EQUIDISTRIBUTION ESTIMATES FOR FEKETE POINTS ON COMPLEX MANIFOLDS

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Abstract. We study the equidistribution of Fekete points in a compact complex manifold. These are extremal point configurations defined through sections of powers of a positive line bundle. Their equidistribution is a known result. The novelty of our approach is that we relate them to the problem of sampling and interpolation on line bundles, which allows us to estimate the equidistribution of the Fekete points quantitatively. In particular we estimate the Kantorovich-Wasserstein distance of the Fekete points to its limiting measure. The sampling and interpolation arrays on line bundles are a subject of independent interest, and we provide necessary density conditions through the classical approach of Landau, that in this context measures the local dimension of the space of sections of the line bundle. We obtain a complete geometric characterization of sampling and interpolation arrays in the case of compact manifolds of dimension one, and we prove that there are no arrays of both sampling and interpolation in the more general setting of semipositive line bundles.

1. Introduction

1.1. Let L be a holomorphic line bundle on a compact complex manifold X of dimension n. The space of global holomorphic sections to L is denoted by $H^0(L)$. If $s_1, \dots, s_N$ is a basis for $H^0(L)$ and $x_1, \dots, x_N$ are N points in X, then the Vandermonde-type determinant

$$\det(s_i(x_j)), \quad 1 \le i, j \le N,$$

is a section to the pulled-back line bundle $L^{\boxtimes N}$ over the manifold X^N. If L is endowed with a smooth hermitian metric ϕ, then it also induces a natural metric on $L^{\boxtimes N}$.

A configuration of N points $x_1, \dots, x_N$ in X is called a *Fekete configuration* for (L, ϕ) if it maximizes the pointwise norm $|\det(s_i(x_j))|_\phi$. It is easy to check that the definition of a Fekete configuration does not depend on the particular choice of the basis $s_1, \dots, s_N$ for $H^0(L)$. The compactness of X ensures the existence of Fekete configurations (but in general there need not be a unique one).

It is interesting to study the distribution of Fekete points with respect to high powers L^k of the line bundle L, where L^k is endowed with the product metric $k\phi$. The model example is the complex projective space $X = \mathbb{CP}^n$ with the hyperplane bundle $L = \mathcal{O}(1)$, endowed with the Fubini-Study metric. The k'th power of L is denoted $\mathcal{O}(k)$, and the holomorphic sections to $\mathcal{O}(k)$ can be identified with the homogeneous polynomials of degree k in $n+1$

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variables. This is in fact the prime example, and it covers in particular the classical theory of weighted orthogonal multivariate polynomials.

For each $k = 1, 2, 3, \dots$ let $\mathcal{F}_k$ be a Fekete configuration for $(L^k, k\phi)$. The goal is to provide information on the distribution of Fekete points $\mathcal{F}_k$ in geometrical terms of the line bundle (L, ϕ), showing that they are "equidistributed" on X. We will consider the case when L is an ample line bundle with a smooth positive metric ϕ. The problem has already been solved by Berman, Boucksom and Witt [BBWN11] in an even more general context, when L is a big line bundle with an arbitrary continuous metric on a compact subset $K \subset X$. The redeeming feature of our approach is that our new proof provides a quantitative version of the equidistribution.

Theorem 1. *If the line bundle (L, ϕ) is positive then*

$$\frac{\#(\mathcal{F}_k \cap B(x,r))}{\#\mathcal{F}_k} = \Big(1 + O\Big(\big(r\sqrt{k}\,\big)^{-\frac{1}{2n+1}}\Big)\Big) \frac{\int_{B(x,r)} (i\partial\bar\partial\phi)^n}{\int_X (i\partial\bar\partial\phi)^n}$$

for $0 < r \le \operatorname{diam}(X)$, uniformly in $x \in X$.

Here $\partial\bar\partial\phi$ is the *curvature form* of the metric ϕ, which is a globally defined $(1,1)$-form on X, and $(i\partial\bar\partial\phi)^n$ is the corresponding volume form on X. By $B(x, r)$ we mean the ball of radius r centered at the point x in X. To define the balls we endow the manifold X with an arbitrary hermitian metric, and use the associated distance function.

The result shows, in particular, that the weak limit as $k \to \infty$ of the probability measures

$$\mu_k = \frac{1}{\#\mathcal{F}_k} \sum_{\lambda \in \mathcal{F}_k} \delta_\lambda \tag{1}$$

is the measure $(i\partial\bar\partial\phi)^n$ normalized to have total mass 1. This is a special case of the main theorem proved in [BBWN11] in the setting of positive line bundles.

Theorem 1 allows to get an even more precise result quantifying the convergence. We can estimate the distance of the Fekete measure μ_k to its limit ν in the Kantorovich-Wasserstein metric W, which metrizes the weak convergence of measures (see Section 7).

Corollary 1. *If the line bundle (L, ϕ) is positive then*

$$k^{-1/2} \lesssim W(\mu_k, \nu) \lesssim k^{-1/(4n+4)}$$

as $k \to \infty$.

1.2. The scheme that we propose to study this problem is similar to the one initiated in [MOC10] where the Fekete points are related to another array of points, the sampling and interpolation points. This has been pursued further in the one-dimensional setting [AOC12] or even in the real setting of compact Riemannian manifolds, see [OCP12].

For each $k = 1, 2, 3, \dots$ let Λ_k be a finite set of points in X. We assume that $\{\Lambda_k\}$ is a separated array, which means that the distance between any two distinct points in Λ_k is bounded below by a positive constant times $k^{-1/2}$. We say that $\{\Lambda_k\}$ is a *sampling array* for (L,ϕ) if there are constants $0<A,B<\infty$ such that, for each large enough k and any section $s\in H^0(L^k)$ we have

$$Ak^{-n}\sum_{\lambda\in\Lambda_k}|s(\lambda)|^2\le\int_X\left|s(x)\right|^2\le Bk^{-n}\sum_{\lambda\in\Lambda_k}|s(\lambda)|^2.$$

We say that $\{\Lambda_k\}$ is an *interpolation array* for (L,ϕ) if there is a constant $0<C<\infty$ such that, for each large enough k and any set of values $\{v_\lambda\}_{\lambda\in\Lambda_k}$, where each v_λ is an element of the fiber of λ in L^k, there is a section $s\in H^0(L^k)$ such that $s(\lambda)=v_\lambda$ $(\lambda\in\Lambda_k)$ and

$$\int_X\left|s(x)\right|^2\le Ck^{-n}\sum_{\lambda\in\Lambda_k}|v_\lambda|^2.$$

In order to integrate over X in these definitions we endow X with an arbitrary volume form. It is easy to see that the definitions of the sampling and interpolation arrays do not depend on the particular choice of the volume form on X.

We can use a classical technique due to Landau [Lan67] to get necessary geometric conditions for an array of points to be sampling or interpolation. The use of Landau's concentration operator, which measures the local dimension of the sections of the line bundle, to obtain necessary density conditions for sampling arrays was suggested earlier by Berndtsson [Ber03] and Lindholm [Lin01] in the context of holomorphic line bundles. We have opted to present this proof in detail because we need a more precise estimate to get the quantitative equidistribution of the Fekete points.

Let $\nu_\Lambda^-(R)$ (respectively $\nu_\Lambda^+(R)$) denote the infimum (respectively supremum) of the ratio

$$\frac{k^{-n}\#(\Lambda_k\cap B(x,r))}{\int_{B(x,r)}(i\partial\bar\partial\phi)^n}\tag{2}$$

over all $x\in X$, and all k,r such that $\frac{R}{\sqrt{k}}\le r\le\operatorname{diam}(X)$. As before, to define the balls $B(x,r)$ we have fixed an arbitrary hermitian metric on the manifold X.

Theorem 2. *Let the line bundle (L,ϕ) be positive, and $\Lambda=\{\Lambda_k\}$ be a separated array.*

(i) *If Λ is a sampling array then*

$$\nu_\Lambda^-(R)>\frac{1}{\pi^n n!}-O(R^{-1}),\quad R\to\infty.$$

(ii) *If Λ is an interpolation array then*

$$\nu_\Lambda^+(R)<\frac{1}{\pi^n n!}+O(R^{-1}),\quad R\to\infty.$$

Our result on the distribution of Fekete points (Theorem 1 above) is obtained by a combination of Theorem 2 with the observation that Fekete points are "almost" sampling and interpolation (see Section 4).

1.3. We believe that the sampling and interpolation arrays on holomorphic line bundles are a subject of interest independently of its application to Fekete points, so we proceed to a more detailed study of them. Theorem 2 yields necessary conditions in terms of the lower and upper Beurling-Landau densities, defined by

$$D^-(\Lambda) = \liminf_{R\to\infty} \nu_\Lambda^-(R), \quad \text{and} \quad D^+(\Lambda) = \limsup_{R\to\infty} \nu_\Lambda^+(R).$$

Corollary 2. *Let the line bundle (L,ϕ) be positive and $\Lambda = \{\Lambda_k\}$ be a separated array. If Λ is a sampling array then*

$$D^-(\Lambda) \geq \frac{1}{\pi^n n!}\,,$$

while if Λ is an interpolation array then

$$D^+(\Lambda) \leq \frac{1}{\pi^n n!}\,.$$

When the complex manifold X is one-dimensional, i.e. we are dealing with a compact Riemann surface, we have a more precise result. In this case there is a complete geometric characterization of the sampling and interpolation arrays in terms of the above densities.

Theorem 3. *Let (L,ϕ) be a positive line bundle over a compact Riemann surface X, and let $\Lambda = \{\Lambda_k\}$ be a separated array. Then Λ is a sampling array if and only if*

$$D^-(\Lambda) > \frac{1}{\pi}\,,$$

while it is an interpolation array if and only if

$$D^+(\Lambda) < \frac{1}{\pi}\,.$$

We remark that the assumption that $\{\Lambda_k\}$ is separated is not essential, and similar results hold in the general case. This can be done with standard techniques, see e.g. [Mar07], so we will not go into these details in the paper.

1.4. As pointed out in [MOC10], not only the sampling and interpolation arrays can be used to obtain information on Fekete points, but also the converse direction is useful. Fekete points provide a construction of an "almost" sampling and interpolation array, with the critical density. In particular this shows that the density threshold in Corollary 2 is sharp (see Corollary 4 in Section 7).

In this context a natural question is whether the Fekete points, or possibly some other array of points, is simultaneously sampling and interpolation for (L,ϕ). In the case when the manifold X is one-dimensional, this question is settled in the negative by Theorem 3 above. For $n > 1$ we do not have strict density conditions, and Corollary 2 does not exclude the existence of simultaneously sampling and interpolation arrays. Nevertheless, we will show that such arrays do not exist, even in the more general setting when the metric ϕ is semi-positive and has at least one point with a strictly positive curvature.

Theorem 4. *Let L be a holomorphic line bundle over a compact projective manifold X, and ϕ be a semi-positive smooth hermitian metric on L. If there is a point in X where ϕ has a strictly positive curvature, then there are no arrays which are simultaneously sampling and interpolation for (L, ϕ).*

Here we need to assume that the manifold X is projective. When the line bundle is positive this is automatically the case, according to the Kodaira embedding theorem [Kod54].

The non-existence of simultaneously sampling and interpolation sequences is a recent result in the classical Bargmann-Fock space [AFK11, GM11]. To prove Theorem 4 we use the fact that near a point of positive curvature, the sections of high powers of the line bundle resemble closely the functions in the Bargmann-Fock space. Also our proof of Theorem 3 is guided with the same principle.

1.5. The plan of the paper is the following. In Sections 2 and 3 we provide the basic properties of the Fekete points, and of the Hilbert space of holomorphic sections that will be the main tool to study them. In Section 4 we introduce the sampling and interpolation arrays and discuss their relationship with the Fekete points. In Section 5 we study Landau's concentration operator, that will allow us to measure the local dimension of the space of sections essentially concentrated in a given ball, and use this local dimension to estimate the number of points in an interpolation or sampling array. In Section 6 we estimate the density of the interpolation and sampling arrays in terms of the volume form associated to the curvature of the line bundle. In Section 7 this is used to estimate from above and below the number of Fekete points that lie in a given ball. This provides an upper bound for the Kantorovich-Wasserstein distance between the Fekete measure (1) and its limiting measure.

Next we proceed to a more detailed study of the sampling and interpolation arrays. In Section 8 we prove that in a big line bundle with a semipositive metric, whenever there is a point of positive curvature there are no arrays that are simultaneously sampling and interpolation. Finally in Section 9 we obtain a geometric characterization of sampling and interpolation arrays for positive line bundles over compact manifolds of dimension one.

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2. Preliminaries

In this section we recall some basic properties of holomorphic line bundles over complex manifolds. For these and other elementary facts on this subject, stated below without proofs, the reader may consult [Ber10].

2.1. **Line bundles.** Below X will be a compact complex manifold of dimension n, endowed with a smooth hermitian metric ω. The metric ω induces a distance function $d(x, y)$ on X, which will be used to define the balls $B(x, r) = \{y \in X : d(x, y) < r\}$. The hermitian metric ω also induced a volume form V on X, which will be used to integrate over X. We emphasize that the choice of the metric ω is arbitrary, and the results will not depend on the particular choice made.

By L we denote a holomorphic line bundle over the manifold X. We assume that L is endowed with a smooth hermitian metric ϕ, which is a smoothly varying norm on each fiber. It has to be understood as a collection of functions ϕ_i defined on trivializing open sets U_i which cover X, and satisfying the compatibility conditions

$$\phi_i - \phi_j = \log |g_{ij}|^2,$$

where g_{ij} are the transition functions of the line bundle L on $U_i \cap U_j$. If s is a section to L represented by a collection of local functions s_i such that $s_i = g_{ij} s_j$, then

$$|s(x)|^2 = |s_i(x)|^2 e^{-\phi_i(x)}.$$

We also have an associated scalar product, defined in a similar way by

$$\langle u(x), v(x)\rangle = u_i(x)\overline{v_i(x)} e^{-\phi_i(x)}.$$

If ϕ is the hermitian metric on L, then $\partial\bar\partial\phi$ is a globally defined $(1,1)$-form on X, which is called the *curvature form* of the metric ϕ. The line bundle L with the metric ϕ is called *positive* if $i\partial\bar\partial\phi$ is a positive form. Equivalently, L with the metric ϕ is positive if the representative of ϕ with respect to any local trivialization is a strictly plurisubharmonic function. We remark that in the case when ϕ is positive, the curvature form $\partial\bar\partial\phi$ may be used to define a natural metric on X, which in turn induces a distance function and a volume form on X. However, we find it convenient to work with an arbitrary metric ω, which is not necessarily related to the curvature form.

