On Knots and Temporality

ABSTRACT

In a class of quantum gravity approaches it is indicated that our observable world emerges out of a fundamental structure that appears highly resistant to any clear spatial or temporal interpretation. In this work we are examining an analogue quantum system that appears to simulate such an unintuitive structure: the emergence of the so called *topolgoical* phase of matter depicted by the Chern-Simons gauge theory. By investigating the proposed analogy from the lens of category theory, we offer a clear interpretation of the way in which space and time act at the fundamental level. In this way, we put forward an analogical argument in support of a relational view of space and time.

Contents

1 Introduction

The mathematical frameworks linked with a class of quantum gravity research programs including loop quantum gravity (Rovelli, 2010), group field theory (Gielen et al., 2013), and causal set theory (Bombelli et al., 1987) point to a discrete and non-geometric structure at the fundamental level, from which one can recover the continuous and geometric fabric of space and time. This picture gives rise to a conceptual difficulty in regards to the existence at the fundamental level: to exist is to be situated within a *region* of space and time[1], which does not appear to be intelligible in a structure without a metric. The typical move to bypass this difficulty is to resort to emergence (Lam and Wüthrich, 2021); locality - or existence in an open region of space and time - is an emergent phenomenon. The emergence of space and time regions from discrete, non-geometric structures calls for not only a mathematically consistent framework, but also a coherent conceptual understanding. One way to make sense of emergence conceptually (depending on the particular theory at hand) is to treat a region of space and time not as an entity but merely as a collection of relations that *can* be codified within the underlying non-geometric setting as well.

[1] Here by existence we mean physical existence linked with concrete entities.

In this work we are examining an analogue quantum system that appears to mimic such non-geometric setting: the emergence of the so called *topological* phase of matter depicted by the Chern-Simons gauge theory (Witten, 1989; Nayak et al., 2008), where metrical concepts such as distance and duration dissolve as the observable properties of the theory are insensitive to smooth deformations of the background space and time. Given the functorial correspondence between Chern-Simons dynamics and knot theory (Kauffman, 2012), we interpret time at the topological phase as a collection of relations encoded in the topology of a knot. A knot encodes a non-geometric relation as it is insensitive to smooth deformations. Put differently, what characterises a knot is its tangled profile, not its length, curvature, nor what it is made out of. A knot when understood as a category, can be deconstructed as a collection of objects *e.g.* distinguishable configurations, with morphisms (gluing edges) between them. From the aforementioned correspondence it follows that at the topological phase temporality can be reduced to a collection of morphisms between distinguishable states. As the system transforms to a geometric phase - at higher energy scales - we argue that the function of time, namely, the connection across distinguishable states remains the same, and what accounts for the appearance of a metric is the outburst of a continuum of distinct states that were rendered indistinguishable at the topological phase due to the dominance of long range patterns of entanglement. In other words, the Chern-Simons gauge theory acts as an analogue system that mimics a non-geometric and discrete structure, which undergoes a transformation to an ordinary geometric and continuous field theory at higher energy scales. In this mode of analogy, the target system - the non-spatiotemporal fundamental fabric - is mimicked by the topological phase of matter. In this way, we provide an argument from analogy (Bartha, 2010; Dardashti et al., 2017) in support of a relational view of time: time acts as a relation across distinguishable states, and it is this functional role of time that closes the metaphysical gap between a geometric and a non-geometric setting. Having construed time as a mere relation between two distinguishable states, we provide an interpretation of atemporality in terms of indistinguishablity across states. We elaborate this point further using the machinery of category theory: if two objects have the same morphisms with respect to all other objects in a category, then the two objects are *the same*, and thus no temporal relation between them can be established.

This paper is organised as follows: in section 2, we present a brief introduction to the topological phase of matter, and its mathematical description in terms of Chern-Simons gauge theory as an effective field theory of which the observable properties are insensitive to smooth deformations of spacetime. In section 3, on the ground of the functorial correspondence between topological field theories and knots, we offer an analogical argument in support of the relational view on time. In section 4, Using the mathematical machinery of category theory, we offer a relational interpretation of atemporality in terms of distinctiveness, and finally in the section 5 we conclude.

2 Topology: a phase of matter

Theoretical physics within its current paradigm is a framework in which matter and its behaviour is studied in a geometric mode. Within the theoretical models that explain natural phenomena, space and time are treated as an entity that lends itself to metrical concepts such as distances and durations. Put differently, modern physics depicts a world that lives in a geometric phase, of which the observable properties depend on space and time in a metrical manner. However, there exist situations in which certain phenomena appear to break such geometric dependencies and exhibit a purely *topological* behaviour. A basic instance of such phenomena is the Aharonov-Bohm effect (Aharonov and Bohm, 1959) where a magnetic field contained inside an impenetrable tube - a flux tube - can have a measurable influence on a charged particle that orbits outside the flux tube, which does not depend on the geometric details of the orbit. More concretely, when the charged particle is moved (adiabatically) around the flux tube the wave function of the particle acquires a phase shift that is insensitive to any deformation of the particle's path, and only depends on the topology of the trajectory - the winding number (see figure 1). This shift in dependency on geometry to topology can occur in many-body quantum systems at thermodynamic limit, which induces what is known as the topological phase of matter.

