# THE FOURIER TRANSFORM OF A FUNCTION OF BOUNDED VARIATION

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## Contents



## 1. Introduction

The definitions of the Fourier transform vary from one function space to another, and sometimes drastically. To illustrate this, one may compare the definitions of the Fourier transform for Lebesgue integrable functions, for Lebesgue $p$-integrable functions with $1 < p \leq 2$, the distributional approach, and so on. The space of functions of bounded variation has its own features. An important initial reference to harmonic analysis on this space is Bochner's celebrated book [10]. It gives us the first basis of our treatment: the Fourier transform will mainly

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be understood in the improper sense. Since we will mostly work on the half-axis $\mathbb{R}_+ = [0,\infty)$, we recall this notion just for $\mathbb{R}_+$:

$$\int_{\mathbb{R}_+} = \lim_{\varepsilon\to 0+, N\to\infty} \int_\varepsilon^N .$$

Of course, there are other classical sources where the reader can find important results from harmonic analysis of functions of bounded variation, see, e.g., [41], [45], [44], etc.

We will study how the fact that a function belongs to the space of functions of bounded variation or to certain of its subspaces affects integrability and summability properties of the Fourier transform. The above illustration demonstrates that it is meaningless to try to define the Fourier transform immediately and once and for all; in each problem we need to specify the definition.

Very classical results are given without proofs, with references to classical sources. More recent results and/or less known results are given with proof. In any case, we try to keep a balance between the details and the picture as a whole. We shall try to make the presentation reasonably self-contained.

This is a survey of main results on the topic in question, from classical to those quite recent; many of them have been obtained by the author, alone and with collaborators.

Let us outline the main subjects we will elucidate. First, we cannot avoid the classical Fourier inverse result which will be presented in Section 2. We then discuss in Section 3 less known results on connections between periodic and non-periodic cases. A summability result in Section 4 is related to multipliers. The greatest part of the paper, Section 5, is devoted to problems of integrability of Fourier transforms.

Let us give the two notations that we will use throughout. The space of functions of bounded variation will be denoted by $BV$ and the norm of a function in it $\|\cdot\|_{BV}$. When we have to fix a set $D$ on which the total variation of functions is calculated, we write $BV_D$. By $C$ we will denote absolute constants, which may be different in various occurrences.



## 2. Preliminaries on functions of bounded variation

Let us give certain basics that will be a starting point for us. As for the definition of bounded variation, we are not going to concentrate on various details. On the contrary, following Bochner, we will mainly restrict ourselves to functions with Lebesgue integrable derivative, since every such function is of bounded variation in the sense that it is representable as a linear combination (generally, with complex coefficients) of monotone functions. Without loss of generality, it suffices to prove this for real-valued functions. Indeed, let $f$ have integrable derivative in

$[a,b]$. Then it is representable as

$$\begin{aligned} f(x) &= f(b) + \int_x^b \frac{|f'(t)| - f'(t)}{2}\, dt - \int_x^b \frac{|f'(t)| + f'(t)}{2}\, dt \\ &= f(b) + h_1(x) - h_2(x). \end{aligned}$$

Both functions $h_1(x)$ and $h_2(x)$ are monotone decreasing. If $b = \infty$, we just consider $\lim\limits_{x\to\infty} f(x)$. Since in that case,

$$\lim_{x\to\infty} h_1(x) = \lim_{x\to\infty} h_2(x) = 0,$$

a function of bounded variation vanishing at infinity can be represented as a difference of two monotone decreasing functions, each of them tending to zero at infinity. Of course, the usual definition that applies to the uniform boundedness of the sums of oscillations of a function over all possible systems of non-overlapping intervals might be helpful.

There is a characterization of functions of bounded variation by means of integral smoothness. It main part is due to Hardy and Littlewood [25, 26]; the reader may also consult [40, Ch.III, §3.6]. In these sources the result is proved on a finite interval, however, the proof goes along the same lines on an infinite set as well. For simplicity, we restrict ourselves to the case of $\mathbb{R}_+$.

**Theorem 1.** *Let $f$ be of bounded variation on $\mathbb{R}_+$. Then it satisfies the Lipschitz condition in the integral metrics*

$$\int_0^\infty |f(t+h) - f(t)|\, dt \le Mh \tag{1}$$

*for any $h$ and absolute constant $M$.*

*Conversely, if $f$ satisfies* (1)*, then almost everywhere on $\mathbb{R}_+$ it coincides with a function of bounded variation.*

*Proof.* The first part is obvious. Indeed,

$$\begin{aligned} \int_0^\infty |f(t+h) - f(t)|\, dt &\le \int_0^\infty \int_t^{t+h} |df(u)|\, dt \\ &= \int_0^h |df(u)| \int_0^u dt + \int_h^\infty |df(u)| \int_{u-h}^u dt \\ &\le h \int_0^\infty |df(u)|. \end{aligned}$$

Here the constant $M$ is just the total variation of $f$ on $\mathbb{R}_+$.

To prove the second part, let us assume that $f$ satisfies (1) and denote

$$f_n(x) = n \int_0^{1/n} f(x+t)\, dt. \tag{2}$$

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We then have

$$\begin{aligned}\int_0^\infty & |f_n(x+h)-f_n(x)|\,dx\\ &= n\int_0^\infty \left|\int_0^{1/n}[f(x+t+h)-f(x+t)]\,dt\right|\,dx\\ &\le n\int_0^n\int_0^\infty |f(x+t+h)-f(x+t)|\,dx\,dt \le Mh.\end{aligned}$$

We mention that (1) ensures also the possibility of application of Fubini's theorem.

If $[x_k, x_k+h_k]$, $k=1,2,...,m$, is a finite system of mutually disjoint intervals, then

$$\begin{aligned}\sum_{k=1}^m |f_n(x_k+h_k)-f_n(x_k)| &\le \sum_{k=1}^m \int_{x_k}^{x_k+h_k} |f_n'(x)|\,dx\\ &\le \int_0^\infty |f_n'(x)|\,dx.\end{aligned}$$

Fatou's lemma implies

$$\int_0^\infty |f_n'(x)|\,dx \le \underset{h\to 0}{\underline{lim}}\int_0^\infty \left|\frac{f_n(x+h)-f_n(x)}{h}\right|\,dx \le M,$$

and, correspondingly,

$$\sum_{k=1}^m |f_n(x_k+h_k)-f_n(x_k)| \le M. \tag{3}$$

Since $f$ is locally integrable, almost every point is its Lebesgue point. Therefore $f_n(x)\to f(x)$ as $n\to\infty$ for all $x$ except maybe a set $Q$ of measure zero. If the points $x_k$ and $x_k+h_k$ do not belong to $Q$, then passing to the limit as $n\to\infty$ in (3), we obtain

$$\sum_{k=1}^m |f(x_k+h_k)-f(x_k)| \le M. \tag{4}$$

This means that $f$ is of bounded variation on the complement of $Q$ and consequently can be represented as the difference of two monotone functions. To let it take values on $Q$, one can define it as the right limit at every point of $Q$. This completes the proof. □

Theorem 1 in the book [10] states that the Fourier transform, no matter whether cosine, sine or complex, of a function monotone decreasing to zero, and correspondingly of bounded variation, exists everywhere, maybe except at zero. This can easily be seen by integration by parts, which also shows that similarly to the Riemann-Lebesgue lemma for integrable functions the Fourier transform in this case also tends to zero at infinity.

Let us give a representation theorem from Chapter 1 in [44, 1.1.4] concerning functions of bounded variation.

**Proposition 1.** Let $g$ be locally integrable on $[0,+\infty)$, i.e., $g \in L[0,b]$ for any $b>0$, and let $f$ be of bounded variation on $[0,+\infty)$. Then

$$\lim_{\lambda\to+0}\int_0^\infty f(\lambda t)\,g(t)\,dt = f(+0)\int_0^\infty g(t)\,dt,$$

provided the integral on the right-hand side converges.

*Proof.* Here $g$ need not be (Lebesgue) integrable on $[0,+\infty)$, written $g\notin L[0,+\infty)$, therefore the integral in $\langle f,g\rangle_\lambda$ is understood in the improper sense. Since for a constant function $f$ the formula in question is valid, and for each $f\in BV_{[0,+\infty)}$ the limit exists as $t\to+\infty$, one may assume, without loss of generality, that this limit is zero. Denoting

$$G(t)=\int_0^t g(u)\,du \quad\text{and}\quad G(+\infty)=\int_0^\infty g(u)\,du,$$

and integrating by parts in the Stieltjes integral yield

$$\int_0^\infty f(\lambda t)\,g(t)\,dt = -\int_0^\infty G(t)\,df(\lambda t).$$

The usual estimate of the Stieltjes integral and the fact that homothety does not change the total variation of the function $f$ imply $|\langle f,g\rangle_\lambda|\le \|G\|_\infty\,\|f\|_{BV}$.