We will use the notation $\lesssim$ to indicate an implicit multiplicative constant which may depend only on the hermitian manifold (X, ω) and the hermitian line bundle (L, ϕ).

The space of global holomorphic sections to L will be denoted $H^0(L)$. This is a finite-dimensional space, satisfying the estimate

$$\dim H^0(L^k) \lesssim k^n.$$

While the latter estimate holds for an arbitrary line bundle on a compact manifold, in the case when the line bundle L is big there is also a similar estimate from below, i.e.

$$k^n \lesssim \dim H^0(L^k) \lesssim k^n. \tag{3}$$

In particular this holds whenever the line bundle L is positive.

If L is a line bundle over X and M is a line bundle over Y, we denote by $L \boxtimes M$ the line bundle over the product manifold $X \times Y$ defined as $L \boxtimes M = \pi_X^*(L) \otimes \pi_Y^*(M)$, where $\pi_X \colon X \times Y \to X$ is the projection onto the first factor and $\pi_Y \colon X \times Y \to Y$ is the projection onto the second.

2.2. **Bergman kernel.** The space $H^0(L)$ admits a Hilbert space structure when endowed with the scalar product

$$\langle u, v\rangle = \int_X \langle u(x), v(x)\rangle, \quad u, v \in H^0(L),$$

where the integration is taken with respect to the volume form V.

The Bergman kernel $\Pi(x, y)$ associated to this space is a section to the line bundle $L \boxtimes \bar{L}$ over the manifold $X \times X$, defined by

$$\Pi(x, y) = \sum_{j=1}^{N} s_j(x) \otimes \overline{s_j(y)}, \tag{4}$$

where $s_1, \ldots, s_N$ is an orthonormal basis for $H^0(L)$. It is easy to check that this definition does not depend on the particular choice of the orthonormal basis $s_1, \ldots, s_N$. The Bergman kernel $\Pi(x, y)$ is in a sense the reproducing kernel for the space $H^0(L)$, satisfying the reproducing formula

$$s(x) = \int_X \big\langle s(y), \Pi(x, y) \big\rangle \, dV(y)$$

for $s \in H^0(L)$. The pointwise norm of the Bergman kernel is symmetric,

$$|\Pi(x, y)| = |\Pi(y, x)|. \tag{5}$$

The function $|\Pi(x, x)|$ is called the Bergman function of $H^0(L)$. It can be expressed as

$$|\Pi(x, x)| = \sum_{j=1}^{N} \big|s_j(x)\big|^2, \tag{6}$$

and it satisfies

$$|\Pi(x, x)| = \int_X |\Pi(x, y)|^2 \, dV(y). \tag{7}$$

Lemma 1. *Let $y \in X$. There is a section $\Phi_y \in H^0(L)$ such that*

$$\big|\Phi_y(x)\big| = \big|\Pi(x, y)\big|, \quad x \in X.$$

Proof. Let $s_1, \ldots, s_N$ be an orthonormal basis for $H^0(L)$. Fix a frame $e(x)$ in a neighborhood U of the point y, then in this neighborhood each s_j is represented by a holomorphic function f_j such that $s_j(x) = f_j(x) e(x)$. Define

$$\Phi_y(x) := |e(y)| \sum_{j=1}^{n} \overline{f_j(y)} s_j(x),$$

then Φ_y is a holomorphic section to L, and we have

$$\big|\Phi_y(x)\big| = \left| \Big(\sum_{j=1}^{N} \overline{f_j(y)} s_j(x) \Big) \otimes \overline{e(y)} \right| = \left| \sum_{j=1}^{N} s_j(x) \otimes \overline{s_j(y)} \right| = \big|\Pi(x, y)\big|. \qquad \square$$

We denote by $\Pi_k(x, y)$ the Bergman kernel for the k'th power L^k of the line bundle L (where L^k is endowed with the product metric $k\phi$). The behavior of $\Pi_k(x, y)$ as $k \to \infty$ is of special importance. In the case when the line bundle (L, ϕ) is positive, it is known (see e.g. [Ber03, Lin01]) that

$$k^n \lesssim |\Pi_k(x, x)| \lesssim k^n, \tag{8}$$

and

$$|\Pi_k(x,y)| \lesssim k^n \exp\big(-c\sqrt{k}\,d(x,y)\big), \tag{9}$$

where $c = c(X,\omega,L,\phi)$ is an appropriate positive constant.

2.3. **Sub-mean value property.** Let $s \in H^0(L^k)$. If $z \in X$ and $0 < \delta < 1$, then

$$\big|s(z)\big|^2 \lesssim \delta^{-2n} k^n \int_{B(z,\delta/\sqrt{k})} \big|s(x)\big|^2. \tag{10}$$

This can be deduced easily from the compactness of X and the corresponding fact in $\mathbb{C}^n$, which may be found for example in [Lin01, Lemma 7].

As a consequence we have the following Plancherel-Pólya type inequality.

Lemma 2. *Let $\{x_j\}$ be a set of points in X such that $d(x_i,x_j) \geq \delta/\sqrt{k}$, where $0 < \delta < 1$. Then*

$$k^{-n} \sum_j \big|s(x_j)\big|^2 \lesssim \delta^{-2n} \int_X \big|s(x)\big|^2$$

for any $s \in H^0(L^k)$.

3. Fekete points and their properties

3.1. Let $N = \dim H^0(L)$, and $s_1,\dots,s_N$ be a basis for $H^0(L)$. A configuration of N points $x_1,\dots,x_N$ in X is called a *Fekete configuration* if it maximizes the pointwise norm of the Vandermonde-type determinant

$$\det(s_i(x_j)), \quad 1 \leq i,j \leq N,$$

which is a holomorphic section to the line bundle $L^{\boxtimes N}$ over the manifold X^N (endowed with the metric inherited from L).

If $e_j(x)$ is a frame in a neighborhood U_j of the point x_j, then the sections $s_i(x)$ are represented on each U_j by scalar functions f_{ij} such that $s_i(x) = f_{ij}(x)e_j(x)$. Similarly, the metric ϕ is represented on U_j by a smooth real-valued function ϕ_j such that $|s_i(x)|^2 = |f_{ij}(x)|^2 e^{-\phi_j(x)}$. A Fekete configuration thus maximizes the quantity

$$e^{-\phi_1(x_1)} \cdots e^{-\phi_N(x_N)} \big|\det\big(f_{ij}(x_j)\big)\big|^2. \tag{11}$$

By the compactness of X, Fekete configurations exist, but in general there need not be a unique one. One may check that the norm $|\det(s_i(x_j))|_\phi$ at a Fekete configuration $x_1,\dots,x_N$ is always non-zero. It is also easy to check that the definition of a Fekete configuration does not depend on the particular choice of the basis $s_1,\dots,s_N$ of $H^0(L)$.

The function (11) is a Vandermonde-type determinant that vanishes when two points are equal. It is exactly the familiar Vandermonde determinant in the special case when the sections s_i are the monomials in dimension one, and the weight ϕ is constant. This suggests what it is actually happening – the Fekete points repel each other and tend to be in a sense "maximally spread".

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3.2. The main property of the Fekete points $x_1, \dots, x_N$ that will be used is the existence of "Lagrange sections" with a uniformly bounded norm. Namely, we have sections $l_1, \dots, l_N$ in $H^0(L)$ such that

$$|l_j(x_i)| = \delta_{ij}, \quad 1 \le i, j \le N, \tag{12}$$

and moreover, they satisfy the additional condition

$$\sup_{x \in X} \big| l_j(x) \big| = 1, \quad 1 \le j \le N. \tag{13}$$

To construct these sections we denote by M the matrix $\big(e^{-\frac{1}{2}\phi_j(x_j)} f_{ij}(x_j)\big)$, and define

$$l_j(x) := \frac{1}{\det(M)} \sum_{i=1}^{N} (-1)^{i+j} M_{ij} s_i(x),$$

where M_{ij} is the determinant of the submatrix obtained from M by removing the i-th row and j-th column. Clearly $l_j \in H^0(L)$, and it is not difficult to check that conditions (12) and (13) above hold, where (13) is a consequence of the extremal property of the Fekete configuration $x_1, \dots, x_N$.

We also observe that the system $\{l_j(x)\}$ forms a *basis* of $H^0(L)$. Indeed, the condition (12) implies that the $l_j(x)$ are linearly independent, and since they form a system with N elements, $N = \dim H^0(L)$, they span the whole $H^0(L)$. An element $s \in H^0(L)$ thus has a unique expansion

$$s(x) = \sum_{j=1}^{N} c_j l_j(x),$$

and the coefficients c_j are given by

$$c_j = \big\langle s(x_j), l_j(x_j) \big\rangle, \quad 1 \le j \le N,$$

which again follows from (12).

3.3. One consequence of the construction above is that Fekete points form a separated array.

Lemma 3. *Let $\mathcal{F}_k$ be a Fekete configuration for $(L^k, k\phi)$. Then*

$$d(x, y) \gtrsim \frac{1}{\sqrt{k}}, \quad x, y \in \mathcal{F}_k, \quad x \ne y. \tag{14}$$

Proof. Indeed, if this is not the case, there are points $x_k, y_k \in \mathcal{F}_k$, $\sqrt{k}\, d(x_k, y_k) \to 0$ but $x_k \ne y_k$, for infinitely many k's. By compactness we may assume that x_k, y_k converge to some point $x \in X$. We choose local coordinates z in a neighborhood of x, and a local trivialization of the line bundle L in this neighborhood. The metric on L is represented by a smooth function $\phi(z)$, and the metric on L^k is given by $k\phi(z)$.

For each k, we have a "Lagrange section" vanishing on x_k and having norm one on y_k. Let it be given by a holomorphic function $f_k(z)$ with respect to the local trivialization. Thus

$$\left|f_k(z)\right|^2 e^{-k\phi(z)} = \begin{cases} 0, & z = z(x_k) \\ 1, & z = z(y_k) \end{cases}$$

and $|f_k(z)|^2 e^{-k\phi(z)} \leq 1$ for all other z.

On the other hand, the distance function d is equivalent to the euclidean distance with respect to the local coordinates. Hence,

$$\sqrt{k}\,\left|z(x_k) - z(y_k)\right| \longrightarrow 0 \quad (k \to \infty).$$

This implies that the norm of the gradient of $|f|^2 e^{-k\phi}$ must be, at some point z_k, larger that $\sqrt{k}$ times a magnitude tending to infinity. However, Lemma 4 below shows that the last conclusion is not possible, and this contradiction concludes the proof of Lemma 3. □

Lemma 4. *Let $\phi(z)$ be a smooth, real-valued function in a neighborhood of the point $w \in \mathbb{C}^n$. Then there are constants C and k_0 such that the following holds. Let $k \geq k_0$, and $f(z)$ be a holomorphic function in a neighborhood of the compact set*

$$U_k(w) = \left\{z \in \mathbb{C}^n : |z_j - w_j| \leq 1/\sqrt{k}\ (j = 1, \dots, n)\right\}.$$

Then for each $1 \leq j \leq n$ we have

$$\left|\frac{\partial}{\partial z_j}\left[\left|f\right|^2 e^{-k\phi}\right](w)\right| \leq C\sqrt{k} \sup_{U_k(w)} \left|f\right|^2 e^{-k\phi}.$$

This is proved in dimension one in [AOC12]. The multi-dimensional version above can be proved in a similar way.

If the line bundle (L, ϕ) is positive, the separation condition (14) of the Fekete array is sharp in a sense. The following is true.

Lemma 5. *If (L, ϕ) is positive then there is $R > 0$ not depending on k, with the following property: if $\mathcal{F}_k$ is a Fekete configuration for $(L^k, k\phi)$, then any ball $B(x, R/\sqrt{k})$, $x \in X$, contains at least one point of $\mathcal{F}_k$.*

This result may be deduced from Theorem 2 and Lemma 6 below. However, as it will not be used later on, we do not present the details of the proof. We merely state it to show that the Fekete points $\mathcal{F}_k$ are roughly spread away from each other at a distance $1/\sqrt{k}$.

4. Sampling and Interpolation arrays

4.1. In this section we relate the Fekete arrays to the sampling and interpolation arrays. We will show that if the line bundle (L, ϕ) is positive, then by a "small perturbation" of the Fekete array one obtains a sampling or interpolation array for (L, ϕ).

Definition 1. Let k be a positive integer, and Λ_k be a finite set of points in X. We say that Λ_k is a *sampling set at level k with sampling constants A, B* if the inequalities

$$Ak^{-n} \sum_{\lambda \in \Lambda_k} |s(\lambda)|^2 \leq \int_X \left|s(x)\right|^2 \leq Bk^{-n} \sum_{\lambda \in \Lambda_k} |s(\lambda)|^2 \tag{15}$$

hold for any section $s \in H^0(L^k)$. We say that Λ_k is an *interpolation set at level k with interpolation constant C* if for any set of values $\{v_\lambda\}_{\lambda \in \Lambda_k}$, where each v_λ is an element of the fiber of λ in L^k, there is a section $s \in H^0(L^k)$ such that $s(\lambda) = v_\lambda$ $(\lambda \in \Lambda_k)$ and

$$\int_X \left|s(x)\right|^2 \leq Ck^{-n} \sum_{\lambda \in \Lambda_k} |v_\lambda|^2. \tag{16}$$

Definition 2. Let $\Lambda = \{\Lambda_k\}$ be an array of points, i.e. a sequence of finite sets Λ_k in X. We call Λ a *sampling array* if there are k_0 and positive constants A, B not depending on k, such that Λ_k is a sampling set at each level $k \geq k_0$ with sampling constants A, B. Analogously, Λ is an *interpolation array* if there are k_0 and a positive constants C not depending on k, such that Λ_k is an interpolation set at each level $k \geq k_0$ with interpolation constant C.

Lemma 6. *Suppose that (L, ϕ) is positive. Let k be a positive integer, and ε be a number satisfying $1/k \lesssim \varepsilon \lesssim 1$. If we define*

$$\Lambda_k := \mathcal{F}_{(1+\varepsilon)k}$$

then Λ_k is a sampling set at level k with sampling constants A, B such that $1 \lesssim A < B \lesssim \varepsilon^{-2n}$. On the other hand, if

$$\Lambda_k := \mathcal{F}_{(1-\varepsilon)k}$$

then it is an interpolation set at level k with interpolation constant C satisfying $C \lesssim \varepsilon^{-2n}$.

