GAP PROBABILITIES FOR THE CARDINAL SINE

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Abstract. We study the zero set of random analytic functions generated by a sum of the cardinal sine functions which form an orthogonal basis for the Paley-Wiener space. As a model case, we consider real-valued Gaussian coefficients. It is shown that the asymptotic probability that there is no zero in a bounded interval decays exponentially as a function of the length.

1. Introduction

We study the asymptotic behaviour of the probability of a particular type of simple point processes not having any point in an interval of increasing length (gap probabilities). The simple point process we consider is given by the real zeros of the random function

$$f(z) = \sum_{n\in\mathbb{Z}} a_n \frac{\sin \pi(z-n)}{\pi(z-n)},$$

where a_n are i.i.d. random variables with zero mean and unit variance. Kolmogorov's inequality shows that this sum is almost surely pointwise convergent. In fact, since

$$\sum_{n\in\mathbb{Z}} \left| \frac{\sin \pi(z-n)}{\pi(z-n)} \right|^2$$

converges uniformly on compact subsets of the plane, this series almost surely defines an entire function. If we take a_n to be Gaussian random variables then f is a Gaussian analytic function (GAF). See [3, Lemma 2.2.3] for details.

We are chiefly concerned with the functions given by taking a_n to be real Gaussian random variables. We denote by n_f the counting measure on the set $Z(f)$ of zeros of f. These functions are an example of a stationary symmetric GAF. As a counterpart of the Kac-Rice formula [1, 4], Feldheim [2] has shown that the density of zeros is given by

$$\mathbb{E}[n_f(z)] = S(y)m(x,y) + \frac{1}{2\sqrt{3}}\mu(x), \tag{1}$$

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where $z = x + iy$, m denotes the planar Lebesgue measure, μ is the singular measure with respect to m supported on $\mathbb{R}$ and identical to Lebesgue measure there, and

$$S\Big(\frac{y}{2\pi}\Big) = \pi \left| \frac{d}{dy} \left(\frac{\cosh y - \frac{\sinh y}{y}}{\sqrt{\sinh^2 y - y^2}} \right) \right|.$$

(Here S is defined only for $y \neq 0$, in fact the atom appearing in (1) is the distributional derivative at 0.) We observe that since $S(y) = O(y)$ as y approaches zero there are almost surely zeros on the real line, but that they are sparse close by. Moreover the zero set is on average uniformly distributed on the real line. We are interested in the 'gap probability', that is the probability that there are no zeros in a large interval on the line.

Our result is the following asymptotic estimate.

Theorem 1. *Let f be the symmetric GAF given by the almost surely convergent series*

$$\sum_{n\in\mathbb{Z}} a_n \frac{\sin \pi(z-n)}{\pi(z-n)},$$

where a_n are i.i.d. real Gaussian variables with mean 0 and variance 1. Then, there exist constants $c, C > 0$ such that for all $r \geq 1$,

$$e^{-cr} \leq \mathbb{P}\Big(\#\big(Z(f) \cap (-r,r)\big) = 0\Big) \leq e^{-Cr}.$$

Remark 1. If instead of considering intervals we consider the rectangle $D_r = (-r,r)\times(-a,a)$ for some fixed $a > 0$, then we obtain a similar exponential decay for $\mathbb{P}(\#(Z(f) \cap D_r) = 0)$

Remark 2. Consider the case when the a_n are i.i.d. Rademacher distributed. I.e., each a_n is equal to either -1 or 1 with equal probability. Since $f(n) = a_n$ for $n \in \mathbb{N}$, it follows that if not all a_n for $|n| \leq N$ are of equal sign, then by the mean value theorem, f has to have a zero in $(-N, N)$. Following the proof of Theorem 1, the remaining two choices of the a_n for $|n| \leq N$ each yield an f without zeroes in $(-N, N)$, whence the desired probability is exactly $\mathrm{e}^{-(2N-1)\log 2}$ for $r = N$.

Remark 3. Whereas the Rademacher distribution is in some sense a simplified Gaussian, the Cauchy distribution, given by the density

$$p(x) = \frac{1}{\pi}\frac{1}{x^2+1},$$

is in some sense its opposite: It has neither an expectation, nor a standard deviation. If we suppose that the a_n are i.i.d. Cauchy distributed, it is not hard to see that with probability one the sum $\sum a_n/n$ diverges, whence the related random function diverges everywhere. For a study of random zeros in the polynomial case see [5].

