A combined study of hadronic $D^+ \to K^-\pi^+\pi^+$ and $D_s^+ \to K^+K^+\pi^-$ decays by means of the analysis of semileptonic $D^+ \to K^-\pi^+\ell^+\nu_\ell$ decays

R. Escribano[*a,b], P. Masjuan[†a,b] and Pablo Sanchez-Puertas[‡b,c]

[a] *Grup de Física Teòrica, Departament de Física,*

Universitat Autónoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain

[b] *Institut de Física d'Altes Energies (IFAE) and*

Barcelona Institute of Science and Technology (BIST),

Campus UAB, E-08193 Bellaterra (Barcelona), Spain

[c] *Departamento de Física Atómica, Molecular y Nuclear,*

Universidad de Granada, E-18071, Spain

We perform a combined study of the two hadronic decays $D^+ \to K^-\pi^+\pi^+$ and $D_s^+ \to K^+K^+\pi^-$ using a detailed analysis of the semileptonic decays $D^+ \to K^-\pi^+\ell^+\nu_\ell$ ($\ell = e, \mu$) thanks to the high-statistics dataset provided by the BESIII Collaboration. We propose simple and suitable amplitude parametrizations of the studied reactions that shall be of interest to experimentalists for upcoming analyses. These new parametrizations are based on the naïve factorization hypothesis and the description of the resulting matrix elements in terms of well-known hadronic form factors, with special emphasis on the $K\pi$ scalar and vector cases. Such form factors account for final state interactions which fulfil analyticity, unitarity and chiral symmetry constraints. As a result of our study, we find a good prediction for the P-wave BR in $D^+ \to K^-\pi^+\pi^+$ decays, while adopting a global phase and moderate rescaling for the S-wave, that we ascribe to three-body effects, we find a reasonable description. The resulting model is confronted against $D_s^+ \to K^+K^+\pi^-$ decays, obtaining again a good agreement with data.

1 Introduction

In 2009, one of us presented a model for the decay $D^+ \to K^-\pi^+\pi^+$ where the weak interaction part of the reaction was described using the effective weak Hamiltonian in the factorization approach, while the two-body hadronic final state interactions were taken

*rescriba@ifae.es

†masjuan@ifae.es

‡psanchez@ifae.es

into account through the $K\pi$ scalar and vector form factors, fulfilling analyticity, unitarity and chiral symmetry constraints [1]. Due to the lack of precise data in semileptonic $D^+ \to K^-\pi^+\ell^+\nu_\ell$ decays, the model required two free parameters that were fixed from experimental branching ratios. Allowing for a global phase difference between the S and P waves, the Dalitz plot of the $D^+ \to K^-\pi^+\pi^+$ decay, the $K\pi$ invariant mass spectra and the total branching ratio due to S-wave interactions were well reproduced.

With the advent of new results for semileptonic decays by the BES-III Collaboration [2], the free parameters employed in Ref. [1] can be fixed and some assumptions regarding the form factors can be relaxed. In this study, we carefully analyze semileptonic decays by employing simple yet well-motivated parametrizations fulfilling analyticity and unitarity constraints to fix the relevant hadronic matrix elements, that might be of interest for future experimental analysis. The corresponding matrix element is then used to describe the $D^+ \to K^-\pi^+\pi^+$ decay in the naïve factorization approach. While our framework does not account for genuine three-body effects[1] (see Refs. [3–7] regarding 3-body unitarity and Refs. [8, 9] for previous works), it allows for a simple parametrization fulfilling two-body unitarity and, not least, to connect $D^+ \to K^-\pi^+\pi^+$ decays to the isospin-related $D_s^+ \to K^+K^+\pi^-$ ones. As a result, we find that naïve factorization performs reasonably well for $D^+ \to K^-\pi^+\pi^+$ decays regarding the P-wave contribution. Concerning the S-wave, a reasonable description can be achieved once a global phase and a moderate rescaling factor are allowed, that might be effectively ascribed to effective three-body unitarity effects. Remarkably, the resulting description reproduces well existing results on $D_s^+ \to K^+K^+\pi^-$ decays, thus further supporting our approach.

The article is organized as follows: in Section 2, we outline the naïve factorization approach applied to $D^+ \to K^-\pi^+\pi^+$ decays, recapitulating all the necessary form factors that enter the description; in Section 3, we review the semileptonic decays in detail, putting forward a parametrization that is used to extract the relevant form factors based on BES-III [2] results; in Section 4, we use the form factor from previous section to put forward a description for $D^+ \to K^-\pi^+\pi^+$ decays; in Section 5, this parametrization is applied to the isospin related $D_s^+ \to K^+K^+\pi^-$ decays. Conclusions are given in Section 6.

2 Naïve factorization in $D^+ \to K^-\pi^+\pi^+$ decays

For $D^+ \to K^-\pi^+\pi^+$decays, we closely follow Ref. [1]. The effective weak interactions driving such decay follow from the Lagrangian at low energies

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} V_{ud}V_{cs}^*[C_1(\mu)\mathcal{O}_1 + C_2(\mu)\mathcal{O}_2] + \text{h.c.}\,, \qquad \mathcal{O}_{1(2)} = 4[\bar{s}_L^i\gamma^\mu c_L^{i(j)}][\bar{u}_L^j\gamma^\mu d_L^{j(i)}]\,, \quad (1)$$

where i, j are color indices, and the Wilson coefficients above differ from those at the electroweak scale due to renormalization [10]. In the following, we employ the naïve factorization

[1]The model does not account either for two-body $\pi^+\pi^+$ final state interactions, but these are non-resonant and presumably small, in such a way that naïve factorization should encompass the most relevant two-body interactions.

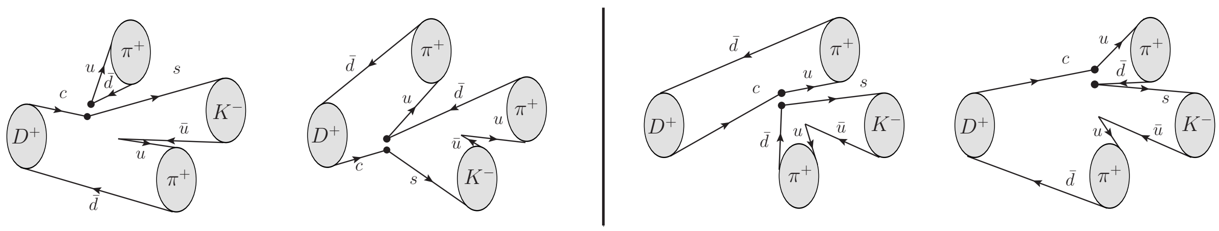

Figure 1: The $\mathcal{O}_1$ (left) and $\mathcal{O}_2$ (right) operator contributions to $D^+ \to K^-\pi^+\pi^+$ decays within naïve factorization. For each operator there is a N_c^0- and N_c^{-1}-suppressed contribution, cf. left and right in each figure.

hypothesis (see Fig. 1), that implies the following decomposition for the process [1]:

$$
\begin{aligned}
i\mathcal{M} = -i\frac{G_F}{\sqrt{2}}V_{ud}V_{cs}^*\Big[& a_1 \langle K^-\pi_1^+|\, \bar{s}\gamma^\mu(1-\gamma^5)c\,|D^+\rangle \langle \pi_2^+|\, \bar{u}\gamma_\mu(1-\gamma^5)d\,|0\rangle + \\
& a_2 \langle K^-\pi_1^+|\, \bar{s}\gamma^\mu(1-\gamma^5)d\,|0\rangle \langle \pi_2^+|\, \bar{u}\gamma_\mu(1-\gamma^5)c\,|D^+\rangle \Big] + (\pi_1^+ \leftrightarrow \pi_2^+)\,, \quad (2)
\end{aligned}
$$

where $a_1 = C_1 + N_c^{-1}C_2 = 1.2(1)$, and $a_2 = C_2 + N_c^{-1}C_1 = -0.5(1)$ have been taken from Ref. [10].[2] Factorization boils down the problem to the description of the hadronic matrix elements in Eq. (2): the matrix element $\langle \pi^+|\, \bar{u}\gamma^\mu(1-\gamma^5)d\,|0\rangle = if_\pi p_\pi^\mu$ with $f_\pi =$ 130.2(1.7) MeV [11]; the matrix element $\langle K^-\pi^+|\, \bar{s}\gamma^\mu(1-\gamma^5)d\,|0\rangle$, that reduces to the well-known scalar and vector $K\pi$ form factors that, following Ref. [1], we take from Refs. [12] and [13]; the $\langle \pi^+|\, \bar{u}\gamma_\mu(1-\gamma^5)c\,|D^+\rangle$ matrix element is connected via isospin symmetry to $D^0 \to \pi^-\ell^+\nu$ decays; finally, the remaining matrix element, $\langle K^-\pi_1^+|\, \bar{s}\gamma^\mu(1-\gamma^5)c\,|D^+\rangle$, corresponds to that appearing in semileptonic $D^+ \to K^-\pi^+\ell^+\nu_\ell$ decays. Indeed, a closer look reveals that all that is required for the current process is[3]

$$
if_\pi p_{\pi_2}^\mu \langle K^-\pi_1^+|\, \bar{s}\gamma_\mu(1-\gamma^5)c\,|D^+\rangle = f_\pi(m_c+m_s)\langle K^-\pi_1^+|\, \bar{s}i\gamma^5c\,|D^+\rangle\,, \quad (3)
$$

that selects a single form factor among those appearing in semileptonic decays. It turns out that such a form factor produces a contribution to the semileptonic decays proportional to the lepton masses, that is irrelevant for $D^+ \to K^-\pi^+e^+\nu_e$ decays (see Eqs. (8), (12) and (43) to (51)).

