigvee_{n=1}^{\infty} (F \vee E_n) \quad \text{and} \quad F \wedge \bigvee_{n=1}^{\infty} E_n = \bigvee_{n=1}^{\infty} (F \wedge E_n),$$

and

$$F \vee \bigwedge_{n=1}^{\infty} E_n = \bigwedge_{n=1}^{\infty} (F \vee E_n) \quad \text{and} \quad F \wedge \bigwedge_{n=1}^{\infty} E_n = \bigwedge_{n=1}^{\infty} (F \wedge E_n).$$

(3) Let $E_{m,n} \in \mathcal{B}$ for $m, n \in \mathbb{Z}_{>0}$. Then

$$\bigvee_{m=1}^{\infty} \bigvee_{n=1}^{\infty} E_{m,n} = \bigvee_{m=1}^{\infty} \bigvee_{n=1}^{\infty} E_{n,m} \quad \text{and} \quad \bigwedge_{m=1}^{\infty} \bigwedge_{n=1}^{\infty} E_{m,n} = \bigwedge_{m=1}^{\infty} \bigwedge_{n=1}^{\infty} E_{n,m}.$$

Proof. These statements all follow easily from the various parts of Lemma 1.33 of [16], or from their duals (stated afterwards). \square

Example 4.8. Let X be a set. Then a σ -algebra of subsets of X is a Boolean σ -algebra.

We introduce σ -homomorphisms and σ -ideals. In [16], they are called σ -complete homomorphisms (Definition 5.1 of [16]) and σ -complete ideals (Definition 5.19 of [16]).

Definition 4.9. Let \mathcal{B} and \mathcal{C} be Boolean σ -algebras. A σ -homomorphism from \mathcal{B} to \mathcal{C} is a function $S: \mathcal{B} \rightarrow \mathcal{C}$ which is a homomorphism of Boolean algebras in the obvious sense, and moreover such that whenever $E_1, E_2, \dots \in \mathcal{B}$ then

$$S \left(\bigvee_{n=1}^{\infty} E_n \right) = \bigvee_{n=1}^{\infty} S(E_n).$$

Lemma 4.10. The σ -homomorphisms of Definition 4.9 have the following properties.

- (1) The composition of two σ -homomorphisms is a σ -homomorphism.
- (2) A σ -homomorphism preserves order and disjointness (as in Definition 4.4).
- (3) A σ -homomorphism S is injective if and only if $S(E) = 0$ implies $E = 0$.

Proof. The first part is obvious, and the second is easy. The nontrivial direction of the third part is proved by considering symmetric differences. (See Lemma 5.3 of [16].) \square

Definition 4.11. Let \mathcal{B} be a Boolean σ -algebra. A σ -ideal in \mathcal{B} is a subset $\mathcal{N} \subset \mathcal{B}$ which is closed under countable unions, such that $0 \in \mathcal{N}$, and such that whenever $E \in \mathcal{B}$ and $F \in \mathcal{N}$ satisfy $E \subset F$, then $E \in \mathcal{N}$.

The standard example is as follows.

Example 4.12. Let X be a set, and let \mathcal{B} be a σ -algebra on X . Let μ be a measure with domain \mathcal{B} . Then

$$\mathcal{N}(\mu) = \{E \in \mathcal{B} : \mu(E) = 0\}$$

is a σ -ideal in \mathcal{B} .

Definition 4.13. Let \mathcal{B} be a Boolean σ -algebra. If $\mathcal{N} \subset \mathcal{B}$ is a σ -ideal, we define $E \sim F$ to mean $E \Delta F \in \mathcal{N}$. We define \mathcal{B}/\mathcal{N} to be the quotient set of \mathcal{B} by \sim . If $E \in \mathcal{B}$, we write $[E]$ for its image in \mathcal{B}/\mathcal{N} .

Lemma 4.14. Let the notation be as in Definition 4.13. Then \sim is an equivalence relation, the obvious induced operations on \mathcal{B}/\mathcal{N} are well defined and make \mathcal{B}/\mathcal{N} a Boolean σ -algebra, and $E \mapsto [E]$ is a surjective σ -homomorphism.

Proof. This is outlined in Section 15.2 of [24], and is contained in Lemma 5.22 of [16] and the remark after its proof. \square

We finish this section with a lemma which will be needed later.

Lemma 4.15. Let \mathcal{B} be a Boolean σ -algebra, and let $\mathcal{N} \subset \mathcal{B}$ be a σ -ideal. Suppose $E_1, E_2, \dots \in \mathcal{B}/\mathcal{N}$ are pairwise disjoint (Definition 4.4). Then there exist disjoint elements $E_1^{(0)}, E_2^{(0)}, \dots \in \mathcal{B}$ such that $[E_n^{(0)}] = E_n$ for all $n \in \mathbb{Z}_{>0}$.

Proof. We first show that if $E, F \in \mathcal{B}/\mathcal{N}$ are disjoint, then there exist disjoint $E_0, F_0 \in \mathcal{B}$ such that $[E_0] = E$ and $[F_0] = F$. Choose any $E_0 \in \mathcal{B}$ such that $[E_0] = E$ and any $K \in \mathcal{B}$ such that $[K] = F$. Then $[K \wedge E_0] = 0$, so we can take $F_0 = K \wedge E_0'$.

Now we prove the statement. For $m \neq n$ choose disjoint elements $S_{m,n}, T_{m,n} \in \mathcal{B}$ such that $[S_{m,n}] = E_m$ and $[T_{m,n}] = E_n$. Then take

$$E_n^{(0)} = \bigwedge_{k \in \mathbb{Z}_{>0} \setminus \{n\}} (S_{n,k} \wedge T_{k,n}).$$

This completes the proof. \square

5. MEASURABLE SET TRANSFORMATIONS

In this section, we consider measure spaces (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) , and a suitable σ -homomorphism $S: \mathcal{B}/\mathcal{N}(\mu) \rightarrow \mathcal{C}/\mathcal{N}(\nu)$. We describe how to use S to produce maps on measurable functions mod equality almost everywhere and on measures. The idea is not new; it can be found in [18] and (for functions, under stronger hypotheses) in Chapter X of [13]. However, we need much more than can be found in these references.

First, we give some definitions and notation which will be frequently used later.

Definition 5.1. Let (X, \mathcal{B}, μ) be a measure space. We denote by $L^0(X, \mu)$ the vector space of all complex valued measurable functions on X , mod the functions which vanish almost everywhere. We follow the usual convention of treating

elements of $L^0(X, \mu)$ as functions when convenient. If $E \in \mathcal{B}$, we denote by χ_E the characteristic function of E , which is a well defined element of $L^0(X, \mu)$.

Definition 5.2. Let (X, \mathcal{B}, μ) be a measure space, and let $\mathcal{N}(\mu)$ be as in Example 4.12. For $\xi \in L^0(X, \mu)$, we define the *support* of ξ to be the element of $\mathcal{B}/\mathcal{N}(\mu)$ given as follows. Choose any actual function $\xi_0: X \rightarrow \mathbb{C}$ whose class in $L^0(X, \mu)$ is ξ , and set

$$\text{supp}(\xi) = [\{x \in X: \xi_0(x) \neq 0\}].$$

Further, for $E \in \mathcal{B}/\mathcal{N}(\mu)$, define $\chi_E \in L^0(X, \mu)$ to be the class of χ_{E_0} for any $E_0 \in \mathcal{B}$ with $[E_0] = E$.

Definition 5.2 is a natural kind of strengthening of Definition 2.6.

Remark 5.3. In Definition 5.2, it is easy to check that $\text{supp}(\xi)$ and χ_E are well defined. Clearly ξ is supported in E , in the sense of Definition 2.6, if and only if $[E] \geq \text{supp}(\xi)$.

The following definition and constructions based on it are adapted from the beginning of Section 1 in Chapter X of [13].

Definition 5.4. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be measure spaces, and let $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$ be as in Example 4.12. A *measurable set transformation* from (X, \mathcal{B}, μ) to (Y, \mathcal{C}, ν) is a σ -homomorphism (in the sense of Definition 4.9) $S: \mathcal{B}/\mathcal{N}(\mu) \rightarrow \mathcal{C}/\mathcal{N}(\nu)$. By abuse of notation, we write $S: (X, \mathcal{B}, \mu) \rightarrow (Y, \mathcal{C}, \nu)$. For $E \in \mathcal{B}$, we also write, by abuse of notation, $S(E)$ for some choice of $F \in \mathcal{C}$ such that $[F] = S([E])$. Injectivity and surjectivity always refer to properties of the map $S: \mathcal{B}/\mathcal{N}(\mu) \rightarrow \mathcal{C}/\mathcal{N}(\nu)$. We denote by $\text{ran}_Y(S)$ the collection of all subsets $F \in \mathcal{C}$ such that $[F] = S([E])$ for some $E \in \mathcal{B}$.

In [13], a set transformation is taken to be a multivalued map from \mathcal{B} to \mathcal{C} , with possible values differing only up to a set of measure zero, and which preserves the appropriate set operations, that is, which defines a σ -homomorphism from $\mathcal{B}/\mathcal{N}(\mu)$ to $\mathcal{C}/\mathcal{N}(\nu)$. Also, $(X, \mathcal{B}, \mu) = (Y, \mathcal{C}, \nu)$, and the map is required to be measure preserving. In our situation, if S is injective, then at least the type of transformation in [13] sends sets of nonzero measure to sets of nonzero measure.

At the beginning of Section 3 of [18], what we call an injective σ -homomorphism is called a regular set isomorphism. The definition specifies a map of sets modulo null sets, although without formally defining an appropriate domain. It omits the requirement (implicit above) that $S(X) = Y$, but this is easily restored by replacing Y by $S(X)$. We do not use the term “regular set isomorphism” because such maps need not be surjective, and we need to consider cases in which they are not. Also, we are more careful with the formalism because we need to use such maps systematically.

Lemma 5.5. Let the notation be as in Definition 5.4. Then $\text{ran}_Y(S)$ is a sub- σ -algebra of \mathcal{B} .

Proof. The proof is easy, and is omitted. \square

Proposition 5.6. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be measure spaces. Let

$$S: (X, \mathcal{B}, \mu) \rightarrow (Y, \mathcal{C}, \nu)$$

be a measurable set transformation (Definition 5.4). Then (following the notation of Definition 5.1) there is a unique linear map $S_*: L^0(X, \mu) \rightarrow L^0(Y, \nu)$ such that:

- (1) $S_*(\chi_E) = \chi_{S(E)}$ for all $E \in \mathcal{B}/\mathcal{N}(\mu)$.
- (2) Whenever $(\xi_n)_{n \in \mathbb{Z}_{>0}}$ is a sequence of measurable functions on X which converges pointwise almost everywhere $[\mu]$ to ξ , then $S_*(\xi_n) \rightarrow S_*(\xi)$ pointwise almost everywhere $[\nu]$.

Moreover:

- (3) Let $n \in \mathbb{Z}_{>0}$, let $f: \mathbb{C}^n \rightarrow \mathbb{C}$ be continuous, and let $\xi_1, \xi_2, \dots, \xi_n \in L^0(X, \mu)$. Set $\eta(x) = f(\xi_1(x), \xi_2(x), \dots, \xi_n(x))$ for $x \in X$. Then

$$S_*(\eta)(y) = f(S_*(\xi_1)(y), S_*(\xi_2)(y), \dots, S_*(\xi_n)(y))$$

for almost all $y \in Y$. (In particular, S_* preserves products and preserves arbitrary positive powers of the absolute value of a function.)

- (4) The range of S_* is $L^0(Y, \nu|_{\text{ran}(S)})$.
- (5) S_* is injective if and only if S is injective.
- (6) If $\xi \in L^0(X, \mu)$ and $B \subset \mathbb{C}$ is a Borel set, then $S([\xi^{-1}(B)]) = [S_*(\xi)^{-1}(B)]$.
- (7) Let $(Z, \mathcal{D}, \lambda)$ be another measure space, and let T be a measurable set transformation from (Y, \mathcal{C}, ν) to $(Z, \mathcal{D}, \lambda)$. Then $(T \circ S)_* = T_* \circ S_*$.

For the proof, we use the following well known lemma.

Lemma 5.7. Let (X, \mathcal{B}, μ) be a measure space, and let $(E_\alpha)_{\alpha \in \mathbb{Q}}$ be a family of measurable sets such that $E_\alpha \subset E_\beta$ whenever $\alpha \leq \beta$, and such that

$$\bigcap_{\alpha \in \mathbb{Q}} E_\alpha = \emptyset \quad \text{and} \quad \bigcup_{\alpha \in \mathbb{Q}} E_\alpha = X.$$

Then there exists a unique measurable function $\xi: X \rightarrow \mathbb{R}$ such that for all $\alpha \in \mathbb{Q}$ we have

$$\{x \in X: \xi(x) < \alpha\} \subset E_\alpha \subset \{x \in X: \xi(x) \leq \alpha\}.$$

Moreover, for $\alpha \in \mathbb{R}$ we have

$$\{x \in X: \xi(x) < \alpha\} = \bigcup_{\beta \in \mathbb{Q} \cap (-\infty, \alpha)} E_\beta$$

and

$$\{x \in X: \xi(x) > \alpha\} = \bigcup_{\beta \in \mathbb{Q} \cap (\alpha, \infty)} (X \setminus E_\beta).$$

Proof. The first part of the statement is Lemma 9 in Chapter 11 of [24]. The last two equations follow easily. \square

Proof of Proposition 5.6. We prove uniqueness. Let $T_1, T_2: L^0(X, \mu) \rightarrow L^0(Y, \nu)$ be linear and satisfy (1) and (2). It follows from linearity and (1) that whenever $\xi \in L^0(X, \mu)$ is a simple function, we have $T_1(\xi) = T_2(\xi)$ almost everywhere $[\nu]$. Since every measurable function is a pointwise limit of simple functions, this conclusion in fact holds for all $\xi \in L^0(X, \mu)$.

We now prove existence. We follow the construction on page 454 of [13], which is done under stronger hypotheses. The verification of linearity will be done as a special case of (3).

First let $\xi \in L^0(X, \mu)$ be real valued. For $\alpha \in \mathbb{Q}$ define

$$(5.1) \quad E_\alpha = \{x \in X : \xi(x) < \alpha\}.$$

Then the sets E_α satisfy the hypotheses of Lemma 5.7. For each $\alpha \in \mathbb{Q}$, choose a set $F_\alpha^{(0)} \in \mathcal{C}$ such that

$$(5.2) \quad [F_\alpha^{(0)}] = S([E_\alpha]).$$

Set

$$(5.3) \quad D_0 = \left(\bigcup_{\alpha < \beta} F_\beta^{(0)} \cap (Y \setminus F_\alpha^{(0)}) \right) \cup \left(\bigcap_{\alpha} F_\alpha^{(0)} \right) \cup \left(Y \setminus \bigcup_{\alpha} F_\alpha^{(0)} \right).$$

Then $\nu(D_0) = 0$ and the sets $F_\alpha^{(0)} \cap (Y \setminus D_0) \subset Y$ satisfy the the hypotheses of Lemma 5.7 with $(Y \setminus D_0, \mathcal{C}|_{Y \setminus D_0}, \nu|_{Y \setminus D_0})$ in place of (X, \mathcal{B}, μ) .

Choose any set $D \in \mathcal{C}$ such that $D_0 \subset D$ and $\nu(D) = 0$. For $\alpha \in \mathbb{Q}$ define

$$(5.4) \quad F_\alpha = \begin{cases} F_\alpha^{(0)} \cup D & \alpha > 0 \\ F_\alpha^{(0)} \cap (Y \setminus D) & \alpha \leq 0. \end{cases}$$

Then the sets $F_\alpha \subset Y$ satisfy the the hypotheses of Lemma 5.7 with (Y, \mathcal{C}, ν) in place of (X, \mathcal{B}, μ) . So Lemma 5.7 provides a measurable function $\eta: Y \rightarrow \mathbb{R}$ such that for all $\alpha \in \mathbb{Q}$, we have

$$(5.5) \quad \{y \in Y : \eta(y) < \alpha\} \subset F_\alpha \subset \{y \in Y : \eta(y) \leq \alpha\}.$$

We define $S_*(\xi) = \eta$.

We claim that, up to equality almost everywhere $[\nu]$, the function η does not depend on the choices made in its construction. First, if we replace D with \tilde{D} , and call the new function $\tilde{\eta}$, then $\tilde{\eta} = \eta$ off $D \cup \tilde{D}$, and $\nu(D \cup \tilde{D}) = 0$. Now suppose that $F_\alpha^{(0)}$ is replaced by some other set $\tilde{F}_\alpha^{(0)} \in \mathcal{C}$ such that $[\tilde{F}_\alpha^{(0)}] = S([E_\alpha])$. Then $\nu(F_\alpha^{(0)} \Delta \tilde{F}_\alpha^{(0)}) = 0$ for all $\alpha \in \mathbb{Q}$. Let D_0 be as in (5.3), and define \tilde{D}_0 analogously, using $\tilde{F}_\alpha^{(0)}$ in place of $F_\alpha^{(0)}$. Let η and $\tilde{\eta}$ be the functions resulting from our construction, with the choice

$$D = \tilde{D} = D_0 \cup \tilde{D}_0 \cup \bigcup_{\alpha \in \mathbb{Q}} (F_\alpha^{(0)} \Delta \tilde{F}_\alpha^{(0)}).$$

We have

$$F_\alpha^{(0)} \cap (Y \setminus D) = \tilde{F}_\alpha^{(0)} \cap (Y \setminus D)$$

for all $\alpha \in \mathbb{Q}$, so also

$$F_\alpha^{(0)} \cup D = \tilde{F}_\alpha^{(0)} \cup D$$

for all $\alpha \in \mathbb{Q}$. It follows that $\eta = \tilde{\eta}$. This completes the proof of the claim.

We next claim that if $\xi = \sum_{k=1}^n \gamma_k \chi_{B_k}$, with $B_1, B_2, \dots, B_n \in \mathcal{B}$, then (using the abuse of notation from Definition 5.4) we have

$$S_*(\xi) = \sum_{k=1}^n \gamma_k \chi_{S(B_k)}.$$

To prove this, first suppose that B_1, B_2, \dots, B_n are disjoint. Using Lemma 4.15, choose disjoint sets $C_1, C_2, \dots, C_n \in \mathcal{C}$ such that $[C_k] = S([B_k])$ for $k = 1, 2, \dots, n$. For $\alpha \in \mathbb{Q}$ choose $F_\alpha^{(0)} = \bigcup_{\gamma_k < \alpha} C_k$. Then the set D_0 in the construction of η is empty. Taking $D = \emptyset$ gives

$$S_*(\xi) = \sum_{k=1}^n \gamma_k \chi_{C_k},$$

as desired. The general case is easily reduced to the disjoint case by using instead of the sets B_k all possible nonempty intersections $E_1 \cap E_2 \cap \dots \cap E_n$ in which E_k is either B_k or $X \setminus B_k$. This proves the claim.

It follows immediately that (3) holds when $\xi_1, \xi_2, \dots, \xi_n$ are real simple functions and f is a continuous function from \mathbb{R}^n to \mathbb{R} .

We now prove (2). Let $(\xi_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of real measurable functions on X , and suppose that $\xi_n(x) \rightarrow \xi(x)$ almost everywhere $[\mu]$. Changing the ξ_n and ξ on a set of measure zero, we may assume that $\xi_n(x) \rightarrow \xi(x)$ for all $x \in X$. For $n \in \mathbb{Z}_{>0}$ and $\alpha \in \mathbb{Q}$, let $E_{n,\alpha}$, $F_{n,\alpha}^{(0)}$, and $D_n^{(0)}$ be the sets of (5.1), (5.2), and (5.3) for the construction of $S_*(\xi_n)$, and let E_α , $F_\alpha^{(0)}$, and D_0 be the corresponding sets for ξ . Let $\alpha, \beta \in \mathbb{Q}$ satisfy $\alpha > \beta$. Since $\xi(x) < \alpha$ implies $\limsup_{n \rightarrow \infty} \xi_n(x) < \alpha$ and $\xi(x) \geq \alpha$ implies $\liminf_{n \rightarrow \infty} \xi_n(x) > \beta$, we get

$$E_\alpha \subset \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_{m,\alpha} \quad \text{and} \quad X \setminus E_\alpha \subset \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} (X \setminus E_{m,\beta}).$$

Set

$$B_\alpha = F_\alpha^{(0)} \cap \left(Y \setminus \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} F_{m,\alpha}^{(0)} \right)$$

and

$$C_{\alpha,\beta} = (Y \setminus F_\alpha^{(0)}) \cap \left(Y \setminus \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} (Y \setminus F_{m,\beta}^{(0)}) \right).$$

Since S is a σ -homomorphism, $\nu(B_\alpha) = 0$ and $\nu(C_{\alpha,\beta}) = 0$.

