# ON SPECTRAL APPROXIMATION, FØLNER SEQUENCES AND CROSSED PRODUCTS

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*Dedicated to Paco Marcellán on his 60th birthday*

ABSTRACT. In this article we review first some of the possibilities in which the notions of Følner sequences and quasidiagonality have been applied to spectral approximation problems. We construct then a canonical Følner sequence for the crossed product of a concrete $C^*$-algebra and a discrete amenable group. We apply our results to the rotation algebra (which contains interesting operators like almost Mathieu operators or periodic magnetic Schrödinger operators on graphs) and the $C^*$-algebra generated by bounded Jacobi operators.

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## 1. INTRODUCTION

Given a sequence of linear operators $\{T_n\}_{n\in\mathbb{N}}$ in a complex separable Hilbert space $\mathcal{H}$ that approximates an operator $T$ in a suitable sense, a natural question is how do the spectral objects of $T$ (the spectrum, spectral measures, numerical ranges, pseudospectra etc.) relate with those of $T_n$ as $n$ grows. (Excellent books that include a large number of examples and references are, e.g., [17, 4].) In such

generality this problem is almost impossible to address. In fact, one can easily produce examples with bad spectral approximation behavior, where the spectrum suddenly expands or contracts in the limit (see, e.g., pp. 289-291 in [37]). Also spectral pollution effects or spurious eigenvalues can may appear as a consequence of the approximation process ([21]). One of the standard methods to treat these problems is to compress $T$ to a finite dimensional subspace $\mathcal{H}_n$ (the so-called finite-section) and, then, analyze the behavior of the eigenvalues of the matrix $P_n T \restriction \mathcal{H}_n$ in the limit of large $n$. This method requires also additional conditions in order to guarantee a good approximation behavior of spectral objects.

The following classical approximation result for Toeplitz operators due to Szegö gives an example where the finite section method can be used to approximate spectral measures: denote by $\mathbb{T}$ the unit circle with normalized Haar measure $d\theta$ and consider the real-valued functions $g$ in $L^\infty(\mathbb{T})$ which can be thought as (self-adjoint) multiplication operators on the complex Hilbert space $\mathcal{H} := L^2(\mathbb{T})$, i.e., $M_g \varphi = g \varphi$, $\varphi \in \mathcal{H}$. Denote by $P_n$ the finite-rank orthogonal projection onto the linear span of $\{z^l \mid z \in \mathbb{T}, l = 0, \dots, n\}$ and let $M_g^{(n)}$ be the corresponding finite section matrix. Write the corresponding eigenvalues (repeated according to multiplicity) as $\{\lambda_{0,n}, \dots, \lambda_{n,n}\}$. Then, for any continuous $f \colon \mathbb{R} \to \mathbb{R}$ one has

$$(1.1) \qquad \lim_{n \to \infty} \frac{1}{n+1} \Big( f(\lambda_{0,n}) + \cdots + f(\lambda_{n,n}) \Big) = \int_{\mathbb{T}} f(g(\theta)) \, d\theta$$

(see [40, Section 8], [23, Chapter 5] and [44] for a careful analysis of this result). This theorem may be also reformulated in terms of weak*-convergence of the corresponding spectral measures and it allows to approximate numerically the spectrum of $M_g$ in terms of the eigenvalues its finite sections (cf. [3]). Szegö's classical result suggests the following question: what is the reason that guarantees the convergence of spectral measures and that can be possibly useful beyond the context of Toeplitz operators? In the last two decades there has been a considerable application of methods from operator algebras (mainly $C^*$-algebras and von Neumann algebras[1]) to this question. This line of research was pioneered by Arveson in the series of papers [1, 2, 3] which were directly inspired by Szegö's theorem (see, e.g., [10, 29] for related developments in numerical analysis). Among other interesting results, Arveson gave conditions that guarantee that the essential spectrum of a large class of selfadjoint operators $T$ may be recovered from the sequence of eigenvalues of certain finite dimensional compressions $T_n$ (see also Subsection 4.2). These results were then refined by Bédos who systematically applied the concept of Følner sequences to spectral approximation problems (see [7, 6, 5] as well as [16]). Hansen extends some of

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[1]For the purposes of this article we will define a $C^*$-algebra to be a *-subalgebra of $\mathcal{L}(\mathcal{H})$ (the set of bounded linear operators in $\mathcal{H}$) which is closed in the topology defined by the operator norm $\| \cdot \|$. If $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ we will denote by $C^*(\mathcal{T})$ the $C^*$-algebra generated by $\mathcal{T}$. A von Neumann is an important subclass of $C^*$-algebras which is closed in the weak operator topology.

the mentioned results to the case of unbounded operators (cf. [26, § 7]; see also [27] for recent developments in the non-selfadjoint case). Brown shows in [12] that abstract results in $C^*$-algebra theory can be applied to compute spectra of important operators in mathematical physics like almost Mathieu operators or periodic magnetic Schrödinger operators on graphs.

We recall next two related notions that are important when addressing spectral approximation problems: let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a separable set of bounded linear operators on the complex separable Hilbert space $\mathcal{H}$. An increasing sequence of non-zero finite rank orthogonal projections $\{P_i\}_{i \in \mathbb{N}}$ is called a Følner sequence for $\mathcal{T}$, if

$$\lim_i \frac{\|TP_i - P_iT\|_2}{\|P_i\|_2} = 0, \quad T \in \mathcal{T},$$

where $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm. The existence of a Følner sequences for a set of operators $\mathcal{T}$ is a weaker notion than quasidiagonality which was introduced by Halmos in the late sixties (cf. [25]). Recall that a set of operators $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ is said to be quasidiagonal if there exists an increasing sequence of finite-rank projections $\{P_i\}_i$ as before such that