Now, as above, we have to check the formula on a set of functions $G$ dense in the space of continuous functions, each function in $G$ having a finite limit at infinity. Let this space be endowed with the norm $\|\cdot\|_\infty$. An appropriate set is that of step-functions which are constant off a finite interval. Consequently, to check the formula, we may restrict ourselves to the indicators of infinite intervals, i.e., to functions of the form $\chi_{[b,+\infty)}$. We have

$$\begin{aligned}-\int_0^\infty \chi_{[b,\infty)}(t)\,df(\lambda t) &= -\int_b^\infty df(\lambda t)\\ &= f(b\lambda)\to f(+0)=f(+0)\lim_{t\to+\infty}\chi_{[b,\infty)}(t).\end{aligned}$$

The first equality holds only if the point $b/\lambda$ is not a discontinuity point of $f$ (otherwise the Stieltjes integral does not exist). Clearly, these $b$ may be taken on the dense subset of $(0,+\infty)$. Then we argue as follows. It suffices to prove the desired formula for any sequence $\lambda_n\to 0$. But then each function $f(\lambda_n t)$, $n=1,2,\dots$, has, as does $f$, no more than a countable number of discontinuity points. Thus, one can choose $b$ on any interval so that the points $b/\lambda_n$, $n=1,2,\dots$, will be continuity points for $f$. $\square$

We now give a well-known classical result on Fourier inversion on which the whole treatment of the Fourier transform of a function of bounded variation is based, in a sense (see, e.g., [45, Vol.II, Ch.XVI, §2] or [44, 3.1.17]); the proof uses the previous result. Let $\widehat{f}$ be the Fourier transform of $f$ defined as

$$\widehat{f}(x) = \int_{-\infty}^{+\infty} f(t)e^{-ixt}\,dt.$$

**Theorem 2.** *If $f$ is a function of bounded variation on $\mathbb{R}$, and $\lim f(x) = 0$ as $|x| \to \infty$, then for all $x_0 \in \mathbb{R}$ there holds*

$$\frac{f(x_0+0)+f(x_0-0)}{2} = \lim_{N\to\infty}\lim_{\delta\to 0}\frac{1}{2\pi}\int_{\delta\le|y|\le N}\widehat{f}(y)e^{iyx_0}dy. \tag{5}$$

*Proof.* In this case the Fourier transform is defined as an improper integral, and is continuous for all $y \in \mathbb{R}\setminus\{0\}$, even without assuming $f \in L(\mathbb{R})$. This is easily checked by integrating by parts. Therefore, the right-hand side equals

$$\begin{aligned}\frac{1}{2\pi}\lim_{N\to\infty}\lim_{\delta\to+0}\int_{-\infty}^{\infty} f(x)\,dx\int_{\delta\le|y|\le N} & e^{iy(x_0-x)}dy\\ &= \frac{1}{\pi}\lim_{N\to\infty}\int_{-\infty}^{\infty} f(x)\frac{\sin N(x_0-x)}{x_0-x}dx\\ &- \frac{1}{\pi}\lim_{\delta\to+0}\int_{-\infty}^{\infty} f(x)\frac{\sin \delta(x_0-x)}{x_0-x}\,dx.\end{aligned}$$

To check that the second integral on the right converges uniformly in $\delta$, we integrate by parts over the segment $[M_1, M_2]$, taking into account that integrating $t^{-1}\sin\delta t$ over any segment gives a value bounded by an absolute constant. Thus we may pass to the limit under the integral sign, which is evidently zero. To calculate the limit of the first integral, we substitute $t = N(x_0 - x)$ and finally apply Proposition 1. □

## 3. From series to integrals and vice versa

In this section, we present two useful results where one sees relations between periodic and non-periodic objects. The first one is apparently quite special and applicable only to functions of bounded variation. The second one is a version of the Poisson summation formula for functions of bounded variation, one of many possible in various settings.

3.1. **Fourier integrals and trigonometric series.** For functions of bounded variation, to pass from series to integrals and vice versa, we will make use of the following result due to Trigub [42, Th. 4] or [44, 4.1.2] (for its extension, see [43]; an earlier version, for functions with compact support, is due to Belinsky [3]):

$$\sup_{0<|x|\le\pi}\left|\int_{-\infty}^{+\infty} f(t)e^{ixt/\varepsilon}\,dt - \varepsilon\sum_{k=-\infty}^{+\infty} f(\varepsilon k)e^{ikx}\right| \le 2\varepsilon\|f\|_{BV}. \tag{6}$$

*Proof.* First, let us note that the series is convergent for $x \not\equiv 0(\text{mod}2\pi)$ according to Dirichlet's test, its sum is $2\pi$-periodic, and the integral is convergent for $x \neq 0$ and has a zero limit as $|x| \to \infty$. Let us prove this inequality for $\varepsilon = 1$, since the general case readily follows from this particular one by replacing $f(\cdot)$ by $f(\varepsilon\cdot)$ and by the appropriate change of variables in the integral. In that case, the total variation of the function is left unchanged.

Thus, keeping in mind that $\varepsilon = 1$, we see that the difference between the sum and the integral can be expressed as

$$\frac{x}{2\sin(x/2)} \sum_{k=-\infty}^{+\infty} \int_{k-1/2}^{k+1/2} [f(k) - f(t)]e^{ixt}\,dt - \left(\frac{x}{2\sin(x/2)} - 1\right) \int_{-\infty}^{+\infty} f(t)e^{ixt}\,dt.$$

Integrating by parts in the last integral, we obtain

$$\left|\int_{-\infty}^{+\infty} f(t)e^{ixt}\,dt - \sum_{k=-\infty}^{+\infty} f(k)e^{ikx}\right| \le \left|\frac{x}{2\sin(x/2)}\right| \sum_{k=-\infty}^{+\infty} \int_{k-1/2}^{k+1/2} |f(k) - f(t)|\,dt + \left|\frac{x}{2\sin(x/2)} - \frac{1}{x}\right| \left|\int_{-\infty}^{+\infty} e^{ixt}\,df(t)\right|.$$

We then observe that the absolute value of the last Stieltjes integral is at most $\|f\|_{BV}$, while

$$\sum_{k=-\infty}^{+\infty} \int_{k-1/2}^{k+1/2} |f(k) - f(t)|\,dt = \int_{-1/2}^{1/2} \sum_{k=-\infty}^{+\infty} |f(k) - f(t)|\,dt \le \|f\|_{BV}.$$

Hence, noting that $\sin x < x$ for $0 < x \le \pi$, we get

$$\left|\int_{-\infty}^{+\infty} f(t)e^{ixt}\,dt - \sum_{k=-\infty}^{+\infty} f(k)e^{ikx}\right| \le \left|\frac{|x|}{2\sin(x/2)} + \frac{1}{2\sin(x/2)} - \frac{1}{x}\right| \|f\|_{BV}.$$

It remains to take into account that the function on the right whose absolute value is taken increases on $[0, \pi]$ from 1 to

$$\frac{\pi + 1}{2} - \frac{1}{\pi} < 2,$$

which completes the proof. □

3.2. **Poisson summation formula.** The Poisson summation formula, discovered by Siméon Denis Poisson, is sometimes called Poisson resummation. A typical form of the Poisson summation formula for integrable functions is given in [37, Ch.VII, Th.2.4].

Moreover, the results are known which show that the Poisson summation characterize, in that or another sense, the Fourier transform (see [14] and [18]).

In [45, Ch.II, §13], one can find certain versions of the Poisson summation formula for the validity of which the boundedness of the variation of a function is assumed, along with additional mild conditions, rather than integrability. In [43, Lemma 2], a somewhat different formula is obtained for functions of bounded variation. It reads as follows.

**Proposition 2.** If $f$ is a function of bounded variation on $\mathbb{R}$,

$$2f(k) = f(k+0) + f(k-0)$$

for all $k \in \mathbb{Z}$, and $\lim f(t) = 0$ as $|t| \to \infty$, then for all $x \not\equiv 0(\mathrm{mod} 2\pi)$ we have

$$\sum_{k=-\infty}^{+\infty} f(k)e^{ikx} = \sum_{k=-\infty}^{+\infty} \widehat{f}(2k\pi - x).$$

*Proof.* By Dirichlet's test, the series on the left is uniformly convergent on $[-\pi, \pi]$ outside any neighborhood of zero, and the integral in the definition of $\widehat{f}$ behaves likewise on $\mathbb{R}$. Let us set

$$s(x) = \sum_{k=-\infty}^{+\infty} f(k)e^{ikx} - \widehat{f}(-x), \qquad \sigma(x) = \sum_{k\neq 0} \widehat{f}(2k\pi - x).$$

We are now going to analyze the function $s(x) - \sigma(x)$. Since the discontinuity at zero is removable, we will show that it is continuous on $[-\pi, \pi]$ and has zero Fourier coefficients. This will yield $s(x) = \sigma(x)$ for $0 < |x| \le \pi$, and hence for all $x \not\equiv 0(\mathrm{mod} 2\pi)$, since each side of the relation is $2\pi$-periodic. For the cut-off function $f_n$, that coincides with $f$ on $[-n, n]$ and vanishes off $[-n, n]$, we have $\|f_n\|_{BV} \le \|f\|_{BV}$. Denoting

$$s_n(x) = \sum_{k=-n}^{n} f(k)e^{ikx} - \int_{-n}^{n} f(t)e^{ixt}dt,$$

we apply (6) to $f_n$ to get $|s_n(x)| \le 2\|f\|_{BV}$ for any $n \in \mathbb{N}$.

