RECONSTRUCTION OF A SUPERSCHEME FROM ITS DERIVED CATEGORY

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ABSTRACT. The aim of this paper is to construct a "super" version of a tensor triangulated category, and to show that super-schemes can be reconstructed from its category of perfect complexes in a way similar to Balmer [Bal05] provided we consider this extra structure.

1. INTRODUCTION

In this paper, we prove that Balmer's construction of Spec of a tensor triangulated category can be used to reconstruct super-schemes from its category of perfect complexes, provided we keep track of the $\mathbb{Z}/2\mathbb{Z}$ grading in the triangulated category. Here the definition of the $\mathbb{Z}/2\mathbb{Z}$-graded perfect complexes on super-scheme is just an adaptation of the definition of perfect complexes on schemes to the super-scheme case. This adaptation is explained in the paper.

The study of derived categories of coherent sheaves on an algebraic variety is a hot topic of research. They have been studied by a lot of mathematicians including Beilinson [Beĭ84], Bondal [BO02] [Bon89] [BO01], Orlov [KO94] [Orl03] [Orl02] [Orl97], [Orl92], Kapranov [Kap88a] [Kap88b] [Kap86], Bridgeland [Bri06] [Bri08] [Bri07] [Bri02] [BM01] [BKR01] [Bri99], Balmer [Bal05] [Bal07] [Bal02] and others. (The list here is in no way complete.) There are many interesting results in this area. One such problem is the reconstruction of a variety in given its derived category of coherent sheaves. There has been a lot of work on reconstruction (for example, [BO01] [Bal05]).

Another construction which is also gaining a lot of importance lately is the concept of a super-scheme [Man88] [DEF$^+$99]. A lot of constructions which can be done on schemes can be extended on a super-scheme. Thus it is natural to ask, "How much information does the derived category of the category of modules on a super-scheme carry?" The aim of this paper is to provide an answer to this question following the lines of Balmer's construction of Spec associated to a triangulated category in [Bal05].

In section 2, we define a $\mathbb{Z}/2\mathbb{Z}$ graded triangulated categories. Then we define Spec of such a category. Section 3 defines support data in this context and proves a universal property which is used to prove that the Spec of the category of perfect complexes on a super-scheme is homeomorphic to the super-scheme.

Perfect complexes on super-schemes are defined in 4. The last section (section 5) contains the proof of the following theorem

Main Theorem. *Let X be a super-scheme with an ample family of line bundles (which means it is also quasi-compact and separated). Then the category $\mathcal{D}^{perf}(X)$ is the category of compact objects in $\mathcal{D}^b(X)$. Also, the locally ringed space* $\operatorname{Spec}\mathcal{D}^{perf}(X)$ *is isomorphic to X as super-schemes.*

Note that in a previous paper [DM10], we had computed the Spec of the category of perfect complexes over a super-scheme. But the definition of perfect complexes there was different from that in this paper. There the category of perfect complexes formed a symmetric monoidal category under the tensor product. Here that is no longer true, but as we show the change in the tensor structure helps us to solve the original problem of reconstruction.

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2. Basic set up

Consider a triangulated category $\mathcal{T}$ whose Hom sets are $\mathbb{Z}/2\mathbb{Z}$ graded.

$$\operatorname{Hom}_{\mathcal{T}}(A,B) = \operatorname{Hom}^0_{\mathcal{T}}(A,B) \oplus \operatorname{Hom}^1_{\mathcal{T}}(A,B)$$

Given $f \in \operatorname{Hom}^i_{\mathcal{T}}(A,B)$, we define the "parity" $|f| = i \in \mathbb{Z}/2\mathbb{Z}$. We call such f's homogeneous.

2.1. The definition.

We define a tensor product in this category which preserves triangles in the category. For objects A, B, C, D, E and F with homogeneous maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$D \xrightarrow{i} E \xrightarrow{j} F,$$

$$\text{we have } (g \circ f) \otimes (j \circ i) = (-1)^{|j||f|}(g \otimes j) \circ (f \otimes i).$$

From now on $\mathcal{T}$ is such a tensor triangulated category and we want to associate a ringed space to it following Balmer [Bal05].