We must provide a clarification concerning the statement of the theorem: we have written $\mathcal{F}_{(1\pm\varepsilon)k}$ as if the numbers $(1 \pm \varepsilon)k$ were integers. In practice, the reader should replace these numbers by an integer approximation. The same is true in other parts of the paper below, where we shall keep using such notation.

It follows from Lemma 6 that by a "small perturbation" of the Fekete array one obtains a sampling or interpolation array for (L, ϕ).

Corollary 3. *Let (L, ϕ) be positive, and $\varepsilon > 0$ be fixed. Then*

(i) $\{\mathcal{F}_{(1+\varepsilon)k}\}$ *is a sampling array for (L, ϕ);*

(ii) $\{\mathcal{F}_{(1-\varepsilon)k}\}$ *is an interpolation array for (L, ϕ).*

The rest of this section is devoted to the proof of Lemma 6.

4.2. We start with the interpolation part of Lemma 6. We fix k and ε satisfying $1/k \lesssim \varepsilon \lesssim 1$ and define the set $\Lambda_k = \mathcal{F}_{(1-\varepsilon)k}$. We will prove that Λ_k is an interpolation set at level k with interpolation constant C satisfying $C \lesssim \varepsilon^{-2n}$.

Denote by $\{x_j\}$ the elements of the finite set Λ_k. Since the points $\{x_j\}$ form a Fekete configuration for the line bundle $L^{(1-\varepsilon)k}$, they have associated Lagrange sections l_j (see

Section 3). The sections l_j are suitable for solving the interpolation problem with nodes x_j, but we also need an estimate for the L^2 norm of the solution. For this reason we need to improve the localization of l_j around the point x_j. We therefore define the auxiliary sections

$$Q_j(x) := l_j(x) \otimes \left[\frac{\Phi_{x_j}^{(\varepsilon/2)k}(x)}{|\Pi_{(\varepsilon/2)k}(x_j, x_j)|} \right]^2 \in H^0(L^k),$$

where $\Phi_y^{(\varepsilon/2)k}$ denotes a holomorphic section to $L^{(\varepsilon/2)k}$ such that

$$|\Phi_y^{(\varepsilon/2)k}(x)| = |\Pi_{(\varepsilon/2)k}(x, y)|, \quad x \in X. \tag{17}$$

The existence of such a section is guaranteed by Lemma 1.

We have thus constructed sections Q_j in $H^0(L^k)$ which are associated to the points $\{x_j\}$. Similar to the Lagrange sections, the sections Q_j satisfy

$$|Q_j(x_i)| = \delta_{ij}, \tag{18}$$

as follows from (7) and (12). We will also need the additional estimates

$$\sup_j \int_X |Q_j(x)| \lesssim (\varepsilon k)^{-n}, \tag{19}$$

and

$$\sup_{x \in X} \sum_j |Q_j(x)| \lesssim \varepsilon^{-n}, \tag{20}$$

that will be proved now. The inequality (19) follows directly from (7), (8) and (13). To prove (20) we recall that Fekete points are separated (Lemma 3), and hence

$$d(x_i, x_j) \gtrsim \frac{1}{\sqrt{(1-\varepsilon)k}} \gtrsim \frac{\delta}{\sqrt{(\varepsilon/2)k}}$$

with $\delta = \sqrt{\varepsilon}$. Thus an application of the Plancherel-Pólya inequality (Lemma 2) to the section $\Phi_x^{(\varepsilon/2)k}$ and to the set of points $\{x_j\}$ yields

$$\sum_j |Q_j(x)| \lesssim (\varepsilon k)^{-2n} \sum_j |\Phi_x^{(\varepsilon/2)k}(x_j)|^2 \lesssim \varepsilon^{-2n} k^{-n} \int_X |\Phi_x^{(\varepsilon/2)k}|^2 \lesssim \varepsilon^{-n},$$

where in these inequalities we have used (5), (7), (8) and (17).

We are now ready to solve the interpolation problem with estimate. Suppose that we are given a set of values $\{v_j\}$, where each v_j is an element of the fiber of x_j in L^k. We will construct a solution $Q(x)$ to the interpolation problem, i.e. a section $Q \in H^0(L^k)$ such that $Q(x_j) = v_j$ for all j. The solution is defined as a linear combination of the Q_j,

$$Q(x) = \sum_j c_j Q_j(x),$$

with the coefficients c_j given by $c_j = \langle v_j, Q_j(x_j) \rangle$. This choice of coefficients and the property (18) imply that $Q(x)$ is indeed a solution to the interpolation problem.

It remains to show that the solution $Q(x)$ is bounded in L^2 with the estimate

$$\int_X \big|Q(x)\big|^2 \lesssim \varepsilon^{-2n} k^{-n} \sum_j |v_j|^2. \tag{21}$$

Indeed, by the Cauchy-Schwartz inequality and (20) we have

$$|Q(x)|^2 \le \Big(\sum_j |c_j|^2 |Q_j(x)|\Big)\Big(\sum_j |Q_j(x)|\Big) \lesssim \varepsilon^{-n} \sum_j |c_j|^2 |Q_j(x)|. \tag{22}$$

Integrating over X and using (19) yields

$$\int_X |Q(x)|^2 \lesssim \varepsilon^{-n} \sum_j |c_j|^2 \int_X |Q_j(x)| \lesssim \varepsilon^{-2n} k^{-n} \sum_j |c_j|^2,$$

and since $|c_j| = |v_j|$ this gives (21).

This complete the proof of the interpolation part of Lemma 6.

4.3. We continue to the proof of the sampling part of Lemma 6. In this case we are dealing with the set $\Lambda_k = \mathcal{F}_{(1+\varepsilon)k}$, and must prove that it is a sampling set at level k with sampling constants A, B such that $1 \lesssim A < B \lesssim \varepsilon^{-2n}$.

Again we denote by $\{x_j\}$ the elements of Λ_k. We will prove the sampling inequality

$$k^{-n} \sum_j |s(x_j)|^2 \lesssim \int_X \big|s(x)\big|^2 \lesssim \varepsilon^{-2n} k^{-n} \sum_j |s(x_j)|^2 \tag{23}$$

for any section $s \in H^0(L^k)$. The left hand side of (23) is a consequence of the Plancherel-Pólya inequality (Lemma 2) and the separation condition

$$d(x_i, x_j) \gtrsim \frac{1}{\sqrt{(1+\varepsilon)k}} \gtrsim \frac{1}{\sqrt{k}}$$

ensured by Lemma 3.

The proof of the right hand side of (23) is similar to the interpolation part. Define

$$P_x(y) := s(y) \otimes \left[\frac{\Phi_x^{(\varepsilon/2)k}(y)}{|\Pi_{(\varepsilon/2)k}(x,x)|} \right]^2 \in H^0(L^{(1+\varepsilon)k}).$$

The space $H^0(L^{(1+\varepsilon)k})$ has a basis of Lagrange sections l_j associated to the Fekete points $\{x_j\}$, so we may expand P_x in terms of this basis. We get

$$P_x(y) = \sum_j \big\langle P_x(x_j), l_j(x_j) \big\rangle \, l_j(y).$$

In particular, if $y = x$ this implies

$$|s(x)| = |P_x(x)| \le \sum_j |P_x(x_j)| = \sum_j |s(x_j)| \, |Q_j(x)|,$$

where now we define

$$Q_j(x) := \left[\frac{\Phi_{x_j}^{(\varepsilon/2)k}(x)}{|\Pi_{(\varepsilon/2)k}(x,x)|} \right]^2 .$$

The estimates (19), (20) are valid in this case as well, and can be proved in the same way. We may therefore continue as in (22). We obtain

$$|s(x)|^2 \le \Big(\sum_j |s(x_j)|^2 |Q_j(x)| \Big) \Big(\sum_j |Q_j(x)| \Big) \lesssim \varepsilon^{-n} \sum_j |s(x_j)|^2 |Q_j(x)|, \tag{24}$$

and integrating over X yields the right hand side of (23).

We have thus proved also the sampling part of Lemma 6, so the lemma is completely proved.

Remark 1. In the proof of Lemma 6 we have not used any off-diagonal estimate such as (9) for the Bergman kernel, but only the asymptotic estimate (8) on the diagonal combined with the L^2 equality (7) (this is in contrast to [AOC12], for example).

5. Landau's inequalities

5.1. In this section we use Landau's method [Lan67] to obtain estimates for the number of points of a separated sampling or interpolation array in a ball.

Let us say that a finite set of points Λ_k is *δ-separated at level k* if

$$d(x,y) \ge \frac{\delta}{\sqrt{k}}, \quad x, y \in \Lambda_k, \quad x \ne y. \tag{25}$$

Our goal is to prove the following two statements.

Lemma 7. *Let Λ_k be a δ-separated sampling set at level k with sampling constants A, B. Then for any $z \in X$ and $r > 0$,*

$$\#\Big(\Lambda_k \cap B\Big(z, \tfrac{r+\delta}{\sqrt{k}}\Big)\Big) \ge \int_\Omega |\Pi_k(x,x)| - M \iint_{\Omega \times \Omega^c} \big|\Pi_k(x,y)\big|^2, \tag{26}$$

where $\Omega = B(z, \frac{r}{\sqrt{k}})$, and the constant M is bounded by the sampling constant B times a constant which may depend on δ but does not depend on k, z, r.

Lemma 8. *Similarly, if Λ_k is a δ-separated interpolation set at level k with interpolation constant C, then for any $z \in X$ and $r > 0$,*

$$\#\Big(\Lambda_k \cap B\Big(z, \tfrac{r-\delta}{\sqrt{k}}\Big)\Big) \le \int_\Omega |\Pi_k(x,x)| + M \iint_{\Omega \times \Omega^c} \big|\Pi_k(x,y)\big|^2, \tag{27}$$

where again $\Omega = B(z, \frac{r}{\sqrt{k}})$, and the constant M is bounded by the interpolation constant C times a constant which may depend on δ but does not depend on k, z, r.

5.2. Let Ω be a measurable subset of X. We denote by T_Ω the linear operator on $H^0(L)$ defined by

$$T_\Omega(s) = P(s \cdot \mathbb{1}_\Omega), \quad s \in H^0(L),$$

where P denotes the orthogonal projection from the Hilbert space of all L^2 sections onto its finite-dimensional subspace $H^0(L)$. It is easy to see that

$$\langle T_\Omega s, s \rangle = \int_\Omega |s|^2, \quad s \in H^0(L),$$

hence T_Ω is self-adjoint, non-negative and $\|T_\Omega\| \le 1$. We may therefore find an orthonormal basis $\{s_j\}$ of $H^0(L)$ consisting of eigensections,

$$T_\Omega(s_j) = \lambda_j(\Omega) s_j.$$

The eigenvalues $\lambda_j(\Omega)$ lie between 0 and 1, and we order them in a non-increasing order,

$$\lambda_1(\Omega) \ge \lambda_2(\Omega) \ge \lambda_3(\Omega) \ge \cdots \ge 0.$$

By using (6) with the basis of eigensections $\{s_j\}$ we can compute the trace of T_Ω,

$$\sum_{j \ge 1} \lambda_j(\Omega) = \sum_{j \ge 1} \langle T_\Omega s_j, s_j \rangle = \sum_{j \ge 1} \int_\Omega \big|s_j(x)\big|^2 = \int_\Omega |\Pi(x,x)|. \tag{28}$$

Similarly, (4) allows us to compute the Hilbert-Schmidt norm of T_Ω (the trace of T_Ω^2) in terms of the Bergman kernel. Indeed,

$$\big|\Pi(x,y)\big|^2 = \sum_{j \ge 1} \sum_{k \ge 1} \langle s_j(x), s_k(x) \rangle \overline{\langle s_j(y), s_k(y) \rangle},$$

hence integrating over $\Omega \times \Omega$ gives

$$\sum_{j \ge 1} \lambda_j^2(\Omega) = \sum_{j,k} \big|\langle T_\Omega s_j, s_k \rangle\big|^2 = \sum_{j,k} \left| \int_\Omega \langle s_j, s_k \rangle \right|^2 = \iint_{\Omega \times \Omega} \big|\Pi(x,y)\big|^2. \tag{29}$$

Using (28) and (29) one may obtain some information on the distribution of the eigenvalues. This is done in the following lemma.

Lemma 9. *Let $0 < \gamma < 1$ and denote by $n(\Omega, \gamma)$ the number of eigenvalues $\lambda_j(\Omega)$ which are strictly greater than γ. Then we have the lower bound*

$$n(\Omega,\gamma) \ge \int_\Omega |\Pi(x,x)| - \frac{1}{1-\gamma} \iint_{\Omega \times \Omega^c} |\Pi(x,y)|^2, \tag{30}$$

and the upper bound

$$n(\Omega,\gamma) \le \int_\Omega |\Pi(x,x)| + \frac{1}{\gamma} \iint_{\Omega \times \Omega^c} |\Pi(x,y)|^2. \tag{31}$$

Proof. We have

$$\mathbb{1}_{(\gamma,1]}(x) \geq x - \frac{x(1-x)}{1-\gamma} \quad (0 \leq x \leq 1),$$

hence

$$n(\Omega,\gamma) = \sum_j \mathbb{1}_{(\gamma,1]}\big(\lambda_j(\Omega)\big) \geq \sum_j \lambda_j(\Omega) - \frac{1}{1-\gamma}\sum_j \big(\lambda_j(\Omega) - \lambda_j^2(\Omega)\big).$$

Using (28),(29) and (7) this implies

$$\begin{aligned} n(\Omega,\gamma) &\geq \int_\Omega |\Pi(x,x)| - \frac{1}{1-\gamma}\Big[\int_\Omega |\Pi(x,x)| - \iint_{\Omega\times\Omega} \big|\Pi(x,y)\big|^2\Big] \\ &= \int_\Omega |\Pi(x,x)| - \frac{1}{1-\gamma}\Big[\iint_{\Omega\times X} \big|\Pi(x,y)\big|^2 - \iint_{\Omega\times\Omega} \big|\Pi(x,y)\big|^2\Big] \end{aligned}$$

which proves (i). To prove (ii) one may argue similarly using the inequality

$$\mathbb{1}_{(\gamma,1]}(x) \leq x + \frac{x(1-x)}{\gamma} \quad (0 \leq x \leq 1).$$ □

5.3. Now consider powers L^k of the line bundle L. We obtain an operator $T_\Omega^{(k)}$ acting on $H^0(L^k)$ with corresponding eigenvalues

$$\lambda_1^{(k)}(\Omega) \geq \lambda_2^{(k)}(\Omega) \geq \cdots \geq 0,$$

and we let $n_k(\Omega,\gamma)$ denote the number of eigenvalues strictly greater than γ $(0<\gamma<1)$.