We now give a short description of the main motivation for our work which comes from the 'hole theorems' proved by Sodin and Tsirelson [10] for point processes uniformly distributed in the plane, and by Peres and Virag [9] for the, so called, hyperbolic GAF in the disk.

The GAF considered by Sodin and Tsirelson [10] is given by

$$F(z)=\sum_{n=0}^{\infty} a_n \frac{z^n}{\sqrt{n!}},$$

where a_n are i.i.d. standard complex Gaussian variables. For this function, the density of zeros is proportional to the planar Lebesgue measure and they show that there are no zeros in a disc of radius r to be e^{-cr^4}, where $c>0$. It was later shown by A. Nishry [8] that $c=3\mathrm{e}^2/4+o(1)$ as $r\to\infty$. Observe that $\{\frac{z^n}{\sqrt{n!}}\}_{n=0}^{\infty}$ constitutes an orthonormal basis for the Bargmann-Fock space

$$\mathcal{F}=\Big\{f\in H(\mathbb{C}): \|f\|_{\mathcal{F}}^2=\int_{\mathbb{C}}|f(z)|^2 e^{-2|z|^2}\frac{dm(z)}{\pi}<+\infty\Big\},$$

where m is the planar Lebesgue measure. For a generalisation to several variables see [11].

The hyperbolic GAF considered in [9] is the determinantal process defined in $\mathbb{D}$ by the zeros of

$$F(z)=\sum_{n=0}^{\infty} a_n z^n,$$

where a_n are i.i.d. standard complex Gaussian variables. In this case, the asymptotic probability that there are no zeros in a disk of radius $r<1$ centered at zero is $e^{-ca(r)}$, where $c>0$ and $a(r)$ stands for the area of the disk in the hyperbolic metric. The set $\{z^n\}_{n=0}^{\infty}$ is an orthonormal basis for the Hardy space $H^2(\mathbb{D})$.

In analogy with these two previous cases, we replace the corresponding orthonormal basis with the sinc functions, which constitute an orthonormal basis for the Paley-Wiener space entire functions in $L^2(\mathbb{R})$ of exponential type at most π.

An important caveat is that, though the GAFs above are constructed from orthonormal bases, almost surely they do not belong to their respective spaces, since the sequence of coefficients a_n is almost surely not in $\ell^2(\mathbb{Z})$. In our case, however, it is not hard to see however that f belongs almost surely to the Cartwright class of entire functions of exponential type such that

$$\int_{-\infty}^{\infty}\frac{\log^+|f(x)|}{1+x^2}dx<\infty.$$

Finally, we mention that a generalisation of the Paley-Wiener space is given by the class of de Branges spaces

$$H(E)=\Big\{f \text{ entire }: \int_{\mathbb{R}}\left|\frac{f(x)}{E(x)}\right|^2 \mathrm{d}x<\infty, \text{ and } f/E, f/E^*\in H^2_+\Big\},$$

where H^2_+ is the Hardy space of the upper-half plane, $f^*(z) = \overline{f(\bar{z})}$ and E is an entire function such that $|E(z)| > |E^*(z)|$ for whenever $\operatorname{Im} z \geq 0$. Analog to the Paley-Wiener space, these spaces admit natural orthonormal bases (consisting of reproducing kernels). An important measure of the behavior of these spaces is the phase function $\phi(x) = -\operatorname{Arg} E(x)$. As was shown by Lyubarskii and Seip [6], when $a \leq \phi'(x) \leq b$, for $a, b > 0$ these spaces are in some sense similar to the Paley-Wiener space in terms of sampling and interpolating sequences. This analogy continues to hold under the weaker assumption that $\phi'(x)\mathrm{d}x$ is a doubling measure, as was shown in [7]. Under the former assumption, it is not hard to show that our results continue to hold. We include our results in this direction in a forthcoming paper.

2. Proof of Theorem 1

2.1. **Upper bound.** We want to compute the probability of an event that contains the event of not having any zeroes on $(-N, N)$, for $N \in \mathbb{N}$. One such event is that the values $f(n)$ have the same sign for $|n| \leq N$. The probability of this event is

$$\mathbb{P}\Big(a_n > 0 \text{ for } |n| \leq N \quad \text{or} \quad a_n < 0 \text{ for} |n| \leq N\Big) = 2(1/2)^{2N+1} = \mathrm{e}^{-CN},$$

for some constant $C > 0$.

