HOLOMORPHIC MAPS BETWEEN MODULI SPACES

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Abstract. We prove that every non-constant holomorphic map $\mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ between moduli spaces of Riemann surfaces is a forgetful map, provided that $g \geq 6$ and $g' \leq 2g-2$.

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Let $\mathcal{M}_{g,p}$ be the moduli space of Riemann surfaces of genus g with p labelled marked points. Moduli space has a natural structure as a complex orbifold. In this paper we study holomorphic maps, in the category of orbifolds, between distinct moduli spaces. Examples of such maps are the so-called *forgetful maps* [7, 10]: given $(i_1, \ldots, i_{p'})$ with $i_j \in \{1, \ldots, p\}$ and $i_j \neq i_k$ for $j \neq k$, set

$$\mathcal{M}_{g,p} \to \mathcal{M}_{g,p'}, \quad (X, x_1, \ldots, x_p) \mapsto (X, x_{i_1}, \ldots, x_{i_{p'}}). \tag{1.1}$$

We prove that under suitable genus bounds, there are no other non-constant holomorphic maps:

Theorem 1.1. *Suppose that $g \geq 6$ and $g' \leq 2g-2$. Every non-constant holomorphic map $\mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ is a forgetful map.*

As a direct consequence of Theorem 1.1 we obtain:

Corollary 1.2. *Suppose that $g \geq 6$ and $g' \leq 2g-2$. If there is a non-constant holomorphic map $\mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$, then $g = g'$ and $p \geq p'$.* □

Theorem 1.1 remains true under slightly more generous conditions - compare with the discussion at the end of section 6. On the other hand, some genus bounds are necessary for Theorem 1.1 to hold. For instance, it follows from [2] that for all $g \geq 2$ there are $g' > g$ and a holomorphic embedding $\mathcal{M}_{g,0} \hookrightarrow \mathcal{M}_{g',0}$.

Moduli space is not compact, but a natural compactification $\bar{\mathcal{M}}_{g,p}$, a projective variety, was constructed in [9] by Deligne and Mumford. Morphisms between Deligne-Mumford compactifications have been studied by several authors (see for example [8, 14]). Notice that in Theorem 1.1 we assume neither that the holomorphic maps in question extend to the Deligne-Mumford compactification, nor that they are surjective or have connected fibers.

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We sketch briefly the proof of Theorem 1.1. Since we are working in the category of orbifolds, every continuous map $f\colon \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ induces a homomorphism $f_*\colon \mathrm{Map}_{g,p} \to \mathrm{Map}_{g',p'}$ between the associated mapping class groups. In [3] we classified all homomorphisms between mapping class groups under the genus bounds in Theorem 1.1; it follows easily from this classification that f is either homotopically trivial or homotopic to a forgetful map h. The following result, the main technical observation of this note, yields immediately that, if f is holomorphic, then $f = h$ unless f is constant.

Proposition 1.3. *Let $M \to \mathcal{M}_{g,p}$ and $N \to \mathcal{M}_{g',p'}$ be finite covers and suppose that $f_1, f_2\colon M \to N$ are homotopic holomorphic maps. If f_1 is not constant, then $f_1 = f_2$.*

To prove Proposition 1.3 we proceed as follows. Endowing the target N with the Weil-Petersson metric and the domain M with McMullen's Kähler hyperbolic metric [19], we derive from a result of Royden [21] that the maps f_1, f_2 have finite energy. Let $(f_t)_{t\in[1,2]}$ be the straight homotopy between f_1 and f_2, and note that the energy function $t \mapsto E(f_t)$ is convex because the Weil-Petersson metric is negatively curved. Proposition 1.3 follows easily once we prove that $E(f_t)$ attains its minimum at $t \in \{1, 2\}$. To see that this is the case, we adapt an argument due to Eells-Sampson [12], who derived from a version of Wirtinger's inequality and Stokes' theorem that holomorphic maps between Kähler manifolds are harmonic, meaning that they minimize energy for every deformation with compact support. A priori, the straight homotopy (f_t) does not have compact support and so we have to control the boundary terms that appear when applying Stoke's theorem - this is what we do.

The content of this paper was originally included in our paper [3]. Following the suggestion of the journal and the referee, we decided to rewrite it and present Theorem 1.1 independently.