Lemma 10. *Let Λ_k be a δ-separated sampling set at level k with sampling constants A, B. Then for any $z \in X$ and $r > 0$,*

$$\#\Big(\Lambda_k \cap B\Big(z, \tfrac{r+\delta}{\sqrt{k}}\Big)\Big) \geq n_k\Big(B\Big(z, \tfrac{r}{\sqrt{k}}\Big), \gamma\Big)$$

where γ is some constant lying between 0 and 1, such that $1/(1-\gamma)$ is bounded by the sampling constant B times a constant which may depend on δ but does not depend on k, z, r.

Proof. Let $\{s_j\}$ be the orthonormal basis of $H^0(L^k)$ which is associated to the eigenvalues $\lambda_j^{(k)}(\Omega)$, where $\Omega = B\big(z, \frac{r}{\sqrt{k}}\big)$. Let $N := \#\big(\Lambda_k \cap B(z, \frac{r+\delta/2}{\sqrt{k}})\big)$. We may restrict to the case when N is strictly smaller than $\dim H^0(L^k)$, since otherwise the inequality holds trivially. In this case, we may choose a linear combination

$$s = \sum_{j=1}^{N+1} c_j s_j$$

of the first $N+1$ eigensections, such that

$$s(\lambda) = 0, \quad \lambda \in \Lambda_k \cap B\big(x, \tfrac{r+\delta/2}{\sqrt{k}}\big)$$

and the c_j are not all zero. Since Λ_k is a sampling set, we have

$$\|s\|^2 \leq Bk^{-n} \sum_{\lambda \in \Lambda_k} |s(\lambda)|^2 = Bk^{-n} \sum_{\lambda \in \Lambda_k \setminus B\left(x, \frac{r+\delta/2}{\sqrt{k}}\right)} |s(\lambda)|^2.$$

Using the inequality (10) and the fact that $B(\lambda, \frac{\delta/2}{\sqrt{k}})$ are disjoint balls, we get

$$\|s\|^2 \leq KB \sum_{\lambda} \int_{B\left(\lambda, \frac{\delta/2}{\sqrt{k}}\right)} |s|^2 \leq KB \int_{X \setminus \Omega} |s|^2,$$

where the constant K may depend on δ but does not depend on k, z, r. This implies

$$\lambda_{N+1}(\Omega)\, \|s\|^2 = \lambda_{N+1} \sum_{1}^{N+1} |c_j|^2 \leq \sum_{1}^{N+1} \lambda_j |c_j|^2 = \langle T_\Omega^{(k)} s, s \rangle = \int_\Omega |s|^2 \leq \gamma \|s\|^2,$$

where $\gamma := 1 - (KB)^{-1}$. This shows that $\lambda_{N+1}(\Omega) \leq \gamma$ and hence $n_k(\Omega, \gamma) \leq N$. □

Lemma 11. *Let Λ_k be a δ-separated interpolation set at level k with interpolation constant C. Then for any $z \in X$ and $r > 0$,*

$$\#\left(\Lambda_k \cap B\left(z, \tfrac{r-\delta}{\sqrt{k}}\right)\right) \leq n_k\left(B\left(z, \tfrac{r}{\sqrt{k}}\right), \gamma\right)$$

where γ is some constant lying between 0 and 1, such that $1/\gamma$ is bounded by the interpolation constant C times a constant which may depend on δ but does not depend on k, z, r.

Proof. Let W denote the orthogonal complement in $H^0(L^k)$ of the subspace of sections vanishing on Λ_k. Since Λ_k is an interpolation set at level k, for any set of values $\{v_\lambda\}_{\lambda \in \Lambda_k}$, where each v_λ is an element of the fiber of λ in L^k, there is a section $s \in H^0(L^k)$ such that $s(\lambda) = v_\lambda$ $(\lambda \in \Lambda_k)$ and

$$\|s\|^2 \leq Ck^{-n} \sum_{\lambda \in \Lambda_k} |s(\lambda)|^2. \tag{32}$$

By taking the orthogonal projection of s onto W we obtain another solution to the interpolation problem, which in addition belongs to W (the projection neither changes the values of s on Λ_k nor increases its norm).

On the other hand, a section in W is uniquely determined by its values on Λ_k, as follows from the definition of W. Hence if s is an arbitrary section in W, then it is the unique interpolant in W to the values $\{s(\lambda)\}_{\lambda \in \Lambda_k}$. This implies that (32) holds for any $s \in W$.

Now let us denote by $x_1, \dots, x_N$ the elements of $\Lambda_k \cap B\left(z, \frac{r-\delta}{\sqrt{k}}\right)$. For each $1 \leq j \leq N$ we can find $s_j \in W$ such that $|s_j(x_j)| = 1$ and s_j vanishes on $\Lambda_k \setminus \{x_j\}$. Certainly, the s_j form a linearly independent set of vectors. We denote by F the N-dimensional linear subspace spanned by the sections $s_1, \dots, s_N$.

Now take any $s \in F$, then we have

$$\|s\|^2 \leq Ck^{-n} \sum_{\lambda \in \Lambda_k} |s(\lambda)|^2 = Ck^{-n} \sum_{\lambda \in \Lambda_k \cap B\left(x, \frac{r-\delta/2}{\sqrt{k}}\right)} |s(\lambda)|^2 < KC \int_\Omega |s|^2,$$

where $\Omega = B\big(z, \frac{r}{\sqrt{k}}\big)$, and the constant K may depend on δ but does not depend on k, z, r. The last inequality holds by (10) and the fact that $B(\lambda, \frac{\delta/2}{\sqrt{k}})$ are disjoint balls. Hence

$$\frac{\langle T_\Omega^{(k)} s, s\rangle}{\|s\|^2} = \frac{\int_\Omega |s|^2}{\|s\|^2} > \frac{1}{KC} =: \gamma,$$

for any section s in the N-dimensional linear subspace F. By the min-max theorem this implies that $\lambda_N(\Omega) > \gamma$ and hence $n_k(\Omega, \gamma) \geq N$. □

6. Curvature and density

In the previous section we have used Landau's method to estimate the number of points of a sampling or interpolation set in a ball, where the estimate obtained was given in terms of the Bergman kernel $\Pi_k(x, y)$. In the present section we will prove Theorem 2 by relating the latter estimate to geometric properties of the positive line bundle (L, ϕ), namely, to the volume form associated with the curvature of the line bundle.

6.1. Given a point $x \in X$, let $\xi_1, \dots, \xi_n$ be a basis for the holomorphic cotangent space at x, orthonormal with respect to the hermitian metric ω on X. With respect to this basis, the form $\partial\bar\partial\phi$ is given at the point x by

$$\partial\bar\partial\phi = \sum_{j,k} \phi_{j,k}\, \xi_j \wedge \bar{\xi_k},$$

where $(\phi_{j,k})$ is a hermitian $n \times n$ matrix. The eigenvalues $\lambda_1(x), \dots, \lambda_n(x)$ of this matrix are called *the eigenvalues of the curvature form $\partial\bar\partial\phi$ with respect to the hermitian metric ω*.

Recall that the line bundle L with the metric ϕ is said to be positive if $i\partial\bar\partial\phi$ is a positive form. This is equivalent to all of the eigenvalues $\lambda_1(x), \dots, \lambda_n(x)$ being strictly positive, for every $x \in X$.

If the form $i\partial\bar\partial\phi$ is positive, then the (n, n)-form $(i\partial\bar\partial\phi)^n$ is a volume form on X. Our goal is to provide geometrical information on a sampling or interpolation array $\Lambda = \{\Lambda_k\}$, by relating the mass distribution of the measure

$$k^{-n} \sum_{\lambda \in \Lambda_k} \delta_\lambda$$

to the volume distribution of $(i\partial\bar\partial\phi)^n$ in a quantitative manner. We emphasize that the volume form $(i\partial\bar\partial\phi)^n$ is a characteristic of the hermitian metric ϕ on the line bundle only, and does not depend on the arbitrary hermitian metric ω that we have chosen on the manifold X. However, the curvature volume form $(i\partial\bar\partial\phi)^n$ is related to the volume form V associated with ω through the eigenvalues, and we have

$$(i\partial\bar\partial\phi)^n = n!\, \lambda_1(x) \cdots \lambda_n(x)\, dV(x). \tag{33}$$

The eigenvalues of the curvature form are related also to the asymptotics of the Bergman function $|\Pi_k(x, x)|$. When the line bundle is positive, it was proven in [Tia90], see [Zel98] that

$$|\Pi_k(x,x)| = \pi^{-n}\lambda_1(x)\cdots\lambda_n(x)k^n + O(k^{n-1}). \tag{34}$$

This a more precise result than (8). In fact, this is only the first term in a complete asymptotic expansion obtained in [Zel98] into a power series in k (see also [BBS08] for a different proof).

6.2. The main ingredient which we need for the proof of Theorem 2 is to show that the "error terms" in Landau's inequalities (26) and (27) are indeed small with respect to the main term. This is done in the following lemma.

Lemma 12. *Let the line bundle (L,ϕ) be positive. If $\Omega = B(z, \frac{r}{\sqrt{k}})$, $z \in X$, then*

$$\iint_{\Omega\times\Omega^c} \big|\Pi_k(x,y)\big|^2 \lesssim r^{2n-1}.$$

For the proof we will use the asymptotic off-diagonal estimate (9) for the Bergman kernel, which holds when the line bundle (L,ϕ) is positive. In fact, we do not need the precise exponential decay given by (9). It will be enough to use the fact that

$$|\Pi_k(x,y)| \le k^n \varphi(\sqrt{k}\, d(x,y)), \tag{35}$$

where φ is a smooth decreasing function on $[0,\infty)$ such that

$$\varphi(u) = O(u^{-\alpha}) \text{ as } u \to \infty, \quad \text{for some } \alpha > n + \tfrac{1}{2}. \tag{36}$$

Proof of Lemma 12. We partition Ω into "dyadic shells" defined by

$$\Omega_j := \left\{ x \in X : \big(1 - 2^{-j+1}\big)\frac{r}{\sqrt{k}} \le d(x,z) < \big(1-2^{-j}\big)\frac{r}{\sqrt{k}} \right\} \quad (j \ge 1).$$

If $x \in \Omega_j$ and $y \in \Omega^c$ then $d(x,y) > 2^{-j}\frac{r}{\sqrt{k}}$, and thus we have

$$\iint_{\Omega\times\Omega^c} \big|\Pi_k(x,y)\big|^2 \le \sum_{j=1}^{\infty} \iint_{\Omega_j \times B(x, 2^{-j}\frac{r}{\sqrt{k}})^c} \big|\Pi_k(x,y)\big|^2.$$

To estimate the right hand side we use (35). For any $A > 0$ we have

$$\int_{B\left(x,\frac{A}{\sqrt{K}}\right)^c} \big|\Pi_k(x,y)\big|^2 dV(y) =$$

$$= \int_0^\infty V\Big(\big\{y : \big|\Pi_k(x,y)\big| > \lambda\big\} \setminus B\Big(x, \frac{A}{\sqrt{k}}\Big)\Big) 2\lambda\, d\lambda$$

$$\le \int_0^{k^n\varphi(0)} V\Big(\big\{y : \varphi\big(\sqrt{k}\, d(x,y)\big) \ge k^{-n}\lambda\big\} \setminus B\Big(x, \frac{A}{\sqrt{k}}\Big)\Big) 2\lambda\, d\lambda.$$

Since φ is decreasing we may apply the change of variable $\lambda = k^n\varphi(u)$, and we get

$$\begin{aligned}
&= \int_0^\infty V\Big(B\Big(x, \frac{u}{\sqrt{k}}\Big) \setminus B\Big(x, \frac{A}{\sqrt{k}}\Big)\Big)\,(2k^n\varphi(u))\,|k^n\varphi'(u)|\,du \\
&\lesssim \int_A^\infty \Big(\frac{u}{\sqrt{k}}\Big)^{2n}\,(2k^n\varphi(u))\,|k^n\varphi'(u)|\,du \\
&\lesssim k^n \int_A^\infty u^{2n}\,\varphi(u)\,|\varphi'(u)|\,du.
\end{aligned}$$

We also use an estimate for the volume of shells, namely

$$V\big(B(x,\rho+\delta) \setminus B(x,\rho)\big) \lesssim \rho^{2n-1}\,\delta \quad (0 < \delta < \rho), \tag{37}$$

which can be proved using the exponential map. In particular, this implies

$$V(\Omega_j) \lesssim 2^{-j}\,\frac{r^{2n}}{k^n}\,.$$

Combining all the estimates above yields

$$\begin{aligned}
\iint_{\Omega\times\Omega^c} \big|\Pi_k(x,y)\big|^2 &\lesssim \sum_{j=1}^\infty \Big(2^{-j}\,\frac{r^{2n}}{k^n}\Big)\,k^n \int_{2^{-j}r}^\infty u^{2n}\,\varphi(u)\,|\varphi'(u)|\,du \\
&= r^{2n} \int_0^\infty \Big[\sum_{j=1}^\infty 2^{-j}\mathbb{1}_{[2^{-j}r,\infty)}(u)\Big]\,u^{2n}\,\varphi(u)\,|\varphi'(u)|\,du \\
&\le r^{2n} \int_0^\infty (2u/r)\,u^{2n}\,\varphi(u)\,|\varphi'(u)|\,du \\
&\lesssim r^{2n-1} \int_0^\infty u^{2n}\,\varphi(u)^2\,du,
\end{aligned}$$

where the integration by parts used is justified by (36). Since the last integral converges, again due to (36), this proves the lemma. □

6.3. We can now finish the proof of Theorem 2. It is an immediate consequence of the following result.

Lemma 13. *Let (L,ϕ) be positive. If Λ_k be a δ-separated sampling set at level k with sampling constants A, B, then for any $z \in X$ and $r > 0$,*

$$\frac{k^{-n}\#(\Lambda_k \cap \Omega)}{\int_\Omega (i\partial\bar\partial\phi)^n} > \frac{1}{\pi^n n!} - \frac{M}{r}, \tag{38}$$

where $\Omega = B(z, \frac{r}{\sqrt{k}})$, and the constant M is bounded by the sampling constant B times a constant which may depend on δ but does not depend on k, z, r.