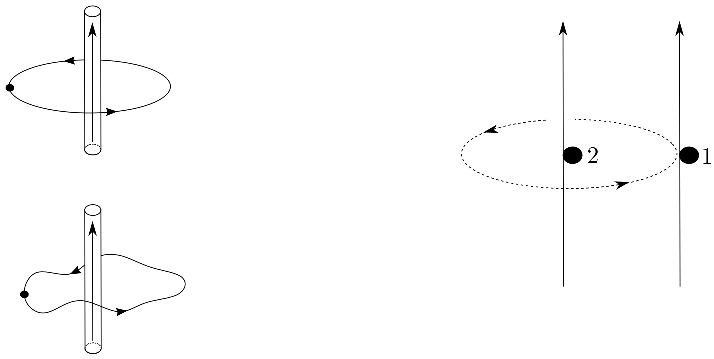

Figure 1: The diagram on the left illustrates the Aharonov-Bohm effect where the phase shift of the charged particle induced by the moving of the particle around a flux tube is insensitive with respect to the geometry of the particle's trajectory. The diagram on the right illustrates the motion (braiding) of the charge-flux composites around each other.

In a situation where the behaviour of a many-body quantum system is robust to any local perturbation, and is only responsive to global effects, it is typically the case that the system dynamic has topological significance. More precisely, a system is said to be in a topological phase when at low energy and long distance scales[2] its observable properties are invariant under local smooth deformations (diffeomorphisms) of the space time manifold in which the system is situated (Nayak et al., 2008; Witten, 2016). Therefore, when a quantum system is in a topological phase there exists sets of distinct states that appear indistinguishable to any local observation, and these degeneracies can not be lifted by local perturbations of the system. In other words, measurements of local observables can not reveal in which state the system happens to be. Global effects on the other hand can induce measurable changes by driving the system from one state to another distinct one.

Such collective effects that one observes in the topological phase of matter are quite distinct and are of a more subtle nature than those occurring in traditional phases since the former relies on *long-range entanglement* across the many-body system and not on conventional correlations. That is, for a system at the topological phase despite that the correlation functions decay exponentially, the entanglement between different parts of the system persists, and this results in the global effects with remarkable consequences. For instance, the aforementioned ground state degeneracy of a system at the topological phase is induced by the topology of the space in which the system is embedded. In this way, any information regarding the system is encoded in

[2]Low energy and short distance means that deformation invariance is only violated by terms that vanish as $\exp(-\Delta/T)$, where Δ denotes some energy gap in the system. Therefore, generally systems at topological phase admit energy gap that separates (degenerate) ground states from the lowest excited states.

the topology of the ambient space, which can not be retrieved by local observation. Therefore, in a topological phase, any measurable change in the state of affairs entails a collective effect - a catastrophe - that generates a topological phase transition.

As the behaviour of the many-body quantum system at the thermodynamic limit in a topological phase does not admit any local degrees of freedom[3] and only lends itself to collective analysis, an emergent field theoretic perspective becomes essential in providing an effective description of the system dynamics. A topological field theory (Atiyah, 1988) as a field theory in which observable properties such as expectation values for processes are independent under smooth deformations can be thought of as a natural framework to study topological phases of matter. More concretely, in a topological field theory any local time evolution is trivial *i.e.* any local dynamics is proportional to identity operator, and the only nontrivial dynamics is when some topological property of the system changes *e.g.* when the system undergoes a topological phase transition. Therefore, a topological phase of matter can be interpreted as a situation where the low-energy effective description of the underlying many-body quantum system furnishes a topological field theory. The canonical example of a topological field theory is the Chern-Simons theory: an emergent gauge theory that codifies the behaviour of a quantum system situated in a (2+1) dimensional spacetime manifold.