Now, we calculate the $m$-th Fourier coefficient of $s$:

$$\begin{aligned} c_m(s) &= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} s_n(t)e^{-imt}dt \\ &= f(m) - \frac{1}{2\pi} \lim_{n\to\infty} \int_{-\pi}^{\pi} e^{-imt} \int_{-n}^{n} f(u)e^{ixu}du\, dt. \end{aligned}$$

To estimate the limit, let us split $[-\pi, \pi]$ into two parts: $[-\delta, \delta]$ and $[-\pi, -\delta) \cup (\delta, \pi]$. After certain calculations we will let $\delta$ tend to zero. On $[-\delta, \delta]$ we have

$$\lim_{n\to\infty} \int_{-\delta}^{\delta} e^{-imx}dx \int_{-n}^{n} f(t)e^{ixt}dt = 2\int_{-\infty}^{\infty} f(t)\frac{\sin(t-m)\delta}{t-m}\, dt.$$

Since the integral of $\dfrac{\sin t\delta}{t}$ over any interval is bounded by an absolute constant, the last integral converges uniformly in $\delta$ and hence its limit as $\delta \to 0$ is zero.

The integral in the definition of $\widehat{f}$ converges uniformly off $[-\delta, \delta]$, which again gives a possibility to pass to the limit under the integral sign. By this,

$$c_m(s) = f(m) - \lim_{\delta \to 0} \frac{1}{2\pi} \int_{\delta \le |x| \le \pi} \widehat{f}(x) e^{imx} dx.$$

Let us now proceed to $\sigma(x)$. Since it is independent of any correction of $f$ on a set of measure zero, we can think of $f$ as right continuous. Integrating by parts, we obtain

$$\sigma_n(x) = \sum_{1 \le |k| \le n} \widehat{f}(2k\pi - x) = \sum_{1 \le |k| \le n} \frac{-i}{2k\pi - x} \int_{-\infty}^{\infty} e^{-it(2k\pi - x)} df(t).$$

All the summands are continuous when $|x| \le \pi$. Using the equality

$$\frac{1}{2k\pi - x} = \frac{1}{2k\pi} + \frac{x}{2k\pi(2k\pi - x)},$$

we split $\sigma_n$ into two sums, while $\sigma$ into two series, respectively. The second series converges uniformly on $[-\pi, \pi]$ in virtue of the Weierstrass test. The sequence of partial sums of the first series

$$\sum_{1 \le |k| \le n} \frac{-i}{2k\pi - x} \int_{-\infty}^{\infty} e^{-it(2k\pi - x)} df(t) = - \int_{-\infty}^{\infty} e^{itx} \left( \sum_{k=1}^{n} \frac{\sin 2k\pi t}{k\pi} \right) df(t)$$

converges boundedly to the continuous function

$$\int_{-\infty}^{\infty} e^{itx} \varphi_0(t) df(t);$$

with $\varphi_0(t) = t - [t] - \frac{1}{2}$ for $t$ non-integer and $\varphi_0(k) = 0$, $k \in \mathbb{Z}$. Here Lebesgue's theorem on dominated convergence is applied to the integral in measure generated by the function $f$ of bounded variation. Thus, $\sigma_n$ converges boundedly to $\sigma$ on $[-\pi, \pi]$, with $\sigma \in C[-\pi, \pi]$, and

$$\begin{aligned} 2\pi c_m(\sigma) &= \lim_{n \to \infty} \sum_{1 \le |k| \le n} \int_{-\pi}^{\pi} \widehat{f}(2k\pi - x) e^{-imx} dx \\ &= \lim_{n \to \infty} \int_{\pi \le |t| \le \pi(2n+1)} \widehat{f}(t) e^{imt} dt. \end{aligned}$$

The needed equality $c_m(s) = c_m(\sigma)$ follows now from the Fourier inversion for the function of bounded variation (5), which completes the proof. $\square$

## 4. Bounded variation, summability and multipliers

The main object of this topic are various spaces $X^\wedge$ of the Fourier integrals

$$f^\circ(t) \sim \frac{1}{2\pi}\int_{\mathbb{R}} \varphi(x)e^{ixt}dx$$

such that $\varphi$ is the Fourier transform $\widehat{f}$ of $f \in X$

$$\varphi(x) = \widehat{f}(x) = \int_{\mathbb{R}} f(t)e^{-ixt}dt.$$

Given $X$ a normed space, we define $\|f^\circ\|_{X^\wedge} = \|f\|_X$.

For more details and references, see [2]. In particular, our scope is such that we avoid the distributional approach.

For functions of bounded variation, the notion of the Fourier-Stieltjes transform proved to be sometimes more natural in many respects than the usual Fourier transform:

$$\widehat{dF}(x) = \int_{\mathbb{R}} e^{-ixt}dF(t).$$

We study conditions under which $f^\circ$ belongs to the space $d^\alpha V^\wedge$, $\alpha \ge 0$, that is, when

$$(ix)^{1-\alpha}\varphi(x) = \widehat{dF}(x),$$

where $F$ is a function of bounded variation. For this we need an appropriate notion of fractional derivative. The one corresponding to our scope is naturally defined via the Fourier transform: $g^{(\alpha)}$, the $\alpha$th derivative of $g$, is the function for which

$$\widehat{g^{(\alpha)}}(x) = (ix)^\alpha \widehat{g}(x).$$

We shall study these spaces in connection with summability. Let the summability method be defined by a single function $\lambda$, a multiplier, as

$$(\Lambda_N f)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \lambda\left(\frac{t}{N}\right)\varphi(t)e^{ixt}dt. \tag{7}$$

It is clear that $\lambda$ should be defined at each point, so let $\lambda$ be continuous. The following representation is useful in many cases:

$$(\Lambda_N f)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} N\widehat{\lambda}(N(t-x))f(t)\,dt. \tag{8}$$

When $f, \lambda \in L^1(\mathbb{R})$ it is merely equivalent to (7), see, e.g., [37, Ch.I, Th.1.16]; moreover, this is true under any assumptions which ensure the validity of the Parseval identity. In what follows we assume that

$$\text{both } \lambda \text{ and } \widehat{\lambda} \text{ are integrable on } \mathbb{R} \tag{9}$$

and

$$\lambda(0) = 1. \tag{10}$$

We shall make use of either (7) or (8) as a definition of the linear means $\Lambda_N$ just according to which of the two formulas is valid.

When a function is already represented by a Fourier integral, to take its $\alpha$th derivative means multiplying the integrand by $(it)^\alpha$. Therefore, by $(\Lambda_N f)^{(1-\alpha)}$ we understand

$$(\Lambda_N f)^{(1-\alpha)}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \lambda\left(\frac{t}{N}\right)\varphi(t)(it)^{(1-\alpha)}e^{ixt}dt.$$

The above-mentioned conditions are given in the following

**Theorem 3.** *Let $\varphi$ and $\lambda$ be such that $\lambda(\frac{t}{N})\varphi(t)(it)^{(1-\alpha)}$ are integrable for all $N$. In order that $f^\circ$ belong to $d^\alpha V^\wedge$, it is necessary and sufficient that*

$$\|(\Lambda_N f)^{(1-\alpha)}\|_{L^1(\mathbb{R})} = O(1). \tag{11}$$

*Remark* 4. Clearly, $\alpha = 0$ and $\alpha = 1$ are two main cases. Slightly less general versions of these cases are known from [17] and [16], respectively. For more information, see [13] or [29].

Recalling that a function $s$ is called a *multiplier of class* $(X, Y)$ if for each $f^\circ \in X^\wedge$ there holds $sf^\circ \in Y^\wedge$, we see that Theorem 3 is a criterion for $(ix)^{1-\alpha}\varphi$ to be a multiplier of class $(L^1, L^1)$ (see, e.g., [13, Th.6.5.6]).

Let

$$\varphi(t) = Si(|t|) = \int_0^{|t|} \frac{1}{u}\sin u\, du,$$

which is bounded and non-negative, and $\lambda(t) = (1-|t|)$ for $|t| < 1$ and 0 otherwise. We wish to check that $(\Lambda_N f)$ are uniformly integrable for this choice. The values

$$\int_0^{1/N} |(\Lambda_N f)(x)|\, dx$$

are uniformly bounded; the same is true for any bounded $\varphi$. Given $x > 1/N$, we obtain for $y = Nx > 1$ by changing the order of integration

$$2\pi(\Lambda_N f)(y) = N\int_0^1 \frac{1}{u}\left[-\frac{1}{y^2}\cos y - \frac{1}{y}(1-u)\sin(yu) + \frac{1}{y^2}\cos(yu)\right]\sin Nu\, du.$$

The main term for

$$\int_{\mathbb{R}} |(\Lambda_N f)(x)|\, dx = \frac{1}{N}\int_{\mathbb{R}} |(\Lambda_N f)(y)|\, dy$$

is

$$\int_1^\infty \frac{1}{y}\left|\int_0^1 \frac{1}{u}\sin(Nu)\sin(yu)\, du\right| dy,$$

the uniform integrability of the rest is rather obvious. For $y > 2N$ and $1 < y < N/2$, we obtain the needed bounds by estimating the inner integral as

$$\frac{1}{2}\left|\int_{|y-N|}^{y+N} \frac{1}{u}\cos u\, du\right| \le \frac{1}{2}\ln\left|\frac{y+N}{y-N}\right|,$$

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and straightforward calculations. For instance, when $y > 2N$ just use that

$$\ln\left|\frac{y+N}{y-N}\right| \le \frac{2N}{y-N}\,.$$

For the integrals from $N/2$ to $N-1$ and $N+1$ to $2N$, use that $\displaystyle\int_1^\infty \frac{\cos u}{u}\,du$ exists and $\displaystyle\int_{N/2}^{2N}\frac{1}{y}\,dy = \ln 4$. The estimate over $N-1 \le y \le N+1$ is simple. Hence $Si(|t|)$ is a multiplier of class $(L^1, L^1)$.