Remark 2.1.1. Note that in the above definition, we do not specify what the tensor product of objects are. We note that in the proofs below all that we use is that, if $f\colon A \to B$ and $g\colon C \to D$ are two morphisms, $f \otimes g$ is a morphism from $A \otimes C$ to $B \otimes D$ satisfying the above condition.

2.2. **An example.** Consider a super-ring A. Let M, M', N, N', P and P' be three (super)-modules defined over A. Consider the following maps

$$M \xrightarrow{f} N \xrightarrow{g} P$$
$$M' \xrightarrow{f'} N' \xrightarrow{g'} P'$$

For two A-super-modules M and M', we define $M \otimes M'$ as usual. We just change the definition of $f \otimes f'$ for two morphisms as follows. Suppose for homogeneous elements $m \in M$ and $m' \in M'$, we define $(f \otimes f')(m \otimes m') = (-1)^{|f'||m|} f(m) \otimes f'(m')$ where $|a|$ denotes the parity of the homogeneous element a. Note that under this definition of tensor of morphisms, the category is no longer symmetric. But it is easy to verify that $((f \otimes f') \otimes f'')((m \otimes m') \otimes m'') = (f \otimes (f' \otimes f''))(m \otimes (m' \otimes m''))$. Thus the category is still monoidal.

Here we have

$$\begin{aligned}
&(g \otimes g')(f \otimes f')(m \otimes m') \\
&\quad = (-1)^{|f'||m|}(g \otimes g')(f(m) \otimes f'(m')) \\
&\quad = (-1)^{|f'||m|}(-1)^{|g'||f(m)|}(g(f(m)) \otimes g'(f'(m'))) \\
&\quad = (-1)^{|f'||m|}(-1)^{|g'||f(m)|}(-1)^{|g' \circ f'||m|}((g \circ f) \otimes (g' \circ f'))(m \otimes m') \\
&\quad = (-1)^{|g'||f|}((g \circ f) \otimes (g' \circ f'))(m \otimes m'),
\end{aligned}$$

since $|f(m)| = |f| + |m|$ and $|g' \circ f'| = |g'| + |f'|$.

2.3. **Thick tensor ideals.** A *triangulated ideal* $\mathcal{A}$ of $\mathcal{T}$ is a subcategory of $\mathcal{T}$ such that for any distinguished triangle $a \to b \to c \to Ta$ in $\mathcal{T}$, with any two of a, b or c in $\mathcal{A}$, the third one is also in $\mathcal{A}$.

A triangulated ideal $\mathcal{A}$ of $\mathcal{T}$ is said to be *thick* if $a = b \oplus c$ in $\mathcal{T}$ and $a \in \mathcal{A}$, then both b and c are in $\mathcal{A}$.

A thick triangulated ideal $\mathcal{T}$ is said to be a *tensor ideal* if for $a \in \mathcal{A}$ and $t \in \mathcal{T}$, $a \otimes t \in \mathcal{A}$.

Definition 2.3.1. *A proper thick tensor ideal $\mathcal{P}$ of $\mathcal{A}$ is said to be* prime *if*

$$a \otimes b \in \mathcal{P} \iff \text{either } a \in \mathcal{P} \text{ or } b \in \mathcal{P}.$$

2.4. **The Spec construction.** In this subsection, we give a definition of Spec of a super triangulated category $\mathcal{T}$. As a set it is the set of prime ideals of $\mathcal{T}$.