Set

$$D = D_0 \cup \left(\bigcup_{n=1}^{\infty} D_n^{(0)} \right) \cup \left(\bigcup_{\alpha \in \mathbb{Q}} B_\alpha \right) \cup \left(\bigcup_{\alpha > \beta} C_{\alpha, \beta} \right).$$

Then $\nu(D) = 0$. Using this choice of D , define sets $F_{n, \alpha}$ as in (5.4) for ξ_n and F_α for ξ , and let η_n and η be the representatives the construction gives for $S_*(\xi_n)$ and $S_*(\xi)$.

Let $y \in Y \setminus D$. We claim that $\eta_n(y) \rightarrow \eta(y)$. This will complete the proof of (2) for real functions.

To prove the claim, let $\varepsilon > 0$, and choose $\alpha, \beta \in \mathbb{Q}$ such that

$$\eta(y) - \varepsilon < \beta < \eta(y) < \alpha < \eta(y) + \varepsilon.$$

Then $y \in F_\alpha$ by (5.5). Since $y \notin B_\alpha$, we have

$$y \in \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} F_{m, \alpha},$$

so (5.5) for η_n implies that there is $n_1 \in \mathbb{Z}_{>0}$ such that for all $m \geq n_1$ we have $\eta_m(y) \leq \alpha$. Then also $\eta_m(y) < \eta(y) + \varepsilon$. Furthermore, (5.5) implies $y \notin F_\beta$, and $y \notin C_{\alpha, \beta}$, so

$$y \in \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} (Y \setminus F_{m, \beta}).$$

Using (5.5) for η_n , we get $n_2 \in \mathbb{Z}_{>0}$ such that for all $m \geq n_2$ we have $\eta_m(y) \geq \beta > \eta(y) - \varepsilon$. We have thus shown that $\eta_n(y) \rightarrow \eta(y)$, as desired.

It now follows that (3) holds whenever $\xi_1, \xi_2, \dots, \xi_n$ are real measurable functions and f is a continuous function from \mathbb{R}^n to \mathbb{R} , because $\xi_1, \xi_2, \dots, \xi_n$ are pointwise limits of real simple functions.

We define S_* on complex functions ξ by $S_*(\xi) = S_*(\text{Re}(\xi)) + iS_*(\text{Im}(\xi))$. It is easy to check that S_* satisfies (1) and (2). For (3), first let $f: \mathbb{C}^n \rightarrow \mathbb{R}$ be continuous. Define $g: \mathbb{R}^{2n} \rightarrow \mathbb{R}$ by

$$g(s_1, t_1, s_2, t_2, \dots, s_n, t_n) = f(s_1 + it_1, s_2 + it_2, \dots, s_n + it_n)$$

for $s_1, t_1, s_2, t_2, \dots, s_n, t_n \in \mathbb{R}$. Set $\eta(x) = f(\xi_1(x), \xi_2(x), \dots, \xi_n(x))$ for $x \in X$. Then one checks that

$$\begin{aligned} S_*(\eta)(y) &= g(S_*(\text{Re}(\xi_1))(y), S_*(\text{Im}(\xi_1))(y), \\ &\quad \dots, S_*(\text{Re}(\xi_n))(y), S_*(\text{Im}(\xi_n))(y)) \\ &= f(S_*(\xi_1)(y), S_*(\xi_2)(y), \dots, S_*(\xi_n)(y)) \end{aligned}$$

for $y \in Y$, as desired. The extension to continuous functions $f: \mathbb{C}^n \rightarrow \mathbb{C}$ is now easy.

We now prove (4). It suffices to show that a real valued measurable function $\lambda \in L^0(Y, \nu|_{\text{ran}(S)})$ is in the range of S_* . For $\alpha \in \mathbb{Q}$, define

$$G_\alpha = \{y \in Y : \lambda(y) < \alpha\} \in \text{ran}(S).$$

Choose $E_\alpha^{(0)} \in \mathcal{B}$ such that $S([E_\alpha^{(0)}]) = [G_\alpha]$. Define

$$D = \left(X \setminus \bigcup_{\alpha \in \mathbb{Q}} E_\alpha^{(0)} \right) \cup \left(\bigcap_{\alpha \in \mathbb{Q}} E_\alpha^{(0)} \right),$$

and set

$$E_\alpha = \begin{cases} D \cup \bigcup_{\beta < \alpha} E_\beta^{(0)} & \alpha > 0 \\ (X \setminus D) \cap \bigcup_{\beta < \alpha} E_\beta^{(0)} & \alpha \leq 0. \end{cases}$$

These sets satisfy the hypotheses of Lemma 5.7. Let ξ be the function obtained from Lemma 5.7. One easily checks that $E_\alpha = \bigcup_{\beta < \alpha} E_\beta$ for all $\alpha \in \mathbb{Q}$, so the second part of Lemma 5.7 implies that $E_\alpha = \{x \in X : \xi(x) < \alpha\}$ for all $\alpha \in \mathbb{Q}$.

Since S is a σ -homomorphism and $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$ for all $\alpha \leq \beta$, one easily checks that $S([E_\alpha]) = [G_\alpha]$ for all $\alpha \in \mathbb{Q}$. In the construction of $S(\xi)$ at the beginning of the proof, we may therefore take $F_\alpha^{(0)} = G_\alpha$ for all $\alpha \in \mathbb{Q}$, giving $D_0 = \emptyset$, and then we may take $D = \emptyset$. Thus $F_\alpha = G_\alpha$ for all $\alpha \in \mathbb{Q}$. So $\eta = S_*(\xi)$ satisfies (5.5) for all $\alpha \in \mathbb{Q}$. Such a function is unique by Lemma 5.7, and λ is such a function, so $S_*(\xi) = \lambda$. This completes the proof of (4).

We next prove (6). Suppose first that ξ is real valued. For $\alpha \in \mathbb{Q}$, let

$$E_\alpha = \{x \in X : \xi(x) < \alpha\}$$

(as in (5.1)). Choose $G_\alpha \subset Y$ such that $[G_\alpha] = S([E_\alpha])$ (as in (5.2), where the set is called $F_\alpha^{(0)}$). Set

$$H = \left(Y \setminus \bigcup_{\alpha \in \mathbb{Q}} G_\alpha \right) \cup \left(\bigcap_{\alpha \in \mathbb{Q}} G_\alpha \right),$$

and set

$$F_\alpha^{(0)} = \begin{cases} H \cup \bigcup_{\beta < \alpha} G_\beta & \alpha > 0 \\ (Y \setminus H) \cap \bigcup_{\beta < \alpha} G_\beta & \alpha \leq 0. \end{cases}$$

Since $E_\alpha = \bigcup_{\beta < \alpha} E_\beta$ for all $\alpha \in \mathbb{Q}$ and S is a σ -homomorphism, we have $[F_\alpha^{(0)}] = S([E_\alpha])$ for all $\alpha \in \mathbb{Q}$. Therefore we may use the sets $F_\alpha^{(0)}$ in the construction of $S_*(\xi)$. This gives $D_0 = \emptyset$. Take $D = \emptyset$. So $F_\alpha = F_\alpha^{(0)}$. One easily checks that $F_\alpha^{(0)} = \bigcup_{\beta < \alpha} F_\beta^{(0)}$ for all $\alpha \in \mathbb{Q}$. The second part of Lemma 5.7 therefore implies that $F_\alpha = \{y \in Y : S_*(\xi)(y) < \alpha\}$ for all $\alpha \in \mathbb{Q}$. We have verified that

$$S([\{x \in X : \xi(x) < \alpha\}]) = [\{y \in Y : S_*(\xi)(y) < \alpha\}]$$

for all $\alpha \in \mathbb{Q}$. Since the collection $\{(-\infty, \alpha) : \alpha \in \mathbb{Q}\}$ generates the Borel σ -algebra and S is a σ -homomorphism, we have (6) for real valued ξ and all Borel subsets of \mathbb{R} .

By considering real and imaginary parts separately, one sees that (6) holds for complex ξ whenever B is the product of two Borel subsets of \mathbb{R} . Such products generate the Borel subsets of \mathbb{C} , so (6) holds for arbitrary B .

For (5), first suppose that S is not injective. Then there is $E \in \mathcal{B}$ such that $\mu(E) \neq 0$ but $S([E]) = [\emptyset]$. So $\chi_E \neq 0$ but $S_*(\chi_E) = 0$.

On the other hand, suppose S_* is not injective. Then there is a nonzero $\xi \in L^0(X, \mu)$ such that $S_*(\xi) = 0$. By considering the positive or negative part of the real or imaginary part of ξ , we may assume that ξ is nonnegative. Since $\xi \neq 0$, there is $\varepsilon > 0$ such that the set $E = \{x \in X : \xi(x) > \varepsilon\}$ satisfies $\mu(E) \neq 0$. Using part (6) at the first step, we get

$$[S(E)] = [\{y \in Y : S_*(\xi) > \varepsilon\}] = [\emptyset].$$

Therefore S is not injective.

Part (7) follows from uniqueness, since $T_* \circ S_*$ and $(T \circ S)_*$ are both linear and satisfy (1) and (2). \square

Corollary 5.8. Let the notation be as in Proposition 5.6, and assume in addition that μ and ν are σ -finite. Then the following are equivalent:

- (1) S is surjective.
- (2) $S_* : L^0(X, \mu) \rightarrow L^0(Y, \nu)$ is surjective.
- (3) The range $S_*(L^0(X, \mu))$ contains χ_F for every $F \in \mathcal{C}$ such that $\nu(F) < \infty$.
- (4) For every $F \in \mathcal{C}$ there are disjoint sets $E_1, E_2, \dots \in \mathcal{B}$ with finite measure such that, with $E = \bigcup_{n=1}^{\infty} E_n$, we have $S_*(\chi_E) = \chi_F$.

Proof. That (1) implies (2) is Proposition 5.6(4). That (2) implies (3) is trivial.

We prove that (3) implies (4). Since ν is σ -finite and S_* preserves pointwise almost everywhere limits (Proposition 5.6(2)), it suffices to consider sets F with $\nu(F) < \infty$. Choose $\xi \in L^0(X, \mu)$ such that $S_*(\xi) = \chi_F$. Set $E = \{x \in X : \xi(x) = 1\}$. Then $S([E]) = [F]$ by Proposition 5.6(6). Now use σ -finiteness of μ to write $E = \bigsqcup_{n=1}^{\infty} E_n$ with $\mu(E_n) < \infty$ for all n .

Finally, we prove that (4) implies (1). Let $F \in \mathcal{C}$. Then (4) implies that there is $\xi \in L^0(X, \mu)$ such that $S_*(\xi) = \chi_F$. Set $E = \{x \in X : \xi(x) = 1\}$. Then $E \in \mathcal{B}$ and $S([E]) = [F]$ by Proposition 5.6(6). \square

Lemma 5.9. Let the notation be as in Definition 5.4. Let λ be a measure on \mathcal{C} such that $\lambda \ll \nu$. Then there exists a unique measure $S^*(\lambda)$ on \mathcal{B} such that whenever $E \in \mathcal{B}$ and $F \in \mathcal{C}$ satisfy $[F] = S([E])$, then $S^*(\lambda)(E) = \lambda(F)$. Moreover:

- (1) $S^*(\lambda) \ll \mu$.
- (2) If σ is another measure on \mathcal{C} and $\sigma \ll \lambda$, then $S^*(\sigma) \ll S^*(\lambda)$.
- (3) If $S^*(\lambda)$ is σ -finite, then λ is σ -finite.
- (4) For every nonnegative function $\xi \in L^0(X, \mu)$, and for every function $\xi \in L^1(X, S^*(\lambda))$, we have

$$\int_X \xi d(S^*(\lambda)) = \int_Y S_*(\xi) d\lambda.$$

- (5) If $(X, \mathcal{B}, \mu) = (Y, \mathcal{C}, \nu)$ and S is the identity map, then $S^*(\lambda) = \lambda$.

- (6) Let (Z, \mathcal{D}, ρ) be another measure space, and let T be a measurable set transformation from (Y, \mathcal{C}, ν) to (Z, \mathcal{D}, ρ) . Suppose σ is a measure on \mathcal{D} such that $\sigma \ll \rho$. Then $(T \circ S)^*(\sigma) = S^*(T^*(\sigma))$.

Proof. Uniqueness of $S^*(\lambda)$ is obvious. For existence, we have to prove that $S^*(\lambda)$ is well defined, satisfies $S^*(\emptyset) = 0$, and is countably additive. The first follows from $\lambda \ll \nu$, the second is immediate, and the third follows from the first together with Lemma 4.15 and Lemma 4.10(2).

Parts (1), (2), (5), and (6) are clear.

To prove (3), write $X = \bigcup_{n=1}^{\infty} E_n$ with $S^*(\lambda)(E_n) < \infty$ for all $n \in \mathbb{Z}_{>0}$. Choose $F_n \in \mathcal{C}$ such that $S([E_n]) = [F_n]$. Then $\lambda(F_n) < \infty$. Therefore $F = \bigcup_{n=1}^{\infty} F_n$ is a subset of Y on which λ is σ -finite. Also $[Y \setminus F] = [\emptyset]$. Since $\lambda \ll \nu$, this implies that $\lambda(Y \setminus F) = 0$. Therefore λ is σ -finite.

It remains to verify (4). It suffices to do so when ξ is nonnegative. The result is immediate for simple functions. Choose a sequence $(\xi_n)_{n \in \mathbb{Z}_{>0}}$ of nonnegative simple functions such that $\xi_n \rightarrow \xi$ pointwise and $\xi_1 \leq \xi_2 \leq \dots$ pointwise. Proposition 5.6(2) implies that $S_*(\xi_n) \rightarrow S_*(\xi)$ pointwise almost everywhere $[\nu]$, and Proposition 5.6(6) implies that $S_*(\xi_1) \leq S_*(\xi_2) \leq \dots$ pointwise almost everywhere $[\nu]$. Using $\lambda \ll \nu$, $S^*(\lambda) \ll \mu$, and the Monotone Convergence Theorem (twice), we get

$$\int_X \xi dS^*(\lambda) = \lim_{n \rightarrow \infty} \int_X \xi_n dS^*(\lambda) = \lim_{n \rightarrow \infty} \int_Y S_*(\xi_n) d\lambda = \int_Y S_*(\xi) d\lambda.$$

This completes the proof. \square

The converse to Lemma 5.9(3) is false. Take $X = \mathbb{Z}$, take μ to be counting measure, take $Y = \mathbb{Z} \times \mathbb{Z}$, take ν to be counting measure, take $\lambda = \nu$, and take $S(E) = E \times \mathbb{Z}$ for $E \subset X$.

We really need to push measures forwards rather than pull them back.

Definition 5.10. Let the notation be as in Definition 5.4, and assume that S is injective. Let λ be a measure defined on \mathcal{B} which is absolutely continuous with respect to μ . Then we define $S_*(\lambda)$ to be the measure $(S^{-1})^*(\lambda)$ on the σ -algebra $\text{ran}(S)$.

The measure $S_*(\mu)$ is called μ^* in the statement of Theorem 3.1 of [18].

We need some notation.

Notation 5.11. Let (X, \mathcal{B}, μ) be a measure space, and let $E \in \mathcal{B}$. We denote by $\mathcal{B}|_E$ the σ -algebra on E consisting of all $F \in \mathcal{B}$ such that $F \subset E$. We call it the *restriction* of \mathcal{B} .

If λ is a measure defined on a σ -algebra \mathcal{B} , and $\mathcal{C} \subset \mathcal{B}$ is a σ -algebra on a set $E \in \mathcal{B}$, we write $\lambda|_{\mathcal{C}}$ for the restriction of λ . If $\mathcal{C} = \mathcal{B}|_E$, we just write $\lambda|_E$. When no confusion can arise, we often just write λ . For example, for a measure space (X, \mathcal{B}, μ) , we often write $L^p(E, \mu)$ rather than $L^p(E, \mu|_E)$. Also,

we identify without comment $L^p(E, \mu)$ as the subspace of $L^p(X, \mu)$ consisting of those functions which vanish off E .

Lemma 5.12. In the situation of Definition 5.10, the measures $S_*(\mu)$ and $\nu|_{\text{ran}(S)}$ are mutually absolutely continuous.

Proof. We have

$$S_*(\mu) \ll \nu|_{\text{ran}(S)}$$

by Lemma 5.9(1). Also, $S_*(S^*(\nu|_{\text{ran}(S)})) = \nu|_{\text{ran}(S)}$ by Parts (5) and (6) of Lemma 5.9, and $S^*(\nu|_{\text{ran}(S)}) \ll \mu$ by Part (1) of Lemma 5.9, so $\nu|_{\text{ran}(S)} \ll S_*(\mu)$ by Part (2) of Lemma 5.9. \square

We summarize some standard computations.

Corollary 5.13. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces, and let S be a bijective measurable set transformation from (X, \mathcal{B}, μ) to (Y, \mathcal{C}, ν) . Set

$$h = \left[\frac{d(S_*(\mu))}{d\nu} \right] \in L^0(Y, \nu).$$

Then h has values in $(0, \infty)$ almost everywhere $[\nu]$, and

$$\left[\frac{d((S^{-1})_*(\nu))}{d\mu} \right] = \frac{1}{(S^{-1})_*(h)} = (S^{-1})_* \left(\frac{1}{h} \right).$$

Moreover, whenever $\xi \in L^0(X, \mu)$ is nonnegative, or one of the integrals in the following exists (in which case they all do), we have

$$\int_X \xi d\mu = \int_Y S_*(\xi) d(S_*(\mu)) = \int_Y S_*(\xi) h d\nu.$$

Proof. This follows from Lemma 5.9(4) for S and S^{-1} , combined with standard properties of Radon-Nikodym derivatives and Proposition 5.6(3). \square

Corollary 5.14. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces, and let S be a bijective measurable set transformation from (X, \mathcal{B}, μ) to (Y, \mathcal{C}, ν) .

- (1) Let λ be a σ -finite measure on (X, \mathcal{B}) such that $\lambda \ll \mu$. Then $S_*(\lambda)$ is σ -finite and $S_*(\lambda) \ll \nu$.
- (2) Let λ and σ be σ -finite measures on (X, \mathcal{B}) such that σ, λ , and μ are all mutually absolutely continuous. Then $S_*(\sigma), S_*(\lambda)$, and $S_*(\mu)$ are all mutually absolutely continuous, and

$$\left[\frac{dS_*(\sigma)}{dS_*(\lambda)} \right] = S_* \left(\left[\frac{d\sigma}{d\lambda} \right] \right)$$

almost everywhere $[S_*(\lambda)]$.

Proof. For (1), set $\tau = S_*(\lambda)$. Then $\tau = (S^{-1})^*(\lambda)$, so $S^*(\tau) = \lambda$ by Lemma 5.9(6). Therefore Lemma 5.9(3) implies that τ is σ -finite, and it follows from Lemma 5.9(1), applied to S^{-1} , that $S_*(\lambda) \ll \nu$.

We prove (2). Mutual absolute continuity follows from Lemma 5.9(2), applied to S^{-1} . In particular, we can use Corollary 5.13 with σ and λ in place of μ . Next, since the measures are σ -finite by (1), the required Radon-Nikodym derivatives exist. We then prove the result by showing that

$$\int_Y \eta \cdot S_* \left(\left[\frac{d\sigma}{d\lambda} \right] \right) dS_*(\lambda) = \int_Y \eta dS_*(\sigma)$$

for all nonnegative $\eta \in L^0(Y, \nu)$. By Corollary 5.8(2), we may assume that $\eta = S_*(\xi)$ for some $\xi \in L^0(X, \mu)$. We may take ξ to be nonnegative by Proposition 5.6(6). Using Corollary 5.13 for (X, \mathcal{B}, σ) and Proposition 5.6(3) at the first step, and Corollary 5.13 for $(X, \mathcal{B}, \lambda)$ at the third step, we have

$$\int_Y S_*(\xi) S_* \left(\left[\frac{d\sigma}{d\lambda} \right] \right) dS_*(\lambda) = \int_X \xi \left[\frac{d\sigma}{d\lambda} \right] d\lambda = \int_X \xi d\sigma = \int_Y S_*(\xi) dS_*(\sigma).$$

This completes the proof. \square

There is a more general version of Corollary 5.14(2), in which we only assume $\sigma \ll \lambda \ll \mu$. Since it has a more complicated statement and we don't need it, we omit it.

6. SPATIAL PARTIAL ISOMETRIES AND LAMPERTI'S THEOREM

We will need a systematic theory of isometries and partial isometries on L^p spaces. The main result is Lamperti's Theorem [18], according to which, for $p \in (0, \infty) \setminus \{2\}$, every isometry between L^p spaces is "semispatial" in a sense which we describe below. The material we need in order to make effective use of Lamperti's Theorem seems not to be in the literature, so we describe it here.

There are two choices of how to formulate the definitions, giving different results on $L^2(X, \mu)$. We adopt the stricter choice.