Taking the same $\lambda$ and $\varphi(t) = 1$ for $|t| \le 1$ and $\varphi(t) = 0$ otherwise, the partial Fourier integrals, we easily derive that this function cannot be a multiplier of class $(L^1, L^1)$. Indeed, we immediately arrive at

$$\int_0^1 \left(1 - \frac{t}{N}\right)\cos xt\,dt = \left(1 - \frac{1}{N}\right)\frac{1}{x}\sin x + \frac{1}{Nx^2}(1-\cos x),$$

which is non-integrable for any $N > 1$.

The other negative example is delivered, again with the same $\lambda$, by $\varphi(t) = 1$ for $t > 0$ and $\varphi(t) = -1$ for $t < 0$. The calculations are simple:

$$\int_0^N \left(1 - \frac{t}{N}\right)\sin xt\,dt = \frac{1}{x} - \frac{1}{Nx^2}\sin Nx,$$

and the right-hand side is non-integrable for any $N$. This gives a different proof that the Hilbert transform is not a bounded operator on $L^1$. We recall that the Hilbert transform $\widetilde{g}$ of an integrable function $g$ is defined in the principal value sense as

$$\widetilde{g}(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{g(t)}{x-t}\,dt = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-t|\ge\varepsilon>0}\frac{g(t)}{x-t}\,dt. \tag{12}$$

**Proof of Theorem 3.** *Necessity.* Let $f^\circ \in d^\alpha V^\wedge$. Then

$$\begin{aligned}(\Lambda_N f)^{(1-\alpha)}(x) &= \int_{\mathbb{R}} \lambda\left(\frac{t}{N}\right)(it)(it)^{-\alpha}\varphi(t)e^{ixt}dt\\ &= \int_{\mathbb{R}} \lambda\left(\frac{t}{N}\right)\widehat{dF}(t)e^{ixt}dt.\end{aligned}$$

By the Stieltjes version of (3) (see, e.g., [13, Th.6.5.6]) we have

$$(\Lambda_N f)^{(1-\alpha)}(x) = \int_{\mathbb{R}} N\widehat{\lambda}(N(t-x))\,dF(t).$$

Hence

$$\|(\Lambda_N f)^{(1-\alpha)}\|_{L^1(\mathbb{R})} \le \int_{\mathbb{R}}\int_{\mathbb{R}} N|\widehat{\lambda}(N(t-x))|\,|dF(t)|\,dx.$$

Since $\widehat{\lambda}$ is integrable on $\mathbb{R}$, and $F$ is of bounded variation, we may use here and in what follows the Fubini theorem freely. This yields

$$\begin{aligned}\|(\Lambda_N f)^{(1-\alpha)}\|_{L^1(\mathbb{R})} &\le \int_{\mathbb{R}} |dF(t)| \int_{\mathbb{R}} N|\widehat{\lambda}(N(t-x))|\,dx \\ &= \|F\|_{BV}\,\|\widehat{\lambda}\|_{L^1(\mathbb{R})},\end{aligned}$$

and we are done.

*Sufficiency.* Denote

$$\Phi_N(x) = \int_{-\infty}^{x} (\Lambda_N f)^{(1-\alpha)}(t)\,dt.$$

This sequence possesses the following properties. First, (11) provides that both the family $\{\Phi_N\}$ and the family of variations of these functions are uniformly bounded (by the same constant $\|(\Lambda_N f)^{(1-\alpha)}\|_{L^1(\mathbb{R})}$). By virtue of the boundedness of $\{\Phi_N\}$ along with their variations, the first Helly's theorem (see, e.g., [29, Th.9.1.1]) ensures the existence of a subsequence $\{N_k\}$, $N_k \to \infty$ as $k \to \infty$, such that

$$\lim_{k\to\infty} \Phi_{N_k} = \Phi(x)$$

at any point $x$ on the whole $\mathbb{R}$, where $\Phi$ is a function of bounded variation (bounded by $\|\Lambda_N f\|_{L^1(\mathbb{R})}$ along with its total variation). Our next (and final) step is to show that

$$\widehat{d\Phi}(x) = \int_{\mathbb{R}} e^{-ixt} d\Phi(t) = \lim_{k\to\infty} \int_{\mathbb{R}} e^{-ixt} d\Phi_{N_k}(t). \tag{13}$$

Generally speaking, the second Helly theorem is true only under additional assumptions (see, e.g., [29, Th.9.1.3]). For example, it holds true if the last limit is uniform on every finite interval. As in [16], we have

$$\lambda\left(\frac{t}{N_k}\right)(it)^{1-\alpha}\varphi(t) = \int_{\mathbb{R}} e^{-ixt}\,d\Phi_{N_k}(x).$$

The right-hand side is continuous as well as $\lambda(\frac{t}{N_k})$, hence $(it)^{1-\alpha}\varphi(t)$ almost everywhere coincides with a continuous function $\psi(t)$. Now, by (9) and (10), we get that $\lambda(\frac{t}{N_k})\psi(t)$ converges, as $N \to \infty$, to $\psi(t)$ uniformly on every finite interval. Thus, (13) is true, which completes the proof. □

## 5. Integrability of the Fourier transform

Finding conditions for the integrability of the Fourier transform has long been the subject of research. Such conditions are mostly given in terms of the transformed function belonging to certain space. However, searching for such spaces as subspaces of the space of functions of bounded variation apparently was started in [30]. The "candidates" were analogs of sequence spaces from the theory of integrability of trigonometric series. Though in [30] or in greater detail in [33] the whole story is presented, let us give some important references to the papers where the mentioned sequence spaces were either introduced or studied to a great extent: [28], [8], [38], [39], [20], [22].

On the other hand, what is given below is not fixed, there are interesting problems there and continuing activity. Just a few references: [31], [32], [21], [36].

5.1. **Function spaces.** In this subsection we will introduce a scale of subspaces of $L^1$, and when the derivative of a transformed function belongs to such a subspace will characterize behavior of the Fourier transform. Correspondingly, we will study the Fourier transform of functions from subspaces of the space of functions of bounded variation.

For $1 < q < \infty$, set

$$\|g\|_{A_q} = \int_0^\infty \left( \frac{1}{u} \int_{u \le |t| \le 2u} |g(t)|^q dt \right)^{1/q} du.$$

These spaces and their sequence analogs first appeared in the paper by D. Borwein [12], but became - for sequences - widely known after the paper by G. A. Fomin [20]; see also [22, 23]. On the other hand, these spaces are a partial case of the so-called Herz spaces (see first of all the initial paper by C. Herz [27], and also a relevant paper of Flett [19]).

Further, for $q = \infty$ let

$$\|g\|_{A_\infty} = \int_0^\infty \text{ess }\sup_{u \le |t| \le 2u} |g(t)| du.$$

The role of integrable monotone majorant for problems of almost everywhere convergence of singular integrals is known from the work of D.K. Faddeev (see, e.g., [1, Ch.IV, §4]; also [37, Ch.I]); for spectral synthesis problems it was used by A. Beurling [7], for more details see [6].

Finally, let

$$\|g\|_{H_{BT}} = \int_{\mathbb{R}} |g(t)|\, dt + \int_{\mathbb{R}} \left| \int_0^{u/2} \frac{g(u-t) - g(u+t)}{t} \right| du. \tag{14}$$

This space was first introduced in [30]. In [21], the inner integral in the last summand on the right of (14) was called the Telyakovskii transform; we shall write it as $Tg$.

Recall that $H := H(\mathbb{R})$ is the space of functions $g \in L^1(\mathbb{R})$ for which their Hilbert transform (12) belongs to $L^1(\mathbb{R})$ as well. A different way to define the Hilbert transform is

$$\widetilde{g}(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(t)(x-t)}{(x-t)^2 + \varepsilon^2}\, dt. \tag{15}$$

In many cases these two definitions are equivalent, but sometimes either one is more convenient for concrete applications.

**Lemma 1.** *The following embeddings hold:*

$$A_\infty \hookrightarrow A_{p_1} \hookrightarrow A_{p_2} \hookrightarrow H_{BT} \hookrightarrow L^1 \ (p_1 > p_2 > 1). \tag{16}$$

*Proof.* The first two embeddings are merely the results of applying the Hölder inequality. Indeed, with $p_1/p_2 > 1$

$$\begin{aligned}
&\int_0^\infty \left(\frac{1}{u}\int_{u\le|t|\le 2u} |g(t)|^{p_1} dt\right)^{1/p_1} du \\
&\le \int_0^\infty \left(\frac{1}{u}\left(\int_{u\le|t|\le 2u} |g(t)|^{p_2} dt\right)^{p_1/p_2} \left(\int_{u\le|t|\le 2u} dt\right)^{1-p_1/p_2}\right)^{1/p_1} du \\
&= 2^{1/p_1-1/p_2} \int_0^\infty \left(u^{-p_2/p_1+(1-p_1/p_2)p_2/p_1} \int_{u\le|t|\le 2u} |g(t)|^{p_2} dt\right)^{1/p_2} du \\
&= 2^{1/p_1-1/p_2} \int_0^\infty \left(\frac{1}{u}\int_{u\le|t|\le 2u} |g(t)|^{p_2} dt\right)^{1/p_2} du,
\end{aligned}$$

and we are done; in the case $q = \infty$, the left embedding obviously goes along the same lines.