In order to obtain an intuitive interpretation of the function that the Chern-Simons term plays let us have a deeper look at the Aharonov-Bohm effect. In the original experiment, the acquired phase shift can be computed in terms of the expectation value of the observable $W = e^{iq\oint dl_\mu A^\mu}$, which can be expressed as a simple path integral:

$$\langle W \rangle = \sum_{\text{loops}} e^{iS_0} W = \sum_{\text{loops}} e^{iS_0} e^{iq\oint dl_\mu A^\mu} \tag{1}$$

where S_0 denotes the action corresponding to a free charged particle, and q and A denote the charge and the gauge field associated with flux tube respectively. W encodes the process of transporting the charged particle on a closed trajectory enclosing the flux tube, the expectation value of which is precisely the phase shift. Now in order to extend this topological behaviour to a system with large number of particles, one can imagine a two-dimensional system comprising multiple charged particles, where to each particle is attached an infinitely thin flux tube perpendicular to the plane. Now if we simply move one particle (1) around another one (2) the entire state of the many-body system acquires an Aharonov-Bohm phase shift *i.e.* the phase acquired by moving of charge (1) around the flux tube attached to (2) plus moving the flux tube attached to (1) around the charge (2) (see figure 1). This phase shift however, is not reducible to the effect of the original electromagnetic gauge field A, but instead, it is induced by a distinct gauge field $\mathcal{A}$ that emerges as a collective degree of freedom[4]. This is because the isolation of magnetic flux tubes to infinitely narrow regions necessitates the many-body system to be in low energy state with long-range patterns of entanglement that leads to the deformation of A as a local gauge to $\mathcal{A}$ as a collective gauge field (Nayak et al., 2008):

$$\langle W \rangle = \int_M \mathcal{D}\mathcal{A} e^{i\mathcal{S}[\mathcal{A}]} W = \int_M \mathcal{D}\mathcal{A} e^{i\mathcal{S}[\mathcal{A}]} e^{iq\oint dl_\mu \mathcal{A}^\mu} \quad , \quad \mathcal{S}[\mathcal{A}] = k/4\pi \int_M \mathcal{A} \wedge dA$$

[3]Here by a degree of freedom we refer to any dynamical variable or observable the perturbation of which induces a measurable change in the state of the system.

[4]Here the gauge field $\mathcal{A}$ is not a local degree of freedom, but rather emerges from the collective behaviour of the underlying many-body system. Ideally $\mathcal{A}$ can be thought of as analogous to the way quasi-particles such as phonon degrees of freedom emerge from the collective motion of particles

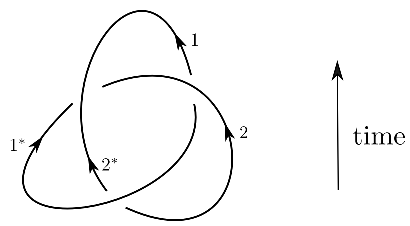

Figure 2: The diagram depicts the history of a two-dimensional system comprising two pairs of quasi-particles that are created at an initial time (state preparation), moved around each other (braiding operation), and finally annihilated (state measurement.)

Here $\mathcal{S}[\mathcal{A}]$ denotes the Chern-Simons action of the emergent gauge field $\mathcal{A}$, which encodes precisely a field theoretic description of the flux-charge binding process and nothing more, and where the expectation value for a certain braiding process $W = e^{iq\oint dl_\mu \mathcal{A}^\mu}$ is given by the path integral $\langle W \rangle$. Furthermore, the path integral in the absence of any process W encodes the system's topological degeneracy that is associated with (or induced by) the topology of the ambient 3-space M. Therefore, the two-dimensional many-body system composed of flux-charge composites is a basic example of a system, the effective behaviour of which is purely topological. This basic example of Aharonov-Bohm effect induced by braiding flux-charge composite around each other is an instance of an *abelian* Chern-Simons theory where $\mathcal{A}$ is a $U(1)$ gauge.

Let us have a look at the general *non-abelian* Chern-Simons action:

$$\mathcal{S}_{CS}[\mathcal{A}] = \frac{k}{4\pi} \int_M \mathrm{Tr}(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}). \tag{2}$$

This action[5], in a similar fashion to the abelian case, encodes the information -topological invariant- associated with the 3-manifold M:

$$Z[M] = \int_M \mathcal{D}\mathcal{A} e^{i\mathcal{S}_{CS}[\mathcal{A}]}. \tag{3}$$

In addition to the free field term one might introduce a suitable class of gauge invariant observables that are equally insensitive to the metric of the underlying manifold. There exists a class of observables that do not require a choice of a metric, the expectation value of which encodes some topological information. These are called Wilson loops W that in a similar fashion to the Aharonov-Bohm effect can be constructed in terms of the holonomy of the gauge field $\mathcal{A}$:

$$W_j = \mathrm{Tr}(P e^{iq\oint_j dl_\mu \mathcal{A}^\mu}) \tag{4}$$

where P denotes path ordering as the gauge field A is non-abelian, and the index j denotes the (equivalent class of) knot generated by the holonomy. The Wilson loop observable can be interpreted as a process in which pairs of collective degrees of freedom such as quasi-particles/quasi-holes[6] (analogous to charge-flux composites) are created, moved around one another, and finally annihilated on a 2-dimensional surface (see Figure 2.) As one can see the

[5]In the abelian case $\mathcal{A}$ is a vector of numbers for which Tr is trivial and $\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}$ vanishes.