We will need the following three computations.

Lemma 6.1. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be measure spaces, let $p \in [1, \infty]$, let S be an injective measurable set transformation from (X, \mathcal{B}, μ) to (Y, \mathcal{C}, ν) such that $\nu|_{\text{ran}(S)}$ is σ -finite, and let g be a measurable function on Y such that $|g(y)| = 1$ for almost all $y \in Y$.

- (1) The Radon-Nikodym derivative

$$h = \left[\frac{dS_*(\mu)}{d(\nu|_{\text{ran}(S)})} \right] \in L^0(Y, \nu|_{\text{ran}(S)})$$

exists and satisfies $h(y) \in (0, \infty)$ for almost all $y \in Y$ with respect to $\nu|_{\text{ran}(S)}$.

- (2) Let h be as in (1). Let ξ be a measurable function on X . Define a measurable function η on Y by

$$\eta = gh^{1/p} S_*(\xi).$$

Then $\|\eta\|_p = \|\xi\|_p$.

- (3) Let ξ be a measurable function on X , let η be as in (2), and suppose that S is bijective. Then for almost all $x \in X$, we have

$$\xi = (S^{-1})_* \left(\frac{1}{g} \right) \left[\frac{d(S^{-1})_*(\nu)}{d\mu} \right]^{1/p} (S^{-1})_*(\eta).$$

Remark 6.2. Lemma 6.1(2) says that $\xi \mapsto gh^{1/p}S_*(\xi)$ defines an isometry in $L(L^p(X, \mu), L^p(Y, \nu))$. (This will be made explicit in Lemma 6.5 below.) The measure $\nu|_{\text{ran}(S)}$ need not be σ -finite, and in this case we do not get an element of $L(L^p(X, \mu), L^p(Y, \nu))$. Example: take X to consist of one point x , take $Y = \mathbb{Z}_{>0}$, take μ and ν to be counting measure, take $S(\emptyset) = \emptyset$ and $S(X) = Y$, and take $g = 1$.

Proof of Lemma 6.1. We prove (1). The measures $S_*(\mu)$ and $\nu|_{\text{ran}(S)}$ are mutually absolutely continuous by Lemma 5.12. The measure $\nu|_{\text{ran}(S)}$ is σ -finite by hypothesis.

We claim that $S_*(\mu)$ is σ -finite. Write $X = \bigcup_{n=1}^{\infty} X_n$ with $\mu(X_n) < \infty$. Choose $Y_n \in \mathcal{C}$ such that $S([X_n]) = [Y_n]$. Then $S_*(\mu)(Y_n) = \mu(X_n) < \infty$. Since $[Y \setminus \bigcup_{n=1}^{\infty} Y_n] = S([\emptyset])$, we have $S_*(\mu)(Y \setminus \bigcup_{n=1}^{\infty} Y_n) = 0$. The claim follows, as does (1).

For the remaining two parts, we present the proof for $p \neq \infty$. (The case $p = \infty$ is simpler.) For (2), we have, using $|g| = 1$ almost everywhere at the first step and Corollary 5.13 at the third step,

$$\|\eta\|_p^p = \int_Y h|S_*(\xi)|^p d\nu = \int_Y |S_*(\xi)|^p dS_*(\mu) = \int_X |\xi|^p d\mu = \|\xi\|_p^p.$$

For (3), use Corollary 5.13 at the first step, Proposition 5.6(3) at the second step, and Proposition 5.6(7) at the third step, to get, with equalities almost everywhere $[\mu]$,

$$\begin{aligned} & (S^{-1})_* \left(\frac{1}{g} \right) \left[\frac{d(S^{-1})_*(\nu)}{d\mu} \right]^{1/p} (S^{-1})_*(\eta) \\ &= (S^{-1})_* \left(\frac{1}{g} \right) (S^{-1})_* \left(\frac{1}{h} \right)^{1/p} (S^{-1})_*(\eta) = (S^{-1})_*(S_*(\xi)) = \xi. \end{aligned}$$

This completes the proof. □

Definition 6.3. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces. A *semispatial system* for (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) is a quadruple (E, F, S, g) in which $E \in \mathcal{B}$, in which $F \in \mathcal{C}$, in which S is an injective measurable set transformation from $(E, \mathcal{B}|_E, \mu|_E)$ to $(F, \mathcal{C}|_F, \nu|_F)$ such that $\nu|_{\text{ran}(S)}$ is σ -finite, and in which g is a \mathcal{C} -measurable function on F such that $|g(y)| = 1$ for almost all $y \in F$.

We say that (E, F, S, g) is a *spatial system* if, in addition, S is bijective.

Definition 6.4. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces, and let $p \in [1, \infty]$. A linear map $s \in L(L^p(X, \mu), L^p(Y, \nu))$ is called a *semispatial partial isometry* if there exists a semispatial system (E, F, S, g) such that, for every $\xi \in L^p(X, \mu)$, we have

$$(s\xi)(y) = \begin{cases} g(y) \left(\left[\frac{dS_*(\mu|_E)}{d(\nu|_{\text{ran}(S)})} \right] (y) \right)^{1/p} S_*(\xi|_E)(y) & y \in F \\ 0 & y \notin F. \end{cases}$$

(When $p = \infty$, we take

$$\left[\frac{dS_*(\mu|_E)}{d(\nu|_{\text{ran}(S)})} \right]^{1/p}$$

to be the constant function 1.)

We call s a *spatial partial isometry* if (E, F, S, g) is in fact a spatial system.

We call (E, F, S, g) the *(semi)spatial system* of s . (We will see in Lemma 6.6 below that it is essentially unique.) We call E and F the *domain support* and the *range support* of s . The sub- σ -algebra $\text{ran}(S) \subset \mathcal{C}|_F$ is called the *range σ -algebra*, the measurable set transformation S is called the *(semi)spatial realization* of s , and g is called the *phase factor*. If $\mu(X \setminus E) = 0$, we call s a *(semi)spatial isometry*.

Lemma 6.5. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be measure spaces, let $p \in [1, \infty]$, and let (E, F, S, g) be a semispatial system for (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) .

- (1) There exists a unique semispatial partial isometry

$$s \in L(L^p(X, \mu), L^p(Y, \nu))$$

whose semispatial system is (E, F, S, g) .

- (2) Let s be as in (1). Then $\|s\xi\|_p = \|\xi|_E\|_p$ for every $\xi \in L^p(X, \mu)$, and $\|s\| \leq 1$.
- (3) Let s be as in (1). Then for any set $B \in \mathcal{C}$, the range of s is contained in $L^p(Y, \nu|_B)$ if and only if B contains F up to a set of measure zero.
- (4) Suppose s as in (1) is a semispatial isometry. Then s is isometric as a linear map, that is, $\|s\xi\|_p = \|\xi\|_p$ for every $\xi \in L^p(X, \mu)$.

Proof. Lemma 6.1(2), applied with E in place of X and F in place of Y , implies existence of s in (1). Uniqueness is obvious. Part (2) follows from Lemma 6.1(2). Part (4) is a special case of (2).

For (3), it is obvious that if B contains F up to a set of measure zero, then $\text{ran}(s) \subset L^p(Y, \nu|_B)$. Now suppose $\text{ran}(s) \subset L^p(Y, \nu|_B)$. Choose $\xi \in L^p(E, \mu)$ such that $\xi(x) \neq 0$ for all $x \in X$. It follows from Proposition 5.6(6) that $S_*(\xi|_E)$ is nonzero almost everywhere on F , from the hypotheses that g is nonzero almost everywhere on F , and from Lemma 6.1(1) that

$$\left[\frac{dS_*(\mu|_E)}{d(\nu|_{\text{ran}(S)})} \right]$$

is nonzero almost everywhere on F . Therefore $s\xi$ is nonzero almost everywhere on F , whence B contains F up to a set of measure zero. \square

Lemma 6.6. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces, let $p \in [1, \infty]$, and let $s \in L(L^p(X, \mu), L^p(Y, \nu))$ be a semispacial partial isometry. Then its spatial system is unique up to changes in the domain and range supports by sets of measure zero and up to equality almost everywhere $[\nu]$ for the phase factors.

Remark 6.7. In Lemma 6.6, we identify $\mathcal{B}|_E/\mathcal{N}(\mu|_E)$ with a subset of $\mathcal{B}/\mathcal{N}(\mu)$. This subset does not change if E is modified by a set of measure zero. We treat $\mathcal{C}|_F/\mathcal{N}(\nu|_F)$ similarly. With these identifications, it makes sense to say that the measurable set transformation component is actually unique. In particular, he part about the domain and range supports says they are uniquely determined as elements of $\mathcal{B}/\mathcal{N}(\mu)$ and $\mathcal{C}/\mathcal{N}(\nu)$.

Proof of Lemma 6.6. Suppose that for $j = 1, 2$, the sets $E_j \subset X$ and $F_j \subset Y$ are measurable, S_j is an injective measurable set transformation from $(E_j, \mathcal{B}|_{E_j}, \mu|_{E_j})$ to $(F_j, \mathcal{C}|_{F_j}, \nu|_{F_j})$, and g_j is a measurable function on F_j with $|g_j(y)| = 1$ for almost all $y \in F_j$. Let s_1 and s_2 be the semispacial partial isometries obtained from Definition 6.4. We have to prove that if $s_1 = s_2$, then

$$[E_1] = [E_2], \quad [F_1] = [F_2], \quad S_1 = S_2, \quad \text{and} \quad g_1 = g_2 \quad \text{almost everywhere} \quad [\nu].$$

It follows from Lemma 6.5(3) that $[F_1] = [F_2]$. It is clear from Lemma 6.5(2) that we must have $[E_1] = [E_2]$.

For $j = 1, 2$, set

$$h_j = \left[\frac{d(S_j)_*(\mu|_{E_j})}{d(\nu|_{\text{ran}(S_j)})} \right].$$

Let $B \in \mathcal{B}|_{E_1}$ satisfy $\mu(B) < \infty$. Then $\chi_B \in L^p(X, \mu)$. From the formula for $s_1(\chi_B)$, and since h_1 and g_1 are nonzero almost everywhere on F , it follows that $S_1([B]) = [C]$ if and only if $s_1(\chi_B)$ is nonzero almost everywhere on C and zero almost everywhere on $Y \setminus C$. Of course, the same applies to s_2 and S_2 . Since μ is σ -finite and S_1 and S_2 are σ -homomorphisms, it follows that $S_1 = S_2$, and also that $h_1 = h_2$.

It remains to prove that $g_1 = g_2$ almost everywhere $[\nu]$. Choose $\xi \in L^p(E, \mu)$ such that $\xi(x) \neq 0$ for all $x \in X$. It follows from Proposition 5.6(6) that $(S_1)_*(\xi|_E)$ is nonzero almost everywhere on F_1 , and from Lemma 6.1(1) that h_1 is nonzero almost everywhere on F_1 . From $s_1\xi = s_2\xi$ almost everywhere, we therefore get $g_1 = g_2$ almost everywhere on F_1 , as desired. \square

Remark 6.8. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be measure spaces, let $p \in [1, \infty]$, and let $s \in L(L^p(X, \mu), L^p(Y, \nu))$ be a semispacial partial isometry with domain support $E \subset X$, range support $F \subset Y$, semispacial realization S , and range σ -algebra \mathcal{C}_0 . Then s is spatial if and only if $\mathcal{C}_0 = \mathcal{C}|_F$. If this is the case, then s is a spatial isometry if and only if $\mu(X \setminus E) = 0$.

Using the terminology we have introduced, we can now state Lamperti's Theorem as follows.

Theorem 6.9 (Lamperti). Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces, let $p \in [1, \infty) \setminus \{2\}$, and let $s \in L(L^p(X, \mu), L^p(Y, \nu))$ be an isometric (not necessarily surjective) linear map. Then s is a semispacial isometry in the sense of Definition 6.4.

Proof. The proof is the same as that of Theorem 3.1 of [18]. In [18] it is assumed that $(X, \mathcal{B}, \mu) = (Y, \mathcal{C}, \nu)$, but this is never used in the proof. \square

Remark 6.10. Let $p \in [1, \infty) \setminus \{2\}$, and let $s \in L(L^p(X, \mu), L^p(Y, \nu))$ be an isometry. Theorem 6.9 states that there exists a semispacial system (E, F, S, g) , with $E = X$, such that the construction of Definition 6.4 yields s . The function g was only required to be measurable with respect to $\mathcal{C}|_F$, not with respect to $\text{ran}(S)$. In general, the stronger condition fails. Take X to consist of one point x and Y to consist of two points y_1 and y_2 . Let μ and ν be the counting measures. Identify $L^p(X, \mu)$ with \mathbb{C} and $L^p(Y, \nu)$ with \mathbb{C}^2 in the obvious way. Define s by $s(\lambda) = 2^{-1/p}(\lambda, -\lambda)$. The semispacial system must be (X, Y, S, g) , with $S(\emptyset) = \emptyset$ and $S(X) = Y$, and with $g(y_1) = 1$ and $g(y_2) = -1$. The function g is then not $\text{ran}(S)$ -measurable.

We are primarily interested in partial isometries which are spatial rather than merely semispacial.

We will need notation for multiplication operators on $L^p(X, \mu)$.

Notation 6.11. Let (X, \mathcal{B}, μ) be a measure space, and let $p \in [1, \infty]$. We let $m_{X, \mu, p}$ (or, when no confusion should arise, just m_X or m) be the homomorphism $m_{X, \mu, p}: L^\infty(X, \mu) \rightarrow L(L^p(X, \mu))$ defined by

$$(m_{X, \mu, p}(f)\xi)(x) = f(x)\xi(x)$$

for $f \in L^\infty(X, \mu)$, $\xi \in L^p(X, \mu)$ and $x \in X$.

Lemma 6.12. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be measure spaces, let $p \in [1, \infty]$, and let (E, F, S, g) be a spatial system for (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) (in the sense of Definition 6.3).

- (1) There exists a unique spatial partial isometry

$$s \in L(L^p(X, \mu), L^p(Y, \nu))$$

whose spatial system is (E, F, S, g) .

- (2) Let s be as in (1). Then the range of s is $L^p(F, \nu) \subset L^p(Y, \nu)$.

- (3) Let s be as in (1). There exists a unique spatial partial isometry

$$t \in L(L^p(Y, \nu), L^p(X, \mu))$$

whose semispacial system is $(F, E, S^{-1}, (S^{-1})_*(g)^{-1})$. Moreover, using Notation 6.11, we have $ts = m(\chi_E)$ and $st = m(\chi_F)$.

- (4) Let s be as in (1) and let t be as in (3). Let $u \in L(L^p(Y, \nu), L^p(X, \mu))$ satisfy $us = m(\chi_E)$ and $u|_{L^p(Y \setminus F, \nu)} = 0$. Then $u = t$.

Proof. Part (1) follows from Lemma 6.5(1) and Definition 6.4. The existence of t in (3) follows from part (1), and the formulas for ts and st in (3) follow from Lemma 6.1(3). Lemma 6.5(4) and the formula for st imply (2).

For (4), let $\eta \in L^p(Y, \nu)$. Write $\eta = \eta_1 + \eta_2$ with $\eta_1 \in L^p(F, \nu)$ and $\eta_2 \in L^p(Y \setminus F, \nu)$. We show that $u\eta_1 = t\eta_1$ and $u\eta_2 = t\eta_2$. Clearly $u\eta_2$ and $t\eta_2$ are both zero. Also, using $\text{ran}(t) = L^p(E, \mu)$ at the last step, we have

$$u\eta_1 = um(\chi_F)\eta_1 = ust\eta_1 = m(\chi_E)t\eta_1 = t\eta_1.$$

This completes the proof. \square

Definition 6.13. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be measure spaces, let $p \in [1, \infty]$, and let $s \in L(L^p(X, \mu), L^p(Y, \nu))$ be a spatial partial isometry. The spatial partial isometry t of Lemma 6.12(3) is called the *reverse* of s .

Remark 6.14. When $p = 2$, the reverse of s is of course s^* . However, we can't define it this way when $p \neq 2$.

Lemma 6.15. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces, let $p \in [1, \infty]$, and let $s \in L(L^p(X, \mu), L^p(Y, \nu))$ be a semispatial partial isometry with domain support $E \subset X$, range support $F \subset Y$, and range σ -algebra \mathcal{C}_0 . Then the following are equivalent:

- (1) s is spatial.
- (2) $\text{ran}(s) = L^p(F, \mu)$.
- (3) $\mathcal{C}_0 = \mathcal{C}|_F$.

Proof. That (3) implies (1) is clear from the definitions, and (1) implies (2) by Lemma 6.12(2). So assume (2). Let S be the semispatial realization of s . Thus S is an injective measurable set transformation from $(E, \mathcal{B}|_E, \mu|_E)$ to $(F, \mathcal{C}|_F, \nu|_F)$ such that $\text{ran}(S_*)$ contains χ_B for every $B \in \mathcal{C}|_F$ with $\nu(B) < \infty$. It follows from Corollary 5.8 that S is surjective, which is (3). \square

Lemma 6.16. Let $p \in [1, \infty) \setminus \{2\}$, let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces, let $E \subset X$ and $F \subset Y$ be measurable subsets, and let $s \in L(L^p(X, \mu), L^p(Y, \nu))$ satisfy the following conditions:

- (1) The range of s is $L^p(F, \nu) \subset L^p(Y, \nu)$.
- (2) $s|_{L^p(E, \mu)}$ is isometric.
- (3) $s|_{L^p(X \setminus E, \mu)} = 0$.

Then s is a spatial partial isometry in the sense of Definition 6.4, and has domain support E and range support F .

Proof. Apply Theorem 6.9 with E in place of X , to conclude that s is a semispatial partial isometry. Then s is spatial by Lemma 6.15. \square

In the rest of this section, we describe some operations which give new spatial partial isometries from old ones. Most of the statements have analogs for semispatial partial isometries, but we don't need them and don't prove them.

We begin by showing that the product of two spatial partial isometries is again a spatial partial isometry. On a Hilbert space, the product of two partial isometries is usually not a partial isometry, unless the range projection of the second commutes with the domain projection of the first. With spatial partial isometries, this commutation relation is automatic.

Lemma 6.17. Let $(X_1, \mathcal{B}_1, \mu_1)$, $(X_2, \mathcal{B}_2, \mu_2)$, and $(X_3, \mathcal{B}_3, \mu_3)$ be σ -finite measure spaces. Let $p \in [1, \infty]$. Let

$$s \in L(L^p(X_1, \mu_1), L^p(X_2, \mu_2)) \quad \text{and} \quad u \in L(L^p(X_2, \mu_2), L^p(X_3, \mu_3))$$

be spatial partial isometries, with reverses t and w , and with spatial systems (E_1, E_2, S, g) and (F_2, F_3, V, h) . Then vs is a spatial partial isometry. Its domain support is $E = S^{-1}(E_2 \cap F_2)$, its range support is $F = V(E_2 \cap F_2)$, and its reverse is tw . Its spatial realization is the composite $V_0 \circ S_0$ of the restriction S_0 of S to $\mathcal{B}_1|_{S^{-1}(E_2 \cap F_2)}$ and the restriction V_0 of V to $\mathcal{B}_2|_{E_2 \cap F_2}$. Its phase factor is $k = (V_0)_*(g|_{E_2 \cap F_2})(h|_{V(E_2 \cap F_2)})$.

Proof. It is immediate that $(E, F, V_0 \circ S_0, k)$ is a spatial system for $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_3, \mathcal{B}_3, \mu_3)$. Let $y \in L(L^p(X_1, \mu_1), L^p(X_3, \mu_3))$ be the corresponding spatial partial isometry. We claim that $y = vs$, and it is enough to prove that $y\xi = vs\xi$ for ξ in each of the three spaces $L^p(X_1 \setminus E_1, \mu_1)$, $L^p(E_1 \setminus E, \mu_1)$, and $L^p(E, \mu_1)$. In the first case, $y\xi = 0$ and $s\xi = 0$. In the second case, $y\xi = 0$. Also $s\xi \in L^p(X_2 \setminus F_2, \mu_2)$, so $vs\xi = 0$.

So let $\xi \in L^p(E, \mu_1)$. Then

$$y\xi = (V_0)_*(g|_{E_2 \cap F_2})(h|_{V(E_2 \cap F_2)}) \left[\frac{d(V_0 \circ S_0)_*(\mu_1|_E)}{d(\mu_3|_F)} \right]^{1/p} (V_0 \circ S_0)_*(\xi)$$

and

$$vs\xi = (h|_F) \left[\frac{d(V_0)_*(\mu_2|_{E_2 \cap F_2})}{d(\mu_3|_F)} \right]^{1/p} \cdot (V_0)_* \left((g|_{E_2 \cap F_2}) \left[\frac{d(S_0)_*(\mu_1|_E)}{d(\mu_2|_{E_2 \cap F_2})} \right]^{1/p} (S_0)_*(\xi) \right).$$

One sees that these are equal by combining Corollary 5.14 with several applications of Proposition 5.6(3) and standard properties of Radon-Nikodym derivatives. This completes the proof that $y = vs$, and thus the proof that vs is as claimed.