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Throughout this paper we make use of standard facts about Teichmüller spaces and mapping class groups. We refer to [13, 15, 16, 20] for an extensive treatment of these subjects, and to [6] for a nice survey.

Let $S_{g,p}$ be the closed surface of genus g with p distinct labeled marked points. The mapping class group $\mathrm{Map}_{g,p}$ is the group of isotopy classes of orientation preserving self-homeomorphisms of $S_{g,p}$ fixing each marked point. It acts discretely on Teichmüller space and this action preserves the standard complex structure on $\mathcal{T}_{g,p}$ (see [1]). Moduli space is the complex orbifold

$$\mathcal{M}_{g,p} = \mathcal{T}_{g,p} / \operatorname{Map}_{g,p}.$$

We stress that we consider $\mathcal{M}_{g,p}$ always as an orbifold - in a language that we are not going to use, $\mathcal{M}_{g,p}$ is the moduli stack of algebraic curves of genus g with p points labelled. In particular, maps and covers are taken in the category of orbifolds. For instance, for every continuous map

$$f\colon \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$$

there are a homomorphism

$$f_*\colon \operatorname{Map}_{g,p} \to \operatorname{Map}_{g',p'}$$

and a continuous f_*-equivariant map

$$\tilde{f}\colon \mathcal{T}_{g,p} \to \mathcal{T}_{g',p'}$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{T}_{g,p} \arrow[r, "\tilde{f}"] \arrow[d] & \mathcal{T}_{g',p'} \arrow[d] \\ \mathcal{M}_{g,p} \arrow[r, "f"] & \mathcal{M}_{g',p'}. \end{tikzcd}$$

The homomorphism f_* and the lift $\tilde{f}$ are unique up to simultaneous conjugation by a mapping class.

Example 1. For the forgetful map $f\colon \mathcal{M}_{g,p} \to \mathcal{M}_{g,p'}$ defined in (1.1), the lift $\tilde{f}\colon \mathcal{T}_{g,p} \to \mathcal{T}_{g,p'}$ is given by the same formula, and the homomorphism $f_*\colon \operatorname{Map}_{g,p} \to \operatorname{Map}_{g,p'}$ is the one given by forgetting the marked points $\{x_1,\ldots,x_p\} \setminus \{x_{i_1},\ldots,x_{i_{p'}}\}$ [3, 13]. In fact, both f and $\tilde{f}$ are holomorphic fiber bundles, and the long exact sequence of homotopy groups corresponding to the fiber bundle f yields the Birman exact sequence for f_*.

The above discussion amounts to saying that Teichmüller space is the (orbifold) universal cover of moduli space. In fact, more is true: Teichmüller space is a classifying space for proper actions $\underline{E}(\operatorname{Map}_{g,p})$ of the mapping class group. In particular, the homotopy class of f is determined by f_*. We give a simple proof of this fact. Suppose that

$$f, h\colon \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$$

have lifts $\tilde{f}, \tilde{h}\colon \mathcal{T}_{g,p} \to \mathcal{T}_{g',p'}$ that are equivariant under the same homomorphism $\rho\colon \operatorname{Map}_{g,p} \to \operatorname{Map}_{g',p'}$. For each $X \in \mathcal{T}_{g,p}$ let $\gamma_X\colon [0,1] \to \mathcal{T}_{g',p'}$ be the unique Weil-Petersson (or Teichmüller) geodesic with $\gamma_X(0) = \tilde{f}(X)$ and $\gamma_X(1) = \tilde{h}(X)$. The uniqueness of γ_X implies that the map

$$F\colon [0,1] \times \mathcal{T}_{g,p} \to \mathcal{T}_{g',p'}, \quad F(X,t) = \gamma_X(t) \tag{2.1}$$

is continuous and ρ-equivariant, and thus descends to a homotopy between f and h.

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A key ingredient in the proof of Proposition 1.3 is the existence of a compact exhaustion of $\mathcal{M}(X)$ that behaves well with respect to the Teichmüller metric d_{T}; see [15, 16] for a discussion of this metric.