$$\lim_i \|TP_i - P_iT\| = 0, \quad T \in \mathcal{T}.$$

It is easy to show that if $\{P_i\}_{i \in I}$ quasidiagonalizes the set of operators $\mathcal{T}$, then it is also a Følner sequences for $\mathcal{T}$ (for details see the next section). Moreover, the Følner condition above can be understood as a quasidiagonality condition, but relative to the growth of the dimension of the corresponding subspaces. Quasidiagonality is an important property in the analysis of the structure of $C^*$-algebras (see, e.g., Chapter 7 in [15] or [13, 8, 11, 43]) and is also a very useful notion in spectral approximation problems. E.g. quasidiagonality is assumed to prove the converge of spectra (in the selfadjoint case) and pseudospectra (cf. [14] and [26, § 2]). Følner sequences were introduced in the context of operator algebras by Alain Connes in his seminal paper [18, Section V] (see also [19, 35, 36]). This notion is an algebraic analogue of Følner's characterization of amenable discrete groups (see Section 2 for precise definitions) and was used by Connes as an essential tool in the classification of injective type $II_1$ factors.

The third expression in the title of this article refers to crossed products. This is a basic operator algebraic construction which is interesting in its own right (see Section 3 for details). The crossed product may be seen as a new $C^*$-algebra constructed from a given $C^*$-algebra which carries an action of a group $\Gamma$. Many important operators in mathematical physics with very interesting spectral properties can be identified as elements of certain crossed products (see, e.g., [9, 30] and Subsection 4.1 below). The question when a crossed product of a quasidiagonal $C^*$-algebra by an amenable group is again quasidiagonal has been addressed several times in the past (see, e.g., Section 11 in [13] for some partial answers).

In Lemma 3.6 of [34] it is shown that if $\mathcal{A}$ is a unital separable quasidiagonal $C^*$-algebra with almost periodic group action $\alpha\colon \mathbb{Z} \to \mathrm{Aut}\mathcal{A}$, then the $C^*$-crossed-product $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ is again quasidiagonal. This result has been extended recently by Orfanos to $C^*$-crossed products $\mathcal{A} \rtimes_\alpha \Gamma$, where now $\Gamma$ is a discrete, countable, amenable, residually finite group (cf. [31]). In this more general case there is again a certain condition on the group action of $\Gamma$ on $\mathcal{A}$ is needed. This result has been applied to generalized Bunce-Deddens algebras in [32].

The existence of Følner sequences may be established in abstract terms, but this gives in general no clues of what are the concrete matrix approximations. The aim of the present paper is to give conditions and construct explicitly Følner sequences for the crossed product of a $C^*$-algebra $\mathcal{A}$ that has a Følner sequence and a discrete countable amenable group $\Gamma$ (see Theorem 3.4 for a precise statement). Our results partly extend those of Bédos for group von Neumann algebras and are related to the articles of Orfanos mentioned above. In Section 4 we will apply our results in two concrete situations: Theorem 3.4 can be applied to the rotation algebra (also known as non-commutative torus), since it can be seen as a crossed product of $C(\mathbb{T})$ by $\mathbb{Z}$. This algebra contains interesting examples from the spectral point of view, like almost Mathieu operators or discrete Schrödinger operators with magnetic potentials. Finally, we will also construct explicit Følner sequences for Jacobi operators.

## 2. Følner sequences and quasidiagonality

The notion of Følner sequences for operators has its origins in group theory. Recall that a discrete countable group $\Gamma$ is amenable if it has an invariant mean, i.e. there is a continuous linear functional $\psi$ on $\ell^\infty(\Gamma)$ with norm one and such that

$$\psi(u_\gamma f) = \psi(f) \,, \quad \gamma \in \Gamma \,, \quad f \in \ell^\infty(\Gamma) \,,$$

where $u$ is the left-regular representation on $\ell^2(\Gamma)$. A Følner sequence for $\Gamma$ is a sequence of non-empty finite subsets $\Gamma_i \subset \Gamma$ that satisfy

$$(2.1) \qquad\qquad \lim_i \frac{|(\gamma\Gamma_i)\triangle\Gamma_i|}{|\Gamma_i|} = 0 \qquad \text{for all} \quad \gamma \in \Gamma \,,$$

where $\triangle$ denotes the symmetric difference and $|\Gamma_i|$ is the cardinality of $\Gamma_i$. Then, $\Gamma$ has a Følner sequence if and only if $\Gamma$ is amenable (cf. Chapter 4 in [33]). Some authors require, in addition to Eq. (2.1), that the sequence is increasing and complete, i.e. $\Gamma_i \subset \Gamma_j$ if $i \leq j$ and $\Gamma = \cup_i\Gamma_i$. We will not need these additional assumptions here.

The counterpart of the previous definition in the context of operators is given as follows:

**Definition 2.1.** *Let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a set of bounded linear operators on the complex separable Hilbert space $\mathcal{H}$. An increasing sequence of non-zero finite*

rank orthogonal projections $\{P_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ is called a Følner sequence for $\mathcal{T}$ if

$$(2.2) \qquad \lim_n \frac{\|TP_n - P_nT\|_2}{\|P_n\|_2} = 0 \ , \quad T \in \mathcal{T} \ ,$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm.

We will state next some immediate consequences of the definition that will be used later on. To simplify expressions in the rest of the article we introduce the commutator of two operators: $[A, B] := AB - BA$.