Potentially, $D^+ \to K^-\pi^+\mu^+\nu_\mu$ decays could probe such a form factor. At the moment, there is available data from FOCUS [14] and CLEO [15]. Regarding FOCUS, the available statistics cannot discern a non-vanishing value for the form factor in Eq. (3). Concerning CLEO, their results are controversial regarding the q^2-dependency. Thereby, some modelling is required. In the following we employ known relations due to Ward identities to suggest a plausible low-q^2 description based on existing results from semileptonic decays. To that end, we revise the model put forward in Ref. [1] to describe the semileptonic matrix element, taking advantage of the precise results from BES-III not available at the time.

[2]In principle, $a_i \equiv a_i(\mu)$ are scale-dependent parameters, but this does not appear at the leading order factorization scheme employed here.

[3]where, in the last step, $i(m_s - m_c)\langle K^-\pi^+|\, \bar{s}c\,|D^+\rangle = 0$ has been used based on parity arguments.

3 $D^+ \to K^-\pi^+\ell^+\nu_\ell$ decays

In this section we address the semileptonic decay in detail, carefully reviewing the relevant form factors, and paying special attention to the known restrictions that follow from Ward identities that, under reasonable assumptions, allow to extract the relevant form factor entering hadronic decays. Our phenomenological description generalizes that in Ref. [1] by incorporating free parameters previously identified with those appearing in $K\pi$ form factors —a necessary assumption back then in the absence of data that can be relaxed now by using the recent results from BES-III [2].

3.1 General definitions

The matrix element for semileptonic decays is given as [16][4]

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}} V_{cs}^* \langle \pi^+ K^- | \, \bar{s}\gamma^\mu (1-\gamma^5) c \, | D^+ \rangle \, [\bar{u}_\nu \gamma_\mu (1-\gamma^5) v_\ell] \to |\mathcal{M}|^2 = 4G_F^2 |V_{cs}|^2 H^{\mu\nu} L_{\mu\nu} \,, \quad (4)$$

where we used ($p_{\ell\nu} = p_\ell + p_\nu$ and $\bar{p}_{\ell\nu} = p_\ell - p_\nu$),

$$H^{\mu\nu} = \langle \pi^+ K^- | \, \bar{s}\gamma^\mu (1-\gamma^5) c \, | D^+ \rangle \langle \pi^+ K^- | \, \bar{s}\gamma^\nu (1-\gamma^5) c \, | D^+ \rangle^\dagger \,, \quad (5)$$

$$L^{\mu\nu} = \frac{1}{2} \left[p_{\ell\nu}^\mu p_{\ell\nu}^\nu - \bar{p}_{\ell\nu}^\mu \bar{p}_{\ell\nu}^\nu - (s_{\ell\nu} - m_\ell^2 - m_\nu^2) g^{\mu\nu} + i\epsilon^{p_{\ell\nu}\mu\bar{p}_{\ell\nu}\nu} \right] . \quad (6)$$

As such, the central quantity is the matrix element in Eq. (2) which, using the variables $p \equiv p_K + p_\pi$, $\bar{p} \equiv p_K - p_\pi$, and $q = p_D - p$, can be expressed as [16, 17]

$$\begin{aligned} \langle K^-\pi^+ | \, \bar{s}\gamma^\mu (1-\gamma^5) c \, | D^+ \rangle &= i w_+ p^\mu + i w_- \bar{p}^\mu + i r q^\mu - h\epsilon^{\mu q p \bar{p}} \\ &= i w_+ (p^\mu - q^\mu \frac{p\cdot q}{q^2}) + i w_- (\bar{p}^\mu - q^\mu \frac{\bar{p}\cdot q}{q^2}) + \frac{i\tilde{r}}{q^2} q^\mu - h\epsilon^{\mu q p \bar{p}} \,, \end{aligned} \quad (7)$$

where the four form factors have an implicit dependence on q^2, p^2, and $\bar{p}\cdot q$. Note that corresponding quantities in D^- decays are related via appropriate CP transformations, that amount to flip signs for the antisymmetric tensor. In addition, the Ward identities (i.e., Eq. (3) and finiteness at $q^2 = 0$) following

$$\tilde{r} = -(m_c + m_s) \langle K^-\pi^+ | \, \bar{s} i\gamma^5 c \, | D^+ \rangle \,, \qquad \lim_{q^2\to 0} [(p\cdot q) w_+ + (\bar{p}\cdot q) w_- - \tilde{r}] = 0 \,, \quad (8)$$

In particular, their dependence on $\bar{p}\cdot q \sim \cos\theta_{K\pi}$ (see Appendix A) means that the relation should be fulfilled for each partial wave (see also Ref. [16]), a property that we will employ when constructing the form factors. Note that the appearance of $\bar{p}$ in the tensor structure accompanying w_- and h requires partial-wave contributions with $\ell \geq 1$. To make contact with experiment, it is customary to employ the following form factors [16, 17][5] (see

[4] Note our $\epsilon^{0123} = 1$ convention, leading to opposite signs compared to Ref. [16] wherever the antisymmetric tensor appears (the sign can be inferred from $L^{\mu\nu}$). We also employ $\epsilon^{\mu k p q} \equiv \epsilon^{\mu\nu\alpha\beta} k_\nu p_\alpha q_\beta$.

[5] Note in this respect that, for the kinematic variables chosen for the semileptonic decay, $X^2 = (p\cdot q)^2 - p^2 q^2$, while $[(p\cdot q)(\bar{p}\cdot q) - q^2 (p\cdot\bar{p})] = X(z\beta_{K\pi}\cos\theta_{K\pi} + X\Delta_{K\pi})$, reproducing the result in Ref. [17]. However, we keep it general in order to use it in $D^+ \to K^-\pi^+\pi^+$ decays.

definitions in Appendix A)

$$F_1 = \frac{1}{X}\left(X^2 w_+ + [(p\cdot q)(\bar{p}\cdot q) - q^2(p\cdot\bar{p})]w_-\right)\,, \tag{9}$$

$$F_2 = \beta_{K\pi}\sqrt{s_{K\pi}s_{\ell\nu}}w_-\,, \tag{10}$$

$$F_3 = \beta_{K\pi}X\sqrt{s_{K\pi}s_{\ell\nu}}h\,, \tag{11}$$

$$F_4 = \tilde{r}\,. \tag{12}$$

where $F_i \equiv F_i(q^2, p^2, \bar{p}\cdot q)$. F_4 was not defined in Ref. [16] and is only relevant for finite lepton masses, so that results in Appendix A might be of some interest. Further, Eq. (8) implies for these form factors that

$$\lim_{q^2\to 0}[F_1(q^2,p^2,\bar{p}\cdot q) - F_4(q^2,p^2,\bar{p}\cdot q)] = 0\,, \tag{13}$$

that relates again the normalization at $q^2 = 0$ that, as mentioned, must be fulfilled for each partial wave. This is as far as can be reached in a model-independent way and we refer to Appendix A for the differential decay width expressed in terms of the previous form factors. In the following section, we present the model that was used in Ref. [1] to parametrize F_4 that, in essence, assumes the $K\pi$ spectra to be dominated by intermediate resonances with roles parallel to those in $\langle K^-\pi^+|\,\bar{s}\gamma^\mu d\,|0\rangle$ form factors. In doing so, we employ a more flexible description compared to that in Ref. [1], that will prove convenient to make contact with the standard phenomenological analysis.

3.2 Resonance model

With the lack of precise data for semileptonic decays, Ref. [1] assumed a model for the F_4 form factor saturated by the lightest $K^*_{(0)}$ resonances. Assuming a similar model for the $K\pi$ scalar and vector form factor allowed them to relate the p^2 dependence of the previous form factors to that of the $K\pi$ scalar and vector form factors. While this was a necessary assumption back then, the current available data for semileptonic decays from BES-III [2] allows to relax this assumption and to provide a more flexible model that might be of interest for experimental analysis. In particular, in the following we assume that the S- and P-waves contributions share the same phase as the scalar and vector form factor (that holds below threshold due to Watson's theorem) but have, in general, different subtraction constants that can be determined thanks to the available data.