Proposition 3.1. *There are $c > 0$ and a collection $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets of $\mathcal{M}_{g,p}$ with the following properties:*

1. $\mathcal{M}_{g,p} = \bigcup_{n\in\mathbb{N}} K_n$, *and* $K_n \subset K_{n+1}$ *for all* n,
2. $\mathrm{vol}_{\mathrm{T}}(\partial K_n) \leq ce^{-n}$ *for all* n, *and*
3. K_n *is contained within* d_{T}*-distance* $c + n$ *of* K_0 *for all* n.

In the statement of Proposition 3.1, $\mathrm{vol}_{\mathrm{T}}(\cdot)$ denotes the co-dimension one d_{T}-volume.

The key tool of the proof of Proposition 3.1 is a result due to Royden [21] asserting that the Teichmüller metric d_{T} agrees with the Kobayashi metric of $\mathcal{T}_{g,p}$. Recall that if M is a complex manifold, the Kobayashi pseudometric [18] is the largest pseudometric on M such that every holomorphic map $\mathbb{D} \to M$ is 1-Lipschitz; here $\mathbb{D}$ is the unit disk in $\mathbb{C}$ endowed with the Poincaré metric.

Remark. In general, holomorphic maps between complex manifolds endowed with the Kobayashi pseudometric are 1-Lipschitz. Thus it follows from Royden's theorem that holomorphic maps between Teichmüller spaces equipped with the Teichmüller metric are 1-Lipschitz.

Proof. To simplify notation, we write $\mathcal{M} = \mathcal{M}_{g,p}$ and let $\bar{\mathcal{M}}$ be its Deligne-Mumford compactification. Let also $\mathbb{D}^*$ and $\mathbb{D}$ be, respectively, the punctured and unpunctured open unit disks in $\mathbb{C}$ endowed with their complete hyperbolic metrics.

By a result of Wolpert [23], every point in $\bar{\mathcal{M}} \setminus \mathcal{M}$ has a neighborhood $\bar{U}$ in $\bar{\mathcal{M}}$ whose intersection $U = \bar{U} \cap \mathcal{M}$ with moduli space is bi-holomorphic to

$$U \simeq ((\mathbb{D}^*)^k \times \mathbb{D}^{d-k})/G$$

where $k \geq 1$, $d = \dim_{\mathbb{C}} \mathcal{M} = 3g - 3 + p$, and G is a finite group. Let $d_{\mathcal{H}}$ be the distance on U induced by the product of hyperbolic metrics.

Let $D_0 \subset \mathbb{D}$ and $D_0^* \subset \mathbb{D}^*$ be, respectively, the disk and punctured disk of Euclidean radius $\frac{1}{2}$, and denote by $D_n^* \subset D_0^*$ the punctured disk such that the hyperbolic distance between ∂D_0^* and ∂D_n^* is equal to n. We set

$$W_n^k = (D_n^* \times D_0^* \times \cdots \times D_0^*) \cup \overset{k}{\ldots} \cup (D_0^* \times \cdots \times D_0^* \times D_n^*) \subset (D_0^*)^k$$

and

$$U_n = (W_n^k \times D_0^{d-k})/G \subset U.$$

By construction ∂U_n is at $d_{\mathcal{H}}$-distance n from ∂U_0 and its co-dimension one volume decreases exponentially: $\mathrm{vol}_{\mathcal{H}}(\partial U_n) \simeq e^{-n}$.

Since $\bar{\mathcal{M}} \setminus \mathcal{M}$ is compact, we can pick finitely many sets $U^1, \ldots, U^r$ such that $\mathcal{M} \setminus \cup_i U_0^i$ is compact. For $n \in \mathbb{N}$ we set

$$K_n = \mathcal{M} \setminus \cup_i U_n^i.$$

By construction, K_n is compact, $\mathcal{M} = \cup_n K_n$, and $K_n \subset K_{n+1}$ for all n. In other words, $\{K_n\}_{n\in\mathbb{N}}$ satisfies (1).

To prove (2), note that Royden's theorem [21] implies that the inclusion

$$(U^i, d_{\mathcal{H}}) \hookrightarrow (\mathcal{M}, d_{\mathrm{T}})$$

is 1-Lipschitz for all i. In particular we have

$$\mathrm{vol}_{\mathrm{T}}(\partial K_n) \leq \sum_{i=1}^{r} \mathrm{vol}_{\mathrm{T}}(\partial U_n^i) \leq \sum_{i=1}^{r} \mathrm{vol}_{\mathcal{H}}(\partial U_n^i).$$

As mentioned, each summand on the right side decreases exponentially in n and thus there is a constant c with

$$\mathrm{vol}_{\mathrm{T}}(\partial K_n) \leq ce^{-n} \quad \text{for all } n,$$

as claimed in (2). It remains to prove that (3) is satisfied.