**Proposition 2.2.** *Let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a set of operators and $\{P_n\}_{n\in\mathbb{N}}$ an increasing sequence of non-zero finite rank orthogonal projections.*

(i) *$\{P_n\}_{n\in\mathbb{N}}$ is a Følner sequence for $\mathcal{T}$ iff it is a Følner sequence for $C^*(\mathcal{T})$ (the $C^*$-algebra generated by $\mathcal{T}$).*

(ii) *Let $\mathcal{T}$ be a self-adjoint set (i.e., $\mathcal{T}^* = \mathcal{T}$). Then $\{P_n\}_{n\in\mathbb{N}}$ is a Følner sequence for $\mathcal{T}$ if one of the four following equivalent conditions holds for all $T \in \mathcal{T}$:*

$$(2.3) \qquad \lim_n \frac{\|TP_n - P_nT\|_p}{\|P_n\|_p} = 0, \qquad p \in \{1, 2\}$$

*or*

$$(2.4) \qquad \lim_n \frac{\|(I - P_n)TP_n\|_p}{\|P_n\|_p} = 0, \qquad p \in \{1, 2\} \ ,$$

*where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the trace-class and Hilbert-Schmidt norms, respectively.*

*Proof.* (i) We just have to show that if $\{P_n\}_{n\in\mathbb{N}}$ is a Følner sequence for $\mathcal{T}$, then it is a Følner sequence for $C^*(\mathcal{T})$. For $R, T \in \mathcal{T}$ the following elementary relations

$$\|[RT, P_n]\|_2 \leq \|R[T, P_n]\|_2 + \|[R, P_n]T\|_2 \leq \|R\| \|[T, P_n]\|_2 + \|[R, P_n]\|_2 \|T\|$$

$$\|[T^*, P_n]\|_2 = \|[T, P_n]^*\|_2 \ = \ \|[T, P_n]\|_2$$

show that $\{P_n\}_{n\in\mathbb{N}}$ is a Følner sequence for the *-algebra $\widetilde{\mathcal{T}}$ generated by $\mathcal{T}$. Then it is a standard $\frac{\varepsilon}{2}$-argument to show that $\{P_n\}_{n\in\mathbb{N}}$ is still a Følner sequence for the norm closure of $\widetilde{\mathcal{T}}$.

(ii) By the previous item we have that $\{P_n\}_{n\in\mathbb{N}}$ is a Følner sequence for $\mathcal{T}$ iff it is a Følner sequence for $C^*(\mathcal{T})$ and we can apply Lemma 1 in [5]. $\qquad\square$

The existence of a Følner sequences has important structural consequences. For the next result we need to recall the following notion: a hypertrace for a $C^*$-algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ is a state $\Psi$ on $B(\mathcal{H})$ that is centralized by $\mathcal{A}$, i.e.

$$\Psi(XA) = \Psi(AX) \ , \quad X \in B(\mathcal{H}) \ , \ A \in \mathcal{A} \ .$$

Hypertraces are the algebraic analogue of the invariant mean mentioned at the begining of this section (cf. [18, 19, 5]).

**Proposition 2.3.** *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a $C^*$-algebra. Then $\mathcal{A}$ has a Følner sequence iff $\mathcal{A}$ has a hypertrace.*

**Examples 2.4.**     (i) *The unilateral shift is a canonical example that shows the difference between the notions of Følner sequences and quasidiagonality. On the one hand, it is a well-known fact that the unilateral unilateral shift $S$ is not a quasidiagonal operator. (This was shown by Halmos in [24]; in fact, in this reference it is shown that $S$ is not even quasitriangular.) If $\mathcal{A}$ is a $C^*$-algebras containing a proper (i.e. non-unitary) isometry, then it not quasidiagonal (see, e.g. [13, 15]). Finally, it can be shown that certain weighted shifts are quasidiagonal (cf. [38]).*

*On the other hand, it is easy to find a canonical Følner sequence for $S$. In fact, define $S$ on $\mathcal{H} := \ell^2(\mathbb{N}_0)$ by $Se_i := e_{i+1}$, where $\{e_i \mid i = 0, 1, 2, \dots\}$ is the canonical basis of $\mathcal{H}$ and consider for any $n$ the orthogonal projections $P_n$ onto $span\{e_i \mid i = 0, 1, 2, \dots, n\}$. Then*

$$\big\|[P_n, S]\big\|_2^2 = \sum_{i=1}^{\infty} \Big\|[P_n, S]e_i\Big\|^2 = \|e_{n+1}\|^2 = 1$$

*and*

$$\frac{\big\|[P_n, S]\big\|_2}{\|P_n\|_2} = \frac{1}{\sqrt{n+1}} \underset{n\to\infty}{\longrightarrow} 0 \ .$$

*A similar argument shows directly that $\{P_n\}_n$ is a Følner sequence for any power $S^k$, $k \in \mathbb{N}$. By Proposition 2.2 (i) it follows that $\{P_n\}_n$ is also a Følner sequence for the Toeplitz algebra $C^*(S)$.*

(ii) *Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator. If a sequence of non-zero finite rank orthogonal operators $\{P_n\}_n$ satisfies*

$$\sup_{n\in\mathbb{N}} \|(\mathbb{1} - P_n)TP_n\|_2 < \infty \ ,$$

*then Eq. (2.4) implies that $\{P_n\}_n$ is clearly a Følner sequence for $T$. Concrete examples satisfying the preceding condition are self-adjoint operators with a band-limited matrix representation (see, e.g., [2, 3]). Band limited operators together with quasidiagonal operators are the essential ingredients in the solution of Herrero's approximation problem, i.e. the characterization of the closure of block diagonal operators with bounded blocks (see Chapter 16 in [15] for a comprehensive presentation).*

## 3. Følner sequences for crossed products

The crossed product may be seen as a new $C^*$-algebra constructed from a given $C^*$-algebra which carries an action of a group $\Gamma$. This procedure goes back to the pioneering work of Murray and von Neumann on rings of operators. Algebraically, this construction has some similarities with the semi-direct product of groups. Standard references which present the crossed product construction

with small variations are [20, Chapter VIII], [41, Chapter 4], [42, Section V.7] or [28, Section 8.6 and Chapter 13]. Since all groups $\Gamma$ considered will be amenable (and countable) we will not distiguish between crossed products and reduced crossed products.