It remains to identify the reverse. We use Lemma 6.12(4). First observe that $w\eta = 0$ for $\eta \in L^p(X_3 \setminus F_3, \mu_3)$ and $tw\eta = 0$ for $\eta \in L^p(F_3 \setminus F, \mu_3)$. Also, for

$\xi \in L^p(X_1, \mu_1)$, the first paragraph of the proof shows that $vs\xi = vsm(\chi_E)\xi$. It is easy to check that $sm(\chi_E) = m(\chi_{E_2 \cap F_2})s$. Therefore

$$twvs = t(wv)sm(\chi_E) = tm(\chi_{F_2})m(\chi_{E_2 \cap F_2})s = tsm(\chi_E) = m(\chi_E).$$

This completes the proof. \square

Lemma 6.18. Let (X, \mathcal{B}, μ) be a σ -finite measure space, let $p \in [1, \infty]$, and let $e \in L(L^p(X, \mu))$. Then the following are equivalent:

- (1) e is an idempotent spatial partial isometry.
- (2) e is a spatial partial isometry, and there is $E \in \mathcal{B}$ such that the spatial system of e is $(E, E, \text{id}_{\mathcal{B}|_E}, \chi_E)$.
- (3) There is $E \in \mathcal{B}$ such that $e = m(\chi_E)$.

Proof. It is clear that (2) and (3) are equivalent, and that (2) implies (1). So assume (1). Let the spatial system of e be (E, F, S, g) , and let f be the reverse of e (Definition 6.13). Lemma 6.17 implies that $f^2 = f$. Therefore

$$(6.1) \quad m(\chi_F)m(\chi_E) = (ef)(fe) = efe = em(\chi_E) = e.$$

Multiplying (6.1) on the left by $m(\chi_E)$ gives

$$(6.2) \quad m(\chi_E)m(\chi_F)m(\chi_E) = m(\chi_E)e = (fe)e = fe = m(\chi_E).$$

The left hand sides of (6.1) and (6.2) are clearly equal, so $e = m(\chi_E)$. This is (3). \square

We now consider the tensor product of two spatial partial isometries. We need a particular case of the product of two σ -homomorphisms, which we can get from Lamperti's Theorem. Quite possibly something more general is true, but we don't need it.

Lemma 6.19. Let $(X_1, \mathcal{B}_1, \mu_1)$, $(X_2, \mathcal{B}_2, \mu_2)$, $(Y_1, \mathcal{C}_1, \nu_1)$, and $(Y_2, \mathcal{C}_2, \nu_2)$ be σ -finite measure spaces. Let

$$S: \mathcal{B}_1/\mathcal{N}(\mu_1) \rightarrow \mathcal{B}_2/\mathcal{N}(\mu_2) \quad \text{and} \quad V: \mathcal{C}_1/\mathcal{N}(\nu_1) \rightarrow \mathcal{C}_2/\mathcal{N}(\nu_2)$$

be bijective σ -homomorphisms. Let $\mathcal{B}_1 \times \mathcal{C}_1$ and $\mathcal{B}_2 \times \mathcal{C}_2$ be the product σ -algebras on $X_1 \times Y_1$ and $X_2 \times Y_2$, or their completions. Then there is a unique σ -homomorphism

$$S \times V: (\mathcal{B}_1 \times \mathcal{C}_1)/\mathcal{N}(\mu_1 \times \nu_1) \rightarrow (\mathcal{B}_2 \times \mathcal{C}_2)/\mathcal{N}(\mu_2 \times \nu_2)$$

such that, whenever $E_1 \in \mathcal{B}_1$, $E_2 \in \mathcal{B}_2$, $F_1 \in \mathcal{C}_1$, and $F_2 \in \mathcal{C}_2$ satisfy $S([E_1]) = [E_2]$ and $V([F_1]) = [F_2]$, we have

$$(6.3) \quad (S \times V)([E_1 \times F_1]) = [E_2 \times F_2].$$

Proof. It does not matter whether we use the product σ -algebras or their completions, because the Boolean σ -algebras

$$(\mathcal{B}_1 \times \mathcal{C}_1)/\mathcal{N}(\mu_1 \times \nu_1) \quad \text{and} \quad (\mathcal{B}_2 \times \mathcal{C}_2)/\mathcal{N}(\mu_2 \times \nu_2)$$

are the same with either choice.

Uniqueness of $S \times V$ follows from the fact that the measurable rectangles generate the product σ -algebra.

We prove existence. Fix any $p \in [1, \infty) \setminus \{2\}$. Let s and v be the spatial isometries with spatial systems $(X_1, X_2, S, 1)$ and $(Y_1, Y_2, V, 1)$. Then s and v are isometric bijections. Applying Theorem 2.16(5) to s and v , and to their inverses, and applying Theorem 2.16(6), we see that

$$s \otimes v: L^p(X_1 \times Y_1, \mu_1 \times \nu_1) \rightarrow L^p(X_2 \times Y_2, \mu_2 \times \nu_2)$$

is an isometric bijection. Lemma 6.16 implies that it is spatial. We take $S \times V$ to be its spatial realization. The relation (6.3) is easily deduced from $(s \otimes v)(\xi \otimes \eta) = s\xi \otimes v\eta$, Fubini's Theorem, and σ -finiteness of all the measures involved. \square

In the next lemma, we exclude $p = \infty$ because we use Theorem 2.16.

Lemma 6.20. Let $(X_1, \mathcal{B}_1, \mu_1)$, $(X_2, \mathcal{B}_2, \mu_2)$, $(Y_1, \mathcal{C}_1, \nu_1)$, and $(Y_2, \mathcal{C}_2, \nu_2)$ be σ -finite measure spaces. Let $p \in [1, \infty)$. Let

$$s \in L(L^p(X_1, \mu_1), L^p(X_2, \mu_2)) \quad \text{and} \quad v \in L(L^p(Y_1, \nu_1), L^p(Y_2, \nu_2)).$$

be spatial partial isometries, with reverses t and w , and with spatial systems (E_1, E_2, S, g) and (F_2, F_3, V, h) . Then

$$s \otimes v \in L(L^p(X_1 \times Y_1, \mu_1 \times \nu_1), L^p(X_2 \times Y_2, \mu_2 \times \nu_2))$$

(as in Theorem 2.16) is a spatial partial isometry. With $S \times V$ as in Lemma 6.19 and $g_1 \otimes g_2$ as in Theorem 2.16, its spatial system is

$$(6.4) \quad (E_1 \times F_1, E_2 \times F_2, S \times V, g_1 \otimes g_2),$$

and its reverse is $t \otimes w$.

Proof. Let y be the spatial partial isometry with spatial system given by (6.4), and let z be its reverse.

It is clear that $S_*(\mu_1) \times V_*(\nu_1)$ and $(S \times V)_*(\mu_1 \times \nu_1)$ agree on measurable rectangles. The measures $S_*(\mu_1)$ and $V_*(\nu_1)$ are σ -finite by Corollary 5.14(1). The product of σ -finite measures is uniquely determined by its values on measurable rectangles. (In Chapter 12 of [24], see Theorem 8 and the discussion after Lemma 14.) Therefore $S_*(\mu_1) \times V_*(\nu_1) = (S \times V)_*(\mu_1 \times \nu_1)$. Set

$$k = \left[\frac{d(S \times V)_*(\mu_1|_{E_1} \times \nu_1|_{F_1})}{d(\mu_2|_{E_2} \times \nu_2|_{F_2})} \right] \quad \text{and} \quad l = \left[\frac{dS_*(\mu_1|_{E_1})}{d(\mu_2|_{E_2})} \right] \otimes \left[\frac{dV_*(\nu_1|_{F_1})}{d(\nu_2|_{F_2})} \right].$$

Then for every measurable rectangle $R \subset E_2 \times F_2$, we have

$$\int_R k d(\mu_2|_{E_2} \times \nu_2|_{F_2}) = \int_R l d(\mu_2|_{E_2} \times \nu_2|_{F_2}).$$

Since

$$G \mapsto \int_G l d(\mu_2|_{E_2} \times \nu_2|_{F_2})$$

is a product of σ -finite measures on $E_2 \times F_2$, it follows from uniqueness of product measures (as above) that $k = l$ almost everywhere $[\mu_2|_{E_2} \times \nu_2|_{F_2}]$. One now checks that $s \otimes v = y$ and $t \otimes w = z$ by showing, in each case, that they agree on elementary tensors. \square

Lemma 6.21. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces, let $p \in [1, \infty)$, and let $s \in L(L^p(X, \mu), L^p(Y, \nu))$ be a spatial partial isometry with spatial system (E, F, S, g) and with reverse t . Let $q \in (1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, and for any σ -finite measure space $(Z, \mathcal{D}, \lambda)$, identify $L^p(Z, \lambda)'$ with $L^q(Z, \lambda)$ in the usual way. Then $s' \in L(L^q(Y, \nu), L^q(X, \mu))$ is a spatial partial isometry with spatial system $(F, E, S^{-1}, (S^{-1})_*(g))$ and with reverse t' .

Proof. To simplify the notation, let

$$h = \left[\frac{dS_*(\mu)}{d\nu} \right] \quad \text{and} \quad k = \left[\frac{d(S^{-1})_*(\nu)}{d\mu} \right].$$

Then $h = S_*(k)^{-1}$ by Corollary 5.13. Let $u \in L(L^q(Y, \nu), L^q(X, \mu))$ be the spatial partial isometry with the spatial system specified for s' . Then for $\xi \in L^p(X, \mu)$ and $\eta \in L^q(Y, \nu)$, we have, using Corollary 5.13 at the second step,

$$\begin{aligned} \int_X \xi \cdot u\eta \, d\mu &= \int_X \xi(S^{-1})_*(g)k^{1/q}(S^{-1})_*(\eta) \, d\mu = \int_Y S_*(\xi)gS_*(k)^{1/q}\eta h \, d\nu \\ &= \int_Y S_*(\xi)gh^{-1/q}\eta h \, d\nu = \int_Y S_*(\xi)gh^{1/p}\eta h \, d\nu = \int_Y s\xi \cdot \eta \, d\nu. \end{aligned}$$

Thus $s' = u$.

To identify the reverse of s' , we calculate:

$$t's' = (st)' = m_{Y,\nu,p}(\chi_F)' = m_{Y,\nu,q}(\chi_F)$$

and

$$t'm_{X,\mu,q}(\chi_{X \setminus E}) = [m_{X,\mu,p}(\chi_{X \setminus E})t]' = 0.$$

Now apply Lemma 6.12(4). \square

We finish this section with a lemma on homotopies that will be needed later. We do not know whether surjectivity is necessary in the hypotheses.

Lemma 6.22. Let $p \in [1, \infty) \setminus \{2\}$. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces, and let $\lambda \mapsto s_\lambda \in L(L^p(X, \mu), L^p(Y, \nu))$, for $\lambda \in [0, 1]$, be a norm continuous path of surjective isometries. Let S_λ be the spatial realization of s_λ . Then $S_0 = S_1$.

Proof. We prove that if v_0 and v_1 are surjective spatial isometries whose spatial realizations V_0 and V_1 are distinct, then $\|v_0 - v_1\| \geq 1$. (It follows that the spatial realization must be constant along a homotopy.) The measurable set transformations V_0 and V_1 are bijective by Lemma 6.15.

Choose a set $E \in \mathcal{B}$ such that $V_0([E]) \neq V_1([E])$. Without loss of generality we may assume that $V_0([E])$ and $V_1([E])$ have representatives $F_0, F_1 \in \mathcal{C}$ such that F_0 does not contain F_1 up to sets of measure zero. That is, there is $F \subset Y$ such that

$$\nu(F) > 0, \quad F \subset F_0, \quad \text{and} \quad F \cap F_1 = \emptyset.$$

Since V_0 is bijective, the set $Q = V_0^{-1}(F)$ satisfies $\mu(Q) > 0$. Since μ is σ -finite, replacing F by a suitable subset allows us to also assume that $\mu(Q) < \infty$. Correcting by a set of measure zero, we may further assume that $Q \subset E$.

Define $\xi = \mu(Q)^{-1/p} \chi_Q \in L^p(X, \mu)$. Then $\|\xi\|_p = 1$. Moreover, $v_0\xi$ is supported in F and, since $Q \subset E$, the function $v_1\xi$ is supported in F_1 . Therefore

$$\|v_0\xi - v_1\xi\|_p^p = \|v_0\xi\|_p^p + \|v_1\xi\|_p^p = 2,$$

so $\|v_0 - v_1\| \geq 2^{1/p} \geq 1$. □

7. SPATIAL REPRESENTATIONS

In this section, we define and characterize spatial representations on spaces of the form $L^p(X, \mu)$, first of M_d and then of L_d for finite d . In each case, we give a number of equivalent conditions for a representation to be spatial, some of them quite different from each other. In particular, some characterizations are primarily in terms of how the representation interacts with X , while others make sense for a representation on any Banach space. We consider M_d first because we use the results about M_d in the theorem for L_d .

We will see in Theorem 7.2 that, for fixed p , any two spatial representations of M_d determine the same norm on M_d , and we will see in Theorem 8.7 (in the next section) that, for fixed p and when $d < \infty$, any two spatial representations of L_d determine the same norm on L_d .

Definition 7.1. Let $d \in \mathbb{Z}_{>0}$, and let $(e_{j,k})_{j,k=1}^d$ be the standard system of matrix units in M_d . Let $p \in [1, \infty]$, let (X, \mathcal{B}, μ) be a σ -finite measure space, and let $\rho: M_d \rightarrow L(L^p(X, \mu))$ be a representation. (Recall that, by convention, representations are unital. See Definition 2.8.) We say that ρ is *spatial* if $\rho(e_{j,k})$ is a spatial partial isometry, in the sense of Definition 6.4, for $j, k = 1, 2, \dots, d$.

Theorem 7.2. Let $d \in \mathbb{Z}_{>0}$, let $p \in [1, \infty) \setminus \{2\}$, let (X, \mathcal{B}, μ) be a σ -finite measure space, and let $\rho: M_d \rightarrow L(L^p(X, \mu))$ be a representation. Let $(e_{j,k})_{j,k=1}^d$ be the standard system of matrix units in M_d . Then the following are equivalent:

- (1) ρ is spatial.
- (2) For $j, k = 1, 2, \dots, d$, the operator $\rho(e_{j,k})$ is a spatial partial isometry with reverse $\rho(e_{k,j})$.
- (3) ρ is isometric as a map from M_d^p (as in Notation 2.4) to $L(L^p(X, \mu))$.
- (4) ρ is contractive as a map from M_d^p to $L(L^p(X, \mu))$.

- (5) $\|\rho(e_{j,k})\| \leq 1$ for $j, k = 1, 2, \dots, d$, and there exists a measurable partition $X = \coprod_{j=1}^d X_j$ such that for $j = 1, 2, \dots, d$, the matrix unit $e_{j,j}$ acts (following Notation 6.11) as $\rho(e_{j,j}) = m(\chi_{X_j})$.
- (6) There exists a measurable partition $X = \coprod_{j=1}^d X_j$ such that for $j = 1, 2, \dots, d$ the operator $\rho(e_{j,1})$ is a spatial partial isometry with domain support X_1 and range support X_j .
- (7) There exists a measurable partition $X = \coprod_{j=1}^d X_j$ such that for $j, k = 1, 2, \dots, d$ the operator $\rho(e_{j,k})$ is zero on $L^p(X \setminus X_k, \mu)$ and restricts to an isometric isomorphism from $L^p(X_k, \mu)$ to $L^p(X_j, \mu)$.
- (8) With γ being counting measure on $N_d = \{1, 2, \dots, d\}$, there exists a σ -finite measure space (Y, \mathcal{C}, ν) and a bijective isometry

$$u: L^p(N_d \times Y, \gamma \times \nu) \rightarrow L^p(X, \mu)$$

such that, following the notation of Theorem 2.16(4), for all $a \in M_d$ we have $\rho(a) = u(a \otimes 1)u^{-1}$.

In the notation of Lemma 2.17, Theorem 7.2(8) says that if $\rho_0: M_d \rightarrow L(l_d^p)$ is the standard representation, then ρ is similar, via an isometry, to $\rho_0 \otimes_p 1$.

We specifically use complex scalars in the proof that (3) and (4) imply the other conditions. We don't know whether complex scalars are necessary.

When $p = 2$, conditions (1) and (3) are certainly not equivalent. We have not investigated what happens when $p = \infty$, but Lamperti's Theorem is not available in this case.

It is essential that the representation be unital. Many of the conditions of Theorem 7.2 are never satisfied for nonunital representations, but (3) and (4) do occur. We show by example that they do not imply that the representation is spatial.

Example 7.3. We adopt the notation of Example 2.5, with $p \in [1, \infty) \setminus \{2\}$. Set

$$\xi = 2^{-1/p}(1, 1) \quad \text{and} \quad \eta = 2^{-1/q}(1, 1).$$

(If $p = 1$, take $\eta = (1, 1)$.) Let $e \in M_2^p$ be the rank one operator called a in Example 2.5. Then

$$e = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

which is an idempotent. Since $\|\xi\|_p = \|\eta\|_q = 1$, we get $\|e\| = 1$.

Now take $(X_0, \mathcal{B}_0, \mu_0)$ to be any σ -finite measure space with a spatial representation $\rho_0: M_d \rightarrow L(L^p(X_0, \mu_0))$. Set $X = X_0 \amalg X_0$, and equip it with the obvious σ -algebra \mathcal{B} and with the measure μ whose restriction to each copy of X_0 is μ_0 . Identify $L^p(X, \mu)$ with $l_2^p \otimes_p L^p(X_0, \mu_0)$ as in Theorem 2.16. Define a nonunital representation $\rho: M_d \rightarrow L(L^p(X, \mu))$ by $\rho(a) = e \otimes \rho_0(a)$ for $a \in M_d$. Then Theorem 2.16(5) implies that, regarded as a map $M_d^p \rightarrow L(L^p(X, \mu))$, the representation ρ is isometric. But it is not spatial, not even in a sense suitable for nonunital representations.

Proof of Theorem 7.2. We first prove the equivalence of (1), (5), (6), and (7), beginning with (1) implies (5).

So assume (1). We have $\|\rho(e_{j,k})\| \leq 1$ because this is true for all spatial partial isometries. For each j and k there is a spatial system for $\rho(e_{j,k})$, say $(E_{j,k}, F_{j,k}, S_{j,k}, g_{j,k})$. Apply Lemma 6.18 to $\rho(e_{j,j})$. We obtain sets $X_j \subset X$ such that

$$(E_{j,j}, F_{j,j}, S_{j,j}, g_{j,j}) = (X_j, X_j, \text{id}_{\mathcal{B}|_{X_j}}, \chi_{X_j}) \quad \text{and} \quad \rho(e_{j,j}) = m(\chi_{X_j})$$

for $j = 1, 2, \dots, d$. Since $\sum_{j=1}^d \rho(e_{j,j}) = 1$, the sets X_j are essentially disjoint and, up to a set of measure zero, $\bigcup_{j=1}^d X_j = X$. Modification by sets of measure zero now gives the rest of (5).

We next prove (5) implies (7). We take the partition $X = \bigsqcup_{j=1}^d X_j$ to be as in (5). Let $j, k \in \{1, 2, \dots, d\}$. The equation $\rho(e_{j,k})(1 - \rho(e_{k,k})) = 0$ translates to $\rho(e_{j,k})\xi = 0$ for all $\xi \in L^p(X \setminus X_k, \mu)$, which is the first part of (7). For $\xi \in L^p(X_k, \mu)$, use

$$\|\rho(e_{j,k})\| \leq 1, \quad \|\rho(e_{k,j})\| \leq 1, \quad \text{and} \quad \rho(e_{k,j})\rho(e_{j,k})\xi = \rho(e_{k,k})\xi = \xi$$

to get $\|\rho(e_{j,k})\xi\| = \|\xi\|$. So $\rho(e_{j,k})$ is isometric on $L^p(X_k, \mu)$. Since

$$\rho(e_{j,k})\rho(e_{k,j})\xi = \xi$$

for $\xi \in L^p(X_j, \mu)$, we have $\text{ran}(\rho(e_{j,k})) \supset L^p(X_j, \mu)$, while the equation

$$m(\chi_{X_j})\rho(e_{j,k})\xi = \rho(e_{j,j})\rho(e_{j,k})\xi = \rho(e_{j,k})\xi$$

for $\xi \in L^p(X_k, \mu)$ implies $\text{ran}(\rho(e_{j,k})) \subset L^p(X_j, \mu)$. This completes the proof of (7).

That (7) implies (6) is immediate from Lemma 6.16.