We now give a topology on the above thing following Balmer [Bal05]. For a family of objects $\mathcal{S} \subset \mathcal{T}$,

$$Z(\mathcal{S}) = \{\mathcal{P} \text{ prime in } \mathcal{T} \mid \mathcal{S} \cap \mathcal{P} = \emptyset\}.$$

This gives a family of closed sets :

$$\bigcap_{j \in \mathcal{J}} Z(\mathcal{S}_j) = \{\mathcal{P} | \mathcal{P} \text{ prime and } \mathcal{S}_j \cap \mathcal{P} = \emptyset \ \forall\ j\}$$

$$
\begin{aligned}
&= \{\mathcal{P} | \mathcal{P} \text{ prime and } (\cup_j \mathcal{S}_j) \cap \mathcal{P} = \emptyset\} \\
&= Z(\cup_j \mathcal{S}_J)
\end{aligned}
$$

$$\bigcup_{j=1}^{n} Z(\mathcal{S}_j) = \{\mathcal{P} | \mathcal{P} \text{ prime and } \mathcal{S}_j \cap \mathcal{P} = \emptyset \ \text{ for some } j\}$$

by thickness,

$$
\begin{aligned}
&= \{\mathcal{P} | \mathcal{P} \text{ prime and } (\oplus_j \mathcal{S}_j) \cap \mathcal{P} = \emptyset\} \\
&= Z(\oplus_j \mathcal{S}_J).
\end{aligned}
$$

This Zariski topology also satisfies $Z(\mathcal{T}) = \emptyset$ and $Z(\emptyset) = \operatorname{Spec} \mathcal{T}$. Obviously the open sets are

$$U(\mathcal{S}) = \operatorname{Spec} \mathcal{T} \setminus Z(\mathcal{S}) = \{\mathcal{P} \text{ prime in } \mathcal{T} \mid \mathcal{S} \cap \mathcal{P} \neq \emptyset\}.$$

Definition 2.4.1. *For an object $a \in \mathcal{T}$, the support of a, denoted by* $\operatorname{supp}(a)$ *is defined to be*

$$\operatorname{supp}(a) := Z(\{a\}) = \{\mathcal{P} \in \operatorname{Spec} \mathcal{T} \mid a \notin \mathcal{P}\}$$

One can check that supp forms a basis of closed sets for the topology on $\operatorname{Spec} \mathcal{T}$.

3. Support data

In this section, we define and study support data. The proofs in this section follow Balmer's proofs very closely. Actually Balmer's proofs work without modification in this general set up. Nevertheless, we provide some of the proofs (originally by Balmer) for the sake of clarity and demonstration.

3.1. Definition and some properties.

Definition 3.1.1 (Support Data)**.** *A* support data *on a tensor triangulated category $(\mathcal{T}, \otimes, 1)$ is a pair (X, σ) where X is a topological space and σ is an assignment*

$$\sigma\colon \ \textit{objects in } \mathcal{T} \longrightarrow \ \textit{closed subsets in } X$$

such that

SD 1. $\sigma(0) = \emptyset$, $\sigma(1) = X$.
SD 2. $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$.
SD 3. $\sigma(Ta) = \sigma(a)$.
SD 4. $\sigma(a) \subset \sigma(b) \cup \sigma(c)$ *for all distinguished triangles* $a \to b \to c \to Ta$.
SD 5. $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$.

Definition 3.1.2. *A* morphism $f\colon (X, \sigma) \to (Y, \tau)$ *of support data on the same category $\mathcal{T}$ is a continuous map $f\colon X \to Y$ such that $\sigma(a) = f^{-1}(\tau(a))$ for all objects a in $\mathcal{T}$.*

Such a morphism is an *isomorphism* if and only if f is a homeomorphism. Now we are ready to prove the following theorem.

Theorem 3.1.3. $(\operatorname{Spec}\mathcal{T}, \operatorname{supp})$ *is a support data on* $\mathcal{T}$. *For any other support data* (X, σ), *there exists a unique continuous map* $f\colon X \to \operatorname{Spec}\mathcal{T}$ *such that* $\sigma(a) = f^{-1}(\operatorname{supp}(a))$ *for all objects* a *in* $\mathcal{T}$.