Throughout this section $\mathcal{A}$ denotes a concrete $C^*$-algebra acting on a complex separable Hilbert space $\mathcal{H}$. We shall assume that $\alpha$ is an automorphic representation of a countable discrete amenable group $\Gamma$ on $\mathcal{A}$, i.e.

$$\alpha \colon \Gamma \to \operatorname{Aut} \mathcal{A} .$$

The crossed product is a new $C^*$-algebra constructed with the previous ingredients and acting on the separable Hilbert space

$$(3.1) \qquad \mathcal{K} := \ell^2(\Gamma) \otimes \mathcal{H} \cong \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma ,$$

where $\mathcal{H}_\gamma \equiv \mathcal{H}$ for all $\gamma \in \Gamma$. To make this notion precise we introduce representations of $\mathcal{A}$ and $\Gamma$ on $\mathcal{K}$: for $\xi = (\xi_\gamma)_\gamma \in \mathcal{K}$ we define

$$(3.2) \qquad (\pi(M)\xi)_\gamma := \alpha_\gamma^{-1}(M)\, \xi_\gamma , \quad M \in \mathcal{A},$$

$$(3.3) \qquad (U(\gamma_0)\,\xi)_\gamma := \xi_{\gamma_0^{-1}\gamma} .$$

The crossed product of $\mathcal{A}$ by $\Gamma$ is the $C^*$-algebra on $\mathcal{K}$ generated by these operators, i.e.,

$$\mathcal{N} = \mathcal{A} \rtimes_\alpha \Gamma := C^* \Big( \{\pi(M) \mid M \in \mathcal{A}\} \cup \{U(\gamma) \mid \gamma \in \Gamma\} \Big) \subset \mathcal{L}(\mathcal{K}) ,$$

where $C^*(\cdot)$ denotes the $C^*$-algebra generated by its argument. A characteristic relation for the crossed product is

$$(3.4) \qquad \pi\left(\alpha_\gamma(M)\right) = U(\gamma)\pi(M)U(\gamma)^{-1} ,$$

that is, $\pi$ is a covariant representation of the $C^*$-dynamical system $(\mathcal{A}, \Gamma, \alpha)$.

**Remark 3.1.** *Later we will need the following useful operator matrix characterization of the elements in the crossed product: consider the identification $\mathcal{K} \cong \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$ with $\mathcal{H}_\gamma \equiv \mathcal{H}$, $\gamma \in \Gamma$. Then, every $T \in \mathcal{L}(\mathcal{K})$ can be written as a matrix $(T_{\gamma'\gamma})_{\gamma',\gamma \in \Gamma}$ with entries $T_{\gamma'\gamma} \in \mathcal{L}(\mathcal{H})$. Any element $N$ in the crossed product $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$ has the form*

$$(3.5) \qquad N_{\gamma'\gamma} = \alpha_\gamma^{-1}\left(A(\gamma'\gamma^{-1})\right) , \quad \gamma', \gamma \in \Gamma,$$

*for some mapping $A \colon \Gamma \to \mathcal{A} \subset \mathcal{L}(\mathcal{H})$. Roughly, this means that the "diagonals" of the operator matrices of elements in the crossed product $\mathcal{N}$ are orbits of the group action on elements in $\mathcal{A}$.*

*For example, the matrix form of the product of generators $N := \pi(M) \cdot U(\gamma_0)$, $M \in \mathcal{A}$, $\gamma_0 \in \Gamma$ is given by*

$(3.6)$

$$N_{\gamma'\gamma} = \alpha_{\gamma'}^{-1}(M)\, \delta_{\gamma',\gamma_0\gamma} = \alpha_{\gamma'}^{-1}\left(A(\gamma'\gamma^{-1})\right) \text{, where } A(\widetilde{\gamma}) := \begin{cases} M & \text{if } \widetilde{\gamma} = \gamma_0 \\ 0 & \text{otherwise.} \end{cases}$$

*This implies that any function $A\colon \Gamma \to \mathcal{A}$ with finite support determines by means of Eq. (3.5) an element in the crossed product.*

3.1. **Construction of a canonical Følner sequence.** The aim of the present section is to give a canonical example of a Følner sequence for the crossed product $C^*$-algebra $\mathcal{N} = \mathcal{A} \rtimes_\alpha \Gamma$ constructed above. Since $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$ with $\mathcal{K} = \ell^2(\Gamma) \otimes \mathcal{H}$, our sequence is canonical in the sense that it uses explicitly a Følner sequences for $\Gamma$ and a sequence of projections on $\mathcal{H}$ (cf. Theorem 3.4). We will also assume that the $C^*$-algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is separable and has a Følner sequence $\{Q_i\}_{i\in\mathbb{N}}$.

We begin recalling some parts of Proposition 4 in [5]:

**Proposition 3.2.** *Assume that the group $\Gamma$ is countable and amenable and denote by $\{P_i\}_{i\in\mathbb{N}}$ the sequence of orthogonal finite-rank projections from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma_i)$ associated to a Følner sequence $\{\Gamma_i\}_{i\in\mathbb{N}}$ for the group $\Gamma$ (cf. Section 2). Then $\{P_i\}_i$ is a Følner sequence for the group $C^*$-algebra*

$$\mathcal{A}_\Gamma := C^*\{\overline{U}(\gamma) \mid \gamma \in \Gamma\} \subset \mathcal{L}(\ell^2(\Gamma)) \,,$$

*where $\overline{U}$ is the left regular representation of $\Gamma$ on $\ell^2(\Gamma)$.*