[6]Such quasi-particles are referred to as (non-abelian) *anyons* which exhibit exotic statistical properties. Unlike bosons and fermions the exchange of two identical anyons can induce any unitary operation on the original state.

history of such a process on a plane generates a knot embedded within a 2+1 dimensional space. W_j is a canonical example of a topological process since its expectation value, $\langle W_j \rangle = \int_M \mathcal{D}\mathcal{A} e^{iS_{CS}[\mathcal{A}]} W_j$, does not depend on how fast the particles moved around each other or on how long their trajectory might have been, and instead it is solely determined by the topology of the knot generated by the history of the interaction. Therefore, the Chern-Simons theory depicts the behaviour of matter at the topological phase where there exist neither any local degrees of freedom nor any notion of local time evolution.

3 Temporality: a relational view

A knot can be thought of as a tangled loop in a three dimensional space. In twentieth century mathematician developed a deep framework to analyse the topological properties of knots, and in particular to determine whether two knots are distinct. In 1983 Jones (Jones, 1985) discovered a simple way to compute a unique number, the Jones polynomial, for any given knot. The Kauffman bracket (Kauffman, 1987), a variant of the Jones polynomial discovered by L. Kauffman in 1987, computes the knot invariant by a recursive application of the following skein relation that relates the original knot to a pair of knots with fewer crossings: $\times = q \asymp + q^{-1} \,)($. See figure (3) where this relation is applied to a trefoil knot 3_1

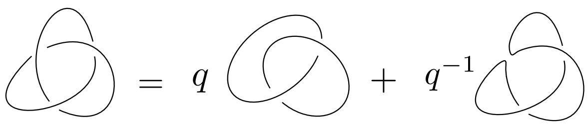

Figure 3: Kauffman brackets applied to a crossing (upper left side) of a trefoil knot generating two loops that each has one crossing less than the original knot.

By following this recipe one will be left with a linear combination of the Kauffman brackets for various disjoint unions of untangled loops, where to each untangled loop one associates: $\langle \bigcirc \rangle = -q^2 - q^{-2}$, where $\bigcirc$ denotes an untangled loop. Then one can easily check that the Kauffman bracket for the trefoil knot reads: $\langle 3_1 \rangle = q^{-7} - q^{-3} - q^5$. This recipe will associate to any given knot a unique number evaluated in terms of q. This number, is a topological property of the knot and does not depend on the order in which the crossings are removed. Said differently, the Kauffman brackets compute the knot invariant.

A knot only possesses a topological significance as long as it is situated in a three dimensional space, and yet the algorithms that are developed to compute its invariance rely on a two dimensional analysis of the knot and do not involve an inherently three dimensional treatment. This matter was resolved through a surprising discovery by Witten (Witten, 1989) where he provided a quantum theoretic interpretation of a knot not as an object in 3-dimension, but instead as a process in $(2+1)$-dimension. More concretely, what Witten showed was that any given knot can be interpreted in terms of a quantum field theory the observable properties of which are not only purely topological, but surprisingly coincide exactly with the knot invariant. In other words, a knot invariant computed by Kauffman brackets can be reproduced as the expectation value associated to a quantum process the spacetime trajectory of which coincides with the topology of that knot (see figure 2). The great insight of Witten was to show that using the Chern-Simons action, the expectation value for any Wilson operator encoding a tangled topology within a 3-sphere is nothing but the invariant of the corresponding knot given by the Kauffman brackets up to multiplication by a constant that depends on k:

$$\langle W_j \rangle_{\mathcal{A}} = c_k \langle j \rangle \tag{5}$$

The topological nature of such process implies that the expectation value $\langle W_j \rangle_{\mathcal{A}}$ is invariant under smooth deformations of the trefoil knot.

The mathematical correspondence between the structure of a knot on the one hand, and the behaviour of a quantum system at its topological phase on the other reveals a deep mathematical and as well conceptual analogy that exists between radically distinct structures such as topological spaces and Hilbert spaces (Baez, 2006) . This correspondence can be seen by interpreting the knot not as an object, but instead as a category whose objects are ordered collection of points (including the empty collection corresponding to the end points of the knot), which are related to each other through tangled strands forming the body of the knot. A knot, therefore, can be thought of essentially as a relationship amongst ordered collection of points. A quantum process, in a similar fashion, is a category the objects of which are Hilbert spaces that are related to each other through linear maps playing the role of "time evolution". This correspondence between knots and processes as categories is a *functorial* relation which implies that each category can be rigorously interpreted in terms of the other. Category theory, as a structural approach in mathematics, is a conceptual framework where objects are determined not in terms of their content in isolation, but instead in terms of their functional relation to other distinct objects in a broader context. A mathematical equivalence[7] between two categories - a functorial correspondence - forms an epistemic tool in order to view and interpret the objects and relations in one category in terms of the objects and the relations in an other.