Explicitly, the map $f\colon X \to \operatorname{Spec}\mathcal{T}$ *is defined by*

$$f(x) = \{a \mid a \text{ is an object in } \mathcal{T},\ x \notin \sigma(a)\}.$$

Proof. The proof is divided into a few obvious steps. First let us check that supp actually defines a support data. We need to check the properties SD 1 - SD 5. Balmer's proofs [Bal05, lemma 2.6] work here without any changes. Still, we shall check SD 5 as a demonstration.

$$\operatorname{supp}(a \otimes b) = \{\mathcal{P} \mid a \otimes b \notin \mathcal{P}\}$$

and since $\mathcal{P}$s are primes

$$= \{\mathcal{P} \mid a \notin \mathcal{P}\} \cap \{\mathcal{P} \mid b \notin \mathcal{P}\} = \operatorname{supp}(a) \cap \operatorname{supp}(b).$$

Thus the $(\operatorname{Spec}\mathcal{T}, \operatorname{supp})$ does form a support data.

Rest is similar to theorem 3.2 in [Bal05]. But we paraphrase the argument below.

Next step is to check that f is a morphism of sets. That is we need to check that $f(x)$ is a prime ideal in $\mathcal{T}$. $f(x)$ is a triangulated ideal since if $a \to b \to c \to Ta$ is a triangle with a and b in $f(x)$, then $x \notin \sigma(a) \cup \sigma(b)$; and therefore $x \notin \sigma(c)$ ($\because$ $\sigma(c) \subset \sigma(a) \cup \sigma(b)$, (X, σ) being a support data); and thus $c \in f(x)$. To see that $f(x)$ is thick, note that if $a \oplus b \in f(x)$, this means $x \notin \sigma(a \oplus b)$, and therefore $x \notin \sigma(a) \cup \sigma(b)$; and hence both a and b are in $f(x)$. Now we need to check that $f(x)$ is a tensor ideal. For $a \in f(x)$ and $t \in \mathcal{T}$, note that $x \notin \sigma(a)$; therefore $x \notin \sigma(a) \cap \sigma(t) = \sigma(a \otimes t)$. In other words $a \otimes t \in f(x)$ proving that $f(x)$ is a tensor ideal. The last step is to prove that it is prime. Let $a \otimes b \in f(x)$. That means $x \notin \sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$. Therefore either $x \notin \sigma(a)$ or $x \notin \sigma(b)$; in other words, either $a \in f(x)$ or $b \in f(x)$. This completes the proof that f is a morphism of sets.

Now we shall establish that f is continuous. As mentioned earlier, $\operatorname{supp}(a)$, $a \in \mathcal{T}$ forms a basis of closed sets. Therefore it is enough to check that $f^{-1}(\operatorname{supp} a)$ is closed in X for all a. But $\{x : f(x) \in \operatorname{supp}(a)\} = \{x : a \notin f(x)\} = \{x : x \in \sigma(a)\} = \sigma(a)$ which is closed.

The above argument also shows that f is a morphism of support data. □

3.2. Classifying support data.

Definition 3.2.1. *A thick tensor ideal* $\mathcal{J}$ *is said to be* radical *if for some* n *and* $a \in \mathcal{T}$, $a^{\otimes n} \in \mathcal{J}$ *then* $a \in \mathcal{J}$.

Definition 3.2.2. *A subset* $Y \subset X$ *of a topological space* X *is said to be* specialization closed *if* $y \in Y$ *implies that the closure* $\overline{\{y\}} \subset Y$. *This is equivalent to saying that* Y *can be written as a union of closed subsets of* X.

A support data (X, σ) is said to be a classifying support data *if the following two conditions hold*

(1) The topological space X is noetherian and any non-empty irreducible closed subset $Z \subset X$ has a unique generic point. That is there exists a unique point $z \in Z$ such that $\overline{\{z\}} = Z$.
(2) We have a bijection between the specialization closed subsets of X and radical thick tensor ideals of $\mathcal{T}$ given by the following two maps

$$\Theta\colon Y \mapsto \{a \in \mathcal{T} : \sigma(a) \subset Y\}$$
$$\Xi\colon \mathcal{J} \mapsto \sigma(\mathcal{J}) := \bigcup_{a \in \mathcal{J}} \sigma(a).$$

Now we can state the main theorem

Theorem 3.2.3. *Suppose (X, σ) is a classifying support data on $\mathcal{T}$. Then the canonical map $f\colon X \to \operatorname{Spec} \mathcal{T}$ is a homeomorphism.*

Proof. This corresponds to [Bal05, theorem 5.2]. The proof is also similar. We give some details below.