Similarly, if Λ_k is a δ-separated interpolation set at level k with interpolation constant C, then for any $z \in X$ and $r > 0$,

$$\frac{k^{-n}\#\big(\Lambda_k \cap \Omega)\big)}{\int_\Omega (i\partial\bar{\partial}\phi)^n} < \frac{1}{\pi^n n!} + \frac{M}{r}, \tag{39}$$

where again $\Omega = B(z, \frac{r}{\sqrt{k}})$, and the constant M is bounded by the interpolation constant C times a constant which may depend on δ but does not depend on k, z, r.

Proof. Assume first that Λ_k is a δ-separated sampling set at level k. Let $\Omega = B\big(z, \frac{r}{\sqrt{k}}\big)$. The separation condition together with (37) imply that the number of points of Λ_k in the shell $B\big(z, \frac{r+\delta}{\sqrt{k}}\big) \setminus B\big(z, \frac{r}{\sqrt{k}}\big)$ is less than $M_1 r^{2n-1}$. Hence by (26) and Lemma 12 we obtain

$$\#(\Lambda_k \cap \Omega) \geq \int_\Omega |\Pi_k(x,x)| - M_2 r^{2n-1}.$$

Using (33) and (34) this implies

$$\#(\Lambda_k \cap \Omega) \geq \frac{k^n}{\pi^n n!}\int_\Omega (i\partial\bar{\partial}\phi)^n - M_2 r^{2n-1} - M_3 k^{n-1} V(\Omega).$$

Since $V(\Omega) \lesssim r^{2n}/k^n$ and $r/\sqrt{k} \leq \operatorname{diam}(X)$ it follows that

$$\#(\Lambda_k \cap \Omega) \geq \frac{k^n}{\pi^n n!}\int_\Omega (i\partial\bar{\partial}\phi)^n - M_4 r^{2n-1},$$

and since $k^n \int_\Omega (i\partial\bar{\partial}\phi)^n$ is of order r^{2n} this proves the claimed inequality. In the second case, when Λ_k is a δ-separated interpolation set at level k, the result is proved in a similar way using (27) instead of (26). □

This concludes the proof of Theorem 2.

Remark 2. One may also define sampling and interpolation arrays with respect to the L^p norm on the line bundle ($1 \leq p \leq \infty$). The necessary density conditions given in Corollary 2 could be extended to this setting as well. This is rather standard and we do not discuss the details, see e.g. [Mar07].

7. Equidistribution of Fekete points

In this section we use the previous results to estimate from above and below the number of Fekete points that lie in a ball. This allows us then to obtain an upper bound for the Kantorovich-Wasserstein distance between the Fekete measure (1) and its limiting measure. We also use the Fekete points to construct a sampling or interpolation array with density arbitrarily close to the critical one, showing that the necessary density conditions in Corollary 2 are sharp.

7.1. We start with the proof of Theorem 1. As before, we denote by $\mathcal{F}_k$ a Fekete configuration for $(L^k, k\phi)$.

Proof of Theorem 1. Let $\Omega = B\big(x, \frac{r}{\sqrt{k}}\big)$, where $r > 0$ is a large number, and k is such that $r/\sqrt{k} \leq \operatorname{diam}(X)$. Lemma 6 says that if ε is a sufficiently small number satisfying $\varepsilon \gtrsim 1/k$, then $\mathcal{F}_k$ is a sampling set at level $k/(1+\varepsilon)$ with sampling constants A, B such that $B \lesssim \varepsilon^{-2n}$. Also, by Lemma 3 there is $\delta > 0$ not depending on k, such that $\mathcal{F}_k$ is a δ-separated set at level $k/(1+\varepsilon)$. Hence by (38) we conclude that

$$\begin{aligned}\#(\mathcal{F}_k \cap \Omega) &\geq \frac{1}{\pi^n n!}\left(\frac{k}{1+\varepsilon}\right)^n \int_\Omega (i\partial\bar{\partial}\phi)^n - O(\varepsilon^{-2n} r^{2n-1}) \\ &= \frac{k^n}{\pi^n n!}\int_\Omega (i\partial\bar{\partial}\phi)^n - O(\varepsilon\, r^{2n}) - O(\varepsilon^{-2n} r^{2n-1}).\end{aligned}$$

We choose $\varepsilon = \varepsilon(r)$ such that the two error terms coincide, that is, $\varepsilon = (1/r)^{\frac{1}{2n+1}}$, and remark that the condition $\varepsilon \gtrsim 1/k$ is indeed satisfied by this choice, since $r/\sqrt{k} \leq \operatorname{diam}(X)$. Hence

$$\#(\mathcal{F}_k \cap \Omega) \geq (1 - O(\varepsilon))\frac{k^n}{\pi^n n!}\int_\Omega (i\partial\bar{\partial}\phi)^n.$$

Applying similar considerations using (39) we also get a similar upper bound. Thus

$$\#(\mathcal{F}_k \cap \Omega) = (1 + O(\varepsilon))\frac{k^n}{\pi^n n!}\int_\Omega (i\partial\bar{\partial}\phi)^n. \tag{40}$$

We also have from (6), (33) and (34) that

$$\#\mathcal{F}_k = \dim H^0(L^k) = \int_X |\Pi_k(x,x)| = (1 + O(k^{-1}))\frac{k^n}{\pi^n n!}\int_X (i\partial\bar{\partial}\phi)^n. \tag{41}$$

Combining (40) with (41) and again using $\varepsilon \gtrsim 1/k$ this proves the theorem. □

7.2. The estimate (40) obtained for the number of Fekete points in a ball shows, in particular, that a Fekete array $\{\mathcal{F}_k\}$ for the positive line bundle has the critical density,

$$D^-(\{\mathcal{F}_k\}) = D^+(\{\mathcal{F}_k\}) = \frac{1}{\pi^n n!}\,.$$

It is easy to check that the density of the perturbed array $\{\mathcal{F}_{(1\pm\varepsilon)k}\}$ will be equal to the critical value multiplied by $(1 \pm \varepsilon)^n$. Combining this with Corollary 3 shows that the density threshold in Corollary 2 is sharp.

Corollary 4. *Let (L, ϕ) be positive. Then*

(i) *For any $\varepsilon > 0$ there is a sampling array Λ with $D^+(\Lambda) < \frac{1}{\pi^n n!} + \varepsilon$.*

(ii) *For any $\varepsilon > 0$ there is an interpolation array Λ with $D^-(\Lambda) > \frac{1}{\pi^n n!} - \varepsilon$.*

7.3. Given two probability measures μ and ν on a metric space X, one defines the Kantorovich-Wasserstein distance between them as

$$W(\mu,\nu) = \inf\left\{\iint_{X\times X} \operatorname{dist}(x,y)\, dp(x,y)\right\}$$

where the infimum is taken over all Borel probability measures p on $X\times X$ with marginals $p(\cdot,X)=\mu$ and $p(X,\cdot)=\nu$. This metric plays a key role in transportation problems, see for instance [Vil09], where one could also find the equivalent dual definition

$$W(\mu,\nu) = \sup\left\{\Big|\int_X fd(\mu-\nu)\Big| : f\in \operatorname{Lip}_{1,1}(X)\right\}, \tag{42}$$

where $\operatorname{Lip}_{1,1}(X)$ is the collection of all functions f on X satisfying $|f(x)-f(y)|\le d(x,y)$.

In our setting we have two probability measures, the first one is the Fekete measure μ_k defined in (1), and the second one is the measure $(i\partial\bar\partial\phi)^n$ normalized to have total mass 1, which we denote by ν. It is known, see [Blü90] for instance, that on a Riemannian manifold if $\mu_k(B(x,r))\to\nu(B(x,r))$ for all balls, as guaranteed by Theorem 1, then μ_k converges weakly to ν as $k\to\infty$, where the latter means that $\int fd\mu_k\to\int fd\nu$ for any continuous function f on X.

The Kantorovich-Wasserstein distance metrizes the weak convergence of measures. Here we will prove Corollary 1 which estimates the rate of convergence in the Kantorovich-Wasserstein distance. In the proof we will use Theorem 1, which already contains a quantitative statement about the convergence.

To prove the lower bound for the Kantorovich-Wasserstein distance we consider the function $f_k(x)=\operatorname{dist}(x,\mathcal{F}_k)$. Then clearly $f_k\in\operatorname{Lip}_{1,1}(X)$, and moreover f_k vanishes on $\mathcal{F}_k$. Hence by (42),

$$W(\mu_k,\nu)\ge\Big|\int_X f_k(d\mu_k-d\nu)\Big| = \int_X f_k d\nu.$$

The function f_k is bounded below by $\delta>0$ outside the balls $B(\lambda,\delta)$, $\lambda\in\mathcal{F}_k$, and so

$$\int_X f_k d\nu \ge \delta\cdot\nu\big(X\setminus\bigcup_{x\in\mathcal{F}_k}B(x,\delta)\big)\ge\delta(1-C\delta^{2n}\#\mathcal{F}_k).$$

We choose $\delta=\delta(k)$ such that $C\delta^{2n}\#\mathcal{F}_k=1/2$. Since $\#\mathcal{F}_k\simeq k^n$ by (3), this implies

$$W(\mu_k,\nu)\gtrsim k^{-1/2}.$$

For the upper estimate we will use the following result.

Lemma 14. [Blü90] *Let X be a compact Riemannian manifold of dimension d, with associated volume measure V. If f is a continuous function on X, μ,ν are two measures on X, and $r>0$, then the following estimate holds,*

$$\begin{aligned}\Big|\int_X f(d\nu-d\mu)\Big| &\le \|f-f_r\|\|\nu-\mu+V\|+Cr^2\|f\|(1+\|\nu-\mu+V\|)\\ &\quad + Cr^{-d}\|f\|\int_X|\nu(B(x,r))-\mu(B(x,r))|dV(x),\end{aligned}\tag{43}$$

where C is a positive constant depending only on the manifold X, and

$$f_r(x) = \frac{1}{V(B(x,r))} \int_{B(x,r)} f(y)\, dV(y), \quad x \in X.$$

Here $\|f\|$ denotes the supremum norm on X, while $\|\mu\|$ is the total variation norm. In our setting X is a hermitian manifold of complex dimension n. If $f \in \mathrm{Lip}_{1,1}(X)$ then $\|f - f_r\| \le r$, and by subtracting a convenient constant from f (which is innocuous in our setting since both μ_k and ν are probability measures) we may also assume that $\|f\| \le \mathrm{diam}(X)$. Theorem 1 then guarantees that

$$|\mu_k(B(x,r)) - \nu(B(x,r))| \lesssim r^{2n} \big(r\sqrt{k}\,\big)^{-\frac{1}{2n+1}},$$

so by Lemma 14 we get

$$\Big| \int_X f(d\nu - d\mu) \Big| \le Cr + Cr^2 + Cr^{-2n} r^{2n} \big(r\sqrt{k}\,\big)^{-\frac{1}{2n+1}}.$$

If we choose $r = k^{-1/(4n+4)}$ we obtain the upper bound in Corollary 1.

8. Simultaneously Sampling and Interpolation arrays

8.1. In this section we assume that X is a *projective* manifold, but we work with a metric ϕ on the line bundle L which is only *semi-positive*. We will show that, if there is a point in X where ϕ has a strictly positive curvature, then the sections of high powers of the line bundle resemble closely the functions in the Bargmann-Fock space. This observation will allow us to establish Theorem 4, showing that in this case there are no arrays which are simultaneously sampling and interpolation for (L, ϕ). The non-existence of simultaneously sampling and interpolation sequences is a recent result in the classical Bargmann-Fock space [AFK11, GM11].

Actually we could have replaced the assumption that X is projective by the apparently weaker condition that X is a Kähler manifold. However, the solution of Siu [Siu84] to the Grauert-Riemenschneieder conjecture shows that, under the hypothesis that L is semi-positive with a point where it has a strictly positive curvature, the base manifold X is Moishezon, and being also Kähler it is automatically projective [Moĭ66].

The proof of Siu also shows that under the hypothesis of the theorem, L is big and thus there is a strictly positive singular metric ϕ_s on L that is in L^1_{loc} and smooth on all points of X outside a proper analytic set E, see [MM07, Theorem 2.3.30].

8.2. We fix a point $x_0 \in X \setminus E$ where the original metric on L had positive curvature.

Definition 3. We say that we have *normalized coordinates* in a neighborhood of $x_0 \in X \setminus E$ if we have a coordinate chart that is mapped to a neighborhood of 0 in $\mathbb{C}^n$ and a local holomorphic frame $e_L(z)$ such that the following conditions hold:

- The curvature form of the line bundle at x_0 is given by $\Theta(0) = \sum_{j=1}^n dz_j \wedge d\bar{z}_j$;
- $h(0) = 1$ and $\frac{\partial h}{\partial z_j}(0) = \frac{\partial^2 h}{\partial z_j \partial z_k}(0) = 0$;

where above $h(z) = |e_L(z)|^2$, and $\Theta(z) = -\partial\bar\partial \log h(z)$ is the curvature form.

This can always be arranged if the curvature of h is smooth and positive at the point x_0, by choosing appropriate coordinates and a convenient local frame. Observe that in normalized coordinates

$$h(z) = e^{-|z|^2+o(|z|^2)}. \tag{44}$$

We fix now a neighborhood $B(0,\delta)$ of the origin at $\mathbb{C}^n$ that is mapped by normal coordinates to a neighborhood U of x_0 in X.

Definition 4. We define the sets $\Sigma_k \subset \mathbb{C}^n$ as follows: $\sigma \in \Sigma_k$ if and only if $\sigma/\sqrt{k}$ is mapped by the normal coordinates to a point in $\Lambda_K \cap U$. By definition $\Sigma_k \subset B(0,\delta\sqrt{k})$.

If Λ_k is both an interpolation and sampling array, we will construct a sequence $\Sigma \subset \mathbb{C}^n$ such that it is both interpolation and sampling for the Bargmann-Fock space.