**Remark 3.3.**
(i) *In Proposition 4 of [5] the author proves a stronger result. He shows that the canonical Følner net $\{P_i\}_{i\in I}$ for the algebra and associated to the Følner net of the (not necessarily countable) amenable group is still a Følner net for the corresponding group von Neumann algebra, i.e., for the weak operator closure of $\mathcal{A}_\Gamma$ in $\mathcal{L}(\ell^2(\Gamma))$. In general, it is not true that a Følner sequence for a concrete $C^*$-algebra is also a Følner sequence for its weak closure.*
(ii) *Recall that the preceding proposition means that the sequence $\{P_i\}_i$ satisfies the four equivalent conditions in Proposition 2.2 (ii) for any element in the group von Neumann algebra.*

**Theorem 3.4.** *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a separable $C^*$-algebra which has a Følner sequence $\{Q_i\}_{i\in\mathbb{N}}$. Consider the countable and amenable group $\Gamma$ and denote by $\{P_i\}_{i\in N}$ the sequence of orthogonal finite-rank projections from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma_i)$ associated to a Følner sequence $\{\Gamma_i\}_{i\in N}$ for the group $\Gamma$. Assume that there is an action of $\Gamma$ on $\mathcal{A}$ that satisfies:*

$$(3.7) \qquad \lim_i \left( \max_{\gamma\in\Gamma_i} \frac{\|\left[Q_i, \alpha_\gamma^{-1}(M)\right]\|_2}{\|Q_i\|_2} \right) = 0 \,, \quad \text{for all } M \in \mathcal{A} \,.$$

*Then the sequence $\{R_i\}_{i\in\mathbb{N}}$ with*

$$R_i := P_i \otimes Q_i$$

*is a Følner sequence for the crossed product $\mathcal{N} = \mathcal{A} \rtimes_\alpha \Gamma$, i.e. the four equivalent conditions in Proposition 2.2 (ii) are satisfied.*

*Proof.* Step 1: we consider the identification $\mathcal{K} \cong \underset{\gamma \in \Gamma}{\oplus} \mathcal{H}_\gamma$, $\mathcal{H}_\gamma \equiv \mathcal{H}$. In this case any element $N$ in the crossed product $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$ can be seen as a matrix of the form

(3.8) $$N_{\gamma'\gamma} = \alpha_\gamma^{-1} \left( A(\gamma'\gamma^{-1}) \right) \, , \quad \gamma', \gamma \in \Gamma \, ,$$

where $\gamma \mapsto A(\gamma)$ is a mapping from $\Gamma \to \mathcal{A}$ (cf. Remark 3.1). Moreover, defining the unitary map

$$W \colon \mathcal{H} \otimes \ell^2(\Gamma) \to \underset{\gamma \in \Gamma}{\oplus} \mathcal{H}_\gamma \, , \quad \varphi \otimes \xi \mapsto (\xi_\gamma \varphi)_{\gamma \in \Gamma}$$

it is straightforward to compute the matrix form of the projections $R_i = P_i \otimes Q_i \in \mathcal{L}(\mathcal{K})$:

$$(\widehat{R_i})_{\gamma'\gamma} := (W R_i W^*)_{\gamma'\gamma} = \begin{cases} Q_i \, \delta_{\gamma'\gamma} \, , & \gamma', \gamma \in \Gamma_i \\ 0 \, , & \text{otherwise} \, . \end{cases}$$

The commutator of $\widehat{R_i}$ with any $N \in \mathcal{N}$ is

$$\left[ \widehat{R_i} \, , \, N \right]_{\gamma'\gamma} = \begin{cases} [Q_i \, , \, N_{\gamma'\gamma}] \, , & \gamma', \gamma \in \Gamma_i \\ Q_i \, N_{\gamma'\gamma} & \gamma \notin \Gamma_i \, , \ \gamma' \in \Gamma_i \\ -N_{\gamma'\gamma} \, Q_i \, , & \gamma \in \Gamma_i \, , \ \gamma' \notin \Gamma_i \\ 0 \, , & \gamma \notin \Gamma_i \, , \ \gamma' \notin \Gamma_i \, . \end{cases}$$

Step 2: we will check first the Følner condition on the product of generating elements $\pi(M) \, U(\gamma_0)$, $\gamma_0 \in \Gamma$, $M \in \mathcal{A}$ (cf. Eqs. (3.2) and (3.3)). The corresponding matrix elements are given according to Eq. (3.6) by

$$N_{\gamma'\gamma} = \alpha_{\gamma'}^{-1}(M) \, \delta_{\gamma', \gamma_0 \gamma} \, .$$

Evaluating the commutator with $\widehat{R_i}$ on the basis elements $\{e_l \, \mathfrak{f}_\gamma\}_{l, \gamma}$, with $\mathfrak{f}_\gamma(\gamma') := \delta_{\gamma\gamma'}$, we get
(3.9)
$$\left[ \widehat{R_i} \, , \, (\pi(M) U(\gamma_0)) \right] e_l \, \mathfrak{f}_\gamma = \begin{cases} \left[ Q_i \, , \, \alpha_{\gamma_0 \gamma}^{-1}(M) \right] e_l \, \mathfrak{f}_{\gamma_0 \gamma} \, , & \gamma \in (\gamma_0^{-1} \Gamma_i) \cap \Gamma_i \\ Q_i \, \alpha_{\gamma_0 \gamma}^{-1}(M) \, e_l \, \mathfrak{f}_{\gamma_0 \gamma} \, , & \gamma \in (\gamma_0^{-1} \Gamma_i) \setminus \Gamma_i \\ -\alpha_{\gamma_0 \gamma}^{-1}(M) \, Q_i \, e_l \, \mathfrak{f}_{\gamma_0 \gamma} \, , & \gamma \in \Gamma_i \setminus (\gamma_0^{-1} \Gamma_i) \\ 0 \, , & \gamma \notin \Gamma_i \, , \ \gamma \notin \gamma_0^{-1} \Gamma_i \, . \end{cases}$$