3.2.1 Scalar contributions

In the original work from Ref. [1], the scalar contribution was assumed to be dominated by the $K_0^*(1430)$ [11] resonance, whose peak dominates the $K\pi$ scalar form factor, $F_0^{K\pi}(s)$, at intermediate energies. Under the assumption that such a resonance is a quasi-stable (e.g. narrow) state, the $D^+\to\bar{K}_0^*\ell^+\nu$ decay can be described via its matrix element (p is the momentum associated to the $\bar{K}_0^*$)

$$\langle\bar{K}_0^*|\,\bar{s}\gamma^\mu(1-\gamma^5)c\,|D^+\rangle = i\left[w_+^{\bar{K}_0^*}(q^2)\left(p^\mu_{\bar{K}_0^*} - \frac{q\cdot p_{\bar{K}_0^*}}{q^2}q^\mu\right) + q^\mu\frac{\tilde{r}^{\bar{K}_0^*}(q^2)}{q^2}\right], \tag{14}$$

where once more

$$\tilde{r}^{\bar{K}_0^*}(q^2) = -(m_c + m_s)\, \langle \bar{K}_0^* |\, \bar{s} i \gamma^5 c \,| D^+ \rangle \,, \qquad \lim_{q^2 \to 0} \left[(q \cdot p) w_+^{\bar{K}_0^*}(q^2) - \tilde{r}^{\bar{K}_0^*}(q^2) \right] = 0 \,. \tag{15}$$

Finally, the q^2 dependence is reduced, as usual, to the closest charmonium resonance. Its subsequent $\bar{K}_0^* \to K^- \pi^+$ decay merely adds the resonance structure, meaning that the full amplitude is given as $\langle K^- \pi^+ | \bar{K}_0^* \rangle \, P_{\bar{K}_0^*} \, \langle \bar{K}_0^* |\, \bar{s} \gamma^\mu (1 - \gamma^5) c \,| D^+ \rangle$, with $P_{\bar{K}_0^*}$ the scalar propagator. Were this fully dominated by the K_0^* resonance both for the semileptonic and $K\pi$ scalar form factors, then $\langle K^- \pi^+ | \bar{K}_0^* \rangle \, P_{K_0^*} \to \chi_{\bar{K}_0^*} F_0^{K\pi}$, where $\chi_{\bar{K}_0^*} = (m_K^2 - m_\pi^2)/(m_s - m_d)/ \langle \bar{K}_0^* |\, \bar{s} d \,| 0 \rangle$ [1]. However, the different interplay of scalar resonances shall in general differ, yet their phase shift below inelasticities must agree by Watson's theorem. We reflect this by shifting $F_0^{K\pi} \to F_0^{D_{\ell 4}}$, that allows for the following *ansatz* for the S-wave contribution

$$w_+(q^2, p^2, \bar{p} \cdot q) = X^{-1} F_1^{\bar{K}_0^*}(q^2, p^2, \bar{p} \cdot q) = 2 \chi_S^{\text{eff}} F_0^{D_{\ell 4}}(p^2) \left(1 - q^2/m_{D_{s1}}^2\right)^{-1} , \tag{16}$$

$$\tilde{r}(q^2, p^2, \bar{p} \cdot q) = F_4^{\bar{K}_0^*}(q^2, p^2, \bar{p} \cdot q) = \chi_S^{\text{eff}} (m_D^2 - p^2) F_0^{D_{\ell 4}}(p^2) \left(1 - q^2/m_{D_s}^2\right)^{-1} . \tag{17}$$

The parametrization in Eq. (17) has been chosen to fulfill Eqs. (8) and (15), and to include the closest pole with appropriate quantum numbers. Regarding the parametrization used in BES III [2], we identify $2\chi_S^{\text{eff}} F_0^{D_{\ell 4}}(p^2) = \mathcal{A}_S(p^2)$ (see Eq. (20) from Ref. [2]).

In order to parametrize $F_0^{D_{\ell 4}}(p^2)$, we follow the approach in Refs. [18–20]. This uses an Omnès representation subtracted at $p^2 = 0$ and the Callan-Treiman point $\Delta_{K\pi} = m_K^2 - m_\pi^2$,

$$F_0^{D_{\ell 4}}(s) = \exp\left[\frac{s[\ln C_{D_{\ell 4}} + G_0(s)]}{\Delta_{K\pi}} \right] , \tag{18}$$

$$G_0(s) = \frac{\Delta_{K\pi}(s - \Delta_{K\pi})}{\pi} \int_{s_{th}}^{\infty} d\eta \frac{\delta_0^{1/2}(\eta)}{\eta(\eta - \Delta_{K\pi})(\eta - s)} , \tag{19}$$

with $\delta_0^{1/2}$ the scalar $I = 1/2$ $K\pi$ phase shift, that preserves the constraints provided by unitarity and analiticity below higher inelasticities. The subtraction constant, $\ln C_{D_{\ell 4}}$, encapsulates high-energy effects that need not be the same as in the $K\pi$ scalar form factor case, thus requiring data on semileptonic decays to fix it. For the phase shift, we take that in Ref. [12] below $\Lambda = 1.67$ GeV, where $\delta_0^{1/2} = \pi$; above, we take a constant phase $\delta_0^{1/2} = \pi$ following Ref. [18–20]. This model allows for a relatively simple and flexible parametrization, that improves the one used by the BES-III Collaboration by incorporating appropriate analyticity and unitarity constraints (up to higher-threshold inelasticities). As such, it might be useful in future experimental analyses.

3.2.2 Vector contributions

The next relevant wave is the P-wave, to narrow $\bar{K}^*$ resonance plays a prominent role both in the $K\pi$ vector form factor and semileptonic decays. Again, assuming them to be narrow states, the $D^+ \to \bar{K}^* \ell^+ \nu$ decay can be described via the corresponding matrix element,[6]

$$\langle \bar{K}^*|\,\bar{s}\gamma^\mu(1-\gamma^5)c\,|D^+\rangle = \Big(A\epsilon^{\mu\nu qp} - i\Big[B(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}) + Cq^\nu(p^\mu - \frac{q\cdot p}{q^2}q^\mu) + \frac{\tilde{D}}{q^2}q^\mu q^\nu\Big]\Big)m_{\bar{K}^*}\varepsilon_\nu\,, \quad (21)$$

where $m_{\bar{K}^*}$ has been used for later convenience. In addition, the Ward identity implies

$$m_{\bar{K}^*}\tilde{D}(q^2)(q\cdot\varepsilon) = (m_c + m_s)\,\langle K^*|\,\bar{s}i\gamma^5 c\,|D^+\rangle\,, \quad \lim_{q^2\to 0}\Big[B(q^2) + (q\cdot p)C(q^2) - \tilde{D}(q^2)\Big] = 0\,. \quad (22)$$

Again, the q^2-dependency can be saturated via the appropriate charmonium resonances. Then, along the lines in Ref. [1], the subsequent $K^-\pi^+$ decay would closely resemble the vector $K\pi$ form factor if both cases were fully dominated by the K^*. Still, as for the scalar case, these will generally differ —even if the phase shift below inelasticities should be the same. Therefore, we replace once more $F_+^{K\pi}(p^2) \to F_+^{D_{\ell 4}}(p^2)$ (for a single resonance contribution $\chi_{\bar{K}^*} = f_{\bar{K}^*}^{-1}$ [1]), obtaining

$$F_1(q^2, p^2, \bar{p}\cdot q) = -\chi_{\bar{K}^*}F_+^{D_{\ell 4}}(p^2)\beta_{K\pi}\cos\theta_{K\pi}\left[X^2C(0) + (q\cdot p)B(0)\right]\left[1 - q^2/m_{D_{s1}}^2\right]^{-1}, \quad (23)$$

$$F_2(q^2, p^2, \bar{p}\cdot q) = -\chi_{\bar{K}^*}F_+^{D_{\ell 4}}(p^2)\beta_{K\pi}\sqrt{s_{K\pi}s_{\ell\nu}}B(0)\left[1 - q^2/m_{D_{s1}}^2\right]^{-1}, \quad (24)$$

$$F_3(q^2, p^2, \bar{p}\cdot q) = -\chi_{\bar{K}^*}F_+^{D_{\ell 4}}(p^2)\beta_{K\pi}X\sqrt{s_{K\pi}s_{\ell\nu}}A(0)\left[1 - q^2/m_{D_s^*}^2\right]^{-1}, \quad (25)$$

$$\begin{aligned} F_4(q^2, p^2, \bar{p}\cdot q) &= -\chi_{\bar{K}^*}F_+^{D_{\ell 4}}(p^2)\frac{N(p^2)}{2}\tilde{D}(q^2, p^2) \\ &= -\chi_{\bar{K}^*}F_+^{D_{\ell 4}}(p^2)\frac{N(p^2)}{2}\left[B(0) + \frac{m_D^2 - p^2}{2}C(0)\right]\left[1 - q^2/m_{D_s}^2\right]^{-1}, \end{aligned} \quad (26)$$

where $N(p^2)/2 = [p^2(\bar{p}\cdot q) - (p\cdot\bar{p})(p\cdot q)]/p^2$ is a variable defined in Ref. [1] that reduces to $X\beta_{K\pi}\cos\theta_{K\pi}$ in semileptonic decays, and the last form factor is chosen to fulfill Eq. (13) and saturated with the closest resonance. The connection to the *ansatz* employed by the BES-III Collaboration [2] can be easily obtained accounting that $2\alpha\sqrt{2}m^{-1}\mathcal{A}(m) = -g_{\bar{K}^*K\pi}\beta_{K\pi}P_{\bar{K}^*}(m^2)$, with $P_{\bar{K}^*}(s)$ the standard propagator. Once again, to obtain a description fulfilling appropriate analyticity and unitarity constraints below higher inelasticities, we take

$$F_+^{D_{\ell 4}}(s) = \exp\left[\lambda_1\frac{s}{m_\pi^2} + G_+(s)\right], \quad G_+(s) = \frac{s^2}{\pi}\int_{s_{th}}^\infty d\eta\frac{\delta_1^{1/2}(\eta)}{\eta^2(\eta - s)}\,, \quad (27)$$

with $\delta_1^{1/2}$ the P-wave $I = 1/2$ $K\pi$ phase shift. The input for the phase shift is taken from the result in Ref. [28] with a single vector resonance and with a single subtraction constant. To match their results, we choose an upper cutoff $s = 4$ GeV^2 and $\lambda_1 = 0.025$, but such parameter could be fitted from the experiment, providing then an useful parametrization for experimentalists. Further details are given in Appendix C.