Remark 4. The same upper bound holds when a_n are i.i.d. random variables with $0 < \mathbb{P}(a_n > 0) < 1$ for which the random function $\sum_{n\in\mathbb{Z}} a_n \mathrm{sinc}(x-n)$ converges.

2.2. **Lower bound.** To compute the lower hole probability, we use the following scheme. First, we introduce the deterministic function

$$f_0(x) = \sum_{n=-2N}^{2N} \mathrm{sinc}(x-n).$$

We show in Lemma 1 that it has no zeroes on $(-N, N)$, and we find an explicit lower bound on $(-N, N)$ for it. This lower bound does not depend on N. Second, we consider the functions

$$f_1(x) = \sum_{n=-2N}^{2N} (a_n - 1)\mathrm{sinc}(x-n) \qquad \text{and} \qquad f_2(x) = \sum_{|n|>2N} a_n \mathrm{sinc}(x-n),$$

which induce the splitting

$$f = f_0 + f_1 + f_2.$$

We show that for all $x \in [-N, N]$ we have $|f_1(x)| \leq \epsilon$ with probability at least e^{-cN} for large N and some constant $c > 0$. Moreover, we show that

$$\mathbb{P}\Big(\sup_{x\in[-N,N]} |f_2(x)| \leq \epsilon\Big)$$

is larger than, say, $1/2$ for big enough N. As the events on f_1 and f_2 are clearly independent, the lower bound now follows by choosing ϵ small enough.

We turn to the first part of the proof.

Lemma 1. *Given* $N \in \mathbb{N}$ *and*

$$f_0(x) = \sum_{n=-2N}^{2N} \operatorname{sinc}(x-n) = \sin \pi x \sum_{n=-2N}^{2N} \frac{(-1)^n}{\pi(x-n)}. \tag{2}$$

Then, there exists a constant $C > 0$ *such that, for* N *big enough,*

$$1 - \frac{C}{N} \le \inf_{|x|\le N} f_0(x) \le \sup_{|x|\le N} f_0(x) \le 1 + \frac{C}{N}.$$

Proof. Let $R = R(N)$ be the boundary of the square of length $4N+1$, centered at the origin. By the residue theorem, it holds that for $-N \le x \le N$ not an integer

$$\frac{1}{2\pi i}\oint_R \frac{dz}{(z-x)\sin \pi z} = \frac{1}{\pi}\sum_{n=-2N}^{2N} \frac{(-1)^n}{n-x} + \frac{1}{\sin \pi x}.$$

Observe that if we shift around the terms, this yields

$$\frac{\sin \pi x}{\pi}\sum_{n=-2N}^{2N} \frac{(-1)^n}{x-n} = 1 + \frac{\sin \pi x}{2\pi i}\oint_R \frac{dz}{(x-z)\sin \pi z}.$$

It is easy now to bound this last integral by C/N. □

Remark 5. The same bound holds for all points z in a strip with fixed height $[-N,N]\times[-C,C]$ for some $C>0$ and N big enough depending on C. We observe though, that the function $f_0(z)$ in (2) is close to zero around $\operatorname{Im} z = \log N$. Indeed, it is smaller than e^{-cN} there, for some $c > 0$ independent of N. Therefore the estimate proved in the lemma above does not work for z on squares.

2.2.1. *The middle terms.* Let $\epsilon > 0$ be given, and consider $N \in \mathbb{N}$ to be fixed. We look at the function

$$f_1(x) = \sum_{n=-2N}^{2N} (a_n - 1)\operatorname{sinc}(x-n) = \frac{\sin \pi x}{\pi}\sum_{n=-2N}^{2N} (a_n-1)\frac{(-1)^n}{x-n}.$$

To simplify the expression, we set $b_n = (a_n-1)(-1)^n$. We want to compute a lower bound for the probability that, for $x \in [-N,N]$,

$$|f_1(x)| \le \epsilon.$$

Since, for $|n| \le N$, we have $f_1(n) = a_n - 1$, the condition

$$|a_n - 1| \le \epsilon \quad \text{for} \quad |n| \le N$$

is necessary.