From this we obtain the following estimates in the Hilbert-Schmidt norm:

$$\left\| \left[ \widehat{R_i}\,, \pi(M)U(\gamma_0) \right] \right\|_2^2$$

$$= \sum_{l,\gamma} \left\| \left[ \widehat{R_i}\,, (\pi(M)U(\gamma_0)) \right] e_l \, \mathfrak{f}_\gamma \right\|^2$$

$$\leq \sum_{\gamma \in (\gamma_0^{-1}\Gamma_i) \cap \Gamma_i} \left\| \left[ Q_i\,, \alpha_{\gamma_0\gamma}^{-1}(M) \right] \right\|_2^2 + 2|(\gamma_0^{-1}\Gamma_i)\triangle\Gamma_i|\, \|M\|^2 \, \|Q_i\|_2^2$$

$$\leq |\Gamma_i| \max_{\gamma \in \Gamma_i} \left\| \left[ Q_i\,, \alpha_\gamma^{-1}(M) \right] \right\|_2^2 + 2|(\gamma_0^{-1}\Gamma_i)\triangle\Gamma_i|\, \|M\|^2 \, \|Q_i\|_2^2 \,.$$

Using now the hypothesis (3.7) as well as the amenability of $\Gamma$ via Eq. (2.1) we get finally

$$\frac{\left\| \left[ \widehat{R_i}\,, \pi(M)U(\gamma_0) \right] \right\|_2^2}{\left\| \widehat{R_i} \right\|_2^2} \leq \max_{\gamma \in \Gamma_i} \frac{\left\| \left[ Q_i\,, \alpha_\gamma^{-1}(M) \right] \right\|_2^2}{\|Q_i\|_2^2} + 2\,\|M\|^2\, \frac{|(\gamma_0^{-1}\Gamma_i)\triangle\Gamma_i|}{|\Gamma_i|}\,,$$

and

$$\lim_i \frac{\left\| \left[ \widehat{R_i}\,, \pi(M)U(\gamma_0) \right] \right\|_2^2}{\left\| \widehat{R_i} \right\|_2^2} = 0\,, \quad M \in \mathcal{A}\,,\ \gamma_0 \in \Gamma\,.$$

Thus, we have shown the Følner condition for the sequence $\{R_i\}_i$ on the product of generating elements of the crossed product. By Proposition 2.2 (i) we have that $\{R_i\}_i$ is also a Følner sequence for their C*-closure

$$\mathcal{N} = \mathcal{A} \rtimes_\alpha \Gamma := C^*\Big( \{U(\gamma_0), \pi(M) \mid \gamma_0 \in \Gamma\,, M \in \mathcal{A}\} \Big)\,.$$

and the proof is concluded.                                                    $\square$

The preceding result extends Proposition 3.2 (proved by Bédos), since in the special case where $\mathcal{H}$ is one-dimensional and $\mathcal{A} \cong \mathbb{C}\mathbb{1}$, the crossed product reduces to the group $C^*$-algebra $\mathcal{A}_\Gamma$.

**Remark 3.5.** *The compatibility condition (3.7) in the choices of the two Følner sequences requires some comments:*

  (i) *Note that the compatibility condition already implies that the sequence $\{Q_i\}_i$ must be a Følner sequence for the $C^*$-algebra $\mathcal{A}$. In fact, this is one of the assumptions in Theorem 3.4 that $\mathcal{A}$ has a Følner sequence.*

  (ii) *Eq. (3.7) is trivially satisfied in some cases: If $\Gamma$ is finite, then the compatibility condition is a consequence of the Følner condtion in Definition 2.1.*
    *Another example is given by the crossed product $\ell^\infty(\Gamma) \rtimes_\alpha \Gamma$, where $\Gamma$ is a discrete amenable group, $\ell^\infty(\Gamma)$ is the von Neumann algebra acting on the Hilbert space $\ell^2(\Gamma)$ by multiplication and the action $\alpha$ of $\Gamma$ on $\ell^\infty(\Gamma)$ is*

*given by left translation of the argument. If $\{\Gamma_i\}_i$ is a Følner sequences for $\Gamma$ and we denote by $\{P_i\}_i$ the sequence of finite rank orthogonal projections from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma_i)$, then it is easy to check*

$$[P_i, g] = 0 \ , \quad g \in \ell^\infty(\Gamma) \ .$$

*Therefore, we have*

$$\sup_{\gamma \in \Gamma} \| \left[ P_i, \alpha_\gamma^{-1}(g) \right] \|_2 = 0 \ , \quad g \in \ell^\infty(\Gamma)$$

*and we may apply Theorem 3.4 to this situation. This particular example is essentially the context in which Bédos studies crossed products in Section 3 of [5]. In fact, in this very special context one can trace back the existence of a Følner sequences for the crossed product to the amenability of the discrete group (see Proposition 14 in [5]).*

## 4. APPLICATIONS

We will apply next the results of the previous sections in two different situations. The following examples should only illustrate the potential applicability of Følner sequences.

4.1. **The example of the rotation algebra.** The rotation algebra $\mathcal{A}_\theta$, $\theta \in \mathbb{R}$, is the (universal) $C^*$-algebra generated by two unitaries $U$ and $V$ that satisfy the equation

$$U V = e^{2\pi i \theta} V U \ .$$

When $\theta$ is an integer, the algebra $\mathcal{A}_\theta$ is isomorphic to the Abelian $C^*$-algebra $C(\mathbb{T}^2)$ of continuous functions on the 2-torus. For this reason, when, e.g., $\theta$ is irrational, $\mathcal{A}_\theta$ is called a non-commutative torus. Moreover, $\mathcal{A}_\theta$ has in this case a unique faithful tracial state $\tau$ which can be interpreted as a non-commutative analogue of the Haar measure on $\mathbb{T}^2$. (See [9] for a thorough presentation.)