We prove that (6) implies (1). From (6), we see that $\text{ran}(\rho(e_{j,1})) = L^p(X_j, \mu)$ for $j = 1, 2, \dots, d$. Moreover, $\rho(e_{1,1}) = m(\chi_{X_1})$ by Lemma 6.18. Now fix j , and consider $\rho(e_{1,j})$. For $k \neq j$ we have $\rho(e_{1,j})|_{L^p(X_k, \mu)} = 0$ since $e_{1,j}e_{k,1} = 0$. Also $\rho(e_{1,j})\rho(e_{j,1}) = \rho(e_{1,1}) = m(\chi_{X_1})$. So Lemma 6.12(4) implies that $\rho(e_{1,j})$ is the reverse of $\rho(e_{j,1})$. In particular, $\rho(e_{1,j})$ is a spatial partial isometry.

For $j, k = 1, 2, \dots, d$, it now follows from Lemma 6.17 that $\rho(e_{j,k}) = \rho(e_{j,1})\rho(e_{1,k})$ is a spatial partial isometry, which is (1).

We next prove that (2) and (8) are equivalent to the conditions we have already considered. We start with (7) implies (8). Let the partition $X = \bigsqcup_{j=1}^d X_j$ be as in (7). Define $Y = X_1$ and $\nu = \mu|_{X_1}$. Identify $L^p(N_d \times Y, \gamma \times \nu)$ with the space of sequences $(\eta_1, \eta_2, \dots, \eta_d) \in L^p(Y, \nu)^d$, with the norm

$$\|(\eta_1, \eta_2, \dots, \eta_d)\| = (\|\eta_1\|_p^p + \|\eta_2\|_p^p + \dots + \|\eta_d\|_p^p)^{1/p}.$$

Define $u: L^p(N_d \times Y, \gamma \times \nu) \rightarrow L^p(X, \mu)$ by

$$u(\eta_1, \eta_2, \dots, \eta_d) = \eta_1 + \rho(e_{2,1})\eta_2 + \rho(e_{3,1})\eta_3 + \dots + \rho(e_{d,1})\eta_d.$$

(Note that $\eta_1 = \rho(e_{1,1})\eta_1$.) Then u is isometric because the summands $\rho(e_{j,1})\eta_j$ are supported in disjoint subsets of X . It is easy to check that u is bijective and that $\rho(e_{j,k}) = u(e_{j,k} \otimes 1)u^{-1}$ for $j, k = 1, 2, \dots, d$. This proves (8).

Now assume (8); we prove (2). It is trivial that the standard representation of M_d on $L^p(N_d)$ satisfies (2). It follows from Lemma 6.20 that the representation $\rho_0(a) = a \otimes 1$ also satisfies (2). The operator u is a spatial isometry by Lemma 6.16, and Lemma 6.12(4) implies that its reverse is u^{-1} . Lemma 6.17 now implies that ρ satisfies (2).

That (2) implies (1) is trivial.

We finish by proving that (3) and (4) are equivalent to the conditions we have already considered. That (8) implies (3) follows from the norm relation $\|a \otimes 1\| = \|a\|$. (See Theorem 2.16(5).) That (3) implies (4) is trivial.

Assume (4); we prove (5). For $j = 1, 2, \dots, d$ and $\zeta \in S_1$, set

$$t_{j,\zeta} = 1 - e_{j,j} + \zeta e_{j,j} \in M_d^p.$$

One checks immediately that $\|t_{j,\zeta}\| = 1$, and that $t_{j,\zeta}^{-1} = t_{j,\zeta^{-1}}$. Since ρ is contractive, it follows that $\rho(t_{j,\zeta})$ is a bijective isometry for all j and ζ . So $\rho(t_{j,\zeta})$ is spatial by Lemma 6.16. Since $\rho(t_{j,\zeta})$ is bijective, its spatial system has the form $(X, X, S_{j,\zeta}, g_{j,\zeta})$. Clearly $S_{j,1} = \text{id}_{\mathcal{B}}$. So Lemma 6.22 implies that $S_{j,-1} = \text{id}_{\mathcal{B}}$. Therefore $\rho(t_{j,-1}) = m(g_{j,-1})$. It follows that there is a unital algebra homomorphism

$$\varphi: l^\infty(\{1, 2, \dots, d\}) \rightarrow L^\infty(X, \mu)$$

such that $\varphi(\chi_{\{j\}}) = \frac{1}{2}(1 - g_{j,-1})$ for $j = 1, 2, \dots, d$. So there is a partition $X = \coprod_{j=1}^d X_j$ such that for $j = 1, 2, \dots, d$ we have $g_{j,-1} = \chi_{X_j}$ almost everywhere $[\mu]$. Condition (5) now follows. \square

We now turn to representations of L_d and C_d . In Definition 2.12, we defined what it means for a representation to be contractive on generators, forward isometric (on the s_j), and strongly forward isometric (in addition, the linear combinations of the s_j are sent to scalar multiples of isometries). We now introduce several further properties of a representation.

Definition 7.4. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3). Let $p \in [1, \infty]$, let (X, \mathcal{B}, μ) be a σ -finite measure space, and let $\rho: A \rightarrow L(L^p(X, \mu))$ be a representation.

- (1) We say that ρ is *disjoint* if there exist disjoint sets $X_1, X_2, \dots, X_d \subset X$ (if $A = L_\infty$, disjoint sets $X_1, X_2, \dots \subset X$) such that $\text{ran}(\rho(s_j)) \subset L^p(X_j, \mu)$ for all j .
- (2) We say that ρ is *spatial* if for each j , the operators $\rho(s_j)$ and $\rho(t_j)$ are spatial partial isometries, with $\rho(t_j)$ being the reverse of $\rho(s_j)$ in the sense of Definition 6.13.

The definition of a spatial representation is quite strong. For $p \neq 2, \infty$, we will see that representations satisfying some much weaker conditions are necessarily spatial.

The representations of L_d in Examples 3.1 and 3.18 are spatial and disjoint. The representations in Examples 3.7 and 3.8 are disjoint, but not spatial; in fact, they are neither contractive on generators nor forward isometric. The representations in Examples 3.10, 3.11, 3.12, and 3.13 are contractive on generators but not disjoint and not spatial. The representations of L_∞ in Examples 3.3, 3.4, and 3.19 are spatial and disjoint. The one in Example 3.5 is disjoint, but it is not spatial. (For $j \in \mathbb{Z}_{>0}$, the operator $\rho(s_j)$ is a spatial partial isometry, but $\rho(t_j)$ is not.) The representation of L_∞ of Example 3.16 is disjoint but not spatial, since the images of neither the s_j nor the t_j are spatial.

Lemma 7.5. Let A be any of L_d , C_d , or L_∞ . Let $p \in [1, \infty]$. Then there is an injective spatial representation of A on $l^p(\mathbb{Z}_{>0})$.

Proof. For L_d (including $d = \infty$), we can use any spatial representation, say that of Example 3.1 for $d < \infty$ and that of Example 3.3 for $d = \infty$, because L_d is simple (Theorem 2 of [21] for $d < \infty$ and Example 3.1(ii) of [3] for $d = \infty$). For C_d , we use the representation π of Example 3.2. We check injectivity. With ρ as in Example 3.1 (for $d + 1$ in place of d) and $\iota_{d,d+1}$ as in Lemma 1.5, we have $\pi = \rho \circ \iota_{d,d+1}$. Moreover, $\iota_{d,d+1}$ is injective by Lemma 1.5 and we saw above that ρ is injective. \square

We give two further conditions, also motivated by Equation (2.3) (before Definition 2.12) for the C*-algebras and by the analogous equation for the adjoints.

Definition 7.6. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2) or L_∞ (Definition 1.3). Let E be a nonzero Banach space, and let $\rho: A \rightarrow L(E)$ be a representation. Let $p \in [1, \infty]$.

- (1) We say that ρ is *p-standard* on $\text{span}(s_1, s_2, \dots, s_d)$ if, following Definition 1.13, the map $\lambda \mapsto \rho(s_\lambda)$ from \mathbb{C}^d to $L(E)$ is isometric from $l^p(\{1, 2, \dots, d\})$ to $L(E)$. (In the case $A = L_\infty$ and $p \neq \infty$, we say that ρ is *p-standard* on $\text{span}(s_1, s_2, \dots)$ if the map $\lambda \mapsto \rho(s_\lambda)$ from \mathbb{C}^∞ to $L(E)$ extends to an isometric map from $l^p(\mathbb{Z}_{>0})$ to $L(E)$. For $p = \infty$, we use $C_0(\mathbb{Z}_{>0})$ in place of l^∞ .)
- (2) Let $q \in [1, \infty]$ be the conjugate exponent, that is, $\frac{1}{p} + \frac{1}{q} = 1$. We say that ρ is *p-standard* on $\text{span}(t_1, t_2, \dots, t_d)$ if, following Definition 1.13, the map $\lambda \mapsto \rho(t_\lambda)$ from \mathbb{C}^d to $L(E)$ is isometric from $l^q(\{1, 2, \dots, d\})$ to $L(E)$. (In the case $A = L_\infty$ and $p \neq 1$, we say that ρ is *p-standard* on $\text{span}(t_1, t_2, \dots)$ if the map $\lambda \mapsto \rho(t_\lambda)$ from \mathbb{C}^∞ to $L(E)$ extends to an isometric map from $l^q(\mathbb{Z}_{>0})$ to $L(E)$. For $p = 1$, we use $C_0(\mathbb{Z}_{>0})$ in place of l^∞ .)

We will see in Theorem 7.7 below that a spatial representation of L_d on a space of the form $L^p(X, \mu)$ is necessarily *p-standard* on both $\text{span}(s_1, s_2, \dots, s_d)$ and

$\text{span}(t_1, t_2, \dots, t_d)$. Example 3.8 shows that a representation which is p -standard on $\text{span}(s_1, s_2, \dots, s_d)$ need not be spatial, or even contractive on generators. If in that example one instead defines

$$\pi(s_j) = \rho(s_j) \oplus_p 2\rho(s_j) \quad \text{and} \quad \pi(t_j) = \rho(t_j) \oplus_p \frac{1}{2}\rho(t_j),$$

the resulting representation is p -standard on $\text{span}(t_1, t_2, \dots, t_d)$ but not spatial. One could fix the difficulty with Example 3.8 by incorporating strongly forward isometric in the definition of p -standard on $\text{span}(s_1, s_2, \dots, s_d)$, but we don't know what the analogous fix for p -standard on $\text{span}(t_1, t_2, \dots, t_d)$ should be.

Theorem 7.7. Let $d \in \mathbb{Z}_{>0}$, let L_d be as in Definition 1.1, let $p \in (1, \infty) \setminus \{2\}$, let (X, \mathcal{B}, μ) be a σ -finite measure space, and let $\rho: L_d \rightarrow L(L^p(X, \mu))$ be a representation. Then the following are equivalent:

- (1) ρ is spatial.
- (2) For $j = 1, 2, \dots, d$, the operator $\rho(s_j)$ is a spatial partial isometry.
- (3) For $j = 1, 2, \dots, d$, the operator $\rho(t_j)$ is a spatial partial isometry.
- (4) ρ is forward isometric and the restriction of ρ to $\text{span}((s_j t_k)_{j,k=1}^d) \cong M_d$ (see Lemma 1.11) is a spatial representation of M_d in the sense of Definition 7.1.
- (5) ρ is contractive on generators and the restriction of ρ to $\text{span}((s_j t_k)_{j,k=1}^d)$ is a spatial representation of M_d .
- (6) ρ is forward isometric and disjoint.
- (7) ρ is contractive on generators and disjoint.
- (8) ρ is strongly forward isometric and disjoint.
- (9) ρ is p -standard on $\text{span}(s_1, s_2, \dots, s_d)$ and is strongly forward isometric.
- (10) ρ is p -standard on $\text{span}(t_1, t_2, \dots, t_d)$ and the representation ρ' of Lemma 2.21 is strongly forward isometric.
- (11) $(\rho(s_1) \ \rho(s_2) \ \cdots \ \rho(s_d))$ is a row isometry, in the sense that, using the notation of Remark 2.15, it defines an isometric linear map

$$L^p(X, \mu) \oplus_p L^p(X, \mu) \oplus_p \cdots \oplus_p L^p(X, \mu) \rightarrow L^p(X, \mu).$$

- (12) ρ is forward isometric and for $j = 1, 2, \dots, d$ there is $X_j \subset X$ such that $\text{ran}(\rho(s_j)) = L^p(X_j, \mu)$.
- (13) ρ is contractive on generators and for $j = 1, 2, \dots, d$ there is $X_j \subset X$ such that $\text{ran}(\rho(s_j)) = L^p(X_j, \mu)$.
- (14) The representation ρ' of Lemma 2.21 is spatial.

For $p = 1$, all the conditions except (3), (10), and (14) are equivalent, and the other conditions imply these three.

Various remarks are in order. First, when $p = 1$, we do not know whether the conditions (3), (10), and (14) imply the representation is spatial. (The last two of these are the ones which for $p = 1$ involve representations on $L^\infty(X, \mu)$.)

Second, some of the equivalences fail for representations of L_∞ and C_d . Assume $p \neq 1$. Then the representation of L_∞ of Example 3.5 satisfies (2), (6), (9), (11),

and (12), but not (3) and hence not (1). The same is true for the restriction to C_d , using the map $\iota_{d,\infty}$ of Lemma 1.5. For $p \in (1, \infty) \setminus \{2\}$, the representation of L_∞ of Example 3.16, and its dual, both satisfy (6), (7), (8), (9), (10), and (11), but none of (2), (3), (12), (13), or (14). The same is true for the representation of C_{d_0} of Example 3.14.

Third, various other implications one might hope for fail. Example 3.10 shows that contractive on generators does not imply that the restriction to M_d is spatial. Example 3.7 shows that the restriction to M_d being spatial does not imply that the whole representation is spatial. If ρ is p -standard on $\text{span}(s_1, s_2, \dots, s_d)$, it does not follow that ρ is p -standard on $\text{span}(t_1, t_2, \dots, t_d)$, or that ρ' is q -standard on $\text{span}(s_1, s_2, \dots, s_d)$, by Example 3.8.

A spatial representation of L_d must be strongly forward isometric, since that is part of Theorem 7.7(8). The converse is not true; Example 3.10 is a counterexample.

Next, we recall from Theorem 7.2 that conditions (4) and (5) actually have many equivalent formulations. Similarly, condition (14) is equivalent to analogs on $L^q(X, \mu)$ of all the other conditions of Theorem 7.7.

If $p > 2$ then p -standard on $\text{span}(s_1, s_2, \dots, s_d)$ can be weakened to $\|\rho(s_\lambda)\| \leq \|\lambda\|_p$ in (9). (See Lemma 7.10(1).) If $p < 2$, then p -standard on $\text{span}(t_1, t_2, \dots, t_d)$ can be weakened to $\|\rho(t_\lambda)\| \leq \|\lambda\|_q$ in (10). (See Corollary 7.11(2).)

Finally, it is again essential that the representation be unital. Many of the conditions in Theorem 7.7 actually do make sense for nonunital representations, but the following example shows that they do not imply that the representation is spatial.

Example 7.8. Let the notation be as in Example 7.3, except take ρ_0 to be a spatial representation of L_d on $L^p(X_0, \mu_0)$. Set $\rho(a) = e \otimes \rho_0(a)$ for $a \in L_d$. Then ρ is not spatial, not even in a sense suitable for nonunital representations. However, it is disjoint, strongly forward isometric, contractive on generators, p -standard on $\text{span}(s_1, s_2, \dots, s_d)$, and isometric (although not spatial) on $\text{span}((s_j t_k)_{j,k=1}^d)$.

It is convenient to break the proof of Theorem 7.7 into several lemmas. Some of them hold in greater generality than needed. The first, in effect, shows that (1) implies all the other conditions.

Lemma 7.9. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3). Let $p \in [1, \infty]$, let (X, \mathcal{B}, μ) be a σ -finite measure space, and let $\rho: L_d \rightarrow L(L^p(X, \mu))$ be a spatial representation. Then:

- (1) ρ is contractive on generators.
- (2) ρ is strongly forward isometric.
- (3) ρ is disjoint.
- (4) ρ is p -standard on $\text{span}(s_1, s_2, \dots, s_d)$ (Definition 7.6(1)).
- (5) ρ is p -standard on $\text{span}(t_1, t_2, \dots, t_d)$ (Definition 7.6(2)).

- (6) $(\rho(s_1) \ \rho(s_2) \ \cdots \ \rho(s_d))$ is a row isometry, in the sense described in Theorem 7.7(11).
- (7) For each j there is $X_j \subset X$ such that $\text{ran}(\rho(s_j)) = L^p(X_j, \mu)$.
- (8) If $A = L_d$ with $d \neq \infty$, the restriction of ρ to $\text{span}((s_j t_k)_{j,k=1}^d) \cong M_d$ (see Lemma 1.11) is a spatial representation of M_d in the sense of Definition 7.1.
- (9) If $p \neq \infty$, the representation ρ' of Lemma 2.21 is spatial.

Proof. Part (1) follows from Lemma 6.5(2), part (7) follows from Lemma 6.15, and part (9) follows from Lemma 6.21. Part (8) follows from the fact (Lemma 6.17) that the product of spatial partial isometries is again a spatial partial isometry.

To prove part (3), let X_j be as in part (7). Suppose $\mu(X_j \cap X_k) \neq 0$. Then $\rho(t_j)\rho(s_k)$ is a nonzero spatial partial isometry by Lemma 6.17, contradicting the relation (1.2) or (1.5), as appropriate, in the definition of A . Part (3) follows.

We prove (2), (4), and (6) together. Let

$$s = (\rho(s_1) \ \rho(s_2) \ \cdots \ \rho(s_d)) \\ \in L(L^p(X, \mu) \oplus_p L^p(X, \mu) \oplus_p \cdots \oplus_p L^p(X, \mu), L^p(X, \mu)).$$

Thus

$$s(\xi_1, \xi_2, \dots, \xi_d) = \sum_{j=1}^d \rho(s_j)\xi_j$$

for $\xi_1, \xi_2, \dots, \xi_d \in L^p(X, \mu)$. We already proved that ρ is disjoint, so

$$\rho(s_1)\xi_1, \rho(s_2)\xi_2, \dots, \rho(s_d)\xi_d$$

have disjoint supports. Since $\|\rho(s_j)\xi_j\| = \|\xi_j\|$ by Lemma 6.5(4), it follows from Remark 2.7 that

$$(7.1) \quad \|s(\xi_1, \xi_2, \dots, \xi_d)\|_p^p = \sum_j \|\rho(s_j)\xi_j\|_p^p = \sum_j \|\xi_j\|_p^p = \|(\xi_1, \xi_2, \dots, \xi_d)\|_p^p.$$

This proves (6). Now let $\lambda \in \mathbb{C}^d$ and let $\xi \in L^p(X, \mu)$. In (7.1), take $\xi_j = \lambda_j \xi$ for $j = 1, 2, \dots, d$, getting

$$\|\rho(s_\lambda)\xi\|_p^p = \sum_j |\lambda_j|^p \|\xi\|_p^p = \|\lambda\|_p^p \|\xi\|_p^p.$$

This equation implies both (2) and (4).

It remains to prove (5). Let $q \in [1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. For $\gamma \in \mathbb{C}^d$ (possibly $d = \infty$, in which case we follow the convention of Definition 1.13), define $\omega_\gamma: \mathbb{C}^d \rightarrow \mathbb{C}$ by $\omega_\gamma(\lambda) = \sum_{j=1}^d \gamma_j \lambda_j$.

We show that for $\gamma \in \mathbb{C}^d$, we have $\|\rho(t_\gamma)\| \geq \|\gamma\|_q$. Let $\varepsilon > 0$, and choose (using the usual pairing between l^p and l^q) an element $\lambda \in \mathbb{C}^d$ such that $\|\lambda\|_p = 1$ and

$|\omega_\gamma(\lambda)| > \|\gamma\|_q - \varepsilon$. By part (4) (already proved), we have $\|\rho(s_\lambda)\| = 1$. Using Lemma 1.14 at the third step, we then get

$$\|\rho(t_\gamma)\| = \|\rho(t_\gamma)\| \cdot \|\rho(s_\lambda)\| \geq \|\rho(t_\gamma)\rho(s_\lambda)\| = \|\omega_\gamma(\lambda) \cdot 1\| > \|\gamma\|_q - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the desired conclusion follows.