We already know that f is a continuous map. So to prove that it is a homeomorphism, under the additional assumptions on (X, σ) being a classifying support data, we need to prove that f is bijective and open, or equivalently closed. Before we do that, we note that all closed subsets X are of the form $\sigma(a)$ for some $a \in \mathcal{T}$. Let Z be any irreducible closed subset of X. Then by assumption, there exists a unique $z \in X$ such that $Z = \overline{\{z\}}$. Now Z being specialization closed,

$$Z = \Xi \circ \Theta(Z) = \bigcup_{a\,:\,\sigma(a) \subset Z} \sigma(a)$$

and hence, there exists an a such that $z \in \sigma(a)$. But then $\sigma(a)$ is closed by definition, and thus $Z = \overline{\{z\}} \subset \sigma(a)$. Therefore, $Z = \sigma(a)$. When Z is not irreducible, since X is a noetherian topological space, $Z = Z_1 \cup \cdots \cup Z_r$ for some finitely many irreducible closed subsets $Z_1, \ldots, Z_r$ of X. Each $Z_i = \sigma(a_i)$ for some $a_i \in \mathcal{T}$. Therefore, $Z = \cup Z_i = \cup \sigma(a_i) = \sigma(a_1 \oplus \cdots \oplus a_r)$ proving the fact that all closed subsets of X are of the form $\sigma(a)$ for some $a \in \mathcal{T}$.

Next we go on to prove that f is injective, surjective and closed. We shall prove injectivity to show similarity with Balmer's proof and leave out surjectivity and closedness. To show injectivity, we write f as a composition of two injective maps. For $x \in X$, define $Y(x) = \{y \in X : x \notin \overline{\{y\}}\}$. To prove that f is injective, we need to prove that Y is injective and $f = \Theta \circ Y$. But Y is injective as $Y(x_1) = Y(x_2)$ implies that $\{y \in X : x_1 \notin \overline{\{y\}}\} = \{y \in X : x_2 \notin \overline{\{y\}}\}$ and therefore, taking complements $\{y \in X : x_1 \in \overline{\{y\}}\} = \{y \in X : x_2 \in \overline{\{y\}}\}$. Taking intersection of the closures of the points in the sets on both sides, we get $\overline{\{x_1\}} = \overline{\{x_2\}}$. By condition (1) in the definition of classifying support data, this means that $x_1 = x_2$. This proves that Y is injective. Now it is also easy to prove that for any x, $Y(x)$ is specialization closed. Thus Y is a map from

$$X \longrightarrow \text{Specialization closed subsets of } X.$$

Thus it makes sense to talk of $\Theta(Y(x)) = \{a \in \mathcal{T} : \sigma(a) \subset Y(x)\}$. Now we claim that $\sigma(a) \subset Y(x)$ if and only if $x \notin \sigma(a)$. But by definition, $\sigma(a) \subset Y(x)$ means that for all $y \in \sigma(a)$, $x \notin \overline{\{y\}}$. Therefore, x cannot be in $\sigma(a)$. On the other hand, if $x \notin \sigma(a)$, since $\sigma(a)$ is closed, for all $y \in \sigma(a)$, $x \notin \overline{\{y\}} \subset \sigma(a)$. In other words, $y \in Y(x)$. Therefore $\sigma(a) \subset Y(x)$. Therefore,

$$\Theta(Y(x)) = \{a \in \mathcal{T} : \sigma(a) \subset Y(x)\} = \{a \in \mathcal{T} : x \notin \sigma(a)\} = f(x)$$

by definition. Therefore f is injective.

Similarly, following Balmer, one can prove surjectivity and closedness of f, and therefore prove that f is a homeomorphism. □

Balmer uses this result along with Thomason's theorem that for a scheme X, supph gives a support data on the category of perfect complexes, to conclude that X is homeomorphic to the Spec of the category of perfect complexes.