To prove the embedding of $A_q$ into $L^1$, we will use a standard expedient

$$2\ln 2 \int_0^\infty |g(t)|\, dt = \int_0^\infty \frac{1}{u}\int_{u\le|t|\le 2u} |g(t)|\, dt\, du.$$

Applying Hölder's inequality to the inner integral on the right yields

$$\begin{aligned}
&\int_0^\infty \frac{1}{u}\int_{u\le|t|\le 2u} |g(t)|\, dt\, du \\
&\le \int_0^\infty \frac{1}{u}\left(\int_{u\le|t|\le 2u} |g(t)|^q\, dt\right)^{1/q} \left(\int_{u\le|t|\le 2u} dt\right)^{1-1/q} du \\
&= 2^{1-1/q} \int_0^\infty \left(\frac{1}{u}\int_{u\le|t|\le 2u} |g(t)|^q dt\right)^{1/q} du.
\end{aligned}$$

A bit more delicate are estimates for the second term on the right-hand side of (14). First, because of Lemma 2 below, we can deal with

$$\ln 3 \int_0^\infty \left|\int_{x/2}^{3x/2} \frac{g(t)}{x-t}\, dt\right| dx = \int_0^\infty \frac{1}{u}\int_{u/2}^{3u/2} \left|\int_{x/2}^{3x/2} \frac{g(t)}{x-t}\, dt\right| dx\, du.$$

The following estimates are similar to those for sequences in [20]; they mainly reduce to M. Riesz's theorem. Thus, the last integral does not exceed

$$\int_0^\infty \frac{1}{u}\int_{u/2}^{3u/2} \left\{\left|\int_{u/2}^{x/2} \frac{g(t)}{x-t}\, dt\right| + \left|\int_{3x/2}^{3u/2} \frac{g(t)}{x-t}\, dt\right| + biggr| \int_{u/2}^{3u/2} \frac{g(t)}{x-t}\, dt\right|\right\} dx\, du.$$

The estimates are the same for the first and the second integrals in braces. For example,

$$\int_0^\infty \frac{1}{u}\int_{u/2}^{3u/2}\left|\int_{u/2}^{x/2}\frac{g(t)}{x-t}\,dt\right|dx\,du$$
$$\le \int_0^\infty \frac{1}{u}\int_{u/2}^{3u/2}\left\{\int_{u/2}^{x/2}|g(t)|^q\,dt\right\}^{\frac1q}\left\{\int_{u/2}^{x/2} x-t^{\frac{q}{1-q}}\,dt\right\}^{1-\frac1q}dx\,du$$
$$\le 2^{1/q}\int_0^\infty \frac{1}{u}\int_{u/2}^{3u/2}\left\{\int_{u/2}^{x/2}|g(t)|^q\,dt\right\}^{\frac1q}(\frac{x}{2})^{(1-\frac{q}{q-1})(1-\frac1q)}\,dx\,du$$
$$= 2^{2/q}\int_0^\infty \frac{1}{u}\int_{u/2}^{3u/2}\left\{\frac1x\int_{u/2}^{x/2}|g(t)|^q\,dt\right\}^{\frac1q}dx\,du$$
$$\le 2^{2/q}\int_0^\infty \frac{1}{u}\int_{u/2}^{3u/2}\left\{\frac1x\int_{x/3}^{x/2}|g(t)|^q\,dt\right\}^{\frac1q}dx\,du$$
$$= 2^{2/q}\ln 3\int_0^\infty \left\{\frac1x\int_{x/3}^{x/2}|g(t)|^q\,dt\right\}^{\frac1q}dx.$$

It now remains to make use of the following simple relation (see [30, (16)]): for any real numbers $\alpha$ and $\beta$, $0<\alpha<\beta$,

$$\alpha^{1-1/q}\int_0^\infty\left(\frac1u\int_{\alpha u}^{\beta u}|g(t)|^q dt\right)^{1/q}du$$
$$\le \int_0^\infty\left(\frac1u\int_{u}^{\infty}|g(t)|^q dt\right)^{1/q}du$$
$$\le (\alpha^{1/q-1}-\beta^{1/q-1})^{-1}\int_0^\infty\left(\frac1u\int_{\alpha u}^{\beta u}|g(t)|^q dt\right)^{1/q}du. \tag{17}$$

The left-hand side inequality in (17) follows immediately by the change of the variable $u$ for $\alpha u$. Indeed,

$$\int_0^\infty\left(\frac1u\int_{u}^{\infty}|g(t)|^q dt\right)^{1/q}du=\alpha^{1-1/q}\int_0^\infty\left(\frac1u\int_{\alpha u}^{\infty}|g(t)|^q dt\right)^{1/q}du,$$

and it remains to discard a part of the inner integral on the right-hand side. Then,

$$\alpha^{1-1/q}\int_0^\infty\left(\frac1u\int_{\alpha u}^{\infty}|g(t)|^q dt\right)^{1/q}du\le \alpha^{1-1/q}\int_0^\infty\left(\frac1u\int_{\alpha u}^{\beta u}|g(t)|^q dt\right)^{1/q}du$$
$$+\alpha^{1-1/q}\int_0^\infty\left(\frac1u\int_{\beta u}^{\infty}|g(t)|^q dt\right)^{1/q}du.$$

But the last summand is equal to

$$\left(\frac{\beta}{\alpha}\right)^{1/q-1}\int_0^\infty\left(\frac{1}{u}\int_u^\infty|g(t)|^q dt\right)^{1/q}du,$$

and the right-hand side of (17) becomes obvious.

For the last summand in braces from (17) applying again Hölder's inequality yields

$$\begin{aligned}\int_0^\infty\frac{1}{u}\int_{u/2}^{3u/2}\left|\int_{u/2}^{3u/2}\frac{g(t)}{x-t}\,dt\right|dx\,du\\ \le\int_0^\infty\frac{1}{u}\left\{\int_{-\infty}^\infty|\widetilde{G}(t)|^q\,dt\right\}^{1/q}\left\{\int_{u/2}^{3u/2}dt\right\}^{1-1/q}du,\end{aligned}\tag{18}$$

where $\widetilde{G}$ is the Hilbert transform, up to a constant, of the function $G$ which is equal to $g$ on $[u/2,3u/2]$ and vanishes otherwise. By the M. Riesz theorem

$$\begin{aligned}\left\{\int_{-\infty}^\infty|\widetilde{G}(t)|^q\,dt\right\}^{1/q}&\le C_q\left\{\int_{-\infty}^\infty|G(t)|^q\,dt\right\}^{1/q}\\&=C_q\left\{\int_{u/2}^{3u/2}|g(t)|^q\,dt\right\}^{1/q},\end{aligned}$$

and the right-hand side of (18) is bounded by

$$\begin{aligned}C_q\int_0^\infty\frac{1}{u}\left\{\int_{u/2}^{3u/2}|g(t)|^q\,dt\right\}^{1/q}\left\{\int_{u/2}^{3u/2}dt\right\}^{1-1/q}du\\=C_q\int_0^\infty\left\{\frac{1}{u}\int_{u/2}^{3u/2}|g(t)|^q\,dt\right\}^{1/q}du.\end{aligned}$$

Now, applying (17) completes the proof of the lemma. □



In [34], the following example of a function $h$ which belongs to $H_{BT}$ but not to the considered subspaces is constructed. Let $h$ be non-zero only on a family of intervals $(a_k,b_k)$, where it is equal to $S_k$, with $S_k$ being a monotone increasing positive sequence. By this,

$$\int_{\mathbb{R}_+}|h(t)|\,dt=\sum_{k=1}^\infty S_kd_k<\infty,\tag{19}$$

where $d_k=b_k-a_k$.

Calculating the norm in $A_p$, we assume $d_k$ to be small enough, while $b_k$ distant enough from $b_{k-1}$, at least so that only one $(a_k,b_k)$ is located on each $(b_{k-1},2b_k)$.

We then get

$$\sum_{k=2}^{\infty}\int_{b_{k-1}}^{b_k}\left(\frac{1}{x}\int_x^{2x}|h(t)|^p dt\right)^{1/p}dx \ge \sum_{k=2}^{\infty}\int_{b_{k-1}}^{b_k}\left(\frac{1}{x}\int_{a_k}^{b_k}|h(t)|^p dt\right)^{1/p}dx.$$

Routine calculations give that this value is not smaller than

$$\sum_{k=2}^{\infty} S_k(b_k-b_{k-1})\left(\frac{d_k}{b_k}\right)^{1/p}. \tag{20}$$

It is clear that when $p=\infty$ we have the same value by formally considering $p=\infty$.