We now show the reverse. Let $\gamma \in \mathbb{C}^d$ and let $\xi \in L^p(X, \mu)$. Let the disjoint sets X_j be as in (3) (already proved). Set $\xi_j = \xi|_{X_j}$. Since $\rho(t_j)$ is a spatial partial isometry with domain support X_j , Lemma 6.5(2) gives $\|\rho(t_j)\xi\|_p = \|\xi_j\|_p$. Now, using Hölder's inequality at the third step and Remark 2.7 at the last step, we get

$$\begin{aligned} \|\rho(t_\gamma)\xi\|_p &\leq \sum_{j=1}^d |\gamma_j| \cdot \|\rho(t_j)\xi\|_p \\ &= \sum_{j=1}^d |\gamma_j| \cdot \|\xi_j\|_p \leq \|\gamma\|_q \left(\sum_{j=1}^d \|\xi_j\|_p^p \right)^{1/p} = \|\gamma\|_q \|\xi\|_p. \end{aligned}$$

So $\|\rho(t_\gamma)\| \leq \|\gamma\|_q$. □

Lemma 7.10. Let A be any of L_d , C_d , or L_∞ . Let $p \in [1, \infty) \setminus \{2\}$, let (X, \mathcal{B}, μ) be a σ -finite measure space, and let $\rho: L_d \rightarrow L(L^p(X, \mu))$ be a representation. Adopt the notational conventions of Definition 7.6 in case $A = L_\infty$.

- (1) Suppose $p > 2$. If ρ is forward isometric and for all $\lambda \in \mathbb{C}^d$ we have $\|\rho(s_\lambda)\| \leq \|\lambda\|_p$, then ρ is disjoint.
- (2) Suppose $p < 2$. If ρ is strongly forward isometric and is p -standard on $\text{span}(s_1, s_2, \dots, s_d)$, then ρ is disjoint.

Proof. For each j , Theorem 6.9 implies that $\rho(s_j)$ is semispatial. Let S_j be its semispatial realization and let X_j be its range support.

Let $\xi \in L^p(X, \mu)$. We now claim that, under either set of hypotheses, $\rho(s_j)\xi$ and $\rho(s_k)\xi$ have essentially disjoint supports for $j \neq k$.

Assume the hypotheses of (1). Let $\delta_1, \delta_2, \dots, \delta_d$ be the standard basis vectors in \mathbb{C}^d . Use the hypotheses with the choices $\lambda = \delta_j + \delta_k$ and $\lambda = \delta_j - \delta_k$ at the first step, and the fact that $\rho(s_j)$ and $\rho(s_k)$ are isometric at the second step, to get

$$(7.2) \quad \begin{aligned} \|\rho(s_j)\xi + \rho(s_k)\xi\|_p^p + \|\rho(s_j)\xi - \rho(s_k)\xi\|_p^p &\leq 2\|\xi\|_p^p + 2\|\xi\|_p^p \\ &= 2\|\rho(s_j)\xi\|_p^p + 2\|\rho(s_k)\xi\|_p^p. \end{aligned}$$

By Corollary 2.1 of [18], we must have in fact

$$\|\rho(s_j)\xi + \rho(s_k)\xi\|_p^p + \|\rho(s_j)\xi - \rho(s_k)\xi\|_p^p = 2\|\rho(s_j)\xi\|_p^p + 2\|\rho(s_k)\xi\|_p^p,$$

and it then follows, by the condition for equality in Corollary 2.1 of [18], that $\rho(s_j)\xi$ and $\rho(s_k)\xi$ have essentially disjoint supports.

Now assume instead the hypotheses of (2). In this case,

$$\|\rho(s_j)\xi + \rho(s_k)\xi\|_p^p + \|\rho(s_j)\xi - \rho(s_k)\xi\|_p^p = \|\delta_j + \delta_k\|_p^p \|\xi\|_p^p + \|\delta_j - \delta_k\|_p^p \|\xi\|_p^p,$$

so we have equality at the first step in (7.2). The condition for equality in Corollary 2.1 of [18] therefore implies that $\rho(s_j)\xi$ and $\rho(s_k)\xi$ have essentially disjoint supports. The claim is proved.

Since μ is σ -finite, there is $\xi \in L^p(X, \mu)$ such that $\xi(x) > 0$ for all $x \in X$. It follows from the definition of a semispacial partial isometry, Proposition 5.6(6), and Lemma 6.1(1) that $\rho(s_j)\xi$ is nonzero almost everywhere on X_j , and similarly that $\rho(s_k)\xi$ is nonzero almost everywhere on X_k . Therefore X_j and X_k are essentially disjoint. The conclusion of the lemma follows. \square

Corollary 7.11. Let A be any of L_d, C_d , or L_∞ . Let $p \in (1, \infty) \setminus \{2\}$, let (X, \mathcal{B}, μ) be a σ -finite measure space, and let $\rho: L_d \rightarrow L(L^p(X, \mu))$ be a representation. Adopt the notational conventions of Definition 7.6 in case $A = L_\infty$. Let $q \in (1, \infty) \setminus \{2\}$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, and let $\rho': A \rightarrow L(L^q(X, \mu))$ be as in Lemma 2.21.

- (1) Suppose $p > 2$. If ρ is p -standard on $\text{span}(t_1, t_2, \dots, t_d)$ and ρ' is strongly forward isometric, then ρ' is disjoint.
- (2) Suppose $p < 2$. If for all $\lambda \in \mathbb{C}^d$ we have $\|\rho(t_\lambda)\| \leq \|\lambda\|_q$, and ρ' is forward isometric, then ρ' is disjoint.

Proof. Under the hypotheses of (1), we have $q < 2$. Also, by duality, for $\lambda \in \mathbb{C}^d$ we have

$$\|\rho'(s_\lambda)\| = \|\rho(t_\lambda)\| = \|\lambda\|_q.$$

Therefore ρ' satisfies the hypotheses of Lemma 7.10(2), with q in place of p , so is disjoint.

A similar argument shows that if ρ satisfies the hypotheses of (2), then ρ' is disjoint by Lemma 7.10(1). \square

Lemma 7.12. Let $d \in \mathbb{Z}_{>0}$, let $p \in [1, \infty) \setminus \{2\}$, let (X, \mathcal{B}, μ) be a σ -finite measure space, and let $\rho: L_d \rightarrow L(L^p(X, \mu))$ be a representation which is disjoint (Definition 7.4(1)) and forward isometric (Definition 2.12(2)). Then ρ is spatial (Definition 7.4(2)).

Proof. Let $X_1, X_2, \dots, X_d \subset X$ be the sets of Definition 7.4(1), so that $\text{ran}(\rho(s_j)) \subset L^p(X_j, \mu)$ for all j . Since $\sum_{j=1}^d \rho(s_j)\rho(t_j) = 1$, the closed linear span of the ranges of the $\rho(s_j)$ is all of $L^p(X, \mu)$, so (up to a set of measure zero, which we may ignore) $\bigcup_{j=1}^d X_j = X$ and $\text{ran}(\rho(s_j)) = L^p(X_j, \mu)$. It follows from Lemma 6.16 that $\rho(s_j)$ is a spatial isometry for $j = 1, 2, \dots, d$. Using Lemma 6.12(3), Definition 6.13, and $\prod_{j=1}^d X_j = X$, we see that there is a representation $\sigma: L_d \rightarrow L(L^p(X, \mu))$ such that, for $j = 1, 2, \dots, d$, we have $\sigma(s_j) = \rho(s_j)$ and $\sigma(t_j)$ is the reverse of $\rho(s_j)$. Clearly σ is spatial. Lemma 2.11 implies that $\rho = \sigma$. \square

Proof of Theorem 7.7. We begin by observing that (1) implies all the other conditions. For (2) and (3), this is trivial. For all the others except (10), this follows from the various conclusions of Lemma 7.9. To get (10), we must also use Lemma 7.9(9) to see that ρ' is spatial, and then apply Lemma 7.9(2) to ρ' .

We now prove that all the other conditions imply (1) (omitting (3), (10), and (14) when $p = 1$), mostly via one of (2), (6), or (14). That (6) implies (1) is Lemma 7.12. To see that (14) implies (1) for $p \neq 1$, we apply Lemma 7.9(9) to ρ' and use $(\rho')' = \rho$. We prove that (2) implies (6) (and hence also implies (1)). For $j = 1, 2, \dots, d$, let $X_j \subset X$ be the range support of $\rho(s_j)$. Then $\text{ran}(\rho(s_j)) = L^p(X_j, \mu)$ because $\rho(s_j)$ is spatial. Therefore $\text{ran}(\rho(s_j t_j)) = L^p(X_j, \mu)$. If now $j \neq k$, then $\rho(s_j t_j)$ and $\rho(s_k t_k)$ are idempotents whose product is zero, so $L^p(X_j, \mu) \cap L^p(X_k, \mu) = \{0\}$. Disjointness of ρ follows, and the other part of (6) is immediate.

For $p \neq 1$, we prove that (3) implies (14). It follows from Lemma 6.21 that ρ' satisfies (2). We have already proved that (2) implies (1), so we conclude that ρ' is spatial, which is (14).

To prove that (4) and (5) imply (6), we use the implication from (1) to (5) in Theorem 7.2 to conclude that ρ is disjoint. If we start with (5), we also use Remark 2.13.

That (7) implies (6) is Remark 2.13, and that (8) implies (6) is clear.

That (9) implies (6) follows from Lemma 7.10. We prove that if $p \neq 1$ then (10) implies (14). It follows from Corollary 7.11 that ρ' is disjoint. We have already proved that (6) implies (1), so we conclude that ρ' is spatial, which is (14).

We now prove that (11) implies (9). Since we already proved that (9) implies (6), this will show that (11) implies (6). Let $\lambda \in \mathbb{C}^d$ and let $\xi \in L^p(X, \mu)$. Using the assumption at the second step, we get

$$\begin{aligned} \|\rho(s_\lambda)\xi\|_p &= \left\| \left(\rho(s_1) \ \rho(s_2) \ \cdots \ \rho(s_d) \right) (\lambda_1\xi, \lambda_2\xi, \dots, \lambda_d\xi) \right\|_p \\ &= \left\| (\lambda_1\xi, \lambda_2\xi, \dots, \lambda_d\xi) \right\|_p = \|\lambda\|_p \|\xi\|_p. \end{aligned}$$

This equation implies both parts of condition (9).

That (12) implies (2) is immediate from Lemma 6.16. To see that (13) implies (2), use in addition Remark 2.13. \square

8. SPATIAL REPRESENTATIONS GIVE ISOMETRIC ALGEBRAS

In this section, we prove that for $d < \infty$, any two spatial representations of L_d (in the sense of Definition 7.4(2)) on $L^p(X, \mu)$, for the same value of p , give isometrically isomorphic Banach algebras. The main technical tools are the notion of a free representation (Definition 8.1) and the spatial realizations of spatial isometries.

Definition 8.1. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3). Let (X, \mathcal{B}, μ) be a σ -finite measure space, let $p \in [1, \infty]$, and let $\rho: A \rightarrow L(L^p(X, \mu))$ be a representation.

- (1) We say that ρ is *free* if there is a partition $X = \coprod_{m \in \mathbb{Z}} E_m$ such that for all $m \in \mathbb{Z}$ and all j , we have

$$\rho(s_j)(L^p(E_m, \mu)) \subset L^p(E_{m+1}, \mu) \quad \text{and} \quad \rho(t_j)(L^p(E_m, \mu)) \subset L^p(E_{m-1}, \mu).$$

- (2) We say that ρ is *approximately free* if for every $N \in \mathbb{Z}_{>0}$ there is $n \geq N$ and a partition $X = \coprod_{m=0}^{n-1} E_m$ such that for $m = 0, 1, \dots, n-1$ and all j , taking $E_n = E_0$ and $E_{-1} = E_{n-1}$, we have

$$\rho(s_j)(L^p(E_m, \mu)) \subset L^p(E_{m+1}, \mu) \quad \text{and} \quad \rho(t_j)(L^p(E_m, \mu)) \subset L^p(E_{m-1}, \mu).$$

When dealing with approximately free representations, we always take the index m in $E_m \bmod n$.

To produce free representations, we follow Lemmas 2.18, 2.19, and 2.20, producing representations of the form $(\rho(\cdot) \otimes 1)^{1 \otimes u}$ for suitable u . To simplify the notation and avoid conflict, we will abbreviate this representation to ρ_u .

Lemma 8.2. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3). Let $p \in [1, \infty)$. Let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces. Let $\rho: A \rightarrow L(L^p(X, \mu))$ be a representation, and let $u \in L(L^p(Y, \nu))$ be invertible. Then there exists a unique representation $\rho_u: A \rightarrow L(L^p(X \times Y, \mu \times \nu))$ such that for all j we have (using the notation of Theorem 2.16(4))

$$\rho_u(s_j) = \rho(s_j) \otimes u \quad \text{and} \quad \rho_u(t_j) = \rho(t_j) \otimes u^{-1}.$$

This construction has the following properties:

- (1) If $a \in A$ is homogeneous of degree k (with respect to the \mathbb{Z} -grading of Proposition 1.7), then $\rho_u(a) = \rho(a) \otimes u^k$.
- (2) If u is isometric, $p \neq 2$, and ρ is spatial (Definition 7.4(2)), then ρ_u is spatial.
- (3) If there is a partition $Y = \coprod_{m \in \mathbb{Z}} F_m$ such that $u(L^p(F_m, \nu)) = L^p(F_{m+1}, \nu)$ for all $m \in \mathbb{Z}$, then ρ_u is free in the sense of Definition 8.1(1).

Proof. Existence and uniqueness of ρ_u follow from Lemma 2.17, combined with the appropriate one of Lemmas 2.18, 2.19, and 2.20, taking $v = u^{-1}$ if $A = C_d$ or L_∞ .

It suffices to prove part (1) when a is a product of generators s_j and t_j . By Lemma 1.10(4), we may assume that there are words $\alpha, \beta \in W_\infty^d$ such that $a = s_\alpha t_\beta$. Then $\deg(a) = l(\alpha) - l(\beta)$ (by Lemma 1.10(2)), and one checks directly that

$$\rho_u(s_\alpha t_\beta) = \rho(s_\alpha t_\beta) \otimes u^{l(\alpha) - l(\beta)}.$$

We prove (2). Lemma 6.16 implies that u is spatial, and Lemma 6.12(4) implies that u^{-1} is the reverse of u . Now apply Lemma 6.20 to conclude that $\rho(s_j \otimes u)$ is spatial with reverse $\rho(t_j) \otimes u^{-1}$.

To prove (3), we use the partition

$$X \times Y = \coprod_{m \in \mathbb{Z}} X \times Y_m.$$

The sets $X \times Y_m$ clearly have the required properties. \square

The construction of Lemma 8.2 preserves various other properties of representations, but we will not need this information.

Proposition 8.3. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3). Let $p \in [1, \infty)$, let (X, \mathcal{B}, μ) be a σ -finite measure space, and let $\rho: A \rightarrow L(L^p(X, \mu))$ be a representation. Let $u \in L(l^p(\mathbb{Z}))$ be the bilateral shift, $(u\eta)(m) = \eta(m-1)$ for $\eta \in l^p(\mathbb{Z})$. Let ρ_u be as in Lemma 8.2. Then for every $a \in A$ we have $\|\rho_u(a)\| \geq \|\rho(a)\|$.

Proof. Let $a \in A$. Let $\varepsilon > 0$; we show that $\|\rho_u(a)\| \geq \|\rho(a)\| - \varepsilon$.

Recall the \mathbb{Z} -grading of Proposition 1.7. Choose $N_0 \in \mathbb{Z}_{>0}$ such that there are homogeneous elements

$$a_{-N_0}, a_{-N_0+1}, \dots, a_{N_0-1}, a_{N_0} \in A$$

such that $\deg(a_k) = k$ for all k and $a = \sum_{k=-N_0}^{N_0} a_k$. Choose $\zeta \in L^p(X, \mu)$ such that

$$\|\zeta\|_p = 1 \quad \text{and} \quad \|\rho(a)\zeta\| > \|\rho(a)\| - \frac{1}{2}\varepsilon.$$

Set $r = \|\rho(a)\zeta\|$. If $r \leq \frac{1}{2}\varepsilon$, then $\|\rho(a)\| < \varepsilon$, and we are done. Otherwise, choose $N \in \mathbb{Z}_{>0}$ such that

$$N > \frac{N_0 r^p}{r^p - (r - \frac{1}{2}\varepsilon)^p}.$$

Let ν be counting measure on \mathbb{Z} . We identify $L^p(X \times \mathbb{Z}, \mu \times \nu)$ with the space of all sequences $\xi = (\xi_m)_{m \in \mathbb{Z}}$ with $\xi_m \in L^p(X, \mu)$ for all m and such that $\sum_{m \in \mathbb{Z}} \|\xi_m\|_p^p < \infty$, with

$$\|\xi\|_p = \left(\sum_{m \in \mathbb{Z}} \|\xi_m\|_p^p \right)^{1/p}.$$

Now define $\xi \in L^p(X \times \mathbb{Z}, \mu \times \nu)$ by

$$\xi_m = \begin{cases} 0 & |n| > N \\ (2N+1)^{-1/p} \zeta & |n| \leq N. \end{cases}$$

Then $\|\xi\|_p^p = 1$.

Set $\eta = \rho_u(a)\xi$. We have $[(1 \otimes u)\xi]_m = \xi_{m-1}$ for all $m \in \mathbb{Z}$. Using Lemma 8.2(1), for all $m \in \mathbb{Z}$ with $-N + N_0 \leq m \leq N - N_0$ we get

$$\eta_m = \sum_{k=-N_0}^{N_0} (2N+1)^{-1/p} \rho(a_k) \xi_{m-k}$$

$$= (2N + 1)^{-1/p} \sum_{k=-N_0}^{N_0} \rho(a_k)\zeta = (2N + 1)^{-1/p} \rho(a)\zeta.$$

There are $2N - 2N_0 + 1$ such values of m . It follows that

$$\begin{aligned} \|\eta\|_p^p &\geq (2N - 2N_0 + 1)(2N + 1)^{-1} \|\rho(a_k)\zeta\|_p^p = \left(\frac{2N - 2N_0 + 1}{2N + 1} \right) r^p \\ &> \left(1 - \frac{N_0}{N} \right) r^p > \left(1 - \frac{r^p - (r - \frac{1}{2}\varepsilon)^p}{r^p} \right) r^p = (r - \frac{1}{2}\varepsilon)^p. \end{aligned}$$

So

$$\|\eta\| > r - \frac{1}{2}\varepsilon = \|\rho(a)\zeta\| - \frac{1}{2}\varepsilon > \|\rho(a)\| - \varepsilon.$$

This shows that $\|\rho_u(a)\| > \|\rho(a)\| - \varepsilon$. \square

Corollary 8.4. Let A be any of L_d , C_d , or L_∞ . Let $p \in [1, \infty) \setminus \{2\}$. Then there exists an injective free spatial representation of A on $l^p(\mathbb{Z}_{>0})$.

Proof. By Lemma 7.5, there is an injective spatial representation ρ of A on $l^p(\mathbb{Z}_{>0})$. Let ρ_u be the representation of A on $l^p(\mathbb{Z}_{>0}) \otimes_p l^p(\mathbb{Z}) \cong l^p(\mathbb{Z}_{>0})$ of Proposition 8.3. The inequality $\|\rho_u(a)\| \geq \|\rho(a)\|$ for all $a \in A$ implies that ρ_u is also injective. Moreover, ρ_u is spatial by Lemma 8.2(2) and free by Lemma 8.2(3). \square

We want to prove an inequality in the opposite direction from that of Proposition 8.3. We need a lemma.

Lemma 8.5. Let (X, \mathcal{B}, μ) be a σ -finite measure space, with $\mu \neq 0$. Let $d \in \{2, 3, 4, \dots, \infty\}$, let $X_1, X_2, \dots, X_d \subset X$ ($X_1, X_2, \dots \subset X$ if $d = \infty$) be disjoint measurable sets, and for each j let S_j be an injective measurable set transformation (Definition 5.4) from (X, \mathcal{B}, μ) to $(X_j, \mathcal{B}|_{X_j}, \mu|_{X_j})$. Then for every $n \in \mathbb{Z}_{>0}$ there exists $E \in \mathcal{B}$ with $\mu(E) \neq 0$ such that the elements

$$S_{\alpha(1)} \circ S_{\alpha(2)} \circ \dots \circ S_{\alpha(m)}([E]) \in \mathcal{B}/\mathcal{N}(\mu),$$

for all $m \in \{0, 1, 2, \dots, d\}$ and all words $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(m)) \in W_m^d$ (see Notation 1.8) are disjoint in the sense of Definition 4.4.

Proof. In this proof, we will write expressions like $S_j(E)$ for measurable subsets $E \subset X$, meaning that $S_j(E)$ is taken to be some measurable subset of X whose image in $\mathcal{B}/\mathcal{N}(\mu)$ is $S_j([E])$ in the sense of Definition 4.13. We remember that such a set is only defined up to sets of measure zero. Also, disjointness and containment of subsets of X will only be up to sets of measure zero. No problem will arise, because we only deal with countably many subsets of X , and we can therefore make the conclusion exact at the end by deleting a set of measure zero.