Definition 5. Given $p \in [1,\infty)$ The Bargmann-Fock space $\mathcal{BF}^p$ consists of entire functions such that

$$\|f\|_p^p := \int_{\mathbb{C}^n} |f(z)|^p e^{-p|z|^2/2} dm(z) < +\infty.$$

When $p = \infty$ the natural norm is

$$\|f\|_\infty := \sup_{\mathbb{C}^n} |f(z)| e^{-|z|^2/2}.$$

A sequence Σ is sampling for the Bargmann-Fock space $\mathcal{BF}^2$ if and only if

$$\|f\|_2^2 \lesssim \sum_\sigma |f(\sigma)|^2 e^{-|\sigma|^2} \lesssim \|f\|_2^2$$

and it is interpolation for $\mathcal{BF}^2$ if given any values $\{v_\sigma\}$ there is a function $f \in \mathcal{BF}^2$ such that $f(\sigma) = v_\sigma$ and with the estimate

$$\|f\|_2^2 \lesssim \sum_\sigma |v_\sigma|^2 e^{-|\sigma|^2},$$

provided that the right hand side is finite.

It is known, see [AFK11] and [GM11], that there do not exist sequences that are simultaneously sampling and interpolation in $\mathcal{BF}^2(\mathbb{C}^n)$.

The key ingredient in the construction of Σ is that the sections of high powers of the (locally positive) line bundle behave as functions in the Bargmann-Fock space when properly rescaled. This is a well known phenomenon that can be illustrated by the fact that the Bergman kernel universally converges to the Bergman kernel of the Bargmann-Fock space in normal coordinates if rescaled properly, see [BSZ00]. The next theorem is another illustration of the same fact. In order to state it we need to introduce the notion of weak limits of sequences. If we have a collection of separated sequences $\Sigma_k \subset \mathbb{C}^n$ with a uniform separation constant for all k and another separated sequence $\Sigma \subset \mathbb{C}^n$ we say that Σ_k converges weakly to Σ if the corresponding measures $\mu_k = \sum_{\sigma\in\Sigma_k} \delta_{\sigma_k}$ converge weakly to $\sum_{\sigma\in\Sigma} \delta_\sigma$. This notion was used extensively by Beurling in his study of sampling sequences in the Paley-Wiener space and it will also be useful in our context.

Theorem 5. *Let Λ_k be a separated sampling array for L^k and let Σ be any weak limit of a partial subsequence Σ_k, then Σ is a sampling sequence for $\mathcal{BF}^2(\mathbb{C}^n)$.*

Let Λ_k be an interpolation array for L^k and let Σ be any weak limit of a partial subsequence of Σ_k, then Σ is an interpolation sequence for $\mathcal{BF}^2(\mathbb{C}^n)$.

Proof. Let us start by the interpolation part. Assume that Σ is the weak limit of a partial subsequences of Σ_k that, with an abuse of notation, will be still denoted by Σ_k. Let us take a sequence of values $\{v_\sigma\}_{\sigma\in\Sigma}$, $v_\sigma \in \mathbb{C}$, with $\sum_{\sigma\in\Sigma} |v_\sigma|^2 e^{-|\sigma|^2} < \infty$. We are going to construct a sequence of functions $f_k \in \mathcal{H}(B(0, M_k))$, with $M_k \to \infty$ such that

$$\sup_k \int_{|z|<M_k} |f_k(z)|^2 e^{-|z|^2} dm(z) < \infty,$$

and for all $\sigma \in \Sigma$, $\lim_k f_k(\sigma) = v_\sigma$. Thus by a normal family argument we conclude that there is an interpolating function $f \in \mathcal{BF}^2$ with $f(\sigma) = v_\sigma$. Actually we may assume without loss of generality that, except for a finite number of points, $v_\sigma = 0$. This is harmless if

$$\limsup_{k\to\infty} \int_{|z|<M_k} |f_k(z)|^2 e^{-|z|^2} dm(z) \le C \sum_\sigma |v_\sigma|^2 e^{-|\sigma|^2}$$

with C a constant independent of the number of non-zero terms.

Since we are assuming that the metric is smooth, and we are using normalized coordinates, we can use Definition 3 and find an increasing sequence M_k, $\lim M_k \to \infty$ (but with $M_k/\sqrt{k} \to 0$) such that around x_0, $h(z)^k \simeq e^{-k|z|^2}$ for all $|z| < M_k/\sqrt{k}$.

Take some given values v_σ. We denote by $\Sigma' \subset \Sigma$ the finite set of points $\sigma \in \Sigma$ such that $v_\sigma \neq 0$. For k big enough $|\sigma/\sqrt{k}| < M_k$ for all $\sigma \in \Sigma'$. For those $\sigma \in \Sigma'$ there is an associated $\lambda_\sigma^k \in \Lambda_k$ such that $\sqrt{k}\lambda_\sigma^k \to \sigma$ because $\Sigma_k \to \Sigma$ weakly (here we are identifying the points in $\mathbb{C}^n$ and in X by its coordinate chart). Consider the interpolation problem with data $v_\sigma e_L^k$ at the points λ_σ^k, $\sigma \in \Sigma'$. By hypothesis there is a section $s \in H^0(L^k)$ such that $s_k(\lambda_\sigma^k) = v_\sigma e_L^k(\lambda_\sigma^k)$ and

$$\|s_k\|^2 \le \frac{C}{k^n} \sum_{\sigma\in\Sigma'} |v_\sigma|^2 h(\lambda_\sigma^k)^k.$$

Near x_0 we may write $s_k(z) = g_k(z) e_L^k(z)$ and thus

$$\int_{|z|\le M_k/\sqrt{k}} |g_k(z)|^2 e^{-k|z|^2} dm(z) \lesssim$$

$$\lesssim \|s_k\|^2 \le \frac{C}{k^n} \sum_{\sigma\in\Sigma'} |v_\sigma|^2 h(\lambda_\sigma^k)^k \le \frac{C}{k^n} \sum_{\sigma\in\Sigma'} |v_\sigma|^2 e^{-k|\lambda_\sigma^k|^2}.$$

The functions $f_k(z) = g_k(\sqrt{k}z)$ are holomorphic in $|z| < M_k$ and they satisfy

$$\int_{|z|<M_k} |f_k(z)|^2 e^{-|z|^2} \le C \sum_{\sigma\in\Sigma'} |v_\sigma|^2 e^{-|\sqrt{k}\lambda_\sigma^k|^2}.$$

If we let $k \to \infty$ in the right hand side of the inequality we obtain:

$$\limsup_{k\to\infty} \int_{|z|<M_k} |f_k(z)|^2 e^{-|z|^2} \lesssim \sum_{\sigma\in\Sigma'} |v_\sigma|^2 e^{-|\sigma|^2}. \qquad \square$$

8.3. The sampling part of Theorem 5 is slightly more involved. We need an approximation lemma that in an informal way shows that one can approximate locally functions in the Bargmann-Fock space by sections of L^k. More precisely, we will work with semipositive holomorphic line bundles L over a projective manifold X that have some point where the metric on L has strictly positive curvature. As we mentioned before, such bundles are big line bundles and therefore they admit a strictly positive singular metric ϕ_s that is in L^1_{loc} and it is smooth away from an analytic exceptional set $E \subset X$, see [MM07, Theorem 2.3.30].

Lemma 15. *Let L be a semipositive holomorphic line bundle over a projective manifold X with some point where the metric on L has positive curvature. We fix a point $x_0 \in X$ where it has strictly positive curvature and that is not contained in the exceptional analytic set E and consider normal coordinates around it and its corresponding frame $e(z)$. Given any function f in the Bargmann-Fock space, and any big $M > 0$, there is a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ there are global holomorphic sections $s_k(z) = f_k(z)e_k(z)$ of L^k such that in the normalized coordinates around x_0:*

$$\int_{|z|<M/\sqrt{k}} |f(\sqrt{k}z) - f_k(z)|^2 e^{-k|z|^2} dz \lesssim \frac{1}{M^2}\|f\|^2/k^n$$

and

$$\int_{|z|>M/\sqrt{k}} |s_k|^2_\phi \lesssim \frac{1}{M^2}\|f\|^2/k^n.$$

In particular $\|s_k\|^2 \simeq \|f\|^2/k^n$ for all $k \geq k_0$.

Thus, in a sense, s_k are global sections that approximate f around x.

This Lemma follows from the L^2, $\bar\partial$-estimates on line bundles for singular metrics. This is a refinement of Hörmander's theorem that is due to Demailly-Nadel, see [Ber10] where a nice exposition can be found. We will use the following theorem.

Theorem 6 (Demailly-Nadel)**.** *Let X be a projective manifold. Let L be a holomorphic line bundle over X which has a possibly singular metric ϕ_s whose curvature satisfies*

$$i\partial\bar\partial\phi_s \geq \varepsilon\omega,$$

where ω is a Kähler form. Let f be an L-valued $\bar\partial$-closed form of bidegree $(n,1)$. Then there is a solution u to the equation $\bar\partial u = f$ satisfying

$$\|u\|^2_{\omega,\phi_s} \lesssim \int_X |f|^2_{\partial\bar\partial\phi_s} e^{-\phi_s}.$$

In this statement $|f(x)|_{\partial\bar\partial\phi_s}$ is the pointwise norm on $(n,1)$ forms induced by the singular Hermitian metric in X. In particular if we have the estimate $i\partial\bar\partial\phi_s \geq M\omega$ in the support of f, then

$$\|u\|^2_{\omega,\phi_s} \lesssim \frac{1}{M}\|f\|^2_{\omega,\phi_s}. \tag{45}$$

We prove now the approximation lemma.

Proof of Lemma 15. Let χ be a cutoff function supported in a ball of radius M centered at the origin and equal to 1 in $B(0, M/2)$. We take M so big that $|\nabla\chi| \leq 4/M$. We put $\chi_k(z) = \chi(z\sqrt{k})$. We define in normal coordinates $g_k(z) = f(\sqrt{k}z)\chi_k(z)e_k(z)$. The section g_k (extended by 0 outside a neighborhood of x_0) defines a global (non-holomorphic) section with the required properties. To make it holomorphic we must correct it with the equation $\bar\partial u_k = \bar\partial g_k$ and define $s_k = g_k - u_k$. We need to make sure that the correction u_k is globally small.

One technical difficulty arises: the Hörmander estimates for the $\bar\partial$-equation deal with $(n,1)$-forms rather than $(0,1)$-forms. We can always twist the line bundle L with the canonical bundle to shift from $(0,1)$-forms to $(n,1)$-forms. In this case this is delicate because while twisting the bundle we could lose its positivity since L is only semipositive and there is no maneuvering room. For this purpose we will need to change the metric on L to make it strictly positive while preserving the estimates in the original metric. This can be achieved by averaging the original metric ϕ on L with the metric ϕ_s that is singular and strictly positive on L. That is the reason we need to work with the more sophisticated Demailly-Nadel estimates on singular metrics rather than the Hörmander estimates. More precisely, let us define a new metric $\widetilde{\phi}_k$ on L^k as follows:

$$\widetilde{\phi}_k = (k-N)\phi + N\phi_s - C, \tag{46}$$

where N and C are big constants, that do not depend on k, to be chosen. This is a well defined singular metric on L^k since $\widetilde{\phi}_k = k\phi + N(\phi_s - \phi) - C$ and the difference of two metrics $\phi_s - \phi$ is a well defined function on X.

The bundle L^k can be expressed as $L^k = K_X \otimes F_k$, where K_X is the canonical line bundle. If we endow L^k with the metric $\widetilde{\phi}_k$ and K_X with the metric inherited from the Hermitian metric on X, the curvature of F_k is

$$c(F_k) = c(\widetilde{\phi}_k) - c(K_X) = (k-N)c(\phi) + Nc(\phi_s) - c(K_X) \geq N\varepsilon\omega - c(K_X),$$

if $k > N$ and thus it has positive curvature taking N big enough, where by $c(\cdot)$ we denote here the curvature form of the corresponding line bundle or metric specified. In fact on the support of $\bar\partial g_k$ the curvature satisfies $c(F_k) \gtrsim k\omega$.

The metric ϕ_s is bounded above because it is in L^1_{loc} and it is plurisubharmonic. Thus we can take the constant C big enough in (46) in such a way that $\widetilde{\phi}_k \leq k\phi$.

The L^2 norm of $\bar\partial g_k$ with the metric $\widetilde{\phi}_k$ is comparable to the L^2 norm with respect to the metric $k\phi$ because ϕ_s is smooth on the support of $\bar\partial g_k$, thus its norm is bounded by $k^{1-n}M^{-2}\|f\|^2$. If we solve the $\bar\partial$ equation using the estimates provided by the Demailly

Nadel theorem with data that is a $(n,1)$-form with values in F_k we get a solution u_k to $\bar\partial u_k = \bar\partial g_k$ (u_k is a global $(n,0)$-form with values in F_k or equivalently a global section of L^k) with L^2 size controlled by a constant times $k^{-n}M^{-2}\|f\|^2$ as desired. A priori the norm control of u_k is with respect to $\widetilde{\phi}_k$ but as $\widetilde{\phi}_k \le k\phi$ we get the desired result. □

We proceed now to prove the sampling part of Theorem 5. Given any function f in the Fock space we take a large $M > 0$ so that

$$\int_{|z|>M} |f|^2 e^{-|z|^2} \le 0.1\|f\|^2.$$

We can construct a sequence of sections s_k such that the conclusions of the approximation lemma hold. For such s_k the sampling property of Λ_k can be applied and we have

$$\|s_k\|^2 \lesssim \frac{1}{k^n}\sum_{\lambda\in\Lambda_k} |f_k(\lambda)|^2 e^{-k\phi(\lambda)}.$$

Since all the f_k have L^2 norm very small outside the region parametrized by $|z| < M/\sqrt{k}$ which we denote by U_k the mean value property implies that

$$\|s_k\|^2 \lesssim \frac{1}{k^n}\sum_{\lambda\in\Lambda_k\cap U_k} |f_k(\lambda)|^2 e^{-k\phi(\lambda)}.$$

We recall that $k^n\|s_k\|^2 \simeq \|f\|^2$, and taking weak limits of Σ_k, implies that

$$\|f\|^2 \lesssim \sum_{|\sigma|\le M} |f(\sigma)|^2 e^{-|\sigma|^2}.$$ □

9. The one-dimensional case

In this section we return to discuss positive line bundles, and focus on the case when $\dim(X) = 1$, i.e. we are dealing with a compact Riemann surface. In this case we have a more precise result, namely a full characterization of the interpolation and sampling arrays given by Theorem 3 above.