Let us now estimate the integral of $|Th|$. For this we separately estimate

$$\int_{2a_k/3}^{2b_k}|Th(x)|\,dx.$$

In fact, there are 5 subintervals of $(2a_k/3, 2b_k)$ on which calculation of $Th$ is slightly different. Integration of the results is similar over each of them, hence we consider one of the most problematic $(2b_k/3, a_k)$. The point is that, as also for $(b_k, 2a_k)$, the length of the interval is not proportional to $d_k$. We get

$$\int_{2b_k/3}^{a_k}\left[\ln(a_k-x)-\ln(b_k-x)\right]dx = d_k\ln\frac{1}{d_k(a_k-2b_k/3)}+\frac{b_k}{3}\ln\left(1-3\frac{d_k}{b_k}\right).$$

Up to logarithmic factors, this is equivalent to $d_k$.

To conclude, we must have (19), even with possible logarithmic factors, such that the series in (20) diverges for each $p\in(1,\infty]$. For instance, let $d_k=2^{-k}$, $b_k=2^k$, and $A_k=k^{-\beta}2^k$. Here $\beta$ should be large enough to ensure not only (19) but the same with possible logarithmic factors which are powers of $k$. For the sum in (20), we will get

$$\sum_{k=2}^{\infty}k^{-\beta}2^{2(1-1/p)k}.$$

Since $1-1/p>0$, we obtain the desired counterexample.

5.2. **Behavior of the Fourier transform.** For functions with derivative in each of the spaces considered in the previous subsection, either $A_q$, $1<q\le\infty$, or $H_{BT}$, the following result, Theorem 5, is obtained in [30] (for convex functions as well as for functions with derivative in $A_\infty$ it was earlier obtained by R.M. Trigub, see, e.g., [42] and [44]). See also [23] and [21].

In view of (16), the assertions of Theorem (5) with $H_{BT}$ replaced by $A_p$ will follow immediately. Nevertheless, though $H_{BT}$ is wider, the conditions of belonging to smaller spaces are more practical.

The space $H_{BT}$ is of importance not only because of (16) but also because of its proximity to the real Hardy space $H$. It is obvious that the inner integral in the second summand on the right-hand side of (14) is very close to $\widetilde{g}$. More precisely, (14) means (cf. Lemma 2) that the odd extension of $g$ belongs to $H(\mathbb{R})$ rather than $g$ itself.

*Let us here denote by $\varphi$ the* **odd** *extension of $f'$ from $[0,\infty)$ to the whole $\mathbb{R}$.*

**Theorem 5.** *Let $f$ be a locally absolutely continuous function on $[0,\infty)$, with $\lim_{t\to\infty} f(t) = 0$, and let $\varphi \in H(\mathbb{R})$. Then for $y > 0$*

$$F_c(y) = \int_0^\infty f(x)\cos yx\, dx = \gamma_1(y),$$

*and*

$$F_s(y) = \int_0^\infty f(x)\sin yx\, dx = \frac{1}{y} f(\frac{\pi}{2y}) + \gamma_2(y),$$

*where $\int_0^\infty |\gamma_j(y)| dy \le C\|\varphi\|_{H(\mathbb{R})}$, $j = 1, 2$.*

The possibility to prove the theorem in this form is justified by the following assertion.

**Lemma 2.** *Let $g$ be an odd function integrable on $\mathbb{R}$. Then*

$$\int_{\mathbb{R}} \frac{g(t)}{x-t}\, dt = \int_{x/2}^{3x/2} \frac{g(t)}{x-t}\, dt + \gamma(x),$$

*where*

$$\int_{\mathbb{R}} |\gamma(x)|\, dx \le C \int_{\mathbb{R}} |g(t)|\, dt.$$

*Proof.* We may assume, without loss of generality, that $x > 0$ since for $x < 0$ the proof is exactly the same. Substituting $t \to -t$ in the integral

$$I_1 = \int_{-\infty}^{-3x/2} \frac{g(t)}{x-t}\, dt,$$

we obtain

$$I_1 = \int_{3x/2}^\infty \frac{g(-t)}{x+t}\, dt = -\int_{3x/2}^\infty \frac{g(t)}{x+t}\, dt.$$

We have

$$\int_{3x/2}^\infty \frac{g(t)}{x-t}\, dt + I_1 = \int_{3x/2}^\infty g(t)\left[\frac{1}{x-t} - \frac{1}{x+t}\right] dt,$$

and by Fubini's theorem

$$\int_0^\infty \left| \int_{3x/2}^\infty g(t)\left[\frac{1}{x-t} - \frac{1}{x+t}\right] dt \right| dx$$

$$\begin{aligned}&\le \int_0^\infty |g(t)|\,dt \int_0^{2t/3}\left[\frac{1}{x+t}-\frac{1}{x-t}\right]dx\\&= \int_0^\infty |g(t)|\,dt \ln\frac{x+t}{t-x}\Big|_0^{2t/3}\\&= \int_0^\infty |g(t)|\,dt \ln\frac{2t/3+t}{t/3} = \ln 5\int_0^\infty |g(t)|\,dt.\end{aligned}$$

In the same way

$$\int_{-x/2}^0 \frac{g(t)}{x-t}\,dt = \int_0^{x/2}\frac{g(-t)}{x+t}\,dt = -\int_0^{x/2}\frac{g(t)}{x+t}\,dt,$$

and

$$\begin{aligned}\int_0^\infty \left|\int_0^{x/2} g(t)\left[\frac{1}{x-t}-\frac{1}{x+t}\right]dt\right| dx&\\ \le \int_0^\infty |g(t)|\,dt\int_{2t}^\infty\left[\frac{1}{x-t}-\frac{1}{x+t}\right]dx&\\ = \int_0^\infty |g(t)|\,dt\ln\frac{x-t}{t+x}\Big|_{2t}^\infty = \ln 3\int_0^\infty |g(t)|\,dt,&\end{aligned}$$

which completes the proof. □

*Remark* 6. Since

$$\begin{aligned}\int_0^\infty\left|\int_{x/2}^{3x/2}\frac{g(t)}{x+t}\,dt\right|dx &\le \int_0^\infty |g(t)|\,dt\int_{2t/3}^{2t}\frac{dx}{x+t}\\&= \int_0^\infty |g(t)|\,dt\ln(x+t)\Big|_{2t/3}^{2t}\\&= \ln(9/5)\int_0^\infty |g(t)|\,dt,\end{aligned}$$

we have

$$\int_{\mathbb{R}}\frac{g(t)}{x-t}\,dt = \int_0^\infty g(t)\left[\frac{1}{x-t}-\frac{1}{x+t}\right]dt + \gamma(x).$$

We are now in a position to prove Theorem 5.

*Proof.* Integrating by parts yields

$$\begin{aligned}F_s(y) = \frac{1}{y}f(0) &+ \frac{1}{y}\int_0^{\pi/(2y)} f'(x)\,dx\\ &+ \frac{1}{y}\int_0^{\pi/(2y)} f'(x)(\cos yx - 1)\,dx\\ &+ \frac{1}{y}\int_{\pi/(2y)}^\infty f'(x)\cos yx\,dx\end{aligned}$$

$$= \frac{1}{y} f(\pi/(2y)) + O\biggl(\int_0^{\pi/(2y)} |f'(x)| x\, dx\biggr) + \frac{1}{y}\int_{\pi/(2y)}^{\infty} f'(x)\cos yx\, dx.$$

Similarly,

$$F_c(y) = O\biggl(\int_0^{\pi/(2y)} |f'(x)| x\, dx\biggr) - \frac{1}{y}\int_{\pi/(2y)}^{\infty} f'(x)\sin yx\, dx.$$

Since

$$\int_0^{\infty} dy \int_0^{\pi/(2y)} |f'(x)| x\, dx = \frac{\pi}{2}\int_0^{\infty} |f'(x)|\, dx,$$

it remains to show that the last integral in both representations, for $F_s$ and $F_c$, satisfies the assumptions imposed on $\gamma_j$. We will prove this in detail for $F_s$, since for $F_c$ computations are similar.

Thus we examine, for sufficiently large $N$,

$$\int_0^N \left|\int_{\pi/(2y)}^{\infty} f'(x)\cos yx\, dx\right| \frac{dy}{y}.$$

Denoting

$$\Phi(y) = \begin{cases} \int_{\pi/(2y)}^{\infty} f'(x)\cos yx\, dx, & 0 \le y \le N, \\ (2-\frac{y}{N})\int_{\pi/(2y)}^{\infty} f'(x)\cos yx\, dx, & N < y \le 2N, \\ 0, & y > 2N, \end{cases}$$

we have

$$\int_0^N \left|\int_{\pi/(2y)}^{\infty} f'(x)\cos yx\, dx\right| \frac{dy}{y} \le \int_0^{\infty} |\Phi(y)|\, \frac{dy}{y}.$$

We need the following extension of the Hardy-Littlewood theorem (see, e.g., [45, Ch.VII, Th.8.7]); though the proof can be found in [30], the following short proof is contained in essence already in [5].