4. Perfect Complexes

The above theorem reduces to Thomason's result. So we try to prove a Thomason like theorem for super-schemes. For that we need to define an analogue of perfect complexes for super-schemes. Before doing that let us recall the definition of perfect complexes on schemes.

4.1. Perfect complexes on schemes.

Definition 4.1.1. *For any integer m, a chain complex $E^\bullet$ of $\mathcal{O}_X$-modules on the scheme X is said to be* strictly m-pseudo-coherent *if E^i is a vector bundle on X for all $i \geq m$ and $E^i = 0$ for all i sufficiently large. A complex $E^\bullet$ is* strictly pseudo-coherent *if it is strictly m-pseudo-coherent for all m, i.e., if it is a bounded above complex of algebraic vector bundles.*

Definition 4.1.2. *A complex $E^\bullet$ of $\mathcal{O}_X$-modules is* strictly perfect *if it is strictly pseudo-coherent and strictly bounded below. That is, a strict perfect complex is a strict bounded complex of algebraic vector bundles.*

Definition 4.1.3. *A complex $E^\bullet$ of $\mathcal{O}_X$ is said to be* perfect *if any of the following equivalent conditions hold*

1. *For each point $x \in X$, there is a neighborhood U of x, a strictly perfect complex $F^\bullet$ on U, and a quasi-isomorphism $F^\bullet \xrightarrow{\sim} E^\bullet|_U$.*
2. *For each point $x \in X$, there is a neighborhood U of x, a strictly perfect complex $F^\bullet$ on U, and an isomorphism in $D(\mathcal{O}_X\text{-}Mod)$ between $E^\bullet|_U$ and $F^\bullet$.*

The perfect complexes have a bunch of properties. We shall define perfect complexes on super-schemes and prove those properties which are required for reconstruction.

4.2. **Perfect complexes on super-schemes.** Let $(X, \mathcal{O}_X = (\mathcal{O}_X)_0 \oplus (\mathcal{O}_X)_1)$ be a super-scheme. Following the scheme case let us define

Definition 4.2.1. *A complex $E^\bullet$ is called* strictly perfect *if it is a bounded complex of locally free (in the super sense) $\mathcal{O}_X$-sheaves. A complex $E^\bullet$ is called perfect if locally it is isomorphic to a strict perfect complex.*

Let $\mathcal{D}^{\text{perf}}(X)$ denote the triangulated category of perfect complexes on $(X, \mathcal{O}_X)$. As in the commutative case we can define

$$\text{supph}(E^\bullet) = \{x \in X \,:\, \text{the stalk complex } E^\bullet_x \text{ is not acyclic.}\}$$

Thus the problem of reconstruction of the scheme as a topological space reduces to proving that (X, supph) is a classifying support data. One way to prove that it is, would be to reduce to the known commutative case. That reduces to the following question. Is $\text{supph}(E^\bullet) = \text{supph}_{(X,\mathcal{O}_{X0})}(E^0 \oplus E^1)$?

The answer to the above question is yes, since $\text{supph}(E^\bullet) = \cup_{n\in\mathbb{Z}} H^n(E^\bullet) = \cup_{n\in\mathbb{Z}} H^n(E^\bullet_0 \oplus E^\bullet_1) = \text{supph}_{(X,\mathcal{O}_{X0})}(E^0 \oplus E^1)$. Therefore, (X, supph) is a classifying support data and hence we can reconstruct X as a topological space by theorem 3.2.3.

5. Reconstruction

5.1. **As a topological space.** We saw above that (X, supph) is a classifying support data and hence we have a homeomorphism (theorem 3.2.3) between X and $\text{Spec}(\mathcal{D}^{\text{perf}}(X))$.