By analogy with Notation 1.9, for a word $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(m)) \in W_m^d$, we define

$$S_\alpha = S_{\alpha(1)} \circ S_{\alpha(2)} \circ \dots \circ S_{\alpha(m)},$$

which is a measurable set transformation from (X, \mathcal{B}, μ) to a suitable subset of X . We take $S_\emptyset = \text{id}_{\mathcal{B}/\mathcal{N}(\mu)}$. As with products of generators of L_d , if $\alpha, \beta \in W_\infty^d$ and $\alpha\beta$ is their concatenation, then $S_\alpha \circ S_\beta = S_{\alpha\beta}$.

We first claim that for all $m \in \mathbb{Z}_{>0}$ and all $\alpha, \beta \in W_m^d$, we have $S_\alpha(X) \cap S_\beta(X) = \emptyset$. To see this, let k be the least integer such that $\alpha(k) \neq \beta(k)$. Set

$$\alpha_0 = (\alpha(k), \alpha(k+1), \dots, \alpha(m)), \quad \beta_0 = (\beta(k), \beta(k+1), \dots, \beta(m)),$$

and

$$\gamma = (\alpha(1), \alpha(2), \dots, \alpha(k-1)).$$

Then $\gamma\alpha_0 = \alpha$ and $\gamma\beta_0 = \beta$. The sets $S_{\alpha_0}(X)$ and $S_{\beta_0}(X)$ are disjoint because they are contained in the disjoint sets $X_{\alpha(k)}$ and $X_{\beta(k)}$. It now follows from Lemma 4.10(2) that $(S_\gamma \circ S_{\alpha_0})(X)$ and $(S_\gamma \circ S_{\beta_0})(X)$ are disjoint, which implies the claim.

Our second claim is that if $D \subset X$ and $n \in \mathbb{Z}_{>0}$ satisfy $\mu(D) > 0$ and $D \cap S_1^n(D) = \emptyset$, then there exists a subset $F \subset D$ such that $\mu(F) \neq 0$ and such that for all $m \in \mathbb{Z}_{>0}$ such that m divides n , we have $F \cap S_1^m(F) = \emptyset$. To prove the claim, first observe that if for some fixed m we have $F \cap S_1^m(F) = \emptyset$, then for every subset $G \subset F$ we also have $G \cap S_1^m(G) = \emptyset$. Thus, if we prove the claim for just one divisor m of n , an induction argument will yield the claim as stated.

Define $F = D \setminus (D \cap S_1^m(D))$. Clearly $F \cap S_1^m(F) = \emptyset$. We need only show that $\mu(F) > 0$. Suppose not. Then (as usual, up to a set of measure zero) we have $D \subset S_1^m(D)$. By induction, we get $D \subset S_1^{km}(D)$ for all $k \in \mathbb{Z}_{>0}$. In particular, $D \subset S_1^n(D)$, which contradicts $D \cap S_1^n(D) = \emptyset$ and $\mu(D) > 0$. The claim is proved.

Our third claim is that for all $n \in \mathbb{Z}_{>0}$, we have

$$\mu(S^n(X) \setminus S^{2n}(X)) \neq 0.$$

Indeed, $X \setminus S_1^n(X)$ contains $X \setminus S_1(X)$, which contains $S_2(X)$, and $\mu(S_2(X)) > 0$ because S_2 is an injective measurable set transformation. Since S_1^n is an injective measurable set transformation, it follows that $\mu(S_1^n(X) \setminus S_1^n(X)) \neq 0$. This proves the claim.

Set $N = n!$. Define $E_0 = S_1^N(X) \setminus S_1^{2N}(X)$. Then $\mu(E_0) > 0$ by the third claim, and also $E_0 \cap S_1^N(E_0) = \emptyset$. The second claim therefore provides a subset $E \subset E_0$ such that $\mu(E) \neq 0$ and such that $E \cap S_1^m(E) = \emptyset$ for $m = 1, 2, \dots, n$.

We show that E satisfies the conclusion of the lemma. So let α and β be distinct words with length at most n . We have to prove that $S_\alpha(E) \cap S_\beta(E) = \emptyset$. Without loss of generality $l(\alpha) \geq l(\beta)$.

Suppose $l(\alpha) = l(\beta)$. Then, using the first claim at the second step,

$$S_\alpha(E) \cap S_\beta(E) \subset S_\alpha(X) \cap S_\beta(X) = \emptyset.$$

Suppose now $l(\alpha) > l(\beta)$. Set $m = l(\alpha)$ and $r = l(\beta)$. Define a new word γ with $l(\gamma) = m$ by

$$\gamma = (\beta(1), \beta(2), \dots, \beta(r), 1, 1, \dots, 1).$$

Thus $S_\gamma = S_\beta \circ S_1^{m-r}$.

We consider two cases, the first of which is $\gamma \neq \alpha$. Then, since $m - r \leq n \leq N$,

$$S_\alpha(E) \subset S_\alpha(X) \quad \text{and} \quad S_\beta(E) \subset S_\beta(S_1^N(X)) \subset S_\gamma(X).$$

So $S_\alpha(E) \cap S_\beta(E) = \emptyset$ by the first claim.

It remains to consider the case $\gamma = \alpha$. Then

$$S_\alpha(E) \cap S_\beta(E) = S_\beta(S_1^{m-r}(E)) \cap S_\beta(E).$$

Since $S_1^{m-r}(E) \cap E = \emptyset$, it follows from Lemma 4.10(2) that $S_\alpha(E) \cap S_\beta(E) = \emptyset$. \square

Proposition 8.6. Let $d \in \{2, 3, 4, \dots\}$, and let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces. Let $p \in [1, \infty) \setminus \{2\}$. Let $\rho: L_d \rightarrow L(L^p(X, \mu))$ be an approximately free spatial representation, and let $\varphi: L_d \rightarrow L(L^p(Y, \nu))$ be a spatial representation. Then for all $a \in L_d$, we have $\|\rho(a)\| \leq \|\varphi(a)\|$.

Proof. We adopt the same conventions with regard to measurable set transformations as described at the beginning of the proof of Lemma 8.5. In particular, set operations and relations are only up to sets of measure zero. Also, as there, for the measurable set transformations S_j and R_j defined below, we write

$$S_\alpha = S_{\alpha(1)} \circ S_{\alpha(2)} \circ \dots \circ S_{\alpha(m)},$$

and define R_α similarly.

By definition, the operators $\varphi(s_j)$ are spatial isometries. Therefore they have spatial systems (Y, Y_j, S_j, g_j) , in which S_j is a bijective measurable set transformation. In particular, $\varphi(s_j)(L^p(E, \mu)) = L^p(S_j(E), \mu)$. By Lemma 7.9(3), the sets Y_j are disjoint. Since $d < \infty$, we get $Y = \coprod_{j=1}^d Y_j$.

Similarly, the operators $\rho(s_j)$ have range supports, say, X_j , and $X = \coprod_{j=1}^d X_j$. Moreover, the spatial realization R_j of $\rho(s_j)$ is a bijective measurable set transformation from X to X_j .

Let $a \in L_d$. We want to prove that $\|\rho(a)\| \leq \|\varphi(a)\|$. By scaling, without loss of generality $\|\rho(a)\| = 1$. Apply Lemma 1.12, obtaining $N_0 \in \mathbb{Z}_{\geq 0}$, a finite set $F_0 \subset W_\infty^d$, and numbers $\lambda_{\alpha,\beta}^{(0)} \in \mathbb{C}$ for $\alpha \in F_0$ and $\beta \in W_{N_0}^d$, such that

$$a = \sum_{\alpha \in F_0} \sum_{\beta \in W_{N_0}^d} \lambda_{\alpha,\beta}^{(0)} s_\alpha t_\beta.$$

Set $N_1 = \max(\{l(\alpha) : \alpha \in F_0\})$. Let τ be any fixed word of length N_0 . Set $b = s_\tau a$. Set $F = \{\tau\alpha_0 : \alpha_0 \in F_0\}$. Thus, for all $\alpha \in F$, we have $N_0 \leq l(\alpha) \leq N_0 + N_1$. For $\alpha \in F$ and $\beta \in W_{N_0}^d$, write $\alpha = \tau\alpha_0$ with $\alpha_0 \in F_0$, and set $\lambda_{\alpha,\beta} = \lambda_{\alpha_0,\beta}^{(0)}$. Then

$$b = \sum_{\alpha \in F} \sum_{\beta \in W_{N_0}^d} \lambda_{\alpha,\beta} s_\alpha t_\beta.$$

Since ρ and φ are both contractive on generators (by Lemma 7.9(1)), we have

$$\begin{aligned}\|\rho(a)\| &= \|\rho(t_\tau)\rho(s_\tau a)\| \leq \|\rho(t_\tau)\| \cdot \|\rho(s_\tau a)\| \\ &\leq \|\rho(b)\| \leq \|\rho(s_\tau)\| \cdot \|\rho(a)\| \leq \|\rho(a)\|,\end{aligned}$$

so $\|\rho(b)\| = \|\rho(a)\| = 1$, and similarly $\|\varphi(b)\| = \|\varphi(a)\|$. It therefore suffices to prove that $\|\rho(b)\| \leq \|\varphi(b)\|$.

Let $\varepsilon > 0$. We prove that $\|\varphi(b)\| > 1 - \varepsilon$. If $N_0 = N_1 = 0$, then b is a scalar, and this inequality is immediate. Otherwise, $N_0 + N_1 > 0$. Choose $r \in \mathbb{Z}_{>0}$ such that

$$r > (N_0 + N_1) \left(\frac{2}{\varepsilon}\right)^p.$$

Choose $\xi^{(0)} \in L^p(X, \mu)$ such that

$$\|\xi^{(0)}\|_p = 1 \quad \text{and} \quad \|\rho(b)\xi^{(0)}\|_p > 1 - \frac{1}{2}\varepsilon.$$

Since ρ is approximately free, there is $N \geq (N_0 + N_1)r$ and a partition $X = \coprod_{m=0}^{N-1} D_m$ such that for $m = 0, 1, \dots, N-1$ and all j , taking $D_N = D_0$ and $D_{-1} = D_{N-1}$, we have

$$\rho(s_j)(L^p(D_m, \mu)) \subset L^p(D_{m+1}, \mu)$$

and

$$\rho(t_j)(L^p(D_m, \mu)) \subset L^p(D_{m-1}, \mu).$$

Write

$$\xi^{(0)} = \sum_{m=0}^{N-1} \xi_m^{(0)}$$

with $\xi_m^{(0)} \in L^p(D_m, \mu)$ for $m = 0, 1, \dots, N-1$.

We claim that there is a set T of $N_0 + N_1$ consecutive values of m such that

$$\left\| \sum_{m \in T} \xi_m^{(0)} \right\|_p < \frac{\varepsilon}{2}.$$

Since the sets D_m are disjoint, Remark 2.7 gives

$$\left\| \sum_{m \in T} \xi_m^{(0)} \right\|_p^p = \sum_{m \in T} \|\xi_m^{(0)}\|_p^p.$$

It is therefore enough to prove that there is a set T of $N_0 + N_1$ consecutive values of m such that for all $n \in T$ we have

$$\|\xi_m^{(0)}\|_p < \frac{\varepsilon}{2} \left(\frac{1}{N_0 + N_1} \right)^{1/p}.$$

Suppose that there is no such set T . Then, in particular, for

$$k = 0, 1, \dots, r-1$$

there is

$$n_k \in [k(N_0 + N_1), (k+1)(N_0 + N_1)] \cap \mathbb{Z}$$

such that

$$\|\xi_{n_k}^{(0)}\|_p \geq \frac{\varepsilon}{2} \left(\frac{1}{N_0 + N_1} \right)^{1/p}.$$

Then

$$\|\xi^{(0)}\|_p^p \geq \sum_{k=0}^{r-1} \|\xi_{n_k}^{(0)}\|_p^p \geq r \left(\frac{\varepsilon}{2} \right)^p \left(\frac{1}{N_0 + N_1} \right) > 1,$$

contradicting $\|\xi_{n_k}^{(0)}\|_p = 1$. This proves the claim.

By cyclically permuting the indices of the sets D_m , we may assume that

$$\left\| \sum_{m=0}^{N_0-1} \xi_m^{(0)} + \sum_{m=N-N_1}^{N-1} \xi_m^{(0)} \right\| < \frac{\varepsilon}{2}.$$

Define

$$\xi_m = \begin{cases} \xi_m^{(0)} & N_0 \leq m \leq N - N_1 - 1 \\ 0 & 0 \leq m \leq N_0 - 1 \text{ and } N - N_1 \leq m \leq N - 1 \end{cases}$$

and

$$\xi = \sum_{m=0}^{N-1} \xi_m = \xi^{(0)} - \sum_{m=0}^{N_0-1} \xi_m^{(0)} - \sum_{m=N-N_1}^{N-1} \xi_m^{(0)}.$$

Then $\|\xi - \xi^{(0)}\| < \frac{1}{2}\varepsilon$, so $\|\rho(b)\xi\| > 1 - \varepsilon$. Also clearly $\|\xi\| \leq \|\xi^{(0)}\| = 1$.

Following Lemma 2.17, except with the factors in the other order, define a representation $\psi: L_d \rightarrow L(L^p(X \times Y, \mu \times \nu))$ by $\psi(c) = 1 \otimes \varphi(c)$ for all $c \in L_d$. It follows from Lemma 2.17 that $\|\psi(b)\| = \|\varphi(b)\|$. We are now going to define an isometry (not necessarily surjective)

$$u: L^p(X, \mu) \rightarrow L^p(X \times Y, \mu \times \nu)$$

which will partially intertwine ρ and ψ .

The bijective measurable set transformations R_j at the beginning of the proof preserve disjointness and finite intersections and unions. Since $X = \coprod_{j=1}^d X_j$, and identifying, as usual, sets with their images in $\mathcal{B}/\mathcal{N}(\mu)$, we get

$$D_{m+1} = \coprod_{j=1}^d R_j(D_m)$$

for $m = 0, 1, \dots, N - 2$. Define

$$W = \bigcup_{m=0}^{N-1} W_m^d.$$

Set $D_\gamma = R_\gamma(D_0)$ for $\gamma \in W$. Then

$$X = \coprod_{\gamma \in W} D_\gamma.$$

For $\gamma \in W$, define $e_\gamma = m(\chi_{D_\gamma}) \in L(L^p(X, \mu))$ (following Notation 6.11). Then the e_γ are idempotents of norm 1 and $\sum_{\gamma \in W} e_\gamma = 1$.

Apply Lemma 8.5 to the injective measurable set transformations S_j at the beginning of the proof, obtaining a set $E \subset Y$ with $\nu(E) \neq 0$ such that the sets $E_\gamma = S_\gamma(E)$, for $\gamma \in W$, are disjoint. Then $\varphi(s_\gamma)(L^p(E, \nu)) = L^p(E_\gamma, \nu)$ for all γ . Choose $\eta_0 \in L^p(E, \nu)$ such that $\|\eta_0\|_p = 1$.

As in Theorem 2.16, identify $L^p(X \times Y, \mu \times \nu)$ with $L^p(X, \mu) \otimes_p L^p(Y, \nu)$. For any $\xi \in L^p(X, \mu)$ (not just the specific element ξ considered above), define

$$u\xi = \sum_{\gamma \in W} \rho(t_\gamma) e_\gamma \xi \otimes \varphi(s_\gamma) \eta_0.$$

Then $u \in L(L^p(X, \mu), L^p(X \times Y, \mu \times \nu))$.

We claim that u is isometric. Let $\xi \in L^p(X, \mu)$ be arbitrary. Since the sets D_γ are disjoint, Remark 2.7 gives

$$\|\xi\|_p^p = \sum_{\gamma \in W} \|e_\gamma \xi\|_p^p.$$

On the other hand, for $\gamma \in W$, we have $L^p(D_\gamma, \mu) \subset \text{ran}(\rho(s_\gamma))$, so

$$\|\rho(t_\gamma) e_\gamma \xi\|_p = \|e_\gamma \xi\|_p.$$

Also, the elements $\rho(t_\gamma) e_\gamma \xi \otimes \varphi(s_\gamma) \eta_0$ are supported in the disjoint sets $X \times E_\gamma$ (in fact, in $D_\gamma \times E_\gamma$), so (using $\|\varphi(s_\gamma) \eta_0\|_p = \|\eta_0\|_p = 1$ at the third step)

$$\begin{aligned} \|u\xi\|_p^p &= \left\| \sum_{\gamma \in W} \rho(t_\gamma) e_\gamma \xi \otimes \varphi(s_\gamma) \eta_0 \right\|_p^p \\ &= \sum_{\gamma \in W} \|\rho(t_\gamma) e_\gamma \xi\|_p^p \cdot \|\varphi(s_\gamma) \eta_0\|_p^p = \sum_{\gamma \in W} \|e_\gamma \xi\|_p^p = \|\xi\|_p^p. \end{aligned}$$

This completes the proof that u is isometric.

Now set

$$G_0 = \bigcup_{m=0}^{N_1} W_m^d \quad \text{and} \quad G = \bigcup_{m=N_0}^{N_0+N_1} W_m^d = \{\alpha_0 \alpha_1 : \alpha_0 \in G_0 \text{ and } \alpha_1 \in W_{N_0}^d\}.$$

Thus, G is the set of all words with lengths from N_0 through $N_0 + N_1$, and $F \subset G$. We can therefore write

$$(8.1) \quad b = \sum_{\alpha \in G} \sum_{\beta \in W_{N_0}^d} \lambda_{\alpha, \beta} s_\alpha t_\beta,$$

by taking $\lambda_{\alpha, \beta} = 0$ for $\alpha \in G \setminus F$.

We claim that for any $\xi \in L^p(X, \mu)$ which is supported in

$$\bigcup_{m=N_0}^{N-N_1-1} \bigcup_{\gamma \in W_m^d} D_\gamma,$$

and any $b \in L_d$ of the form (8.1) (not just the particular b used above), we have

$$(8.2) \quad u\rho(b)\xi = \psi(b)u\xi.$$

By linearity, it suffices to prove that for each $\gamma \in W$ with $N_0 \leq l(\gamma) \leq N - N_1 - 1$, the claim holds for all ξ which are supported in D_γ . Write $\gamma = \gamma_0\gamma_1$ with

$$\gamma_0 \in W_{N_0}^d \quad \text{and} \quad 0 \leq l(\gamma_1) \leq N - N_0 - N_1 - 1.$$

Then $\xi \in \text{ran}(\rho(s_\gamma)) \subset \text{ran}(\rho(s_{\gamma_0}))$, so $\rho(s_{\gamma_0})\rho(t_{\gamma_0})\xi = \xi$. For $\beta \in W_{N_0}^d$, we have $t_\beta s_{\gamma_0} = \delta_{\beta, \gamma_0} \cdot 1$, so

$$\rho(b)\xi = \sum_{\alpha \in G} \sum_{\beta \in W_{N_0}^d} \lambda_{\alpha, \beta} \rho(s_\alpha) \rho(t_\beta) \rho(s_{\gamma_0}) \rho(t_{\gamma_0}) \xi = \sum_{\alpha \in G} \lambda_{\alpha, \gamma_0} \rho(s_\alpha) \rho(t_{\gamma_0}) \xi.$$

Since $\gamma = \gamma_0\gamma_1$, the element $\rho(t_{\gamma_0})\xi$ is supported in D_{γ_1} . Let $\alpha \in G$. Since $l(\alpha) + l(\gamma_1) \leq N - 1$, it follows that $\rho(s_\alpha)\rho(t_{\gamma_0})\xi$ is supported in $D_{\alpha\gamma_1}$. Therefore (recalling that Notation 1.9 gives $t_{\alpha\gamma_1} = t_{\gamma_1}t_\alpha$),

$$\begin{aligned} u\rho(s_\alpha)\rho(t_{\gamma_0})\xi &= \rho(t_{\alpha\gamma_1})\rho(s_\alpha)\rho(t_{\gamma_0})\xi \otimes \varphi(s_{\alpha\gamma_1})\eta_0 \\ &= \rho(t_{\gamma_1}t_\alpha s_\alpha t_{\gamma_0})\xi \otimes \varphi(s_{\alpha\gamma_1})\eta_0 = \rho(t_\gamma)\xi \otimes \varphi(s_{\alpha\gamma_1})\eta_0. \end{aligned}$$

Thus

$$u\rho(b)\xi = \sum_{\alpha \in G} \lambda_{\alpha, \gamma_0} \rho(t_\gamma)\xi \otimes \varphi(s_{\alpha\gamma_1})\eta_0.$$

On the other hand, using $t_\beta s_{\gamma_0} = \delta_{\beta, \gamma_0}$ for $\beta \in W_{N_0}^d$ at the third step,

$$\begin{aligned} \psi(b)u\xi &= (1 \otimes \varphi(b))(\rho(t_\gamma)\xi \otimes \varphi(s_\gamma)\eta_0) \\ &= \sum_{\alpha \in G} \sum_{\beta \in W_{N_0}^d} \lambda_{\alpha, \beta} \rho(t_\gamma)\xi \otimes \varphi(s_\alpha)\varphi(t_\beta)\varphi(s_{\gamma_0})\varphi(s_{\gamma_1})\eta_0 \\ &= \sum_{\alpha \in G} \lambda_{\alpha, \gamma_0} \rho(t_\gamma)\xi \otimes \varphi(s_\alpha)\varphi(s_{\gamma_1})\eta_0 = u\rho(b)\xi. \end{aligned}$$

This completes the proof of the claim.