**Lemma 3.** *Let* $\Phi \in L^1(-\infty,\infty)$, *with* $\mathrm{supp}\Phi \subset [0,\infty)$. *Then*

$$\int_0^{\infty} \frac{|\Phi(y)|}{y}\, dy \le C\int_0^{\infty} |\widehat{\Phi}(u)|\, du. \tag{21}$$

*Proof.* We have

$$\frac{1}{R}\sum_{\alpha R\le k\le \beta R} |\Phi(\frac{k}{R})|\frac{R}{k} \le \sum_{k=1}^{\infty} |\Phi(\frac{k}{R})| k$$

$$\leq C\int_{-1/2}^{1/2}|\sum_{k=1}^{\infty}|\Phi(\frac{k}{R})e^{2\pi ikx}|\,dx\leq C\int_0^\infty|\hat{\Phi}(u)|\,du. \tag{22}$$

For sufficiently large $R$ and an appropriate choice of $\alpha$ and $\beta$, the first inequality is obvious, the second one is the Hardy-Littlewood theorem, and the last one is well known (see, e.g., [4], where it is given under much more general assumptions). We observe that an integral sum for the left-hand side of (21) occurs on the left of (22). Passing then to the limit as $R\to\infty$ completes the proof. □

We are now going to estimate the Fourier transform of $\Phi$. Since $e^{iuy}=\cos uy+i\sin uy$, we restrict ourselves to estimating, say, the sine Fourier transform of $\Phi$; the cosine Fourier transform is estimated in the same way with minor changes. We have

$$\begin{aligned}\int_0^\infty \Phi(y)\sin uy\,dy &= \int_0^N \sin uy\,dy\int_{\pi/(2y)}^\infty f'(x)\cos yx\,dx\\ &\quad+\int_N^{2N}(2-y/N)\sin uy\,dy\int_{\pi/(2N)}^\infty f'(x)\cos yx\,dx.\end{aligned}$$

Changing the order of integration, we arrive at the integral

$$\begin{aligned}\int_{\pi/(2N)}^\infty f'(x)\,dx\Big[\int_{\pi/(2x)}^N &\sin uy\cos yx\,dy\\ &+\int_N^{2N}(2-y/N)\sin uy\cos yx\,dy\Big].\end{aligned}$$

Using known trigonometric formulas for both inner integrals and integrating by parts in the second one, we get

$$\begin{aligned}\frac12\int_{\pi/(2N)}^\infty f'(x)\,dx\Big\{&\Big[-\frac{\cos(u+x)y}{u+x}-\frac{\cos(u-x)y}{u-x}\Big]_{\pi/(2x)}^N\\ &+(2-\frac{y}{N})\Big[-\frac{\cos(u+x)y}{u+x}-\frac{\cos(u-x)y}{u-x}\Big]_N^{2N}\\ &-\frac1N\int_N^{2N}\Big[-\frac{\cos(u+x)y}{u+x}-\frac{\cos(u-x)y}{u-x}\Big]\,dy\Big\}\\ &=\frac12\int_{\pi/(2N)}^\infty f'(x)\Big[\frac{1}{u+x}-\frac{1}{u-x}\Big]\sin\frac{\pi u}{2x}\,dx\\ &-\frac{1}{2N}\int_{\pi/(2N)}^\infty f'(x)\int_N^{2N}\Big[\frac{\cos(u+x)y}{u+x}+\frac{\cos(u-x)y}{u-x}\Big]\,dy\,dx.\end{aligned} \tag{23}$$

We are now going to group certain values on the right-hand side in a special way. In accordance with Lemma 3, we then integrate them modulo over $[1/(2N),\infty)$.

Indeed, integration over $[0,1/(2N)]$ is carried out trivially:

$$\int_0^{\pi/(2N)} |\hat\Phi(u)|\,du \le \int_0^{1/(2N)} \left[\left(\int_0^N + \int_N^{2N}\right) dy \int_0^\infty |f'(x)|\,dx\right] du$$
$$= \pi \int_0^\infty |f'(x)|\,dx.$$

We mostly deal with the terms corresponding to $u-x$ since those corresponding to $u+x$ are handled in an even easier manner. First, grouping

$$\int_{\pi/(2N)}^\infty f'(x)\,\frac{1}{N}\int_N^{2N} \frac{1}{u-x}\left[-\cos 2\pi(u-x)y + \sin\frac{\pi u}{2x}\right] dy\,dx$$

and estimating the expression in the square brackets by

$$2\left|\sin\left(\frac{u-x}{2}\left(\frac{\pi}{2x} - 2\pi y\right)\right)\right|,$$

we proceed with the integral over $[u-\pi/(4N), u+\pi/(4N)]$ as follows:

$$\int_{\pi/(2N)}^\infty \left|\int_{u-\pi/(4N)}^{u+\pi/(4N)} f'(x)\,\frac{1}{N}\int_N^{2N} (2\pi y - \pi/(2x))\,dy\right| du\,dx$$
$$\le \frac{1}{N}\int_{\pi/(4N)}^{3\pi/(4N)} |f'(x)| \int_N^{2N} (2\pi y - \pi/(2x))\,dy \int_{\pi/(2N)}^{x+\pi/(4N)} du\,dx$$
$$+ \frac{1}{N}\int_{3\pi/(4N)}^\infty |f'(x)| \int_N^{2N} (2\pi y - \pi/(2x))\,dy \int_{\pi/(2N)}^{x+\pi/(4N)} du\,dx$$
$$\le C\int_0^\infty |f'(x)|\,dx.$$

When $x \notin [u-\pi/(4N), u+\pi/(4N)]$, we check integrability of the last integral in (23) separately. Integrating in the inner integral and estimating $|\sin(u-x)y|$ roughly by 1, we get the term $(u-x)^{-2}$ to deal with and obtain

$$\frac{1}{N}\int_{\pi/(2N)}^\infty \left|\int_{\pi/(4N)}^{u-\pi/(4N)} |f'(x)|\frac{1}{(u-x)^2}dx\right| du$$
$$\le \frac{1}{N}\int_{\pi/(4N)}^\infty |f'(x)| \int_{x+\pi/(4N)}^\infty \frac{1}{(u-x)^2} du\,dx$$
$$= 4\int_0^\infty |f'(x)|\,dx.$$

Similarly,

$$\frac{1}{N}\int_{\pi/(2N)}^\infty \left|\int_{u+1/(4N)}^\infty |f'(x)|\frac{1}{(u-x)^2}dx\right| du$$
$$\le \frac{1}{N}\int_{3\pi/(4N)}^\infty |f'(x)|\,dx \int_{\pi/(2N)}^{x-\pi/(4N)} \frac{1}{(u-x)^2}du$$

$$= \frac{1}{N}\int_{3\pi/(2N)}^{\infty} |f'(x)|\,dx\Big(\frac{4N}{\pi} - \frac{1}{x-\pi/(2N)}\Big)$$
$$\le 4\int_0^\infty |f'(x)|\,dx.$$

It remains to proceed with (one half of)

$$\Big(\int_{\pi/(4N)}^{u-\pi/(4N)} + \int_{u+\pi/(4N)}^{\infty}\Big) f'(x)\Big[\frac{1}{u-x} - \frac{1}{u+x}\Big]\sin\frac{\pi u}{2x}\,dx.$$

Estimating $|\sin\frac{\pi u}{2x}| \le 1$, we thus obtain

$$\int_{\pi/(2N)}^{\infty}\int_{\pi/(4N)}^{u/2} |f'(x)|\Big[\frac{1}{u-x} - \frac{1}{u+x}\Big]\,dx\,du$$
$$\le \int_{\pi/(4N)}^{\infty} |f'(x)|\int_{2x}^{\infty}\Big[\frac{1}{u-x} - \frac{1}{u+x}\Big]\,du\,dx$$
$$\le C\int_0^\infty |f'(x)|\,dx.$$

Since analogously

$$\int_{\pi/(2N)}^{\infty}\int_{3u/2}^{\infty} |f'(x)|\Big|\frac{1}{u-x} - \frac{1}{u+x}\Big|\,dx\,du$$
$$\le \int_{3\pi/(2N)}^{\infty} |f'(x)|\int_{\pi/(2N)}^{2x/3}\Big[\frac{1}{u+x} - \frac{1}{u-x}\Big]\,du\,dx$$
$$\le C\int_0^\infty |f'(x)|\,dx,$$

we arrive to the estimating (one half of)

$$\Big(\int_{u/2}^{u-1/(4N)} + \int_{u+1/(4N)}^{3u/2}\Big) f'(x)\Big[\frac{1}{u-x} - \frac{1}{u+x}\Big]\sin\frac{\pi u}{2x}\,dx.$$

Further, since

$$\int_{\pi/(2N)}^{\infty}\int_{u/2}^{3u/2} |f'(x)|(u+x)^{-1}dx\,du$$
$$\le \int_0^\infty |f'(x)|\int_{2x/3}^{2x}(u-x)^{-1}du\,dx = \ln 3\int_0^\infty |f'(x)|\,dx$$

and

$$\int_0^\infty\int_{u/2}^{3u/2} |f'(x)|\,|u-x|^{-1}|\sin\frac{\pi u}{2x} - 1|\,dx\,du$$
$$\le C\int_0^\infty |f'(x)|x^{-1}\int_{2x/3}^{2x} du\,dx \le C\int_0^\infty |f'(x)|\,dx,$$

it remains to estimate

$$\left(\int_{u/2}^{u-\pi/(4N)} + \int_{u+\pi/(4N)}^{3u/2}\right)\frac{f'(x)}{u-x}\,dx$$

for $N$ large enough. For convenience, we rewrite the last quantity as

$$\left(\int_{u/2}^{u-\varepsilon} + \int_{u+\varepsilon}^{3u/2}\right)\frac{f'(x)}{u-x}\,dx$$

with small $\varepsilon$, or, equivalently,

$$\int_{\substack{u/2\le x\le 3u/2,\\ |x-u|\ge\varepsilon}} \frac{f'(x)}{u-x}\,dx.$$