Since $K_n \subset K_0 \cup_i (U_0^i \setminus U_n^i)$ and ∂U_0^i is compact for each i, we can enlarge c so that

$$d_{\mathrm{T}}(X, K_0) \leq c + \max_i \max_{Y \in U_0^i \setminus U_n^i} d_{\mathrm{T}}(Y, \partial U_0^i)$$

for every $X \in K_n$. Applying once again Royden's theorem, we get

$$\max_{Y \in U_0^i \setminus U_n^i} d_{\mathrm{T}}(Y, \partial U_0^i) \leq \max_{Y \in U_0^i \setminus U_n^i} d_{\mathcal{H}}(Y, \partial U_0^i) = n,$$

from where we obtain

$$d_{\mathrm{T}}(X, K_0) \leq c + n \quad \text{for all } X \in K_n,$$

as we needed to prove. □

4

In this section we discuss some geometric facts about holomorphic maps $M \to N$ between finite (orbifold) covers of moduli spaces:

$$M \to \mathcal{M}_{g,p}, \quad N \to \mathcal{M}_{g',p'}.$$

We will endow the domain M with McMullen's Kähler hyperbolic metric and the target N with the Weil-Petersson metric.

The Weil-Petersson distance d_{WP} is induced by a negatively curved, although unfortunately incomplete, Riemannian metric. However, d_{WP} is geodesically convex, meaning that any two points in Teichmüller space $\mathcal{T}_{g,p}$ are connected by a unique d_{WP}-geodesic segment. Moreover, the identity map

$$\mathrm{Id}\colon (\mathcal{T}_{g,p}, d_{\mathrm{T}}) \to (\mathcal{T}_{g,p}, d_{\mathrm{WP}}) \tag{4.1}$$

is Lipschitz [19, Proposition 2.4]. McMullen's Kähler hyperbolic metric d_{KH} on $\mathcal{T}_{g,p}$ is again induced by a Riemannian metric, and the identity map

$$\mathrm{Id}\colon (\mathcal{T}_{g,p}, d_{\mathrm{KH}}) \to (\mathcal{T}_{g,p}, d_{\mathrm{T}}) \tag{4.2}$$

is bi-Lipschitz [19, Theorem 1.1]. Hence, $(\mathcal{M}_{g,p}, d_{\mathrm{KH}})$ is complete and has finite volume. See [15, 16] for background on the Weil-Petersson metric d_{WP}, and [19] for the definition and properties of d_{KH}.

As remarked earlier, Royden's theorem [21] implies that holomorphic maps between Teichmüller spaces endowed with the Teichmüller metric are 1-Lipschitz. In particular, (4.1) and (4.2) imply that there is $L > 0$ such that every holomorphic map $f\colon (M, d_{\mathrm{KH}}) \to (N, d_{\mathrm{WP}})$ is L-Lipschitz.

Suppose from now on that

$$f_1, f_2\colon (M, d_{\mathrm{KH}}) \to (N, d_{\mathrm{WP}})$$

are homotopic holomorphic maps, and let

$$\hat{F}\colon [1,2] \times M \to N$$

be a homotopy between them. Since the Weil-Petersson metric is negatively curved and geodesically convex, we can replace $\hat{F}$ with the straight homotopy

$$F\colon [1,2] \times M \to N, \quad F(t, X) = f_t(X) \tag{4.3}$$

determined by the fact that $t \mapsto f_t(X)$ is the d_{WP}-geodesic segment joining $f_1(X)$ and $f_2(X)$ in the homotopy class of $\hat{F}([1,2] \times \{X\})$.

Note that f_t is L-Lipschitz for all t because f_1 and f_2 are. Indeed, for $v \in T_X M$ the vector field $t \mapsto d(f_t)_X v$ is a Jacobi field along $t \mapsto f_t(X)$. Since the Weil-Peterson metric is negatively curved, the length of Jacobi fields is a convex function, and hence attains its maximum at $t \in \{1, 2\}$.