[6]Different parametrizations appear in Refs. [21–27]; the connection reads, up to overall signs,

$$A = -\frac{2V}{m_{\bar{K}^*}(m_D + m_{\bar{K}^*})}\,, \quad B = \frac{m_D + m_{\bar{K}^*}}{m_{\bar{K}^*}}A_1\,, \quad C = -\frac{2A_2}{m_{\bar{K}^*}(m_D + m_{\bar{K}^*})}\,, \quad \tilde{D} = 2A_0\,. \quad (20)$$

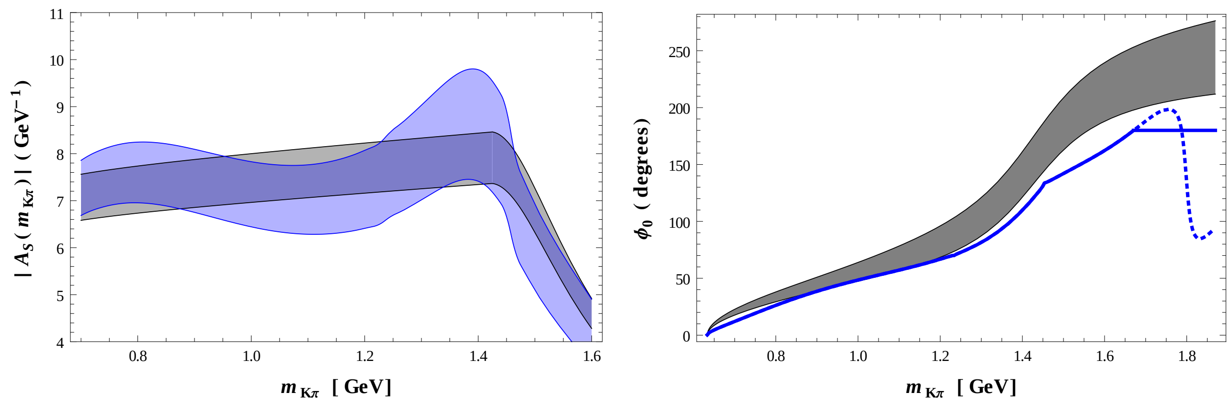

Figure 2: Modulus (left) and phase (right) of the scalar form factor. The gray band stands for BES-III results [2], while the blue band represents our model. The dotted line in the phase plot represents the original input from [12]. We neglect errors from the phase shift that are subleading as compared to BES-III uncertainties on the modulus.

3.3 Obtaining parameters from BES-III

Since there is no available data from experiment, we fit our model to the scalar and vector form factors extracted by BES-III Collaboration. Still, we emphasize that having such data available would allow for a more reliable estimate of our parameters. Regarding the free parameters for the scalar part (cf. Eq. (18)), we fit $2\chi_S^{\rm eff} F_0^{D_{\ell 4}}(s)$, to pseudodata from the $A_S(s)$ form factor from BES-III, obtaining

$$\chi_S^{\rm eff} = 2.13(16)\ \text{GeV}^{-1}\,, \quad \ln C_{D_{\ell 4}} = 0.152(11)\,, \tag{28}$$

with a correlation of -0.27. We show our results in Fig. 2. Note that, in the case of $F_0^{K\pi}(s)$, the chosen parametrization would require $\ln C_{D\ell 4} = 0.206(9)$ [20], based on a combined analysis from $\tau \to K\pi\nu$ and $K_{\ell 3}$ decays. This is not incosistent, but shows that the necessary assumption adopted back in Ref. [1] holds only approximately. Concerning the vector part, we fit the differential decay width distributions obtained from pseudodata from BES-III parametrization with vector contributions only. This way we obtain the parameters

$$\chi_A^{\rm eff} = -3.35(16)\ \text{GeV}^{-3}\,, \quad \chi_B^{\rm eff} = 8.44(23)\ \text{GeV}^{-1}\,, \quad \chi_C^{\rm eff} = -1.64(12)\ \text{GeV}^{-3}\,. \tag{29}$$

where $\chi_X^{\rm eff} \equiv \chi_{\bar{K}^*} X(0)$ in Eqs. (23) to (26). The error to describe the semileptonic decay is fully dominated by that of $\chi_B^{\rm eff}$. The correlation for $\chi_B^{\rm eff}$ and $\chi_C^{\rm eff}$, that enters the F_4 form factor, reads -0.35. In Fig. 3, we show our description for the differential $m_{K\pi}$ spectrum compared to the central values of BES-III, observing nice agreement. We observe that an overall sign cannot be extracted from experiment; to do so, we make use of quark models [25, 29], that allow to choose a sign that is consistent among the different matrix elements here considered. These imply a positive sign for $\chi_B^{\rm eff}$. That this gives the correct interference pattern in the hadronic decays below suggests a reasonable performance of naïve factorization.

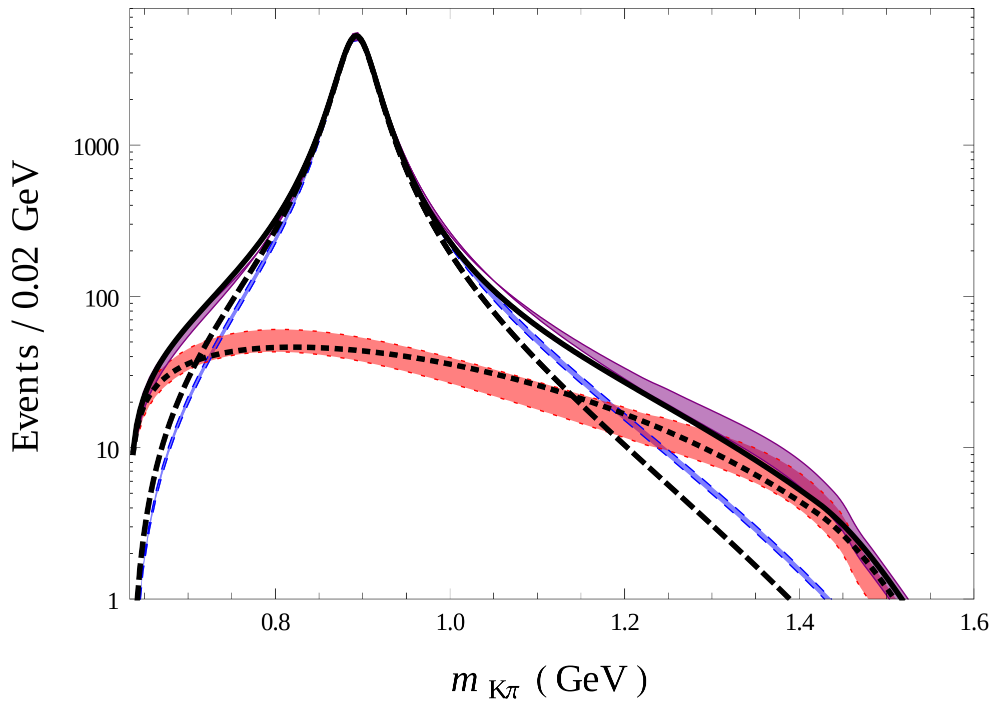

Figure 3: The differential spectra normalized to BES-III events. The red band is our scalar contribution; the blue one is the vector part; the purple band is their combination. The black dotted, dashed, and full lines stands for the scalar, vector and full BES-III model, respectively.

4 Case I: $D^+ \to K^-\pi^+\pi^+$ decays

Having extracted the F_4 semileptonic form factor, and with existing parametrizations for the $F_{+,0}^{K\pi}(s)$ form factors at hand [12, 28], we can continue to the prediction of $D^+ \to K^-\pi^+\pi^+$ decays within the naïve factorization hypothesis.

4.1 Matrix elements

First, we summarize the required matrix elements,

$$\langle K^-\pi^+|\,\bar{s}\gamma^\mu(1-\gamma^5)d\,|0\rangle = \left(\bar{p}^\mu_{K\pi} - \frac{\Delta_{K\pi}}{p^2_{K\pi}}p^\mu_{K\pi}\right)F_+^{K\pi}(p^2_{K\pi}) + \frac{\Delta_{K\pi}}{p^2_{K\pi}}p^\mu_{K\pi}F_0^{K\pi}(p^2_{K\pi})\,, \quad (30)$$

$$\langle \pi^+|\,\bar{u}\gamma_\mu(1-\gamma^5)c\,|D^+\rangle = \left(p^\mu_{D\pi} - \frac{\Delta_{D\pi}}{\bar{p}^2_{D\pi}}\bar{p}^\mu_{D\pi}\right)F_+^{D\pi}(\bar{p}^2_{D\pi}) + \frac{\Delta_{D\pi}}{\bar{p}^2_{D\pi}}\bar{p}^\mu_{D\pi}F_0^{D\pi}(\bar{p}^2_{D\pi})\,, \quad (31)$$

$$i f_\pi p^\mu_{\pi_2}\,\langle K^-\pi_1^+|\,\bar{s}\gamma_\mu(1-\gamma^5)c\,|D^+\rangle = (i)^2 f_\pi F_4 = -f_\pi\Big[\chi_S^{\rm eff}(m_D^2 - p^2_{K\pi})F_0^{D_{\ell 4}}(p^2_{K\pi}) - \frac{1}{2}N(p^2_{K\pi})F_+^{D_{\ell 4}}(p^2_{K\pi})(\chi_B^{\rm eff} + \frac{m_D^2 - p^2_{K\pi}}{2}\chi_C^{\rm eff})\Big]\frac{1}{1-\frac{m_\pi^2}{m^2_{D_s}}}\,, \quad (32)$$

where $p^\mu_{AB} = p^\mu_A + p^\mu_B$ and $\bar{p}^\mu_{AB} = p^\mu_A - p^\mu_B$ and $\Delta_{AB} = m_A^2 - m_B^2$. Concerning $F_0^{K\pi}(s)$, we use that from Ref. [12], while for $F_+^{K\pi}(s)$ we take that from Ref. [28]. For the $D^+ \to \pi^+$

transition, we use isospin symmetry that relates it to that in $D^0 \to \pi^- \ell^+ \nu$ decays, that is parametrized as

$$F_{+(0)}^{D\pi}(s) = \frac{F_{+(0)}^{D\pi}(0)}{1 - s/m^2_{D^{*0}_{(0)}}}\,, \quad F_0^{D\pi}(0) = F_+^{D\pi}(0) = 0.612(35)\ [30]\,. \tag{33}$$