By Lemma 2, we may consider

$$\int_{|x-u|\ge\varepsilon} \frac{\varphi(x)}{u-x}\,dx$$

rather than the last integral. From now on we no longer need to remember that $\varphi$ is odd, the only thing we are interested in is the integrability of $\varphi$. Unfortunately, considering

$$\int_{\mathbb{R}}\left|\int_{|x-u|\ge\varepsilon} \frac{\varphi(x)}{u-x}\,dx\right|du$$

does not lead us directly to the desired Hilbert transform. We will consider the Hilbert transform in the form (15). For this, we estimate, for arbitrary $\varepsilon>0$,

$$\int_{\mathbb{R}}\left|\int_{|x-u|\ge\varepsilon} \frac{\varphi(x)}{u-x}\,dx - \int_{\mathbb{R}}\varphi(x)\frac{u-x}{(u-x)^2+\varepsilon^2}\,dx\right|du.$$

Let us first handle

$$\int_0^\infty \int_{|x-u|\le\varepsilon} |\varphi(x)|\frac{|u-x|}{(u-x)^2+\varepsilon^2}\,dx\,du;$$

the integral over $(-\infty,0)$ is worked out in exactly the same way. We consider three different cases. The first one is extremely simple:

$$\int_0^\varepsilon \int_{|u-x|\le\varepsilon} |\varphi(x)|\frac{|u-x|}{(u-x)^2+\varepsilon^2}\,dx\,du \le \int_0^\varepsilon \int_{-\varepsilon}^{\varepsilon} |\varphi(x)|\varepsilon^{-1}\,dx\,du \le 2\int_{\mathbb{R}}|\varphi(x)|\,dx.$$

For the second one,

$$\int_\varepsilon^{2\varepsilon} \int_{|u-x|\le\varepsilon} |\varphi(x)|\frac{|u-x|}{(u-x)^2+\varepsilon^2}\,dx\,du \le \int_\varepsilon^{2\varepsilon} \int_{u-\varepsilon}^{\varepsilon} |\varphi(x)|\frac{|u-x|}{(u-x)^2+\varepsilon^2}\,dx\,du$$

$$\le \int_0^{\varepsilon} |\varphi(x)| \int_{\varepsilon}^{x+\varepsilon} \varepsilon^{-1} du\, dx \le \int_{\mathbb{R}} |\varphi(x)|\, dx.$$

Finally,

$$\int_{\varepsilon}^{\infty} \int_{|u-x|\le\varepsilon} |\varphi(x)| \frac{|u-x|}{(u-x)^2+\varepsilon^2}\, dx\, du$$
$$= \int_{\varepsilon}^{\infty} |\varphi(x)| \int_{x-\varepsilon}^{x+\varepsilon} \frac{|u-x|}{(u-x)^2+\varepsilon^2}\, du\, dx \le 2\int_0^{\infty} |\varphi(x)|\, dx.$$

We have now to estimate

$$\int_{\mathbb{R}} \left| \int_{|x-u|\ge\varepsilon} \varphi(x) \left[ \frac{1}{u-x} - \frac{u-x}{(u-x)^2+\varepsilon^2} \right] dx \right| du.$$

With

$$\frac{1}{u-x} - \frac{u-x}{(u-x)^2+\varepsilon^2} = \frac{\varepsilon^2}{(u-x)[(u-x)^2+\varepsilon^2]}$$

in hand, and assuming, as above, that $u$ and $x$ are positive, we obtain

$$\int_{\varepsilon}^{\infty} \left| \int_0^{u-\varepsilon} \varphi(x) \frac{\varepsilon^2}{(u-x)[(u-x)^2+\varepsilon^2]}\, dx \right| du$$
$$\le \int_0^{\infty} |\varphi(x)|\, \varepsilon \int_{x+\varepsilon}^{\infty} \frac{du}{(u-x)^2+\varepsilon^2}\, dx \le \frac{\pi}{4} \int_0^{\infty} |\varphi(x)|\, dx.$$

Observing that

$$\int_0^{\infty} \left| \int_{u+\varepsilon}^{\infty} \varphi(x) \frac{\varepsilon^2}{(u-x)[(u-x)^2+\varepsilon^2]}\, dx \right| du$$
$$\le \int_{\varepsilon}^{\infty} |\varphi(x)|\, \varepsilon \int_0^{x-\varepsilon} \frac{du}{(u-x)^2+\varepsilon^2}\, dx \le \frac{\pi}{4} \int_0^{\infty} |\varphi(x)|\, dx,$$

we then need the following

**Lemma 4.** *Let $g$ be an integrable function. Then*

$$\int_{\mathbb{R}} g(x) \frac{u-x}{(u-x)^2+\varepsilon^2}\, dx = \varepsilon \int_{\mathbb{R}} \frac{\widetilde{g}(x)}{(u-x)^2+\varepsilon^2}\, dx.$$

*Proof.* This result is proved in [37, Ch.VI, Lemma 1.5] for functions from $L^p$, $p > 1$, by passing to the Fourier transforms. This idea works here as well but we give a simple direct proof instead. Rewriting the right-hand side and using Fubini's theorem, we have

$$\int_{\mathbb{R}} \frac{1}{(u-x)^2+\varepsilon^2}\, \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(t)}{x-t}\, dt\, dx = \int_{\mathbb{R}} g(t)\, \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x-t}\, \frac{1}{(u-x)^2+\varepsilon^2}\, dx\, dt.$$

Substituting $x-u = z\varepsilon$, we obtain

$$\int_{\mathbb{R}} g(t)\, \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x-t} \frac{1}{(u-x)^2+\varepsilon^2}\, dx\, dt$$

$$= \varepsilon^{-2} \int_{\mathbb{R}} g(t)\, \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{z - (t-u)/\varepsilon}\, \frac{1}{z^2+1}\, dz\, dt.$$

Since

$$\int_{\mathbb{R}} \frac{1}{z-a}\, \frac{1}{z^2+1}\, dz = \frac{1}{a^2+1} \int_{\mathbb{R}} \left[\frac{1}{z-a} - \frac{z+a}{z^2+1}\right] dz = -\frac{a\pi}{a^2+1},$$

we have ($a = (t-u)/\varepsilon$)

$$\int_{\mathbb{R}} \frac{\widetilde{\varphi}(x)}{(u-x)^2+\varepsilon^2}\, dx = \varepsilon^{-2} \int_{\mathbb{R}} g(t)\, \frac{(u-t)/\varepsilon}{((u-t)/\varepsilon)^2+1}\, dt$$

$$= \varepsilon^{-1} \int_{\mathbb{R}} g(t) \frac{u-t}{(u-t)^2+\varepsilon^2}\, dt,$$

which completes the proof. □

To finish the proof of the theorem, it remains to apply Lemma 4 with $g = \varphi$ and observe that

$$\int_{\mathbb{R}} \left| \varepsilon \int_{\mathbb{R}} \frac{\widetilde{\varphi}(x)}{(u-x)^2+\varepsilon^2}\, dx \right| du \le \int_{\mathbb{R}} |\widetilde{\varphi}(x)| \int_{\mathbb{R}} [1+(u-x)^2/\varepsilon^2]^{-1} d((u-x)/\varepsilon)\, dx$$

$$= \pi \int_{\mathbb{R}} |\widetilde{\varphi}(x)|\, dx.$$

The terms estimated above are bounded by $\int_{\mathbb{R}} |\varphi(x)|\, dx$, along with the last one bounded by $\int_{\mathbb{R}} |\widetilde{\varphi}(x)|\, dx$ they can be treated as $\gamma_2$. □

## 6. Concluding remarks

The above is apparently the first attempt to merge into one topic various results on the Fourier transform of a function of bounded variation. Only some representative results are given in this survey. However, let us discuss open problems and prospects rather than going further into details.

A natural question is what is known and/or can be done in the multidimensional case. There are numerous multivariate generalizations. Many results are presented in [44], see also [23] or [35]. However, much is unclear and there an ocean of work remains to be done in that case.

It seems that one of the most important, interesting and challenging problems is to find the widest subspace of the space of functions of bounded variation for which the results from Section 5, especially Theorem 5, hold. The above shows that it is strongly related to the theory of the Hilbert transform and Hardy spaces. The other direction here is to get rid of absolute continuity, apparently by using more Stieltjes integrals rather than just Riemann or Lebesgue integrals. Not every operation with the latter in Section 5 can automatically be transferred to the Stieltjes integration. In case of success, the next idea can be to obtain analogous results for the Fourier transform of measures rather than functions.

It will be interesting to find applications of Theorem 3 or related results to probability, maybe in the spirit of [29].

In these notes, we have tried not only to convince the reader that this is an interesting self-contained topic but also that it is vital and has interesting prospects. If the author has succeeded to interest any of the readers in certain problems that have been presented, then his task will have been fulfilled.

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