5.2. **The structure sheaf.** Now it remains to prove that the structure sheaves are isomorphic. Define the structure sheaf on $\text{Spec}(\mathcal{D}^{\text{perf}}(X))$ by defining it to be $U \mapsto \text{End}_{\mathcal{O}_U}(\mathcal{O}_U)$. For a general $\mathbb{Z}/2\mathbb{Z}$ graded category with a unit $\mathbb{1}$, one can define the structure sheaf to be the following. For each open set $U \subset \text{Spec}(\mathcal{T})$, let Z be the complement. Let $\mathcal{T}_Z$ be the full triangulated subcategory of $\mathcal{T}$ consisting of objects a satisfying $\text{supp}(a) \subset Z$. Let $\mathbb{1}_Z$ be the image of the object $\mathbb{1}$ in $\mathcal{T}/\mathcal{T}_Z$ under the quotient functor. Then the super-ring associated to U is $\text{End}_{\mathcal{T}}(\mathbb{1}_Z)$.

Claim 5.2.1. $\text{End}_{\mathcal{T}}(\mathbb{1})$ *is a super-ring.*

Proof. Suppose $f : \mathbb{1} \to \mathbb{1}$ and $g : \mathbb{1} \to \mathbb{1}$ be two homogeneous morphisms. Then $g \circ f = (g \otimes \text{id}_\mathbb{1}) \circ (\text{id}_\mathbb{1} \otimes f) = (g \circ \text{id}_\mathbb{1}) \otimes (\text{id}_\mathbb{1} \circ f)$ since $|\text{id}_\mathbb{1}| = 0$. Therefore, $g \circ f = g \otimes f = (\text{id}_\mathbb{1} \circ g) \otimes (f \circ \text{id}_\mathbb{1}) = (-1)^{|g||f|}(\text{id}_\mathbb{1} \otimes f) \circ (g \otimes \text{id}_\mathbb{1}) = (-1)^{|g||f|} f \circ g$. This proves the above claim. □

Now it remains to prove that for the $\mathbb{Z}/2\mathbb{Z}$ graded category $\mathcal{D}^{\mathrm{perf}}(X)$, the associated structure sheaf is isomorphic to $\mathcal{O}_X$. We have already proved that $\operatorname{Spec}(\mathcal{D}^{\mathrm{perf}}(X)) \cong X$. We shall use that and confuse between the open sets in $\operatorname{Spec}(\mathcal{D}^{\mathrm{perf}}(X))$ and open sets in X. Let $U \subset X$ be an open set in X. We claim that $\mathcal{D}^{\mathrm{perf}}_Z(X) := (\mathcal{D}^{\mathrm{perf}}(X))_Z$ is such that $\mathcal{D}^{\mathrm{perf}}(X)/\mathcal{D}^{\mathrm{perf}}_Z(X)$ is equivalent to $\mathcal{D}^{\mathrm{perf}}(U)$. For this we use Neeman. Then $\mathbb{1}_Z$ will be nothing but $\mathbb{1}_U$ on U and hence the $\operatorname{End}(\mathbb{1}_Z)$ will be what we want.

Before we can prove the theorems we need the following definition:

Definition 5.2.2. *A quasi-compact, separated super-scheme X is said to have* ample family of line bundles *if there exists a collection of line bundles $\{\mathcal{L}_\alpha\}$ such that for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the evaluation map*

$$\bigoplus_{\alpha, n \geq 1} \Gamma\left(X,\ \mathcal{F} \otimes \mathcal{L}_\alpha^{\otimes n}\right) \otimes \mathcal{L}_\alpha^{\otimes(-n)} \longrightarrow \mathcal{F}$$

is an epimorphism.

Example 5.2.3. Any super-scheme with an ample line bundle will satisfy the above definition.

Theorem 5.2.4. *Let X be a super-scheme with an ample family of line bundles. Then perfect complexes are compact objects in $\mathcal{D}^b(X)$. They generate $\mathcal{D}^b(X)$.*

Recall that compact objects are those objects $c \in \mathcal{T}$ such that $\operatorname{Hom}_{\mathcal{T}}(c, \_)$ preserves coproducts.

Proof. The proof for the scheme case as was proved in [TT90, Theorem 2.4.1(e)], holds here too. The proof depended on the following fact.

Let X be a quasi-compact and quasi-separated super-scheme with an ample family of line bundles. Let $F^\bullet$ be a perfect complex of $\mathcal{O}_X$ modules. Then there is a strict perfect complex $E^\bullet$ and an isomorphism in the derived category of the category of $\mathcal{O}_X$ modules between $E^\bullet$ and $F^\bullet$.