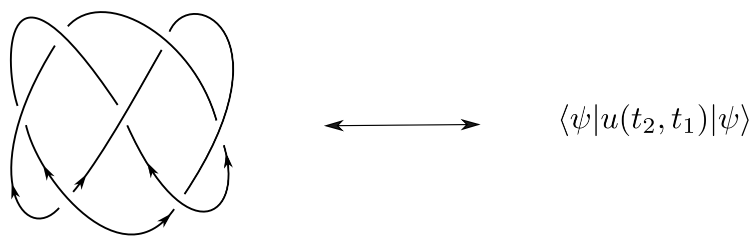

Figure 4: The functorial correspondence between a knot and a quantum process; the function of time in a quantum system at a topological phase is distilled out and codified in a knot. The diagram on the left represents a 7_4 knot that is deconstructed as a category with a collection of ordered points (six points) as its objects, and braidings as its morphisms. The term on the right represents a quantum mechanical evolution in terms of a unitary process.

This strict mathematical analogy reflects the fact that both categories despite their distinct appearances admit the same underlying information. Therefore, this correspondence constitutes an interpretive tool that reads off the functional role of time in a quantum system at a topological phase, and codifies it in terms of a collection of visible topological relations. We note that since we are analysing the behaviour of a quantum system at a topological phase, time does not appear to have a metrical structure, and it looks rather flabby. This can be

[7]Two categories C and D are equivalent if for a functor $\mathcal{F} : C \to D$ there exists an other functor $\mathcal{G} : D \to C$ such that $\mathcal{G}.\mathcal{F}$ is isomorphic to the identity functor from C to itself, and $\mathcal{F}.\mathcal{G}$ is isomorphic to identity functor from D to itself.

seen by noting that the essential property of a knot is not linked with its geometric data such as length and curvature, but rather with its tangled profile. In other words, knots encode non-geometric relations as they are insensitive to deformations. Let us examine the structure of a knot more closely as to get a clearer picture of the function of time in the topological phases of matter. As was mentioned previously a knot considered as a category is a collection of ordered points that are related to each other through braidings forming tangled strands. The function of any braiding - permuting a pair of points - is analogous to a single unitary operation in the category of quantum systems. Furthermore, *any* unitary process $u(t_2, t_1)$ can be codified rigorously in terms of a set of braidings forming a knot (see figure 4).

Having located the essential function of time in matter at a topological phases in terms of a set of relations encoded in a knot, we proceed to address the question regarding the appearance of time as a metrical concept at the geometric phase of matter. As the quantum system behaviour transforms from its topological phase to the geometric phase, the long-range entanglement patterns become dominated by the strong short-range interactions (Kitaev, 2003), and consequently, the underlying geometric data is no longer concealed, at which point the system begins to be responsive with respect to local perturbations. Alternatively, what makes time to appear radically different at the geometric phase is that the system is no longer degenerate with respect to local degrees of freedom. Therefore, at the geometric phase, time appears to perform a new task and that is to specify temporal durations, a metrical concept that is simply not present at the topological phase. This new functional role however, can be deconstructed into a collection of relations that are codified as morphisms in a category, which suggests that similar to the topological phase a relational interpretation of the geometric time should be a manageable position to hold. This *categorification* of an ordinary geometric field theory can be achieved *locally* (Schreiber, 2009) - where the metric can be considered to an arbitrary accuracy as being flat - by deconstructing a bounded region of spacetime into a 2-category of which each object corresponds to a single point in a space, each morphism corresponds to spatial paths between a pair of objects, and each 2-morphism corresponds to a temporal connection between two paths - a homotopy. As depicted in the figure (5), to each path - a 1-morphism that codifies a spatial relation between two points - is associated a distinct state (a field configuration), and to each homotopy a unique time evolution (a propagator). As one can see, at the geometric phase we are faced with an outburst of a continuum of distinct states $\psi, \phi, ...$ that were rendered simply indistinguishable at the topological phase where the notion of a distance relation between a pair of points is absent. In other words, time, construed as a set of relations or morphisms in a suitable category that link distinguishable states, can operate both non-metrically and discrete as well as metrically and continuous depending on the phase in which the matter happens to be.