Define $B_n = b_{-2N} + \dots b_n$ for $|n| \leq 2N$ with $B_{-2N-1} = 0$, and suppose that $x \notin \mathbb{Z}$. With this, summation by parts yields

$$\sum_{-2N}^{2N} \frac{b_n}{x-n} = -\sum_{-2N}^{2N} \frac{B_n}{(x-n)(x-n-1)} + \frac{B_{2N}}{x-2N-1}. \tag{3}$$

We now claim that under the event

$$E = \Big\{ |b_n| \leq \epsilon, \ |B_n| \leq \epsilon \quad \text{for} \quad |n| \leq 2N \Big\}, \tag{4}$$

we have $|f_1(x)| \leq \epsilon$ for $|x| \leq N$, with a bound independent of N. Indeed, the second summand at the right hand side of (3) converges almost surely to zero, because

$$\left| \frac{B_{2N}}{x-2N-1} \right| \leq \frac{\epsilon}{N}.$$

Suppose that $x \in (k, k+1)$ and split the first sum in (3) as

$$\sum_{\substack{n=-2N \\ n \neq k-1,k,k+1}}^{2N} \frac{B_n}{(x-n)(x-n-1)} + \sum_{n=k-1}^{k+1} \frac{B_n}{(x-n)(x-n-1)}.$$

Then

$$\left| \sum_{\substack{n=-2N \\ n \neq k-1,k,k+1}}^{2N} \frac{B_n}{(x-n)(x-n-1)} \right| \leq \sum_{n \geq k+2} \frac{\epsilon}{(k+1-n)^2} + \sum_{n \leq k-2} \frac{\epsilon}{(k-n)^2} \lesssim \epsilon.$$

For the remaining terms, the function $\sin \pi x$ comes into play. E.g., suppose that $|x-k| \leq 1/2$, then

$$\left| \sin \pi x \frac{B_k}{(x-k)(x-k-1)} \right| \lesssim \frac{\epsilon}{|x-k-1|} \left| \frac{\sin \pi (x-k)}{\pi (x-k)} \right| \lesssim \epsilon.$$

The remaining terms are treated in exactly the same way.

What remains is to compute the probability of the event E defined by (4). We recall that the b_n were all defined in terms of the real and independent Gaussian variables a_n. So the event E above defines a set

$$V = \left\{ (t_{-2N}, \dots t_{2N}) \in \mathbb{R}^{4N+1} : |t_n| \leq \epsilon, \ |\sum_{-2N}^{n} t_n| \leq \epsilon, \ |n| \leq 2N \right\}$$

in terms of the values of the b_n. Hence,

$$\mathbb{P}(E) = c^{4N+1} \int_V \mathrm{e}^{-((t_{-2N}-1)^2 + \cdots (t_{2N}-1)^2)/2} \mathrm{d}t_{-2N} \cdots \mathrm{d}t_{2N}.$$

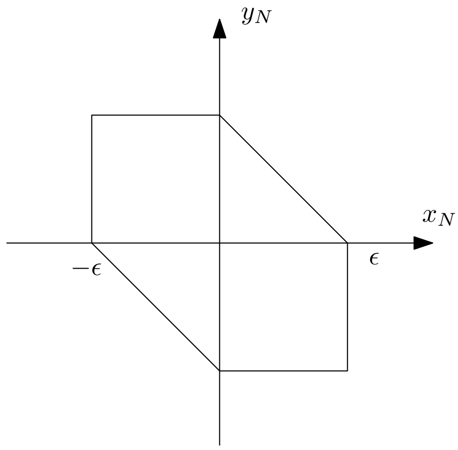

FIGURE 1. Illustration of the solid V_N.

Here, c is the normalising constant of the one dimensional Gaussian. It follows that

$$\mathbb{P}(E) \geq c^{4N+1}\mathrm{e}^{-(4N+1)(1+\epsilon)^2/2}\int_V \mathrm{d}t_{-2N}\cdots\mathrm{d}t_{2N} = c^{4N+1}\mathrm{e}^{-(4N+1)(1+\epsilon)^2/2}\mathrm{Vol}(V).$$

We now seek a lower bound for this euclidean $(4N+1)$-volume.