The final result for the amplitude and differential decay width is given in Appendix B. In addition, we will need to use in the following $|V_{ud}V_{cs}^*| = 0.971(17)$ [11], $G_F = 1.1663787 \times 10^{-5}\text{GeV}^{-2}$, $\Gamma_{D^+} = 6.33 \times 10^{-13}$ GeV and masses in Ref. [11].

4.2 P-wave contribution

Following Ref. [1], we check first the P-wave contribution, which spectrum essentially corresponds to that of a two-body $D^+ \to \bar{K}^{*0}\pi^+$ decay, thus free of relevant genuine three-body problems and theoretically cleanest. The current branching ratio (BR) reads $(1.04(12) + 0.022(11) = 1.06(12))\%$ [11], where the first and second part refer, respectively, to the $K^*(892)$ and the $K^*(1680)$ resonances. With the given values for the form factors and $a_{1,2}$ Wilson coefficients, we find $\text{BR} = (0.19(^{+17}_{-11})_{a_1}(^{+37}_{-16})_{a_2}(^{+9}_{-7})_{\text{BES}}(^{+8}_{-7})_{F^{D\pi}}[^{+42}_{-22}]_{\text{Total}})\%$, that slightly underestimates the corresponding branching fraction below the 2σ level. Accounting the inherent uncertainties from naïve factorization, potential model-dependencies in extracting the P-wave branching ratio, and subtleties regarding the overall normalization in semileptonic decays (cf. discussions in [2] and Ref. [31]), the result seems reasonable. To attain a better prediction, and before accounting for the S-wave component, in the following we allow for an overall re-scaling of the F_4 form factor of $1.23(6)_{\text{BES}}(3)_{\text{PDG}}(12)_{a_1}(16)_{a_2}(5)_{F^{D\pi}}[21]_{\text{Total}}$, that reproduces the central value for the BR and that is compatible at the 1σ level. Conversely, this could be reinterpreted as a fit of the a_1 Wilson coefficient.

4.3 Complete description

With the above rescaling, everything should be fixed and the Dalitz plot should be a prediction. However, as found in Ref. [1], we need an additional (global) phase for the scalar component of $(180-60)^\circ$, that might be ascribed to genuine three-body effects that should be mostly relevant for the S-wave and within the $K^*(892)$ window, where interference is stronger. The additional phase gives the appropriate picture in the Dalitz-plot, but still underestimates the total BR in the PDG [11], BR$= 9.38(16)\%$. Allowing for an additional moderate rescaling for the S-wave part in the semileptonic F_4 form factor, we find that a rescaling factor of $1.55(10)_{\text{BES}}(6)_{P-\text{wave}}(2)_{\text{BR}}(0)_{a_1}(26)_{a_2}(5)_{F^{D\pi}}[30]_{\text{Total}}$ suffices to reproduce the total BR that, once more, could be related to effects beyond two-body unitarity. With these modifications, the P-wave BR is fixed to $1.06(12)\%$ by construction, while the S-wave fraction (e.g., the corresponding S-wave BR normalized to the total one, see [1]) is $85.9(1.6)\%$, with the error fully dominated by the uncertainty of the P-wave BR. The resulting invariant mass distribution and Dalitz plot are shown, respectively, in Fig. 4 and Fig. 5, displaying an overall nice agreement given the simplicity of the approach.

Overall, the naïve factorization approach seems to provide a decent first-order estimate, offering a nice picture regarding the P-wave contribution, while the S-wave necessitates from

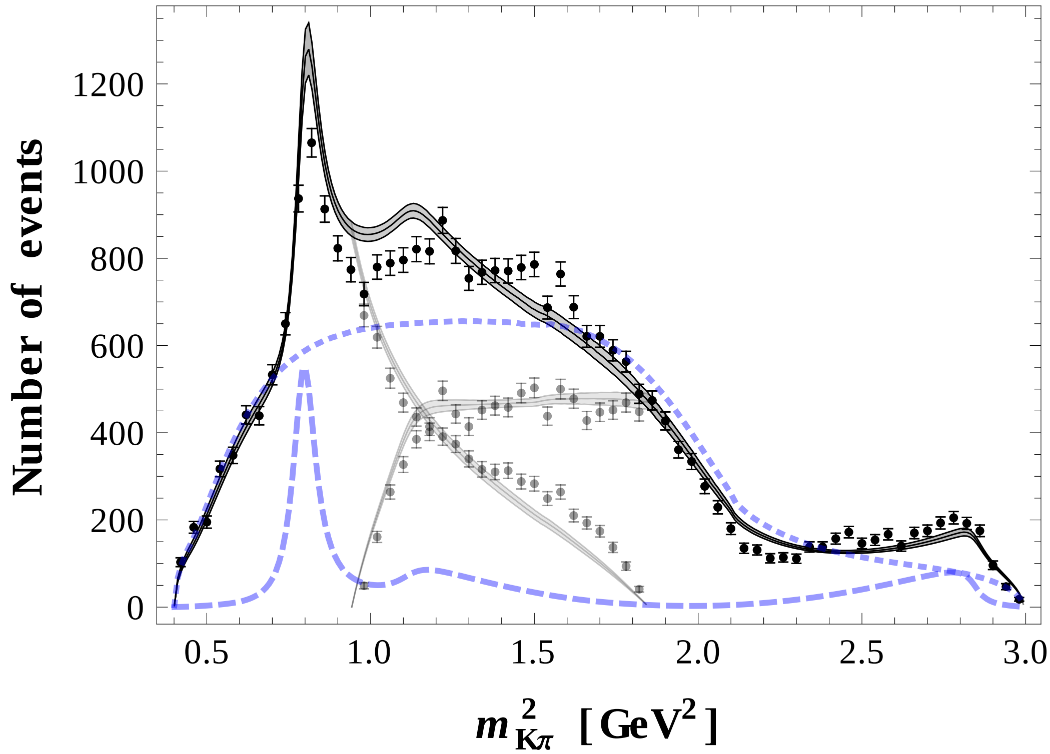

Figure 4: Differential decay width for $D^+ \to K^-\pi^+\pi^+$ compared to E791 data [32]. The dark gray band represents our model, while the light gray band represent the low- and high-mass parts of the spectra. The dotted (dashed) blue lines represent the scalar (vector) components in our model. The bands do not show errors from $a_{1,2}$, nor inherent uncertainties from the naïve factorization hypothesis.

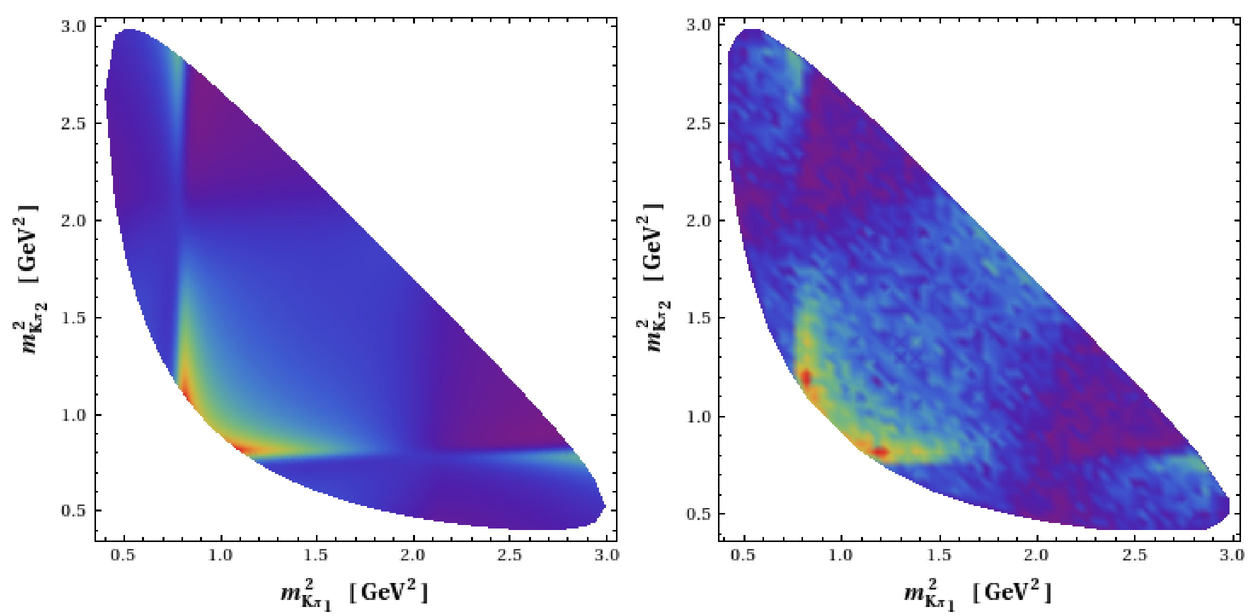

Figure 5: The symmetrized Dalitz plot for $D^+ \to K^-\pi^+\pi^+$ in our model (left) together with the experimental one from E791 [32] (right).