A priori, the map F itself need not be Lipschitz: the norm of $dF_{(t,X)} \frac{\partial}{\partial t}$ is equal to the length of the geodesic arc $t \mapsto F(t, X)$, and there is no reason for this to be bounded, as M is not compact. However, fixing $X_0 \in M$ there is a constant k, independent of X, such that the segment $t \mapsto F(t, X)$ has length at most $k + 2L d_{\mathrm{KH}}(X, X_0)$ because f_1, f_2 are L-Lipschitz. It follows that there are constants A, B with

$$\|dF_{(t,X)}\|^2 \leq A \cdot d_{\mathrm{KH}}(X_0, X)^2 + B$$

for all $(t, X) \in [1,2] \times M$. Here, $\|dF_{(t,X)}\|$ is the operator norm of $dF_{(t,X)}$.

The convexity of Jacobi fields also implies the convexity of the *energy density*

$$t \mapsto E_X(f_t) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i=1}^{\dim_{\mathbb{R}} M} \|d(f_t)_X v_i\|^2_{\text{WP}}$$

where $v_1, \ldots, v_{\dim_{\mathbb{R}} M}$ is an arbitrary orthonormal basis of $T_X M$ with respect to the Kähler hyperbolic metric. This function is strictly convex if one of $d(f_1)_X$ or $d(f_2)_X$ has rank at least 2. In particular, if the holomorphic maps f_1, f_2 are distinct and one of them is non-constant, then the *energy*

$$t \mapsto E(f_t) \stackrel{\text{def}}{=} \int_M E_X(f)\nu_{\text{KH}}$$

is strictly convex. Here ν_{KH} is the Riemannian volume form of (M, d_{KH}), and $E(f_t) < \infty$ because f_t is L-Lipschitz and M has finite d_{KH}-volume.

We summarize this discussion in the following lemma:

Lemma 4.1. *Let $M \to \mathcal{M}_{g,p}$ and $N \to \mathcal{M}_{g',p'}$ be finite covers and*

$$f_1, f_2 \colon (M, d_{\text{KH}}) \to (N, d_{\text{WP}})$$

homotopic holomorphic maps. Consider the straight homotopy

$$F \colon [1,2] \times M \to N, \quad F(t, X) = f_t(X)$$

between them. Then:

(1) *There is $L > 0$ such that $f_t \colon M \to N$ is L-Lipschitz and has finite energy $E(f_t) < \infty$ for all t.*
(2) *For $X_0 \in M$, there are $A, B > 0$ with*

$$\|dF_{(t,X)}\|^2 \leq A \cdot d_{\text{KH}}(X_0, X)^2 + B$$

for all $(t, X) \in [1,2] \times M$.
(3) *The energy function $t \mapsto E(f_t)$ is convex. Moreover, it is strictly convex unless either $f_1 = f_2$ or both are constant.* □

So far, we have only used Royden's theorem, the comparison between the different metrics, and the curvature properties of d_{WP}. We will also make key use of the fact that both the Weil-Petersson metric and McMullen's metric are Kähler. This means that the Kähler form, i.e. the 2-form $\omega = \langle \cdot, J \cdot \rangle$, is closed, where $\langle \cdot, \cdot \rangle$ is the relevant Riemannian metric and J is the endomorphism of the tangent bundle given by complex multiplication. See [5, 22] for facts on Kähler manifolds.

We need an observation due to Eells-Sampson [12]. Suppose that

$$f \colon \mathcal{T}_{g,p} \to \mathcal{T}_{g',p'}$$

is a smooth map, and write ω_{WP} and ω_{KH} for the Kähler forms of the Weil-Petersson metric and of McMullen's metric respectively. The Riemannian volume form induced by d_{KH} may be expressed as

$$\nu_{\mathrm{KH}} = \frac{1}{d!}\omega_{\mathrm{KH}}^d = \frac{1}{d!}\omega_{\mathrm{KH}} \wedge \cdots \wedge \omega_{\mathrm{KH}},$$

where $d = \dim_{\mathbb{C}} \mathcal{T}_{g,p}$. Pulling back the Kähler form ω_{WP} via f we can also consider the top-dimensional form $(f^*\omega_{\mathrm{WP}}) \wedge \omega_{\mathrm{KH}}^{d-1}$ on $\mathcal{T}_{g,p}$. An infinitesimal computation due to Eells and Sampson [12] - valid for maps between arbitrary Kähler manifolds - proves that:

Proposition 4.2 (Eells-Sampson)**.** *Let $f\colon (\mathcal{T}_{g,p}, d_{\mathrm{KH}}) \to (\mathcal{T}_{g',p'}, d_{\mathrm{WP}})$ be a smooth map. For every $X \in \mathcal{T}_{g,p}$ we have*

$$E_X(f)\nu_{\mathrm{KH}} \geq \frac{1}{d!}(f^*\omega_{\mathrm{WP}}) \wedge \omega_{\mathrm{KH}}^{d-1} \tag{4.4}$$

where $d = \dim_{\mathbb{C}} \mathcal{T}_{g,p}$. Moreover, equality holds in (4.4) *if and only if f is holomorphic at X.*

Proposition 4.2 is basically an incarnation of the classical Wirtinger inequality. In other words, it is a purely linear algebra fact which follows from the observation that, whenever $\Lambda\colon \mathbb{C} \to \mathbb{C}^n$ is $\mathbb{R}$-linear, then

$$E_0(\Lambda)\omega_{\mathbb{C}} \geq \det(\Lambda)\omega_{\mathbb{C}} \geq \Lambda^*(\omega_{\mathbb{C}^n})$$

where $\omega_{\mathbb{C}} = dx \wedge dy$ and $\omega_{\mathbb{C}^n} = \sum_{i=1}^n dx_i \wedge dy_i$ are the standard Kähler forms.

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We are now ready to prove Proposition 1.3:

Proposition 1.3. *Let $M \to \mathcal{M}_{g,p}$ and $N \to \mathcal{M}_{g',p'}$ be finite covers and suppose that $f_1, f_2\colon M \to N$ are homotopic holomorphic maps. If f_1 is not constant, then $f_1 = f_2$.*

We recall from the introduction that the proof of Proposition 1.3 uses an argument due to Eells and Sampson [12] based on Proposition 4.2 and Stokes' theorem. Here we have to integrate over moduli space, but as we mentioned above, $\mathcal{M}_{g,p}$ is non-compact. We apply Stoke's theorem to the compact subsets K_n provided by Proposition 3.1 and show that the arising boundary terms tend to 0 when $n \to \infty$.

Proof. Suppose that f_1 is not constant and $f_1 \neq f_2$, and let

$$F\colon [1,2] \times M \to N, \quad F(t,X) = f_t(X)$$

be the straight homotopy (4.3) between them. From Lemma 4.1 we know that the function $t \mapsto E(f_t)$ is strictly convex; we may hence assume that

$$E(f_t) < E(f_1) \tag{5.1}$$

for all $t \in (1,2)$. We are going to contradict this assertion.

Let $\{K_n\}$ be the compact exhaustion of $\mathcal{M}_{g,p}$ provided by Proposition 3.1. Abusing terminology, we denote also by K_n the preimage of K_n under the covering map $M \to \mathcal{M}_{g,p}$. By (4.2), the Teichmüller metric is bi-Lipschitz to McMullen's Kähler hyperbolic metric and thus we get from Proposition 3.1 that there are constants c and L such that:

(1) $M = \bigcup_{n\in\mathbb{N}} K_n$, and $K_n \subset K_{n+1}$ for all n,
(2) $\mathrm{vol}_{\mathrm{KH}}(\partial K_n) \leq ce^{-n}$ for all n, and
(3) K_n is contained within d_{KH}-distance $c + L\cdot n$ of K_0 for all n.

Write ω_{KH} and ω_{WP} for the Kähler forms of McMullen's metric and the Weil-Petersson metric respectively, and set $d = \dim_{\mathbb{C}} M = 3g-3+p$. Since Kähler forms are closed, we deduce from Stokes theorem that

$$\begin{aligned}
0 &= \int_{[1,t]\times K_n} d\left((F^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1}\right)\\
&= \int_{\partial([1,t]\times K_n)} (F^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1}\\
&= \int_{\{t\}\times K_n} (F^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1} - \int_{\{1\}\times K_n} (F^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1}\\
&\quad + \int_{[1,t]\times\partial K_n} (F^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1}\\
&= \int_{K_n} (f_t^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1} - \int_{K_n} (f_1^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1}\\
&\quad + \int_{[1,t]\times\partial K_n} (F^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1}
\end{aligned}$$