To simplify notation, we pose this problem as follows. For real variables $x_1,\ldots,x_N$, we wish to compute the euclidean volume of the solid V_N defined by $|x_i| \leq \epsilon$ for $i = 1,\ldots,N$ and

$$\begin{aligned} |x_1 + x_2| &\leq \epsilon, \\ |x_1 + x_2 + x_3| &\leq \epsilon \\ &\vdots \\ |x_1 + x_2 + \cdots + x_N| &\leq \epsilon. \end{aligned}$$

One way to do this is as follows. Write $y_N = x_1 + \cdots + x_{N-1}$, then

$$\mathrm{Vol}(V_N) = \int_{V_{N-1}}\left(\int_{\max\{-\epsilon,-\epsilon-y_N\}}^{\min\{\epsilon,\epsilon-y_N\}}\mathrm{d}x_N\right)\mathrm{d}x_1\cdots\mathrm{d}x_{N-1}.$$

This is illustrated in Figure 1. Clearly, whenever $y_N < 0$, the upper limit is ϵ, and whenever $y_N > 0$, the lower limit is $-\epsilon$. Hence,

$$\begin{aligned}\mathrm{Vol}(V_N) \geq &\int_{V_{N-1}\cap\{y_N<0\}}\left(\int_0^{\epsilon}\mathrm{d}x_N\right)\mathrm{d}x_1\cdots\mathrm{d}x_{N-1} \\ &+ \int_{V_{N-1}\cap\{y_N>0\}}\left(\int_{-\epsilon}^{0}\mathrm{d}x_N\right)\mathrm{d}x_1\cdots\mathrm{d}x_{N-1} = \epsilon\mathrm{Vol}(V_{N-1}).\end{aligned}$$

Iterating this, we get

$$\mathrm{Vol}(V_N) \geq \epsilon^N.$$

In conclusion,

$$\mathbb{P}(E) \geq \mathrm{e}^{-cN},$$

which concludes this part of the proof.

2.2.2. *The tail.* We now turn to the tail term

$$f_2(x) = \sin \pi x \sum_{|n|>2N} \frac{a_n(-1)^n}{\pi(x-n)}.$$

Clearly, we need only consider the terms for which n is positive. Set $c_n = (-1)^n a_n$. We apply summation by parts, to get

$$\sum_{n>2N}^{L} \frac{c_n}{x-n} = \sum_{2N+1}^{L} C_n \frac{-1}{(x-n)(x-n-1)} + \frac{C_L}{x-L-1}. \tag{5}$$

where

$$C_n = c_{2N+1} + \cdots + c_n, \quad C_{2N} = 0.$$

We want to take the limit as $L \to \infty$. It is easy to see that the last term almost surely tends to zero. Indeed, C_L is a sum of independent Gaussian variables with mean 0 and variance 1, and therefore is itself Gaussian with mean 0 and variance $L - 2N$. Moreover, since for all $|x| \leq N$

$$\left| \frac{C_L}{x-L-1} \right| \lesssim \left| \frac{C_L}{N-L} \right|,$$

and the random variable inside of the absolute values on the right-hand side has variance $L - 2N$, it follows by the law of large numbers that the limit is almost surely equal to 0, whence we are allowed to let $L \to \infty$ in (5).

We prove the following. With a positive probability, we have for $|x| \leq N$

$$\left| \sum_{2N+1}^{L} C_n \frac{1}{(x-n)(x-n-1)} \right| \leq \epsilon.$$

As $n^2 \simeq |(x-n)(x-n-1)|$ for $|x| \leq N$ and $n > 2N$, it is enough to consider the expression

$$\sum_{2N+1}^{\infty} \frac{|C_n|}{n^2}.$$

The absolute value of a Gaussian random variable has the folded-Gaussian distribution. In particular, if $X \sim N(0, \sigma^2)$, then

$$\mathbb{E}(|X|) = \sigma \sqrt{\frac{2}{\pi}}.$$

Since, in our case, $\sigma^2 = n - 2N$, this yields

$$\mathbb{E}\left(\sum_{2N+1}^{\infty} \frac{|C_n|}{n^2} \right) \lesssim \sum_{2N+1}^{\infty} \frac{\sqrt{n-2N}}{n^2} \lesssim \sum_{1}^{\infty} \frac{1}{(n-2N)^{3/2}} \lesssim \frac{1}{\sqrt{N}}.$$

Finally, by Chebyshev's inequality,

$$\mathbb{P}\left(\sum_{2N+1}^{\infty}\frac{|C_n|}{n^2}\leq\epsilon\right)\geq 1-\frac{1}{\epsilon}\mathbb{E}\left(\sum_{2N+1}^{\infty}\frac{|C_n|}{n^2}\right)\geq 1-\frac{C}{\epsilon\sqrt{N}}.$$

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