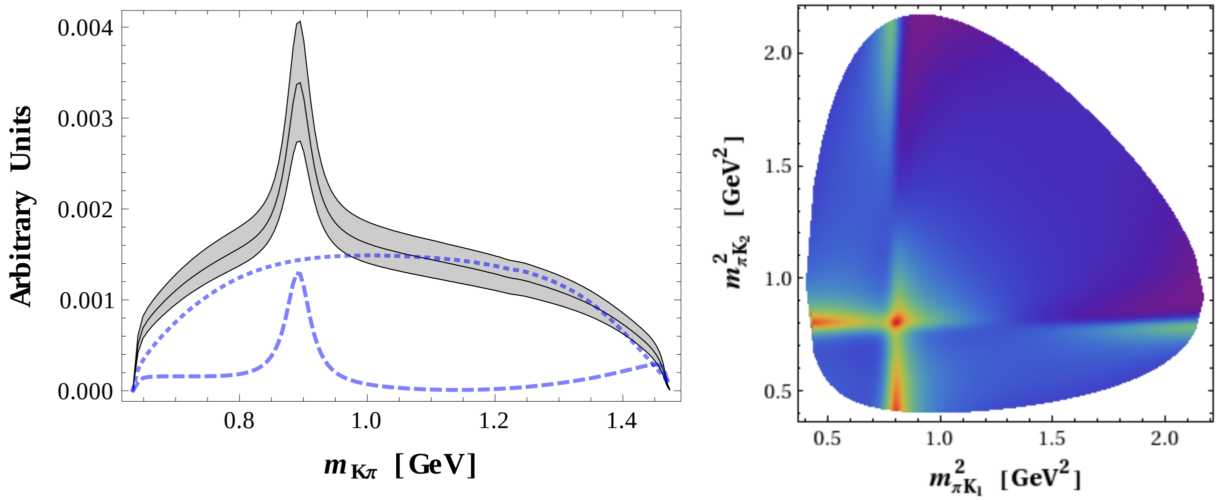

Figure 6: Left: invariant mass distribution (gray band) together with S-wave and P-wave contributions shown as dotted- and dashed-blue lines respectively. Right: Dalitz plot. Note in particular the depleted lower-left corner that requires the same relative scalar phase as in $D^+ \to K^-\pi^+\pi^+$ decays.

an additional phase and a moderate rescaling that might be ascribed to genuine three-body effects. To further test this hypothesis, we address the isospin-related $D_s^+ \to K^+K^+\pi^-$ decays.

5 Case II: $D_s^+ \to K^+K^+\pi^-$ decays

In order to shift to $D_s^+ \to K^+K^+\pi^-$ decays, the following replacements need to be done with respect to $D^+ \to K^-\pi^+\pi^+$ decays:

$$f_\pi \to f_K\,, \quad F_{+(0)}^{D\pi}(s) \to F_{+(0)}^{D_sK}(s)\,, \quad F_4^{D_{\ell 4}}(m_\pi^2, s, t) \to F_4^{D_{s\ell 4}}(m_K^2, s, t)\,, \tag{34}$$

as well as $V_{ud}V_{cs}^* \to V_{us}V_{cd}^*$, $m_{D^+} \to m_{D_s}$ and $m_{K^\pm} \leftrightarrow m_{\pi^\pm}$ where necessary. We take $f_K/f_\pi = 1.193(2)$ [11], $F_{+(0)}^{D_sK}(0) = 0.720(84)(13)$ [33] as well as effective masses $m_{D^{*0}_{(0)}}$. Regarding the semileptonic form factors, there are results in Refs. [33] that show a similar pattern for the relative strengths, but do not report the overall normalization. We assume it to be the same based on approximate U-spin symmetry. With our model above, we predict BR $= 1.5(2) \times 10^{-4}$, fully dominated by the uncertainties for $F_{+(0)}^{D_sK}(0)$. The result is in agreement with the experimental result BR $= 1.28(3) \times 10^{-4}$ [11]. In addition, we show the $m_{K\pi}$ spectra as well as the Dalitz plot in Fig. 6. We emphasize that the $(180-60)^\circ$ global phase in the scalar component brings again results in good agreement with the recent experimental analysis of the Dalitz plot at LHCb [34], that requires a depletion of events for $m_{K\pi}$ values below the $K^*(892)$ resonance, that is reassuring. The obtained branching fraction for P-wave is 13.6(4.7)% and for the S-wave 79.3(5.8)%. Ref. [35] offers estimates for the $K^*(892)$ component to yield a fit fraction of 47(22)(15)%, also in agreement, yet their result relies on their model describing the S wave, for which no branching fraction is given.

6 Conclusions

We have described the hadronic $D^+ \to K^-\pi^+\pi^+$ and $D_s^+ \to K^+K^+\pi^-$ decays using naïve factorization. Compared to previous work in Ref. [1], we have taken advantage of the precise data from semileptonic $D^+ \to K^-\pi^+\ell^+\nu_\ell$ decays from BES-III [2]. To do so, we have used a model for final-state $K\pi$ interactions fulfilling analitycity and unitarity constraints below higher inelasticities —these in general differ from the $K\pi$ scalar and vector form factors, so the corresponding parameters have been fixed based on the BES-III [2] analysis. These parametrizations might be of relevance for experimentalists. With the semileptonic form factor fixed, a parameter-free prediction for the P-wave BR in $D^+ \to K^-\pi^+\pi^+$ decays has been possible. We have obtained a reasonable prediction, and the BR in PDG [11] could be achieved with a minor rescaling of the semileptonic form factor, compatible with 1 at the 1σ level. This is remarkable, as it requires appropriate signs for the interference amongst the two contributions in Eq. (2), that is completely fixed in our approach. Also, we emphasize that uncertainties related to the factorization approach, or model-dependencies in extracting the P-wave BR in PDG [11], were not accounted for. To reproduce the full BR and Dalitz plot, a global phase of $(180-60)^\circ$ among the S and P waves was necessary, analogous to Ref. [1]. Furthermore, a moderate rescaling for the S-wave semileptonic form factor was necessary; both were ascribed to effective three-body effects not accounted for in our approach. The resulting description could be used to describe $D_s^+ \to K^+K^+\pi^-$ decays, satisfactorily describing existing data.

Acknowledgments

P. S-P thanks J. Nieves for discussions on semileptonic form factors. This work has been supported by the European Union's Horizon 2020 Research and Innovation Programme under grants 754510 (EU, H2020-MSCA-COFUND2016) and 824093 (H2020-INFRAIA-2018-1), the Spanish Ministry of Science and Innovation under grants PID2020-112965GB-I00 and PID2020-114767GB-I00, Junta de Andalucía under grants POSTDOC_21_00136 and FQM-225, and the Secretaria d'Universitats i Recerca del Departament d'Empresa i Coneixement de la Generalitat de Catalunya under grant 2021 SGR 00649. IFAE is partially funded by the CERCA program of the Generalitat de Catalunya.

A Definitions for $D^+ \to K^-\pi^+\ell^+\nu_\ell$ decays

A.1 Phase space and kinematics

For this process, we take the conventions in Ref. [36]. Note in particular that our lepton-hadron-plane angle (ϕ in the following) defined in Fig. 7 has opposite sign to that in Refs. [2, 16] (χ in the following). The phase space can be described in terms of the invariant masses $p^2_{K\pi(\ell\nu)} = s_{K\pi(\ell\nu)}$, angles in the hadronic/leptonic reference frames $\theta_{K\pi(\ell\nu)}$ and hadron-lepton planes angle. For the calculation all that is required is ($p_{ij} = p_i + p_j$,

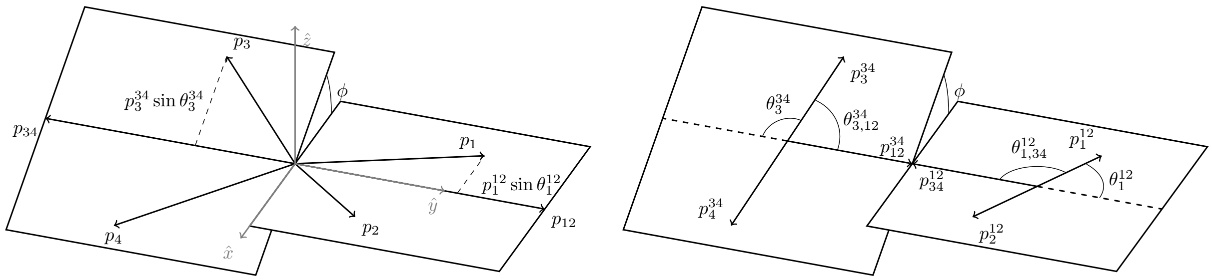

Figure 7: Definitions for the phase space variables in $D^+ \to K^-\pi^+\ell^+\nu_\ell$ decays. The particle labeling reads $\{1,2,3,4\} = \{K^-,\pi^+,\ell^+,\nu\}$.