Below we will prove:

$$\lim_{n\to\infty}\int_{[1,t]\times\partial K_n} (F^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1} = 0. \tag{5.2}$$

Assuming (5.2), we obtain from the computation above that

$$\lim_{n\to\infty}\left(\int_{K_n} (f_t^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1} - \int_{K_n} (f_1^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1}\right) = 0.$$

Taking into account that f_t and f_1 are Lipschitz and that (M, d_{KH}) has finite volume, we deduce that

$$\int_M (f_t^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1} = \int_M (f_1^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1}.$$

From Proposition 4.2 we get

$$\begin{aligned} E(f_t) &\geq \frac{1}{d!}\int_M (f_t^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1} \\ &= \frac{1}{d!}\int_M (f_1^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1} = E(f_1) \end{aligned}$$

where the last equality holds because f_1 is holomorphic. This contradicts (5.1). It remains to prove (5.2).

Fix $(t,X)\in[0,1]\times\partial K_n$ and let $v_1,\dots,v_{2d}$ be an orthonormal basis of $T_{(t,X)}([0,1]\times\partial K_n)$. We have

$$\begin{aligned} &\left|(F^*\omega_{\mathrm{WP}})(v_1,v_2)\cdot\omega_{\mathrm{KH}}(v_3,v_4)\cdot\ldots\cdot\omega_{\mathrm{KH}}(v_{2d-1},v_{2d})\right| \\ &\qquad= \left|\langle dF_{(t,X)}v_1, JdF_{(t,X)}v_2\rangle_{\mathrm{WP}}\cdot\langle v_3,Jv_4\rangle_{\mathrm{KH}}\cdot\ldots\cdot\langle v_{2d-1},Jv_{2d}\rangle_{\mathrm{KH}}\right| \\ &\qquad\leq \|dF_{(t,X)}\|^2 \end{aligned}$$

where $\|dF_{(t,X)}\|$ is the operator norm of $dF_{(t,X)}$. Fixing $X_0\in M$ we get from Lemma 4.1 that there are $A,B>0$ with

$$\|dF_{(t,X)}\|^2\leq A\cdot d_{\mathrm{KH}}(X_0,X)^2+B$$

for all $(t,X)\in[1,2]\times M$. We deduce that

$$\begin{aligned} \left|\int_{[0,t]\times\partial K_n}(F^*\omega_{\mathrm{WP}})\wedge\omega_{\mathrm{KH}}^{d-1}\right| &\leq (2d)!\int_{[0,t]\times\partial K_n}\|dF_{(t,X)}\|^2\nu_{[0,t]\times\partial K_n} \\ &\leq (2d!)\left(A\cdot\max_{X\in\partial K_n} d_{\mathrm{KH}}(X,X_0)^2+B\right)\mathrm{vol}_{\mathrm{KH}}(\partial K_n). \end{aligned}$$

This last quantity tends to 0 as $n\to\infty$ by (2) and (3) above. Having proved (5.2), we are done with the proof of Proposition 1.3. □

6. Proof of Theorem 1.1

In this section we deduce Theorem 1.1 from Proposition 1.3 and a rigidity theorem for homomorphisms between mapping class groups proved in [3]. We remind the reader of some terminology from the said paper.

Let S and S' be compact surfaces, possibly with boundary, and P and P' finite sets of marked points in the interior of S and S' respectively. By an *embedding* $\iota\colon(S,P)\to(S',P')$ we understand a continuous injective map $\iota_{top}\colon S\to S'$ with the property that $\iota_{top}^{-1}(P')\subset P$. Every embedding $\iota\colon(S,P)\to(S',P')$ induces a (continuous) homomorphism $\mathrm{Homeo}(S,P)\to\mathrm{Homeo}(S,P)$ between the corresponding groups of self-homeomorphisms fixing pointwise the boundary and the set of marked points. In particular, ι induces a homomorphism

$$\iota_\#\colon\ \mathrm{Map}(S,P)\to\mathrm{Map}(S',P')$$

between the associated mapping class groups. The main result proved in [3] asserts that, subject to suitable genus bounds, every non-trivial homomorphism between mapping class groups is in fact induced by an embedding.