In our attempt to construe time as a collection of morphisms in a suitable category, it appears that we have evaded a fundamental obstacle, and that is the deconstruction of the topological/geometric properties linked with the *environment* in which the objects of the category are situated. The temporal relations across distinct states do not seem to codify the ambient structure that is superimposed on the matter. One can clearly see this by noting that in the equation (2) the term M that denotes the 3-space containing the topological field theory appears external to the Chern-Simons term, and that the two spaces M and M' with distinct topologies result in different expectation values for the same process. Alternatively, in the corresponding category, the relation encoded in a knot depends on the topology of the ambient space in which the knot is situated *e.g.* a trefoil knot that lives in a 3-sphere is distinct from the one that lives in a 3-torus. In a similar fashion, the behaviour of a matter at the geometric phase depends on the curvature linked with the space in which the matter fields live *e.g.* expectation values for an interaction depends on whether the field theory lives on a sphere or on a flat disk. To further expose this apparent obstacle one can consider the situation where there exists no matter to begin with, and all that is left amounts to nothing but a vacuum state with *some* spatial and temporal topology. In the absence of matter fields and of the relations between them there seems to be nothing left to which such topological structure can be reduced. At the first glance it might seem that time is not entirely reducible to a set of relations amongst objects, and a significant feature of its function must somehow be assumed to lie outside the category. A more deeper look at the Chern-Simons gauge theory however, reveals that indeed the expectation value associated with no process - the absence of any matter field - in an arbitrary 3-space M, is equivalent to the expectation value associated with *some* process W situated in a 3-sphere[8] (Witten, 1989) :

$$\int_M \mathcal{DA} \exp\{i\mathcal{S}[\mathcal{A}]\} = \int_{S^3} \mathcal{DA} \exp\{i\mathcal{S}[\mathcal{A}]\} W \tag{6}$$

This indicates that the topology of any ambient space is indeed reducible to the expectation value of *some* process happening inside a 3-sphere, which itself can be codified in terms of a set of relations within a suitable category. Although the equation (6) seems to provides a strong case for a relational interpretation of the topological property linked with an ambient space, it does not appear to hint at any particular geometry [9]. As we discussed previously the geometric data of a flat region can be reduced into class of relations within a category, which can be analogised in the corresponding category as a class of distinguishable states as shown in

[8]This result is analogous to a celebrated theorem (Wallace, 1960; Lickorish, 1962) in topology stating that that every (closed, connected, orientable) 3-manifold can be obtained from or reduced to a 3-sphere (or indeed any desired 3-manifold) by performing repeated *Dehn surgeries* on a knot.

[9]In a world where the underlying quantum field theory is believed to be geometric, the possibility of reducing the background Lorentzian metrical structure in which quantum systems operate to pure relations in a category is not clear. In attempting to formalise the structure of QFTs in curved space-times, Brunetti, Fredenhagen and Verch (Brunetti et al., 2003; Sanders, 2020) construct a functor that corresponds to a category of Lorentzaian manifolds (Loc) a category of C* algebra (Alg) that encodes the QFT systems. In this approach, the local structure as well as the principle of general covariance is purely encoded in the morphisms of the Loc category whereas the metric structure, is contained in the objects. This might be an implication that the metrical structure is a property that can not be reduced to mere relations in a category. This is not surprising after all since the background metric is essentially linked with the gravitational field that should be treated as a physical systems on its own and be represented in terms of algebraic relation similar to the other fields in the theory. Encoding algebraically the metric structure of spacetime in the category of quantum processes is the grand goal of any theory of quantum gravity, which yet to be constructed.

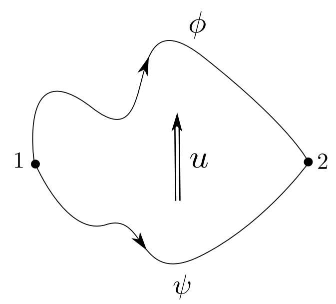

Figure 5: Deconstruction of a region of spacetime as a 2-category: each 1-morphisms between a pair of objects (points) encodes a spatial relation (distance relation), and the 2-morphisms (the homotopy) between a pair of paths encodes the temporal relation. A general quantum field theory is a functor that to each 1-morphism associates a distinct quantum state, and to each 2-morphism a time evolution operator.

figure (5): a pair of points being say "close" or "far" is a situation to which one can associate a distinct state ψ or ϕ respectively. The degree to which ψ differs from ϕ *globally* however, profoundly supervenes on the entire geometry of the ambient space *i.e.* the distance relations between a pair of points vary depending on whether they live on sphere or on a topologically equivalent yet geometrically distinct space such as a disk. — Similarly to the way in which one recovers the topological information of an arbitrary 3-space using some process in a 3-sphere, is it possible to mimic the effect of an arbitrary geometry using some field theory that lives in a 3-sphere? or rather can one fake any geometry with another one at hand such as the 3-sphere? — A positive answer to such question would certainly offer a secure case for a relational interpretation of geometry, and subsequently, for a relational view of time in theoretical physics. Questions like these, or at least questions closely related to these, are typically addressed within the domain of information theory. An instance of a closely related question asked by Feynman reads (Feynman, 1982): is it possible to mimic an arbitrary local quantum dynamics with a quantum computing machine to any degree of accuracy efficiently? The answer is yes (Lloyd, 1996). This means that the observable properties of a quantum field theory coupled to an arbitrary background metric field can be mimicked by a quantum computing machine which itself might be implemented in terms of some field theory living in a space with a fixed geometry such as a 3-sphere.