The rotation algebra is one of the fundamental examples in the theory $C^*$-algebras and has been extensively used in mathematical physics. Interesting examples from the spectral point of view, like the almost Mathieu operators or discrete Schrödinger operators with magnetic potentials (Harper operators), can be identified as elements of the rotation algebra (cf. [39, 9]). In fact, consider for example the following representation of the generators $U, V$ on $\mathcal{H} := \ell^2(\mathbb{Z})$:

$$(U\xi)_k := \xi_{k-1} \quad \text{and} \quad (V\xi)_k := e^{2\pi i \theta k} \, \xi_k \ ,$$

where $\xi = (\xi_k)_k \in \mathcal{H}$. One defines the almost Mathieu operator with real parameters $\theta, \lambda, \beta$ as

$$(4.1) \qquad H_{\theta,\lambda,\beta} := U + U^* + \frac{\lambda}{2} \left( e^{2\pi i \beta} V + e^{-2\pi i \beta} V^* \right) \in \mathcal{A}_\theta \ .$$

These classes of operators have a natural generalization to arbitrary graphs.

An important fact for our purposes is that the rotation algebra can also be expressed as a crossed product

$$\mathcal{A}_\theta \cong C(\mathbb{T}) \rtimes_\alpha \mathbb{Z} \ ,$$

where $C(\mathbb{T})$ are the continuous function on the unit circle and the action $\alpha\colon \mathbb{Z} \to C(\mathbb{T})$ is given by rotation of the argument:

$$(4.2) \qquad \alpha_k(f)(z) := f\left(e^{2\pi ik\theta} z\right) \ , \quad f \in C(\mathbb{T}) \ , \quad z \in \mathbb{T} \ .$$

We will apply our main result to the C*-crossed product $\mathcal{A}_\theta$. Let $\{\epsilon_k(z) := z^k \mid k \in \mathbb{Z}\}$ be an orhtonormal basis of Hilbert space $\mathcal{H} := L^2(\mathbb{T})$ with the normalized Haar measure. We choose (as in [5, p. 216]) a Følner sequence $\{Q_n\}_{n \in \mathbb{N}_0}$, where $Q_n$ denotes the orthogonal projection onto the subspace generated by $\{\epsilon_i \mid i = 0, \dots n\}$. Moreover, we choose for the group $\Gamma = \mathbb{Z}$ the Følner sequence $\Gamma_n := \{-n, -(n-1), \dots, (n-1), n\}$ and denote by $P_n$ the corresponding finite-rank orthogonal projections on $\ell^2(\mathbb{Z})$. First we verify that the compatibility condition (3.7) for our choice of Følner sequences:

**Lemma 4.1.** *Consider the previous Følner sequence $\{Q_n\}_n$ for the Abelian $C^*$-algebra $\mathcal{A} := C(\mathbb{T})$ and the group action $\alpha\colon \mathbb{Z} \to C(\mathbb{T})$ definded in Eq. (4.2). Then for $g \in C(\mathbb{T})$ we have*

$$\left\| \left[Q_n, \alpha_k^{-1}(g)\right] \right\|_2 = \left\| \left[Q_n, g\right] \right\|_2 \ , \quad k \in \mathbb{Z} \ ,$$

*and*

$$\lim_{n\to\infty} \left( \max_{k \in \Gamma_n} \frac{\left\| \left[Q_n, \alpha_k^{-1}(g)\right] \right\|_2}{\|Q_n\|_2} \right) = 0 \ , \quad \text{for all } g \in C(\mathbb{T}) \ .$$

*Proof.* The first equation is a consequence of the some elementary statements in harmonic analysis:

$$\left\| \left[Q_n, \alpha_k^{-1}(g)\right] \right\|_2^2 = \sum_{l=-\infty}^{\infty} \left\| (Q_n \, \alpha_{-k}(g) - \alpha_{-k}(g)\, Q_n)\, \epsilon_l \right\|^2$$

$$= \sum_{l=0}^{n} \left\| (\mathbb{1} - Q_n)\, \alpha_{-k}(g)\, \epsilon_l \right\|^2 \ + \sum_{l \in (\mathbb{Z}\backslash\{0,\dots,n\})} \left\| Q_n \, \alpha_{-k}(g)\, \epsilon_l \right\|^2$$

$$= \sum_{m \in (\mathbb{Z}\backslash\{0,\dots,n\})} \sum_{l=0}^{n} \left| e^{2\pi i \, k\theta(m-l)}\, \widehat{g}(m - l) \right|^2$$

$$+ \sum_{m=0}^{n} \sum_{l \in (\mathbb{Z}\backslash\{0,\dots,n\})} \left| e^{2\pi i \, k\theta(m-l)}\, \widehat{g}(m - l) \right|^2$$

$$= \left\| \left[Q_n, g\right] \right\|_2^2 \ .$$

The second equation follows directly from the first equation and the fact that $\{Q_n\}_n$ is a Følner sequence for the algebra $C(\mathbb{T})$. $\qquad\square$

**Proposition 4.2.** *Let $\mathcal{A}_\theta \cong C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$, with $\theta$ irrational, be the $C^*$-algebra associated to the rotation algebra and acting on $\mathcal{K} = \ell^2(\mathbb{Z}) \otimes \mathcal{H}$.*

(i) *Consider the sequences $\{Q_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ defined before. Then*

$$\{R_n := P_n \otimes Q_n\}_{n\in\mathbb{N}_0}$$

*is a Følner sequence for $\mathcal{A}_\theta$.*

(ii) *Let $T \in \mathcal{A}_\theta$ be a selfadjoint element in the rotation algebra and denote by $\mu_T$ the spectral measure associated with the unique trace of $\mathcal{A}_\theta$. Consider the compressions $T_n := R_n T R_n$ and denote by $\mu_T^n$ the probability measures on $\mathbb{R}$ supported on the spectrum of $(T_n)$. Then $\mu_T^n \to \mu_T$ in the weak\*-topology, i.e.*

$$\lim_{n\to\infty} \frac{1}{d_n}\Big( f(\lambda_{1,n}) + \cdots + f(\lambda_{d_n,n}) \Big) = \int f(\lambda)\, d\mu(\lambda)\,, \quad f \in C_0(\mathbb{R})\,,$$