Theorem 6.1 (Aramayona-Souto)**.** *Suppose that S and S' are compact surfaces, possibly with boundary, and that P and P' are finite sets of marked points in the interior of S and S' respectively. If S has genus $g \geq 6$ and S' has genus $g' \leq 2g-2$, then every nontrivial homomorphism*

$$\mathrm{Map}(S,P) \to \mathrm{Map}(S',P')$$

is induced by an embedding $(S,P) \to (S',P')$.

Recall that $\mathcal{T}_{g,p}$ is a classifying space for proper actions $\underline{E}(\mathrm{Map}_{g,p})$. In particular, as we discussed at the end of section 2, the homotopy type of a map $\mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ is determined by the associated homomorphism between the corresponding mapping class groups. Armed with Theorem 6.1 we prove:

Proposition 6.2. *If $g \geq 6$ and $g' \leq 2g-2$, then every map $f\colon \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ is either homotopically trivial or homotopic to a forgetful map.*

Proof. Let $f_*\colon \mathrm{Map}_{g,p} \to \mathrm{Map}_{g',p'}$ be the homomorphism associated to f and let $\tilde{f}\colon \mathcal{T}_{g,p} \to \mathcal{T}_{g',p'}$ be an f_*-equivariant lift of f. If f_* is trivial, then f is homotopically trivial and we have nothing to prove. Suppose from now on that this is not the case.

Let (S,P) and (S',P') be, respectively, closed surfaces of genus g and g', with p and p' marked points. Identifying $\mathrm{Map}(S,P) = \mathrm{Map}_{g,p}$ and $\mathrm{Map}(S',P') = \mathrm{Map}_{g',p'}$, we obtain from Theorem 6.1 that the homomorphism f_* is induced by an embedding

$$\iota\colon (S,P) \to (S',P').$$

Since S and S' are closed, the underlying injective map $\iota_{top}\colon S \to S'$ is a homeomorphism and $\iota_{top}(P) \supset P'$. In other words, the embedding ι is obtained by forgetting marked points.

In the same way that we have identified mapping class groups, we also identify Teichmüller spaces $\mathcal{T}_{g,p} = \mathcal{T}(S,P)$ and $\mathcal{T}_{g',p'} = \mathcal{T}(S',P')$. The embedding ι induces an f_*-equivariant map

$$\tilde{h}\colon \mathcal{T}_{g,p} \to \mathcal{T}_{g',p'}$$

obtained again by forgetting marked points. By construction, $\tilde{h}$ descends to a forgetful map

$$h\colon \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$$

Since both $\tilde{f}$ and $\tilde{h}$ are f_*-equivariant, (2.1) yields a homotopy between f and h. □

We are now ready to prove Theorem 1.1:

Theorem 1.1. *Suppose that $g \geq 6$ and $g' \leq 2g-2$. Every non-constant holomorphic map $\mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ is a forgetful map.*

Proof. Suppose that $f\colon \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ is holomorphic and not constant. Proposition 1.3 implies that f is not homotopically trivial and, in particular, f is homotopic to a forgetful map $h\colon \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ by Proposition 6.2. Since both f and h are holomorphic and non-constant we get that $f = h$ from Proposition 1.3, as we needed to prove. □

There is a number of rigidity results for homomorphisms between mapping class groups; see for example [4] for a survey of results in this direction. Combining any such theorem with Proposition 1.3 one obtains a rigidity result for holomorphic maps between the corresponding moduli spaces. For instance, the version of Theorem 6.1 proved in [3] covers a few more cases than the ones stated here. From this more general version, it follows that Theorem 1.1 also holds for maps $\mathcal{M}_{g,p} \to \mathcal{M}_{2g-1,p'}$ with $p' \geq 1$, and for maps $\mathcal{M}_{g,p} \to \mathcal{M}_{g,p'}$ as long as $g \geq 4$. In particular, we have:

Corollary 6.3. *Suppose that $g \geq 4$. Every non-constant holomorphic map $\mathcal{M}_{g,p} \to \mathcal{M}_{g,p}$ is induced by a permutation of marked points, and is hence a biholomorphism.* □

Note that the isomorphism, for $g \geq 2$, between the group of biholomorphisms of $\mathcal{M}_{g,p}$ and the symmetric group Σ_p follows also from Royden's characterization of the biholomorphism group of Teichmüller space [21, 11].

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