The proof of this fact follows from the arguments of 2.3.1(d) of Thomason [TT90, Page 293]. In the proof of 2.4.1(e) in Thomason [TT90, Page 296], the step

$$\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(X)}(E^\bullet, F^\bullet) \xrightarrow{\cong} H^0(X, \operatorname{RHom}(E^\bullet, F^\bullet))$$

also holds as that reduces to showing $\_ \otimes E^\bullet$ is a left adjoint to $\operatorname{RHom}(E^\bullet, \_)$. From this the proof follows from the fact that coproducts commute with $\operatorname{RHom}(E, \_)$ and $H^0(X, \_)$. See for example, [Nee96, Lemma 1.4] and [Nee96, example 1.13].

The fact that perfect complexes generate the $\mathcal{D}^{\mathrm{b}}(X)$ can be deduced by adapting the arguments in [Nee96, Example 1.10]. □

Now we are ready to use the following theorem by Neeman [Nee92] [Nee96, Theorem 2.1] For a category $\mathcal{A}$, let $\mathcal{A}^{\mathrm{c}}$ be the full subcategory of compact objects in $\mathcal{A}$.

Theorem 5.2.5 (Neeman). *Let $\mathcal{S}$ be a compactly generated triangulated category. R be a set of compact objects of $\mathcal{S}$ closed under suspension. Let $\mathcal{R}$ be the smallest full subcategory of $\mathcal{S}$ containing R and closed with respect to coproducts and triangles. Let $\mathcal{T}$ be the quotient category $\mathcal{S}/\mathcal{R}$.*

1. *The category $\mathcal{R}$ is compactly generated, with R as a generating set.*
2. *If R happens to be a generating set of $\mathcal{S}$, then $\mathcal{R} = \mathcal{S}$.*
3. *If $R \subset \mathcal{R}$ is closed under formation of triangles and direct summands then it is all of $\mathcal{R}^c$. In any case, $\mathcal{R}^c = \mathcal{R} \cap \mathcal{S}^c$.*
4. *Suppose t is a compact object of $\mathcal{T}$. Then there is an object $t' \in \mathcal{T}^c$, an object $s \in \mathcal{S}^c$ and an isomorphism in $\mathcal{T}$, $s \cong t \oplus t'$. Furthermore, t' can be chosen to be $t[1]$, or any other object whose sum with t is zero in K_0.*
5. *Given an object $s \in \mathcal{S}^c$, an object $s' \in \mathcal{S}$, and a morphism in $\mathcal{T}$ $s \to s'$, there is a diagram in $\mathcal{S}$*

$$\begin{tikzcd} & \tilde{s} \arrow[dl] \arrow[dr] & \\ s & & s' \end{tikzcd}$$

where $\tilde{s}$ is compact; in the triangle $r \to \tilde{s} \to s \to r[1]$, the object r is in $\mathcal{R}^c$, and when we reduce the diagram to $\mathcal{T}$, the composite of the map $\tilde{s} \to s'$ with the inverse of $\tilde{s} \to s$ is the given map $s \to s'$.

Note that $\mathcal{D}^{\mathrm{b}}(X)/\mathcal{D}^{\mathrm{b}}_Z(X)$ is equivalent to $\mathcal{D}^{\mathrm{b}}(U)$ by definition of $\mathcal{D}^{\mathrm{b}}_Z(X)$. Thus we deduce the following result.

Theorem 5.2.6. *The idempotent completion of $\mathcal{D}^{perf}(X)/\mathcal{D}^{perf}_Z(X)$ is $\mathcal{D}^{perf}(U)$.*

Now the quotient functor respect tensor structure. This along with the discussions at the beginning of this subsection leads to the result that the given structure sheaf on $\mathrm{Spec}(\mathcal{D}^{\mathrm{perf}}(X))$ is isomorphic to $\mathcal{O}_X$.

Thus with the notion of tensor structure introduced in this paper, Balmer's construction can be used to reconstruct super-schemes from perfect complexes on them.

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