We now return to our specific choices of ξ and b . They satisfy the hypotheses in the claim, so we have, using (8.2) in the second calculation,

$$\|u\xi\|_p = \|\xi\|_p \leq 1 \quad \text{and} \quad \|\psi(b)u\xi\|_p = \|u\rho(b)\xi\|_p = \|\rho(b)\xi\|_p > 1 - \varepsilon.$$

Therefore $\|\varphi(b)\| = \|\psi(b)\| > 1 - \varepsilon$. This completes the proof. \square

Theorem 8.7. Let $d \in \{2, 3, 4, \dots\}$, let (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) be σ -finite measure spaces, and let $\rho: L_d \rightarrow L(L^p(X, \mu))$ and $\varphi: L_d \rightarrow L(L^p(Y, \nu))$ be spatial representations (Definition 7.4(2)). Then the map $\rho(s_j) \mapsto \varphi(s_j)$ and $\rho(t_j) \mapsto \varphi(t_j)$, for $j = 1, 2, \dots, d$, extends to an isometric isomorphism $\overline{\rho(L_d)} \rightarrow \overline{\varphi(L_d)}$.

Proof. The statement is symmetric in ρ and φ , so it suffices to prove that for all $a \in L_d$, we have $\|\varphi(a)\| \leq \|\rho(a)\|$.

Let $u \in L(l^p(\mathbb{Z}))$ be the bilateral shift, and let $\varphi_u: L_d \rightarrow L(L^p(Y, \nu) \otimes l^p(\mathbb{Z}))$ be as in Lemma 8.2. Proposition 8.3 implies that $\|\varphi(a)\| \leq \|\varphi_u(a)\|$. Since φ_u is free (by Lemma 8.2(3)), it is clearly essentially free. Since φ_u is spatial (by Lemma 8.2(2)), Proposition 8.6 therefore implies that $\|\varphi_u(a)\| \leq \|\rho(a)\|$. \square

Theorem 8.7 justifies the following definition.

Definition 8.8. Let $d \in \{2, 3, 4, \dots\}$ and let $p \in [1, \infty)$. We define \mathcal{O}_d^p to be the completion of L_d in the norm $a \mapsto \|\rho(a)\|$ for any spatial representation ρ of L_d on a space of the form $L^p(X, \mu)$ for a σ -finite measure space (X, \mathcal{B}, μ) . We write s_j and t_j for the elements in \mathcal{O}_d^p obtained as the images of the elements with the same names in L_d , as in Definition 1.1.

When $p = 2$, we get the usual Cuntz algebra \mathcal{O}_d . Indeed, if ρ is a spatial representation on $L^2(X, \mu)$, then, by Remark 6.14, we have $\rho(t_j) = \rho(s_j)^*$ for $j = 1, 2, \dots, d$. Thus, ρ is a *-representation.

Proposition 8.9. Let $d \in \{2, 3, 4, \dots\}$, let $p \in [1, \infty) \setminus \{2\}$, let (X, \mathcal{B}, μ) be a σ -finite measure space, and let $\rho: \mathcal{O}_d^p \rightarrow L(L^p(X, \mu))$ be a unital contractive homomorphism. Then ρ is isometric.

Proof. Let ρ_0 be the composition of ρ with the obvious map $L_d \rightarrow \mathcal{O}_d^p$. By Theorem 8.7, it suffices to prove that ρ_0 is spatial. We prove this by verifying condition (5) of Theorem 7.7. (By the last part of Theorem 7.7, this suffices when $p = 1$ as well as when $p \in (1, \infty) \setminus \{2\}$.) That ρ_0 is contractive on generators is immediate. Also, the obvious map sending the standard matrix unit $e_{j,k} \in M_d$ to $s_j t_k \in \mathcal{O}_d^p$ is isometric from M_d^p to \mathcal{O}_d^p , by Lemma 7.9(8) and the equivalence of conditions (1) and (3) in Theorem 7.2. Therefore the restriction of ρ_0 to $\text{span}((s_j t_k)_{j,k=1}^d)$ is contractive as a map from M_d^p to $L(L^p(X, \mu))$. The equivalence of conditions (1) and (4) in Theorem 7.2 therefore shows that this restriction is spatial. This completes the verification of condition (5) of Theorem 7.7. \square

Corollary 8.10. Let $d \in \{2, 3, 4, \dots\}$, let $p \in [1, \infty) \setminus \{2\}$, let (X, \mathcal{B}, μ) be a σ -finite measure space such that $L^p(X, \mu)$ is separable, and let $\rho: \mathcal{O}_d^p \rightarrow L(L^p(X, \mu))$ be a not necessarily unital contractive homomorphism. Then ρ is isometric.

Proof. It is clear that $e = \rho(1)$ is an idempotent in $L(L^p(X, \mu))$. Set $E = \text{ran}(e)$. Then ρ defines a contractive unital homomorphism from \mathcal{O}_d^p to $L(E)$.

The hypotheses imply that $\|\rho(1)\| = 1$. It follows from Theorem 3 in Section 17 of [17] that there is a measure space (Y, \mathcal{C}, ν) such that E is isometrically isomorphic to $L^p(Y, \nu)$. Since $L^p(X, \mu)$ is separable, so is E , and therefore we may take ν to be σ -finite. Now apply Proposition 8.9. \square

Unfortunately, unlike the case of C*-algebras ($p = 2$), we know of no result which allows us to deduce simplicity of \mathcal{O}_d^p from Proposition 8.9 or Corollary 8.10,

or the other way around. We will prove simplicity of \mathcal{O}_d^p in [22], using methods much closer to those used in the C^* -algebra case in [9].

9. NONISOMORPHISM OF ALGEBRAS GENERATED BY SPATIAL REPRESENTATIONS FOR DIFFERENT p

To what extent do the various algebras \mathcal{O}_d^p differ from each other? Taking $p = 2$, the K-theory computation in [10] shows that, for $d_1 \neq d_2$, we have $\mathcal{O}_{d_1}^2 \not\cong \mathcal{O}_{d_2}^2$. We will show in [23] that the K-theory is the same for $p \neq 2$ as for $p = 2$, giving the analogous nonisomorphism result. Here, we address what happens when one instead varies p . Here, K-theory is of no help. Instead, we show by more direct methods that for $p_1 \neq p_2$ and for d_1 and d_2 arbitrary, there is no nonzero continuous homomorphism from $\mathcal{O}_{d_1}^{p_1}$ to $\mathcal{O}_{d_2}^{p_2}$. In fact, there is no nonzero continuous homomorphism from $\mathcal{O}_{d_1}^{p_1}$ to $L(l^{p_2}(\mathbb{Z}_{>0}))$.

This result gives a different proof of the fact (Corollary 6.15 of [6]) that for distinct $p_1, p_2 \in (1, \infty)$, there is no nonzero continuous homomorphism from $L(l^{p_1}(\mathbb{Z}_{>0}))$ to $L(l^{p_2}(\mathbb{Z}_{>0}))$. (We are grateful to Volker Runde for providing this reference. In this connection, we note that Corollary 2.18 of [6] implies that if E and F are Banach spaces such that $L(E)$ is isomorphic to $L(F)$ as Banach algebras, then E is isomorphic to F as Banach spaces.)

Lemma 9.1. Let $p \in [1, \infty)$. Let (X, \mathcal{B}, μ) be a σ -finite measure space. Let $\rho: L_\infty \rightarrow L(l^p(X, \mu))$ be a spatial representation. Let E be a Banach space. Suppose there is a nonzero continuous homomorphism $\varphi: \overline{\rho(L_\infty)} \rightarrow L(E)$. Then $l^p(\mathbb{Z}_{>0})$ is isomorphic as a Banach space (recall the conventions in Definition 2.2) to a subspace of E .

Proof. The element $\varphi(1) \in L(E)$ is an idempotent. Replacing E by $\varphi(1)E$, we may assume that φ is unital (and E is nonzero).

Let $q \in (1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Let $\lambda \mapsto s_\lambda$ and $\lambda \mapsto t_\lambda$ be as in Definition 1.13 for L_∞ . For $p \in (1, \infty)$, it follows from Lemma 7.9(4) and Lemma 7.9(5) that the maps $\lambda \mapsto \rho(s_\lambda)$ and $\lambda \mapsto \rho(t_\lambda)$ extend to isometric maps $s^\rho: l^p(\mathbb{Z}_{>0}) \rightarrow L(L^p(X, \mu))$ and $t^\rho: l^q(\mathbb{Z}_{>0}) \rightarrow L(L^p(X, \mu))$. For $p = 1$, we similarly get s^ρ as above and $t^\rho: C_0(\mathbb{Z}_{>0}) \rightarrow L(L^p(X, \mu))$.

Fix $\eta_0 \in E$ with $\|\eta_0\| = 1$. Define $v: l^p(\mathbb{Z}_{>0}) \rightarrow E$ as follows. For $\lambda = (\lambda_1, \lambda_2, \dots) \in l^p(\mathbb{Z}_{>0})$, set $v(\lambda) = \varphi(s^\rho(\lambda))\eta_0$. Then v is bounded, because

$$\|v(\lambda)\| \leq \|\varphi\| \cdot \|s^\rho(\lambda)\| \cdot \|\eta_0\| = \|\varphi\| \cdot \|\lambda\|_p.$$

We claim that for all $\lambda \in l^p(\mathbb{Z}_{>0})$, we have $\|v(\lambda)\| \geq \|\varphi\|^{-1} \|\lambda\|_p$. The claim will imply that v is an isomorphism of $l^p(\mathbb{Z}_{>0})$ with some closed subspace of E , completing the proof of the lemma.

We prove the claim. Let $\lambda \in l^p(\mathbb{Z}_{>0})$. Without loss of generality $\lambda \neq 0$. First suppose $p \neq 1$. It is well known that there exists $\gamma \in l^q(\mathbb{Z}_{>0}) \setminus \{0\}$ such that

$$(9.1) \quad \sum_{j=1}^{\infty} \gamma_j \lambda_j = 1$$

and

$$(9.2) \quad \|\gamma\|_q = \|\lambda\|_p^{-1}.$$

(The method of proof can be found, for example, at the beginning of Section 6.5 of [24].)

By Lemma 1.14, for every $n \in \mathbb{Z}_{>0}$ we have

$$\left(\sum_{j=1}^n \gamma_j \rho(t_j) \right) \left(\sum_{j=1}^n \lambda_j \rho(s_j) \right) = \left(\sum_{j=1}^n \gamma_j \lambda_j \right) \cdot 1.$$

Letting $n \rightarrow \infty$ and applying (9.1), we get $t^\rho(\gamma)s^\rho(\lambda) = 1$. Therefore

$$\eta_0 = \varphi(t^\rho(\gamma))\varphi(s^\rho(\lambda))\eta_0 = \varphi(t^\rho(\gamma))v(\lambda).$$

So, using (9.2) at the third step,

$$1 = \|\eta_0\| \leq \|\varphi\| \cdot \|t^\rho(\gamma)\| \cdot \|v(\lambda)\| = \|\varphi\| \cdot \|\lambda\|_p^{-1} \cdot \|v(\lambda)\|.$$

It follows that $\|v(\lambda)\| \geq \|\varphi\|^{-1}\|\lambda\|_p$, as desired.

Now suppose $p = 1$. Let $\varepsilon > 0$. Choose $\gamma \in c_0(\mathbb{Z}_{>0})$ with finite support and such that $\gamma_j \lambda_j \geq 0$ for $j \in \mathbb{Z}_{>0}$,

$$\sum_{j=1}^{\infty} \gamma_j \lambda_j > 1 - \varepsilon, \quad \text{and} \quad \|\gamma\|_q = \|\lambda\|_p^{-1}.$$

Then $t^\rho(\gamma)s^\rho(\lambda) = \alpha \cdot 1$ with $\alpha > 1 - \varepsilon$. We get

$$\alpha \eta_0 = \varphi(t^\rho(\gamma))v(\lambda) \quad \text{and} \quad 1 - \varepsilon \leq \|\varphi\| \cdot \|\lambda\|_p^{-1} \cdot \|v(\lambda)\|.$$

Since $\varepsilon > 0$ is arbitrary, we again get $\|v(\lambda)\| \geq \|\varphi\|^{-1}\|\lambda\|_p$, as desired. \square

Theorem 9.2. Let $p_1, p_2 \in [1, \infty)$ be distinct. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3). Let $\rho: A \rightarrow L(l^{p_1}(\mathbb{Z}_{>0}))$ be a spatial representation. Then there is no nonzero continuous homomorphism from $\overline{\rho(A)}$ to $L(l^{p_2}(\mathbb{Z}_{>0}))$.

Proof. Suppose that $\varphi: \overline{\rho(A_1)} \rightarrow L(l^{p_2}(\mathbb{Z}_{>0}))$ is a continuous homomorphism.

Regardless of what A_1 is, there is a unital homomorphism $\psi: L_\infty \rightarrow A_1$ such that (in the notation of Definition 1.3)

$$\psi(s_j^{(\infty)}) = s_2^j s_1 \quad \text{and} \quad \psi(t_j^{(\infty)}) = t_1 t_2^j$$

for all $j \in \mathbb{Z}_{>0}$. Since ρ is spatial, one easily checks that $\rho \circ \psi$ is a spatial representation of L_∞ on $l^{p_1}(\mathbb{Z}_{>0})$. Set $B = \overline{(\rho \circ \psi)(L_\infty)}$. Then Lemma 9.1, applied to $\varphi|_B$, provides an isomorphism of $l^{p_1}(\mathbb{Z}_{>0})$ with a subspace of $l^{p_2}(\mathbb{Z}_{>0})$.

The remark after Proposition 2.a.2 of [19] (on page 54) therefore implies that $p_1 = p_2$. \square

Corollary 9.3. Let $p_1, p_2 \in [1, \infty)$ be distinct. Let A_1 and A_2 be any two of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3). Let $\rho_1: A_1 \rightarrow L(l^{p_1}(\mathbb{Z}_{>0}))$ be a spatial representation (Definition 7.4(2)), and let $\rho_2: A_2 \rightarrow L(l^{p_2}(\mathbb{Z}_{>0}))$ be an arbitrary representation. Then there is no nonzero continuous homomorphism from $\overline{\rho(A_1)}$ to $\overline{\rho(A_2)}$.

Proof. Combine Theorem 9.2 and Lemma 7.5. \square

In particular, there is no nonzero continuous homomorphism from $\mathcal{O}_{d_1}^{p_1}$ to $\mathcal{O}_{d_2}^{p_2}$. We recover part of Corollary 6.15 of [6].

Corollary 9.4. Let $p_1, p_2 \in [1, \infty)$ be distinct. Then there is no nonzero continuous homomorphism from $L(l^{p_1}(\mathbb{Z}_{>0}))$ to $L(l^{p_2}(\mathbb{Z}_{>0}))$.

Proof. Suppose $\varphi: L(l^{p_1}(\mathbb{Z}_{>0})) \rightarrow L(l^{p_2}(\mathbb{Z}_{>0}))$ is a nonzero continuous homomorphism. Use Lemma 7.5 to choose an injective spatial representation $\rho: L_\infty \rightarrow L(l^{p_1}(\mathbb{Z}_{>0}))$. Then $1 \in \overline{\rho(L_\infty)}$, so $\varphi|_{\overline{\rho(L_\infty)}}$ is nonzero, contradicting Theorem 9.2. \square

Corollary 9.3 does not rule out isomorphism as Banach spaces. In fact, we have the following result.

Proposition 9.5. Let $p \in [1, \infty)$, and suppose $\frac{1}{p} + \frac{1}{q} = 1$. Let A be any of L_d (Definition 1.1), C_d (Definition 1.2), or L_∞ (Definition 1.3), let (X, \mathcal{B}, μ) be a σ -finite measure space, and let $\rho: A \rightarrow L(L^p(X, \mu))$ be any representation. Then $\overline{\rho(A)}$ is isometrically antiisomorphic to a subalgebra of $L(L^q(X, \mu))$, namely $\overline{\rho'(A)}$ with ρ' as in Lemma 2.21.

Proof. Use Lemma 2.21 and the fact that (following Notation 2.3) one always has $\|a'\| = \|a\|$. \square

In particular, when ρ is spatial, $\overline{\rho(A)}$ and $\overline{\rho'(A)}$ are isometrically isomorphic as Banach spaces, even though (for $p \neq 2$) they are not even isomorphic as Banach algebras.

REFERENCES

[1] G. Abrams, P. Ara, and M. Siles Molina, *Leavitt Path Algebras*, Springer-Verlag, to appear.
 [2] G. Abrams and G. Aranda Pino, *The Leavitt path algebra of a graph*, *J. Algebra* **293** (2005), 319–334.
 [3] G. Abrams and G. Aranda Pino, *The Leavitt path algebras of arbitrary graphs*, *Houston J. Math.* **34** (2008), 423–442.
 [4] G. Abrams and D. Funk-Neubauer, *On the simplicity of Lie algebras associated to Leavitt algebras*, preprint ([arXiv:0908.2179v2\[math.RA\]](https://arxiv.org/abs/0908.2179v2)).
 [5] A. Arias and J. D. Farmer, *On the structure of tensor products of l_p -spaces*, *Pacific J. Math.* **175** (1996), 13–37.

- [6] E. Berkson and H. Porta, *Representations of $\mathfrak{B}(X)$* , J. Funct. Anal. **3** (1969), 1–34.
- [7] J. Chaney, *Banach lattices of compact maps*, Math. Z. **129** (1972), 1–19.
- [8] P. M. Cohn, *Some remarks on the invariant basis property*, Topology **5** (1966), 215–228.
- [9] J. Cuntz, *Simple C^* -algebras generated by isometries*, Commun. Math. Phys. **57** (1977), 173–185.
- [10] J. Cuntz, *K -theory for certain C^* -algebras*, Ann. Math. **113** (1981), 181–197.
- [11] K. R. Davidson, *Free semigroup algebras. A survey*, pages 209–240 in: *Systems, Approximation, Singular Integral Operators, and Related Topics (Bordeaux, 2000)*, Oper. Theory Adv. Appl. **129**, Birkhäuser, Basel, 20014.
- [12] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Mathematics Studies no. 176, North-Holland Publishing Co., Amsterdam, 1993.
- [13] J. L. Doob, *Stochastic Processes*, John Wiley & Sons, Inc., New York, 1953.
- [14] T. Figiel, T. Iwaniec, and A. Pełczyński, *Computing norms and critical exponents of some operators in L^p -spaces*, Studia Math. **79** (1984), 227–274.
- [15] W. B. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, pages 1–84 in: *Handbook of the Geometry of Banach Spaces*, Vol. 1, W. B. Johnson and J. Lindenstrauss (eds.), Elsevier, Amsterdam, London, New York, Oxford, Paris, Shannon, Tokyo, 2001.
- [16] S. Koppelberg, *Handbook of Boolean algebras. Vol. 1*, J. D. Monk and R. Bonnet (eds), North-Holland, Amsterdam, 1989.
- [17] H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Grundlehren der mathematischen Wissenschaften no. 208, Springer-Verlag, Berlin, Heidelberg, New York, 1974.
- [18] J. Lamperti, *On the isometries of certain function-spaces*, Pacific J. Math. **8** (1958), 459–466.
- [19] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I. Sequence spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 92, Springer-Verlag, Berlin, New York, 1977.
- [20] W. G. Leavitt, *The module type of a ring*, Trans. Amer. Math. Soc. **103** (1962), 113–130.
- [21] W. G. Leavitt, *The module type of homomorphic images*, Duke Math. J. **32** (1965), 305–311.
- [22] N. C. Phillips, *Simplicity of UHF and Cuntz algebras on L^p spaces*, in preparation.
- [23] N. C. Phillips, *Crossed products of L^p operator algebras and the K -theory of Cuntz algebras on L^p spaces*, in preparation.
- [24] H. L. Royden, *Real Analysis*, 3rd. ed., Macmillan, New York, 1988.
- [25] M. Tomforde, *Uniqueness theorems and ideal structure for Leavitt path algebras*, J. Algebra **318** (2007), 270–299.

DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF OREGON
 EUGENE OR 97403-1222
 USA
 AND
 RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
 KYOTO UNIVERSIT
 KITASHIRAKAWA-OIWAKECHO, SAKYO-KU
 KYOTO 606-8502
 JAPAN.

E-mail address: ncp@darkwing.uoregon.edu