$\bar{p}_{ij} = p_i - p_j$)

$$p_{K\pi}\cdot p_{\ell\nu} = (m_D^2 - s_{K\pi} - s_{\ell\nu})/2 \equiv z\,, \tag{35}$$

$$\bar{p}_{K\pi}\cdot p_{\ell\nu} = \tilde{\Delta}_{K\pi} z + X\beta_{K\pi}\cos\theta_{K\pi} \equiv \zeta\,, \tag{36}$$

$$p_{K\pi}\cdot \bar{p}_{\ell\nu} = \tilde{\Delta}_{\ell\nu} z + X\beta_{\ell\nu}\cos\theta_{\ell\nu}\,, \tag{37}$$

$$\begin{aligned}\bar{p}_{K\pi}\cdot \bar{p}_{\ell\nu} =& \Big[z\left(\tilde{\Delta}_{K\pi}\tilde{\Delta}_{\ell\nu} + \beta_{K\pi}\beta_{\ell\nu}\cos\theta_{K\pi}\cos\theta_{\ell\nu}\right) \\ &+ X\left(\tilde{\Delta}_{K\pi}\beta_{\ell\nu}\cos\theta_{\ell\nu} + \tilde{\Delta}_{\ell\nu}\beta_{K\pi}\cos\theta_{K\pi}\right)\Big] \\ &- \sqrt{s_{K\pi}s_{\ell\nu}}\beta_{K\pi}\beta_{\ell\nu}\sin\theta_{K\pi}\sin\theta_{\ell\nu}\cos\phi\,,\end{aligned} \tag{38}$$

$$\epsilon^{p_{K\pi}\bar{p}_{K\pi}p_{\ell\nu}\bar{p}_{\ell\nu}} = -X\sqrt{s_{K\pi}s_{\ell\nu}}\beta_{K\pi}\beta_{\ell\nu}\sin\theta_{K\pi}\sin\theta_{\ell\nu}\sin\phi\,, \tag{39}$$

where $\tilde{\Delta}_{ij} = (p_i^2 - p_j^2)/p_{ij}^2$, $\beta_{ij} = \lambda_{ij}^{1/2}/p_{ij}^2$, $X = \lambda_{K\pi,\ell\nu}^{1/2}/2$, and $\lambda_{ij} = [p_{ij}^2 - (p_i^2 + p_j^2)]^2 - 4p_i^2 p_j^2$. Finally, the differential phase space can be defined as

$$d\Phi_4 = \frac{1}{(4\pi)^6}\frac{1}{2m_D^2}X\beta_{K\pi}\beta_{\ell\nu}ds_{K\pi}ds_{\ell\nu}d\cos\theta_{K\pi}d\cos\theta_{\ell\nu}d\phi\,. \tag{40}$$

A.2 Decay width

Following Eq. (4) and the notation for the hadronic form factors in Eqs. (7) and (9) to (12), the differential decay width is given by

$$d\Gamma = \frac{G_F^2|V_{cs}|^2}{(4\pi)^6 m_D^3}X\beta_{K\pi}\beta_{\ell\nu}(H^{\mu\nu}L_{\mu\nu})ds_{K\pi}ds_{\ell\nu}d\cos\theta_{K\pi}d\cos\theta_{\ell\nu}d\phi\,. \tag{41}$$

Taking in parallel to Ref. [16] the following decomposition[7] (for corresponding CP-related D^- decays, $\phi\to-\phi$ needs to be taken)

$$\begin{aligned}H^{\mu\nu}L_{\mu\nu} \equiv\;& I_1 + I_2\cos 2\theta_{\ell\nu} + I_3\sin^2\theta_{\ell\nu}\cos 2\phi + I_4\sin 2\theta_{\ell\nu}\cos\phi + I_5\sin\theta_{\ell\nu}\cos\phi \\ &+ I_6\cos\theta_{\ell\nu} - I_7\sin\theta_{\ell\nu}\sin\phi - I_8\sin 2\theta_{\ell\nu}\sin\phi - I_9\sin^2\theta_{\ell\nu}\sin 2\phi\,,\end{aligned} \tag{42}$$

[7] Note in particular the minus sign in the I_{7-9} terms due to our ϕ definition.

the results in Ref. [16] are modified for finite lepton masses ($m_\nu = 0$) as follows

$$I_1 = \frac{1}{4}\beta_{\ell\nu}\left[(1+\frac{m_\ell^2}{s_{\ell\nu}})|F_1|^2 + \frac{3}{2}\sin^2\theta_{K\pi}(1+\frac{m_\ell^2}{3s_{\ell\nu}})(|F_2|^2+|F_3|^2) + \frac{2m_\ell^2}{s_{\ell\nu}}|F_4|^2\right], \tag{43}$$

$$I_2 = -\frac{1}{4}\beta_{\ell\nu}^2\left[|F_1|^2 - \frac{1}{2}\sin^2\theta_{K\pi}(|F_2|^2+|F_3|^2)\right], \tag{44}$$

$$I_3 = -\frac{1}{4}\beta_{\ell\nu}^2\left[|F_2|^2 - |F_3|^2\right]\sin^2\theta_{K\pi}, \tag{45}$$

$$I_4 = \frac{1}{2}\beta_{\ell\nu}^2\,\mathrm{Re}(F_1F_2^*)\sin\theta_{K\pi}, \tag{46}$$

$$I_5 = \beta_{\ell\nu}\,\mathrm{Re}\left[F_1F_3^* + \frac{m_\ell^2}{s_{\ell\nu}}F_4F_2^*\right]\sin\theta_{K\pi}, \tag{47}$$

$$I_6 = \beta_{\ell\nu}\,\mathrm{Re}\left[F_2F_3^*\sin^2\theta_{K\pi} - \frac{m_\ell^2}{s_{\ell\nu}}F_1F_4^*\right], \tag{48}$$

$$I_7 = \beta_{\ell\nu}\,\mathrm{Im}\left[F_1F_2^* + \frac{m_\ell^2}{s_{\ell\nu}}F_4F_3^*\right]\sin\theta_{K\pi}, \tag{49}$$

$$I_8 = \frac{1}{2}\beta_{\ell\nu}^2\,\mathrm{Im}(F_1F_3^*)\sin\theta_{K\pi}, \tag{50}$$

$$I_9 = -\frac{1}{2}\beta_{\ell\nu}^2\,\mathrm{Im}(F_2F_3^*)\sin^2\theta_{K\pi}, \tag{51}$$

which in the $m_\ell \to 0$ coincides with that in Ref. [16]. Note that the hadronic matrix element can also be expressed in terms of the F_i form factors as ($\xi = \Delta_{K\pi}X + z\beta_{K\pi}\cos\theta_{K\pi}$)

$$\begin{aligned}\langle K^-\pi^+|\,\bar{s}\gamma^\mu(1-\gamma^5)c\,|D^+\rangle &= \frac{iF_1}{X}(p_{K\pi}^\mu - p_{\ell\nu}^\mu\frac{z}{s_{\ell\nu}}) + \frac{iF_4}{s_{\ell\nu}}p_{\ell\nu}^\mu \\ &+ \frac{iF_2}{\beta_{K\pi}\sqrt{s_{K\pi}s_{\ell\nu}}}\left[(\bar{p}_{K\pi}^\mu - p_{\ell\nu}^\mu\frac{\zeta}{s_{\ell\nu}}) - \frac{\xi}{X}(p_{K\pi}^\mu - p_{\ell\nu}^\mu\frac{z}{s_{\ell\nu}})\right] - \frac{F_3}{\beta_{K\pi}X\sqrt{s_{K\pi}s_{\ell\nu}}}\epsilon^{\mu p_{\ell\nu}p_{K\pi}\bar{p}_{K\pi}}.\end{aligned} \tag{52}$$

B Definitions in $D^+ \to K^-\pi^+\pi^+$ decays

Following Eq. (2) and the notation in Section 3.2 and Ref. [1], the matrix element of this process can be expressed as $\mathcal{M} = -i\frac{G_F}{\sqrt{2}}V_{ud}V_{cs}^*[\mathcal{M}(s,t)+\mathcal{M}(t,s)]$, where

$$\begin{aligned}\mathcal{M}(s,t) = \frac{-a_1 f_\pi}{1-\frac{m_\pi^2}{m_{D_s}^2}}&\left[\chi_S^{\text{eff}}(m_D^2-s)F_0^{D_{\ell4}}(s) - N(s)F_+^{D_{\ell4}}(s)\frac{1}{2}(\chi_B^{\text{eff}} + \frac{m_D^2-s}{2}\chi_C^{\text{eff}})\right] \\ &+ a_2\left[\frac{(m_D^2-m_\pi^2)(m_K^2-m_\pi^2)}{s}F_0^{K\pi}(s)F_0^{D\pi}(s) + N(s)F_+^{K\pi}(s)F_+^{D\pi}(s)\right],\end{aligned} \tag{53}$$

with $F_{0,+}^{K\pi,D\pi}(s)$ standing for the relevant scalar(vector) form factors as defined in Ref. [1], and $N(s) = t-u-(m_D^2-m_\pi^2)(m_K^2-m_\pi^2)s^{-1}$ defined below Eq. (26). Consequently, the differential decay width can be expressed in terms of the Dalitz variables as

$$d\Gamma = \frac{1}{2}\frac{1}{(2\pi)^3}\frac{1}{32m_D^3}\frac{G_F^2|V_{ud}V_{cs}^*|^2}{2}|\mathcal{M}(s,t)+\mathcal{M}(t,s)|^2. \tag{54}$$