4 Atemporality

Viewing time as a set of relations that can be established across distinct states invites the question of what constitutes *distinctiveness*. Before addressing this however, let us consider a situation that sheds light on the role that distinctiveness plays in the time experience of an observer. Imagine a pair of two-level quantum systems initially prepared in a Bell state, $|\phi_+\rangle$, that is subject to a unitary map that drives the system to another orthogonal state, $u|\phi_+\rangle = |\phi_-\rangle$. Such a change is entirely visible in the eyes of an observer who probes the system using non-local operators $|\phi_\pm\rangle\langle\phi_\pm|$ with which $|\phi_+\rangle$ can be perfectly distinguished from $|\phi_-\rangle$. On the other hand, to an observer who probes the system from the lens of local operators $\{\sigma_i\}$[10], $|\phi_+\rangle$ and $|\phi_-\rangle$ look the same, in which case no change can be noticed and the system would simply appear frozen. Such toy example seems to suggest that distinctiveness is fundamental to the temporality of a system. In order to give a sharp account of what distinctiveness is let us once again resort to the mathematical machinery of category theory. As we pointed out earlier, a recurring motif in category theory is that an object is determined not in terms of its content in isolation, but rather by the network of relations that it enjoys with all the other objects in a broader context. Let us express this idea in a slightly more clear term: given an object X living in a category $\mathcal{C}$ one can define a functor $\mathcal{F}_X$ that assigns to any object Y in $\mathcal{C}$ the set of all morphisms from X to Y, $\mathcal{F}_X(Y) := \mathrm{Hom}(X;Y)$[11]; the elegant fact that follows as a corollary from the Yoneda lemma (Riehl, 2016) - the most fundamental result in category theory - reads : given two objects X and X' in a category, X is isomorphic to X' if and only if $\mathcal{F}_X$ is isomorphic to $\mathcal{F}_{X'}$:

$$X \cong X' \iff \mathcal{F}_X \cong \mathcal{F}_{X'}. \tag{7}$$

That is, the object X in a category $\mathcal{C}$ is fully determined up to a *canonical isomorphism* from the information that is codified in $\mathcal{F}_X$ - the set of all the morphisms from X to each of

[10] σ_i denote the Pauli matrices.

[11] Here $\mathrm{Hom}(X;Y)$ is a set - an object of the category of sets, **Set**.

the objects in $\mathcal{C}$. An alternative interpretation of the statement (7) is that two objects in a category are the *same* if and only if the set of their relation with all the objects in that category are the same. It is this weakening of sameness from equality to some canonical isomorphism that is central to a contextual understanding of systems and their objects. Therefore, a pair of objects X, X' are distinct - one can perfectly distinguish one from the other - if and only if the ways in which they relate to the rest of the category do not admit an isomorphism.

This category theoretic view of distinctiveness is highly desirable when the objects under study are quantum mechanical systems for the way such objects can be "seen" is through their interaction (morphisms) with measuring devices - other similar objects that surround the object of interest *e.g.* a particle *is* the way it appears in interaction with other particles that constitute the measuring apparatus. To see this more concretely, let us have a closer look at the way a measurement process is visually codified in a knot. The act of measuring always involves an interaction between the system of interest and its *dual* - the apparatus. — Where in the knot category is the interaction with the corresponding dual object located?— It seems obvious that any knot intersects a line always at an even number of points, which is simply the consequence of the mathematical fact that a knot is a closed object - a tangled loop. In this way one can regard a knot as a collection of even number of braided strands that fuse pairwise as to form a loop, and it is this fusion operation taking place between objects and their dual that corresponds to the measurement process encoded in the interaction between the system and the apparatus (see figure 6). This way of viewing a knot sheds light on the way that a system's temporal aspect is determined by its relation to the measuring system: time evolution is nothing but a change in the way in which a system relates to the measuring device, and analogously, a knot is nothing but a change in the way in which an object relates to its dual[12]. In this way we can view atemporality not as an existential immobility, but rather as an observational paralysis.

Figure 6: As one can see the way the object $(1, 2, 3)$ relates to the dual $(1^*, 2^*, 3^*)$ determines the form of the knot, and accordingly, the time evolution of the system. The diagram on the left depicts a process in which the object and its dual have undergone different paths through which their order-relation has changed, which results in a nontrivial time evolution that is encoded in the topology of the knot. The diagram on the right however, depicts a situation where the relation between the object and the dual remains the same throughout the trajectory, which results in a trivial topology corresponding to no change.