9.1. The sampling part of Theorem 3 can be reformulated as follows.

Theorem 7. *Let Λ be a separated array and let L be a holomorphic line bundle over a compact Riemann surface X endowed with a smooth positive metric ϕ. Then Λ is a sampling array for the line bundle if and only if there is an $\varepsilon > 0$, $r > 0$ and $k_0 > 0$ such that for all $k \ge k_0$,*

$$\frac{\#(\Lambda_k \cap B(x, r/\sqrt{k}))}{\int_{B(x,r/\sqrt{k})} ik\partial\bar\partial\phi} > \frac{1}{\pi} + \varepsilon \qquad \forall x \in X, \tag{47}$$

Remark that the metric in X used to define the balls in the inequality (47) is irrelevant, since the density inequality is invariant under change of metric. We will prove this invariance in an arbitrary dimension. Assume that we have two different metrics that induce two distances d_1 and d_2 and two volumes V_1 and V_2. Suppose that (47) holds for the first metric. Denote by $\mu_k := \frac{1}{k^n}\sum_{\lambda\in\Lambda_k}\delta_\lambda$ and $\nu := (i\partial\bar\partial\phi)^n$. The hypothesis (47) (in dimension n) can we written as

$$\int_{B_1(y,r/\sqrt{k})} d\mu_k(x) \geq \Big(\frac{1}{\pi^n n!}+\varepsilon\Big)\int_{B_1(y,r/\sqrt{k})} d\nu. \tag{48}$$

We need some notation to check that (48) is invariant under change of metrics. Denote by

$$\widetilde{f}_r(z) := \frac{1}{\lambda_0(r/\sqrt{k})}\int_{B_1(z,r/\sqrt{k})} f(y)dV_1(y) = \frac{1}{\lambda_0(r/\sqrt{k})}\int_X f(y)\mathbb{1}_{B_1(y,r/\sqrt{k})}(z)dV_1(y),$$

where $\lambda_0(r)$ denotes as in [Blü90] the volume of a Euclidean ball in $\mathbb{R}^{2n}$. Thus

$$\int_X \widetilde{f}_r\, d\mu_k = \frac{1}{\lambda_0(r/\sqrt{k})}\int_X f(y)\mu_k(B_1(y,r/\sqrt{k}))dV_1(y).$$

For any $f\geq 0$, we have by (48)

$$\int_X \widetilde{f}_r\, d\mu_k \geq \big(\frac{1}{\pi^n n!}+\varepsilon\big)\int_X \widetilde{f}_r\, d\nu.$$

We choose $f := \mathbb{1}_{B_2(x,R/\sqrt{k})}$, then

$$\mathbb{1}_{B_2(x,(R-cr)/\sqrt{k})} \leq f_r \leq \mathbb{1}_{B_2(x,(R+cr)/\sqrt{k})},$$

where

$$f_r(z) := \frac{1}{V_1(B_1(z,r/\sqrt{k}))}\int_{B_1(z,r/\sqrt{k})} f(y)dV_1(y).$$

The following inequalities are now elementary:

$$\begin{aligned}
\mu_k\big(B_2(x,(R+cr)/\sqrt{k})\big) \geq \int_X f_r d\mu_k = \int_X \widetilde{f}_r d\mu_k + \int_X (f_r-\widetilde{f}_r)d\mu_k \geq \\
\Big(\frac{1}{\pi^n n!}+\varepsilon\Big)\int_X \widetilde{f}_r d\nu + \int_X (f_r-\widetilde{f}_r)d\mu_k = \\
\Big(\frac{1}{\pi^n n!}+\varepsilon\Big)\int_X f_r d\nu + \Big(\frac{1}{\pi^n n!}+\varepsilon\Big)\int_X (\widetilde{f}_r-f_r)d\nu + \int_X (f_r-\widetilde{f}_r)d\mu_k \geq \\
\Big(\frac{1}{\pi^n n!}+\varepsilon\Big)\nu(B_2(x,(R-cr)/\sqrt{k})) + \Big(\frac{1}{\pi^n n!}+\varepsilon\Big)\int_X (\widetilde{f}_r-f_r)d\nu + \int_X (f_r-\widetilde{f}_r)d\mu_k.
\end{aligned}$$

We aim to prove that

$$\mu_k(B_2(x,(R+cr)/\sqrt{k})) \geq \Big(\frac{1}{\pi^n n!}+\frac{\varepsilon}{2}\Big)\nu\big(B_2(x,(R+cr)/\sqrt{k})\big). \tag{49}$$

Clearly if R is big enough ($R >> cr$), then by (37):

$$\nu\Big(\Big\{y : \frac{R-rc}{\sqrt{k}} \leq d_2(y,x) \leq \frac{R+cr}{\sqrt{k}}\Big\}\Big) \leq \frac{\varepsilon}{4}\nu\big(B_2(x,(R-rc)/\sqrt{k})\big).$$

We still need to prove that the terms $\left(\frac{1}{\pi^n n!}+\varepsilon\right)\int_X(\widetilde{f}_r-f_r)d\nu+\int_X(f_r-\widetilde{f}_r)d\mu_k$ are negligible when compared to $\nu\big(B_2(x,(R-rc)/\sqrt{k})\big)\simeq R^{2n}/k^n$ as $k\to\infty$.

Observe that $|f_r-\widetilde{f}_r|\le K_1(r/\sqrt{k})f_r$, where $K_1(s)=\sup_X|1-V_1\big(B_1(x,s)\big)/\lambda_0(s)|$. The distortion function $K_1(s)=O(s^2)$ [Blü90, Lemma 2], and thus

$$\Big|\int_X(f_r-\widetilde{f}_r)d\nu\Big|\le K_1\Big(\frac{r}{\sqrt{k}}\Big)\int_X f_r d\nu\le K_1\Big(\frac{r}{\sqrt{k}}\Big)\nu\Big(B\Big(x,\frac{R+cr}{\sqrt{k}}\Big)\Big)\lesssim\frac{1}{k^{n+1}}.$$

We assume that Λ is separated, thus $\mu_k\big((x,\frac{R+cr}{\sqrt{k}})\big)\lesssim\frac{R^{2n}}{k^n}$, and therefore

$$\Big|\int_X(f_r-\widetilde{f}_r)d\mu_k\Big|\le K_1\Big(\frac{r}{\sqrt{k}}\Big)\int_X f_r d\nu\le K_1\Big(\frac{r}{\sqrt{k}}\Big)\mu_k\Big(B\Big(x,\frac{R+cr}{\sqrt{k}}\Big)\Big)\lesssim\frac{1}{k^{n+1}},$$

and if we take k big enough, we have proved (49) and the invariance of the density condition under changes of metric follows.

9.2. We proceed now to the proof of Theorem 7. We start by proving that under the density hypothesis (47) the array Λ is sampling. We will initially prove that the array is L^∞-sampling.

Definition 6. We say that a separated array $\Lambda=\{\Lambda_k\}$ is an L^∞-*sampling array* if there is k_0 and a constant $0<C<\infty$ such that, for each $k\ge k_0$ and any section $s\in H^0(L^k)$ we have

$$\sup_{x\in X}\big|s(x)\big|\le C\sup_{\lambda\in\Lambda_k}|s(\lambda)|.$$

If this were not true then for infinitely many k's there will be $s_k\in H^0(L^k)$ and points $x_k\in X$ such that

$$\sup_X|s_k|=|s_k(x_k)|=1,$$

and

$$\sup_{\lambda\in\Lambda_k}|s_k(\lambda)|=o(1).$$

We take normal coordinates around x_k, see Definition 3, and we consider as before arrays $\Lambda_k\subset X$ and $\Sigma_k\subset B(0,M_k)$ the dilated sequences in $\mathbb{C}$. Since Σ_k are separated there is a subsequence converging weakly to Σ that for simplicity we keep denoting by Σ_k. The hypothesis implies that

$$\frac{\#\Sigma\cap B(y,r_0)}{r_0^2}\ge\Big(\frac{1}{\pi}+\varepsilon\Big),$$

the balls $B(y,r_0)$ are standard balls in $\mathbb{C}$ because we may choose a metric in X such that when rescaling around x by the normal coordinates it converges to the Euclidean metric in $\mathbb{C}$. By a theorem of Seip, Σ is sampling for the space of functions $\mathcal{BF}^\infty$ consisting of entire functions such that $\sup|f|e^{-|z|^2}<+\infty$. On the other hand we may extract a converging subsequence of functions f_k that represent the sections s_k in normal coordinates to $f\in\mathcal{BF}^\infty$ such that $|f(0)|=1$ and $f|_\Sigma=0$ and this is a contradiction with the fact that Σ is sampling for $\mathcal{BF}^\infty$.

Once we know that Λ is L^∞ sampling it is possible to argue as with the Fekete points that $\{\Lambda_{(1+\varepsilon)k}\}$ is L^2 sampling.

Proposition 1. If $\Lambda = \{\Lambda_k\}$ is L^∞ sampling then $\{\Lambda_{(1+\varepsilon)k}\}$ is L^2 sampling.

Proof. We know by hypothesis that for any $s \in H^0(L^k)$, $\sup_X |s| \le C \sup_{\Lambda_k} |s(\lambda_k)|$. In this case it is elementary to check that $\{\Lambda_{(1+\varepsilon)k}\}$ is also L^∞ sampling. For any $s \in H^0(L^k)$, and $y \in X$ we define the section

$$p_y(x) = s(x) \otimes \left[\frac{\Phi_y^{(\varepsilon/2)k}(x)}{|\Pi_{(\varepsilon/2)k}(y,y)|} \right]^2 \in H^0(L^{(1+\varepsilon)k})$$

Let us take now $y \in X$ to be a point where $|s|$ attains its maximum. Then

$$\sup_X |s| = |s(y)| = |p_y(y)| \le C \sup_{\Lambda_{(1+\varepsilon)k}} |p_y(\lambda)| \le C \sup_{\Lambda_{(1+\varepsilon)k}} |s(\lambda)|. \tag{50}$$

Moreover for any $z \in X$, since Λ is sampling,

$$|s(z)| = |p_z(z)| \lesssim \sup_{\Lambda_{(1+\varepsilon)k}} |s(\lambda)| \left| \frac{\Phi_z^{(\varepsilon/2)k}(\lambda)}{|\Pi_{(\varepsilon/2)k}(z,z)|} \right|^2 \le \sum_{\Lambda_{(1+\varepsilon)k}} |s(\lambda)| \left| \frac{\Phi_z^{(\varepsilon/2)k}(\lambda)}{|\Pi_{(\varepsilon/2)k}(z,z)|} \right|^2$$
$$= \sum_{\Lambda_{(1+\varepsilon)k}} |s(\lambda)| \left| \frac{\Pi_{(\varepsilon/2)k}(z,\lambda)}{|\Pi_{(\varepsilon/2)k}(z,z)|} \right|^2.$$

Recall that $|\Pi_{(\varepsilon/2)k}(z,z)| \simeq \varepsilon k$. Thus if we integrate both sides, we get

$$\int_X |s(z)| \lesssim \frac{1}{\varepsilon k} \sum_{\Lambda_{(1+\varepsilon)k}} |s(\lambda)| \tag{51}$$

If we interpolate between (50) and (51) we obtain

$$\int_X |s(z)|^2 \lesssim \frac{1}{\varepsilon k} \sum_{\Lambda_{(1+\varepsilon)k}} |s(\lambda)|^2,$$

as stated.

Finally, since the hypothesis of Theorem 7 is an open condition we can conclude that actually $\{\Lambda_{(1-\varepsilon)k}\}$ is L^∞-sampling and therefore Λ is L^2-sampling.

We turn now to the necessity of the density condition. We assume that Λ is a sampling array. We know already by Corollary 2 that the density of Λ is bigger or equal than a critical level. We need a strict inequality. We prove now that if Λ is a sampling array there is a $\varepsilon > 0$ such that $\{\Lambda_{(1-\varepsilon)k}\}$ is still and L^2-sampling array.

We know by Theorem 5 than any weak limit $\Sigma \in W(\Lambda)$ is a sampling sequence in $\mathcal{BF}^2(\mathbb{C})$. Thus by the description of sampling sequences for such spaces obtained in [SW92], the lower Beurling density $D^-(\Sigma) > 1$. We will prove that under this circumstances there is a $\varepsilon > 0$ such that $\{\Lambda_{(1-2\varepsilon)k}\}$ is L^∞-sampling. We argue by contradiction. Suppose not, then, for any n there are sections $s_k \in H^0(L^k)$ such that $\|s_k\|_\infty = 1$ and $\|s|_{\Lambda_{(1-1/n)k}}\|_\infty = o(1)$

when k is very big. If we fix n and by passing to a subsequence in normal coordinates around the points x_k where $|s_k|$ takes its maximum value, we construct functions $f_n \in \mathcal{BF}^\infty(\mathbb{C})$ of norm one such that $f_n(0) = 1$ and $f_n|_{\Sigma_n} \equiv 0$, where Σ_n is a weak limit of a subsequence of $\Lambda_{(1-1/n)k}$ as $k \to \infty$ in normal coordinates scaled appropriately. We take another subsequence of the functions f_n and of the separated sequences Σ_n in such a way that Σ_n converge weakly to Σ, $f_n \to f$ and $f \in \mathcal{BF}^\infty(\mathbb{C})$ of norm one, $f(0) = 1$, $f|_\Sigma \equiv 0$ and $\Sigma \in W(\Lambda)$. This is a contradiction since Σ has $D^-(\Sigma) > 1$.

We have proved that $\{\Lambda_{(1-2\varepsilon)k}\}$ is L^∞-sampling. We finish the proof by observing that by Proposition 1 this implies that $\{\Lambda_{(1-\varepsilon)k}\}$ is L^2-sampling.

9.3. We provide now a characterization for the interpolation arrays.

Theorem 8. *Let Λ be a separated array and let L be a holomorphic line bundle with a smooth positive metric ϕ over a compact Riemann surface X. Then Λ is an interpolation array for the line bundle if and only if there is an $\varepsilon > 0$, $r > 0$ and k_0 such that for all $k \geq k_0$,*

$$\frac{\#(\Lambda_k \cap B(x, r/\sqrt{k}))}{\int_{B(x,r/\sqrt{k})} ik\partial\bar{\partial}\phi} < \frac{1}{\pi} - \varepsilon \qquad \forall x \in X, \tag{52}$$

Remark that the density condition (52) is invariant under change of metric, which can be shown in a simiar way as we did above for the condition (47). We will first check that condition (52) implies that Λ is interpolating. We start by the following reduction.