*where $d_n$ is the dimension of the $R_n$ and $\{\lambda_{1,n}, \dots, \lambda_{d_n,n}\}$ are the eigenvalues (repeated according to multiplicity) of $T_n$.*

*Proof.* Part (i) follows from Theorem 3.4 and Lemma 4.1. To prove Part (ii) recall $\mathcal{A}_\theta$ has a unique trace ([9]). The rest of the statement is a direct application of Theorem 6 (iii) in [5] to the example of the rotation algebra. $\qquad\square$

Since almost Mathieu operator $H_{\theta,\lambda,\beta}$ defined in Eq. (4.1) are self-adjoint elements in $\mathcal{A}_\theta$, we can apply part (ii) of the precedent proposition. In this case the discrete measures $\mu_H^n$ are supported on the eigenvalues of the corresponding finite section matrices.

4.2. **Jacobi operators.** Jacobi operators have have been used in many branches of mathematics. E.g., they can be interpreted as a discrete version of Schrödinger operators and appear in the approximation of differential operators by difference operators (see, e.g., [1]). Moreover, the relation between selfadjoint tridiagonal infinite Jacobi matrices and orthogonal polynomials is by now a standard fact. In Chapter 2 of [22] it is shown that there is a one-to-one corresondence between bounded selfadjoint Jacobi operators $J$ and probability measures $\mu$ with with compact support. The purpose of this subsection is to illustrate how naturally the notion of Følner sequence fits into the analysis of this class of operators.

Consider on $\mathcal{H} := \ell^2(\mathbb{N}_0)$ with canonical basis $\{e_i \mid i = 0, 1, 2, \dots\}$ the infinite Jacobi matrix

$$(4.3) \qquad J = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \dots \\ b_0' & a_1 & b_1 & 0 & \dots \\ 0 & b_1' & a_2 & b_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}\,.$$

If the diagonals $\alpha = (a_0, a_1, \dots)$, $\beta = (b_0, b_1, \dots)$, $\beta' = (b'_0, b'_1, \dots)$ are bounded (i.e., $\alpha, \beta, \beta' \in \ell^\infty(\mathbb{N}_0)$), then $J$ is a bounded operator. Moreover, if $D_\alpha = \mathrm{diag}(\alpha)$ denotes the diagonal operator, then $J$ can be decomposed as

$$(4.4) \qquad\qquad J = D_\alpha + D_\beta S^* + S\, D_{\beta'} \,,$$

where $Se_i := e_{i+1}$ is the shift already considered in Example 2.4 (i).

**Proposition 4.3.** *Denote by $\mathcal{J}$ the set of all bounded Jacobi matrices as in Eq. (4.3). Then $\{P_n\}_n$, where $P_n$ are the orthogonal projections onto $\mathrm{span}\{e_i \mid i = 0, 1, 2, \dots, n\}$ is a Følner sequence for $C^*(\mathcal{J})$.*

*Proof.* By Proposition 2.2 (i) it is enough to check Eq. 2.2 for the generating set $\mathcal{J}$. For any $J \in \mathcal{J}$ we have using the decomposition (4.4)

$$\|[J, P_n]\|_2 \leq \underbrace{\|[D_\alpha, P_n]\|_2}_{0} + \|[D_\beta S^*, P_n]\|_2 + \|[SD_{\beta'}, P_n]\|_2$$

$$\leq \|D_\beta\|\,\|[S^*, P_n]\|_2 + \|[S, P_n]\|_2\|D_{\beta'}\| \,.$$

Følner's condition in Eq. (2.2) follows from the computation in Example 2.4 (i). $\qquad\square$

Let $J \in \mathcal{J}$ be selfadjoint and denote by $\mu$ the spectral measure associated to the cyclic vector $e_0$. The support of the (discrete) spectral measures $\mu_n^J$ of the finite sections $J_n = P_n J P_n$ (with $P_n$ as in the preceding propostion) correspond precisely with the zeros of the polynomials $p_n$ which are orthogonal with respect to $\mu$. For any Borel set $\Delta \subset \mathbb{R}$ define $N_n(\Delta)$ to be the number of eigenvalues of $J_n$ counting multiplicities contained in $\Delta$. Note that in this case we have

$$\mu_n^J(\Delta) = \frac{N_n(\Delta)}{n+1} \,.$$

Following Arveson [2, 3] we say that $\lambda \in \mathbb{R}$ is essential if for every open set $\Delta \subset \mathbb{R}$ containing $\lambda$, we have

$$\lim_{n \to \infty} N_n(\Delta) = \infty \,.$$

The set of essential points is denoted by $\Lambda_{\mathrm{ess}}(J)$. Recall finally that in this context the essential spectrum of the selfadjoint Jacobi operator $J$ is defined by

$$\sigma_{\mathrm{ess}}(J) = \sigma(J) \setminus \sigma_{\mathrm{disc}}(J) \,,$$

where $\lambda \in \sigma_{\mathrm{disc}}(J)$ if it is an isolated eigenvalue in the spectrum $\sigma(J)$ whose eigenspace is finite dimensional. Then since for tridiagonal Jacobi matrices we have that

$$\sup_n \mathrm{rank}\,(P_n J - J P_n) \leq 2$$

we can apply a theorem by Arveson to conclude that

$$\sigma_{\mathrm{ess}}(J) = \Lambda_{\mathrm{ess}}(J) \,.$$

This result shows the way in which one can recover the essential spectrum of $J$ out of its finite sections.

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