The relational view of objects and their spatial and temporal aspects, which is codified in a category theoretic style appears to highly resonate with the position that maintains the

[12] It is important to note that here the dual object does not represent the measuring apparatus itself, but rather codifies the effect of a measuring process *i.e.* the mapping from a quantum state $|\psi\rangle$ to a complex number: $\langle\psi|...\rangle : |\psi\rangle \to \mathbb{C}$. This is already implied in Dirac notation of quantum measurement: a state that is represented as a ket state is measured by its projection onto a bra state - the dual state.

fundamental ontology of the world consists in configurations of objects with respect to one another, and not the objects themselves[13]. This view paints a picture in which the world cannot be associated with objects or things, but rather with the connections between them - the facts. The facts are made out of atomic facts - the states of affairs - that are embedded in a *logical space* that, in analogy with the physical space, can be thought of as a single rigid network of interrelated places. Despite a clear emphasis on the significance of relations or connections in constituting the world, there is an implicit hint that the objects as well as the space enjoy an independent mode of existence on which the facts, and consequently the world supervene *i.e.* the structure of the ambient space constrains the facts. However, this position that seems to favor a substantival view of space (and time), is not maintained since objects do not get to exist entirely on their own. Objects are intelligible by the virtue of being placed within a network of relations. That is, in order for a particle - to be a particle - it must be involved in a collection of relations with other particles. For instance, from an empirical standpoint an electron can be seen only in its interaction with a set of particles in a cloud chamber, and it is this relation with the detector particles that constitutes an electron. This view might appear to be challenged by noting that an electron can certainly exist without being detected - a free electron. A deeper look however, suggests that the existence of a free particle - even at a conceptual level - is conditioned on its relation to other systems. this can bee seen by noting that a free particle state is a thing that *responds* in a certain way to a class of symmetry groups[14] (Wigner, 1939; Weinberg, 2002). These symmetry groups however, are nothing but a class of unitary transforms that encode certain interactions between the particle of interest and some other system. Similar to the objects, space (and time) also seem to have lost their privileged mode of existence as an independent entity[15]. The two-directional dependence between the structure of the ambient space and states of affairs seems in resonance with the argument that is distilled in the equation (6) in support of a relational view on space and on time: the existence of space is a necessity as to situate objects and their relations; the form - the topology of the space - however, is not singular, but instead can be reduced to any other form depending on the interactions between the objects within it. In this view therefore, the properties of any object (and of the space in which it is embedded) appear to hinge on a set of relations that the object enjoys with the rest of the world.

What if two objects happen to have their relation with others identical?— On the ground of the above argument, a possible answer would be that the two objects collapse into a single one - they are the same. We can apply this principle of sameness not just to objects but as well to their spatial and temporal aspects. An object can not *move* if it is involved in the same set of relations that it was a moment ago. This is precisely the sense in which we interpreted the idea of atemporality previously: a system is atemporal not because it is ontologically frozen, but rather since all its possible states happen to be indistinguishable - a continuum of degenerate states that can not be distinguished as they respond to measurements in exactly the same way.

[13]Here we interpret the proposed relational view of objects and their spatiotemporal aspects in the light of the Fogelin's commentary (Fogelin, 1987) on Wittgenstein's atomistic ontology put forward in the *Tractatus* (Wittgenstein, 1922).

[14]These symmetry groups form the irreducible unitary representations of the Poincaré group.

[15]This point has been indicated by the paragraph 2.0124 in the *Tractatus*: on the one hand space and time shape the state of affairs, and on the other, all the possible states of affairs are given once one have access to all the objects. In this way, a balance in the dependence relation between space and objects is reached. As Fogelin puts it (Fogelin, 1987): "By establishing a systematic parity between two fundamental principles of atmosim (matter and void or being and non-being), Wittgenstein gives this position its most coherent articulation."

5 Conclusion

The mathematical frameworks linked with some of the quantum gravity approaches including loop quantum gravity, group field theory and causal set theory point to a construct that at the effective level behaves as a spacetime manifold with a metric, whereas at the fundamental level it does not admit any geometry. In this work we have put forward an intuitive interpretation of such construct, which is informed by examining an analogue system. In this mode of analogy, a topological field theory by virtue of not admitting any metrical information such as distance or duration, acts as the fundamental non-geometric system that when probed at higher energy scales gives rise to a geometric field theory. This analogical interpretation can provide a clear view on the conceptual issues facing the aforementioned quantum gravity approaches, particularly in two respects. Firstly, it illustrates a situation in which time and space can function non-metrically as well as metrically depending on the energy scale. This suggests that the emergence of spacetime as a geometric structure out of a system without a metric can be not only mathematically consistent, but also conceptually and perhaps empirically coherent. Secondly, the deconstruction of this analogue system as a category hints at a relational view of time that can close the conceptual gap between the geometric and the non-geometric time. In other words, time as a set of relations that link distinguishable states, can function both non-metrically and discrete as well as metrically and continuous depending on the phase in which the matter happens to be.

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