Proposition 2. Let Λ be separated. If there is a $C > 0$ such that for every $k \geq k_0$ and every $\lambda \in \Lambda_k$ there is a section $s_\lambda \in H^0(L^k)$ with

(i) $|s_\lambda(\lambda)| = 1$;
(ii) $\sup_\lambda \sum_{\lambda' \neq \lambda} |s_\lambda(\lambda')| < 1/2$;
(iii) $\sup_{\lambda'} \sum_{\lambda \neq \lambda'} |s_\lambda(\lambda')| < 1/2$;
(iv) $\| \sum c_\lambda s_\lambda \|_2^2 \leq C k^{-1} \sum |c_\lambda|^2$;

then Λ is an interpolation array.

Proof. Let $\ell^2(\Lambda_k)$ be endowed with the norm $\|v\|^2 := k^{-1} \sum_{\Lambda_k} |v_\lambda|^2$. We consider the following two operators. The first is the restriction operator $R\colon H^0(L^k) \to \ell^2(\Lambda_k)$ defined as $R(s) = \{s(\lambda)\}$. It is bounded from $H^0(L^k)$ endowed with the L^2 norm by the Plancherel-Pólya inequality (Lemma 2) since Λ is separated and its norm $\|R\|$ depends only on the separation constant of Λ.

The second operator is $E\colon \ell^2(\Lambda_k) \to H^0(L^k)$ defined as $E(\{v_\lambda\}) = \sum \langle v_\lambda, s_\lambda(\lambda)\rangle s_\lambda(x)$. It is bounded clearly by properties (i) and (iv). If we prove that $RE\colon \ell^2 \to \ell^2$ is invertible with the norm of the inverse $\|(RE)^{-1}\| \leq C$ bounded independently of k, then clearly Λ is interpolating, because any values $\{v_\lambda\}$ are attained by the section $s = E(RE)^{-1}(\{v_\lambda\})$ with size control.

But the conditions (i),(ii),(iii) imply that the operator $RE - \mathrm{Id}\colon \ell^2(\Lambda_k) \to \ell^2(\Lambda_k)$ by Schur's Lemma has norm bounded by $1/2$. Thus RE is invertible.

To finish the proof of the sufficiency of (52) we are going to construct the sections as in the Proposition 2. Around any given $\lambda \in \Lambda_k$ we can consider normal coordinates. Since by hypothesis the density is small the corresponding sequence Σ_k is an interpolation sequence for the $\mathcal{BF}^2$ space in $\mathbb{C}$. Actually since the separation constant is uniform and the density is uniform then by a theorem of Seip and Walsten the constants of interpolation for all the sequences Σ_k around any point $\lambda \in \Lambda_k$ will be uniformly bounded, for $k \geq k_0$. Thus we can construct functions f_λ^k such that $|f_\lambda^k(0)| = 1$, $\|f_\lambda^k\| \leq C$ and $f_\lambda^k(\sigma) = 0$ for all $\sigma \in \Sigma_k \setminus 0$. Now we can construct a global section $g_\lambda \in H^0(L^k)$ such that near λ, $g_\lambda(z)$ is very close to $f_\lambda(z)^k e_L^k(z)$, where $e_L^k(z)$ is the local frame around λ used for the normal coordinates.

In order to do this we define $g_\lambda = \chi_{\lambda,k}(z) f_\lambda^k(z) e^k(z) + u$, where $\chi_{\lambda,k}$ is a cutoff function around λ such that $g_\lambda(z) = 0$ if $d(z,\lambda) > 2C/\sqrt{k}$ and $g_\lambda(z) = 1$ if $d(z,\lambda) < C/\sqrt{k}$ and u is the solution to the equation $\bar\partial u = \bar\partial \chi_{\lambda,k} f_\lambda^k(z) e^k(z)$ provided by the Hörmander theorem. This theorem ensures that $\|u\|^2 \leq \varepsilon$, provided that the cutoff constant C is big enough.

This is not enough if we want the decay needed in the Proposition 2, in particular in the items (ii),(iii) and (iv). We are again going to use the extra freedom that we have because the hypothesis is an open condition. We could have taken f_λ^k such that $\int |f_\lambda^k|^2 e^{-(1-\varepsilon)|z|^2} < +\infty$ and in this case we could have constructed $g_\lambda \in H^0(L^{(1-\varepsilon)k})$ such that

$$|g_\lambda(\lambda)| = 1, \qquad \|g_\lambda\|^2 \leq C/k, \qquad k^{-1} \sum_{\lambda' \neq \lambda} |g_\lambda(\lambda')|^2 \leq \varepsilon.$$

and we can take in the construction $\varepsilon > 0$ as small as we want without affecting the K.

We define $s_\lambda(x) = g_\lambda(x) \otimes \left[\frac{\Phi_\lambda^{\varepsilon k/2}(x)}{|\Pi_{(\varepsilon/2)k}(\lambda,\lambda)|}\right]^2$ and using (9) it is easy to check that

$$\sup_X |\sum_{\Lambda_k} c_\lambda s_\lambda(x)| \lesssim \sup_{\Lambda_k} |c_\lambda| \quad \text{and} \quad \int_X |\sum_{\Lambda_k} c_\lambda s_\lambda(x)| \lesssim k^{-1} \sum_{\Lambda_k} |c_\lambda|.$$

Thus by interpolation we get $\|\sum c_\lambda s_\lambda\|_2^2 \leq C k^{-1} \sum |c_\lambda|^2$ which gives (iv). Finally (ii),(iii) can be checked in a similar way.

9.4. We turn now to the neccesity of the density condition (52). We need to check that the density condition that we proved that was necessary in Corollary 2 is actually a strict density condition. As a technical tool to prove the necessity of the strict inequality we need to work with L^1 interpolating arrays. The definition is the following:

Definition 7. We say that a separated array $\Lambda = \{\Lambda_k\}$ is an L^1-*interpolation array* if there is k_0 and a constant $0 < C < \infty$ such that, for each $k \geq k_0$ and any set of vectors $\{v_\lambda\}_{\lambda \in \Lambda_k}$ (each v_λ is an element of the fiber of λ in L^k) there is a section $s \in H^0(L^k)$ such that

$$s(\lambda) = v_\lambda, \quad \lambda \in \Lambda_k,$$

and

$$\int_X |s(x)| \leq C k^{-1} \sum_{\lambda \in \Lambda_k} |v_\lambda|. \tag{53}$$

On each level $k \geq k_0$, the best constant C_k such that (53) holds for all $s \in H^0(L^k)$ that interpolate the prescribed values, is called the constant of interpolation at level k. Of course Λ is an interpolation array if all the constants $\{C_k\}$ are uniformly bounded. There is an alternative way of computing C_k by duality.

Proposition 3. The constant of L^1 interpolation at level k is comparable to the smallest constant A_k such that

$$\sup_{x \in X} k^{-1} \Big| \sum_{\Lambda_k} \langle a_\lambda, \Pi_k(x,\lambda)\rangle \Big| \leq A_k \sup_{\Lambda_k} |a_\lambda|,$$

where $\{a_\lambda\}_{\lambda \in \Lambda_k}$ are arbitrary elements on the fiber of λ in L^k.

Proof. This is standard and follows from the fact that the Bergman kernel decays very fast away from the diagonal (9). Thus the Bergman projection from sections of L^k endowed with the L^p norm to holomorphic sections endowed with the L^p norm is bounded for all $p \in [1,\infty]$, and the dual space of $H^0(L^k)$ with the L^1 norm is the space $H^0(L^k)$ endowed with the supremum norm.

It will be convenient to compare interpolating arrays in L^1 and in L^2 and we will use the following proposition

Proposition 4. If $\Lambda = \{\Lambda_k\}$ is an L^1 interpolation array then $\{\Lambda_{(1-\varepsilon)k}\}$ is an L^2 interpolation array.

Proof. If Λ is an L^1 interpolation array then for each $\lambda \in \Lambda_{(1-2\varepsilon)k}$ we can build a "Lagrange type" section $s_\lambda \in H^0(L^{(1-2\varepsilon)k})$ such that $|s_\lambda(\lambda)| = 1$, $|s_\lambda(\lambda')| = 0$ for all $\lambda' \in \Lambda_{(1-2\varepsilon)k} \setminus \{\lambda\}$, and $\|s_\lambda\|_{L^1} \leq C/k$. Then by the sub-mean value property (10) we obtain $\sup_X |s_\lambda(x)| \leq Ck\|s_\lambda\|_{L^1} \leq C$. Thus we can use the same argument as in Theorem 6 and we prove that $\{\Lambda_{(1-\varepsilon)k}\}$ is an L^2-interpolation array.

The proof of strict inequality (52) follows once we establish the following

Proposition 5. Assume that $\dim(X) = 1$. Let Λ be an L^2-interpolating array. There is a $\varepsilon > 0$ such that $\{\Lambda_{(1+\varepsilon)k}\}$ is L^2-interpolating.

Proof. We know by Theorem 5 than any weak limit $\Sigma \in W(\Lambda)$ is an interpolating sequence in $\mathcal{BF}^2(\mathbb{C})$. Thus by the description of interpolating sequences for such spaces obtained in [SW92] and [Sei92], the upper Beurling density $D^+(\Sigma) < 1$. We will prove that under this circumstances there is a $\varepsilon > 0$ such that $\{\Lambda_{(1+2\varepsilon)k}\}$ is L^1-interpolating.

We argue by contradiction. Suppose not, then, for any n the interpolation constants at level k, C_k for $\Lambda_{(1+1/n)k}$ blow up. Thus by the dual description of C_k given in Proposition 3 we can find sequences of vectors $\{a_\lambda\}_{\lambda \in \Lambda_{(1+1/n)k}}$ such that $\sup_{\Lambda_{(1+1/n)k}} |a_\lambda| = 1$ and

$$\sup_{x \in X} k^{-1} \Big| \sum_{\Lambda_{(1+1/n)k}} \langle a_\lambda, \Pi_k(x,\lambda)\rangle \Big| = o(1), \qquad \text{as } k \to \infty.$$

If we fix n and by passing to a subsequence in normal coordinates around the points λ_k^* where $|a_\lambda|$ takes its maximum value, we can extract a subsequence of $\Lambda_{(1+1/n)k}$ as $k \to \infty$ in normal coordinates that scaled appropriately converges weakly to the separated sequence $\Sigma_n \subset \mathbb{C}$. Moreover, after taking a subsequence again, there are subsequences $a_\lambda^k \to a_\sigma^n$ for all $\sigma \in \Sigma_n$. We are going to prove that in this case

$$f_n(z) := \sum_{\sigma\in\Sigma_n} a_\sigma^n e^{\bar\sigma z - 1/2|\sigma|^2} \equiv 0,$$

with $|a_0| = 1$, and $\sup_\sigma |a_\sigma^n| \le 1$.

To see this we will prove that for any $\varepsilon > 0$, $\sup_{|z|<1} |f_n(z)| e^{-|z|^2} \le \varepsilon$.

Observe that since Σ_n is separated and $|a_\sigma^n| \le 1$, the decay of the Bargmann-Fock kernel away from the diagonal implies that for any $\varepsilon > 0$ it is possible to find $R > 0$ such that

$$\sup_{|z|<1} \Bigg| \sum_{\sigma^n\in\Sigma_n, |\sigma|>R} a_\sigma^n e^{\bar\sigma z - \frac12|\sigma|^2} Bigg| e^{-\frac12|z|^2} \le \varepsilon$$

So we only need to care about the points $\sigma \in \Sigma_n \cap D(0,R)$. But this we can deal with because, with certain abuse of notation,

$$k^{-1} \sum_{\lambda\in\Lambda_{(1+1/n)k}\cap D(\lambda_k^*, R/\sqrt{k})} \langle a_\lambda, \Pi_k(x,\lambda)\rangle \longrightarrow \frac{1}{\pi} \sum_{\sigma\in\Sigma_n, |\sigma|<R} a_\sigma^n e^{\bar\sigma z - \frac12|\sigma|^2 - \frac12|z|^2}$$

uniformly in $|z| < 1$ when the section is expressed in appropriately scaled normalized coordinates around λ_k^*. This property is usually called the universality of the reproducing kernels and it is proved in [BSZ00, Theorem 3.1]. Actually in [BSZ00] it is assumed that X is equipped with the metric induced by the curvature of the line bundle, but since the condition (52) is invariant under change of metric we may also assume that this is the case. The sum

$$k^{-1} \Bigg| \sum_{\lambda\in\Lambda_{(1+1/n)k}\cap D(\lambda_k^*, R/\sqrt{k})} \langle a_\lambda, \Pi_k(x,\lambda)\rangle \Bigg| \le \varepsilon,$$

if k is big enough because the global sum for all $\lambda \in \Lambda_{(1+1/n)k}$ converges to zero and the terms λ outside the ball $D(\lambda_k^*, R/\sqrt{k})$ are small when R is big because $\Lambda_{(1+1/n)k}$ is separated and there is a fast decay of the normalized reproducing kernel away from the diagonal (9).

Finally we have proved that $f_n \equiv 0$ and $\{a_\sigma^n\}$ is uniformly bounded sequence with $a_0 = 1$. We can take a subsequence as $n \to \infty$ and we find $\Sigma_n \to \Sigma$ weakly and there is a bounded sequence $\{a_\sigma\}$ such that $f(z) = \sum a_\sigma e^{\bar\sigma z - |\sigma|^2/2} \equiv 0$ and $|a_0| = 1$. This is clearly not possible since $\Sigma \in W(\Lambda)$ and thus it has $D^+(\Lambda) < 1$, thus Λ is interpolating for the L^1 Bargmann-Fock space and this means that by duality

$$\sup_\sigma |a_\sigma| \le C \sup_{z\in\mathbb{C}} \Big| \sum a_\sigma e^{\bar\sigma z - |\sigma|^2/2} \Big| e^{-|z|^2}.$$

We have thus proved that $\{\Lambda_{(1+2\varepsilon)k}\}$ is L^1-interpolation. We finish the proof by observing that by Proposition 4 this implies that $\{\Lambda_{(1+\varepsilon)k}\}$ is L^2-interpolation.

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