C The vector form factor description

The phase for the vector form factor is that of the following [28]

$$\tilde{f}_+^{K\pi} = \frac{m_{K^*}^2 - \left(\frac{192\pi}{\sigma_{K\pi}^3}\frac{\gamma_{K^*}}{m_{K^*}}\right) H_{K\pi}(0) + \gamma s}{m_{K^*}^2 - s - \left(\frac{192\pi}{\sigma_{K\pi}^3(m_{K^*}^2)}\frac{\gamma_{K^*}}{m_{K^*}}\right) H_{K\pi}(s)} - \frac{\gamma s}{m_{K^{*\prime}}^2 - s - \left(\frac{192\pi}{\sigma_{K\pi}^3(m_{K^{*\prime}}^2)}\frac{\gamma_{K^{*\prime}}}{m_{K^{*\prime}}}\right) H_{K\pi}(s)}, \quad (55)$$

with $\sigma_{K\pi}^2(s) = \lambda(s, m_K^2, m_\pi^2)/s^2$, where we used the Kahlén function $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$, and with

$$H_{K\pi}(s) = \frac{1}{(4\pi)^2}\frac{1}{12}\left[s\sigma_{K\pi}^2(s)\bar{B}_0(s; m_\pi^2, m_K^2) - \frac{s}{2}\ln\frac{m_\pi^2 m_K^2}{\mu^4} - \frac{(\Sigma_{K\pi}^2 - \Delta_{K\pi}^2 - \frac{s\Sigma_{K\pi}}{2})\ln\frac{m_K^2}{m_\pi^2}}{\Delta_{K\pi}} + \left(\frac{2}{3}s - 2\Sigma_{K\pi}\right)\right], \quad (56)$$

with $\Delta_{K\pi} = m_K^2 - m_\pi^2$ and $\Sigma_{K\pi} = m_K^2 + m_\pi^2$. The function $\bar{B}$ is defined in terms of the 1-loop two-point function $\bar{B}(s, m_K^2, m_\pi^2) = B(s, m_K^2, m_\pi^2) - B(0, m_K^2, m_\pi^2)$ and reads

$$\bar{B}(s, m_K^2, m_\pi^2) = \frac{1}{2}\left[2 + \left(\frac{\Delta_{K\pi}}{s} - \frac{\Sigma_{K\pi}}{\Delta_{K\pi}}\right)\ln\frac{m_\pi^2}{m_K^2} + 2\sigma_{K\pi}(s)\ln\left(\frac{\Sigma_{K\pi} + s\sigma_{K\pi} - s}{2m_K m_\pi}\right)\right]. \quad (57)$$

In order to match their poles position we use the same parameters $m_{K^*} = 0.94338(69)$ GeV, $\gamma_{K^*} = 0.06666(8)$ GeV and $m_{K^{*\prime}} = 1.379(36)$ GeV, $\gamma_{K^{*\prime}} = 0.196(66)$ GeV. Concerning γ, we choose $\gamma = 0$ instead of $\gamma = -0.034$ since BES-III finds no evidence for a $K^*(1410)$. Still, the model allows for an easy extension to study possible effects of the $K^*(1410)$.

References

[1] D. R. Boito and R. Escribano, Phys. Rev. **D80**, 054007 (2009), arXiv:0907.0189 [hep-ph] .

[2] M. Ablikim et al. (BESIII), Phys. Rev. **D94**, 032001 (2016), arXiv:1512.08627 [hep-ex] .

[3] P. C. Magalhaes, M. R. Robilotta, K. S. F. F. Guimaraes, T. Frederico, W. de Paula, I. Bediaga, A. C. d. Reis, C. M. Maekawa, and G. R. S. Zarnauskas, Phys. Rev. D **84**, 094001 (2011), arXiv:1105.5120 [hep-ph] .

[4] K. S. F. F. Guimarães, O. Lourenço, W. de Paula, T. Frederico, and A. C. dos Reis, JHEP **08**, 135 (2014), arXiv:1404.3797 [hep-ph] .

[5] P. C. Magalhães and M. R. Robilotta, Phys. Rev. D **92**, 094005 (2015), arXiv:1504.06346 [hep-ph] .

[6] S. X. Nakamura, Phys. Rev. D **93**, 014005 (2016), arXiv:1504.02557 [hep-ph] .

[7] F. Niecknig and B. Kubis, JHEP **10**, 142 (2015), arXiv:1509.03188 [hep-ph] .

[8] M. Diakonou and F. Diakonos, Phys. Lett. B **216**, 436 (1989).

[9] J. A. Oller, Phys. Rev. D **71**, 054030 (2005), arXiv:hep-ph/0411105 .

[10] A. J. Buras, Nucl. Phys. **B434**, 606 (1995), arXiv:hep-ph/9409309 [hep-ph] .

[11] P. A. Zyla et al. (Particle Data Group), PTEP **2020**, 083C01 (2020).

[12] M. Jamin, J. A. Oller, and A. Pich, Phys. Rev. **D74**, 074009 (2006), arXiv:hep-ph/0605095 [hep-ph] .

[13] D. R. Boito, R. Escribano, and M. Jamin, JHEP **09**, 031 (2010), arXiv:1007.1858 [hep-ph] .

[14] J. M. Link et al. (FOCUS), Phys. Lett. **B544**, 89 (2002), arXiv:hep-ex/0207049 [hep-ex] .

[15] R. A. Briere et al. (CLEO), Phys. Rev. **D81**, 112001 (2010), arXiv:1004.1954 [hep-ex] .

[16] C. L. Y. Lee, M. Lu, and M. B. Wise, Phys. Rev. **D46**, 5040 (1992).

[17] B. Bajc, S. Fajfer, R. J. Oakes, and T. N. Pham, Phys. Rev. **D58**, 054009 (1998), arXiv:hep-ph/9710422 [hep-ph] .

[18] V. Bernard, M. Oertel, E. Passemar, and J. Stern, Phys. Lett. B **638**, 480 (2006), arXiv:hep-ph/0603202 .

[19] V. Bernard, M. Oertel, E. Passemar, and J. Stern, Phys. Rev. D **80**, 034034 (2009), arXiv:0903.1654 [hep-ph] .

[20] V. Bernard, JHEP **06**, 082 (2014), arXiv:1311.2569 [hep-ph] .

[21] M. Wirbel, B. Stech, and M. Bauer, Z. Phys. **C29**, 637 (1985).

[22] C. W. Bernard, A. X. El-Khadra, and A. Soni, Phys. Rev. D **45**, 869 (1992).

[23] K. C. Bowler, N. M. Hazel, H. Hoeber, R. D. Kenway, D. G. Richards, L. Lellouch, J. Nieves, C. T. Sachrajda, and H. Wittig (UKQCD), Phys. Rev. D **51**, 4905 (1995), arXiv:hep-lat/9410012 .

[24] J. D. Richman and P. R. Burchat, Rev. Mod. Phys. **67**, 893 (1995), arXiv:hep-ph/9508250 [hep-ph] .

[25] D. Melikhov and B. Stech, Phys. Rev. **D62**, 014006 (2000), arXiv:hep-ph/0001113 [hep-ph] .

[26] B. Bajc, S. Fajfer, and R. J. Oakes, Phys. Rev. **D53**, 4957 (1996), arXiv:hep-ph/9511455 [hep-ph] .

[27] A. Khodjamirian, Hadron Form Factors: From Basic Phenomenology to QCD Sum Rules (CRC Press, Taylor & Francis Group, Boca Raton, FL, USA, 2020).

[28] R. Escribano, S. González-Solís, M. Jamin, and P. Roig, JHEP **09**, 042 (2014), arXiv:1407.6590 [hep-ph] .

[29] C. Albertus, E. Hernandez, J. Nieves, and J. M. Verde-Velasco, Phys. Rev. D **71**, 113006 (2005), arXiv:hep-ph/0502219 .

[30] V. Lubicz, L. Riggio, G. Salerno, S. Simula, and C. Tarantino (ETM), Phys. Rev. **D96**, 054514 (2017), [Erratum: Phys. Rev.D99,no.9,099902(2019)], arXiv:1706.03017 [hep-lat] .

[31] P. del Amo Sanchez et al. (BaBar), Phys. Rev. D **83**, 072001 (2011), arXiv:1012.1810 [hep-ex] .

[32] E. M. Aitala et al. (E791), Phys. Rev. Lett. **89**, 121801 (2002), arXiv:hep-ex/0204018 .

[33] M. Ablikim et al. (BESIII), Phys. Rev. Lett. **122**, 061801 (2019), arXiv:1811.02911 [hep-ex] .

[34] R. Aaij et al. (LHCb), JHEP **03**, 176 (2019), arXiv:1810.03138 [hep-ex] .

[35] P. del Amo Sanchez et al. (BaBar), Phys. Rev. **D83**, 052001 (2011), arXiv:1011.4190 [hep-ex] .

[36] K. Kampf, J. Novotný, and P. Sanchez-Puertas, Phys. Rev. **D97**, 056010 (2018), arXiv:1801.06067 [hep-ph] .