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COTORSION PAIRS AND TOR-PAIRS OVER COMMUTATIVE NOETHERIAN RINGS

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ABSTRACT. For a commutative noetherian ring R , we classify all the hereditary cotorsion pairs cogenerated by pure-injective modules of finite injective dimension. The classification is done in terms of integer-valued functions on the spectrum of the ring. Each such function gives rise to a system of local depth conditions which describes the left-hand class in the corresponding cotorsion pair. Furthermore, we show that these cotorsion pairs correspond by explicit duality to hereditary Tor -pairs generated by modules of finite flat dimension.

1. INTRODUCTION

There have been many successful classification theorems of pure-injective modules over specific rings. However, pure-injective modules over even a well-understood ring, such as a noetherian ring of global dimension greater than one, can have pathological behaviour [Jen70]. Given that describing them explicitly can be a daunting task, another approach is instead to associate a class of modules to a given a collection of pure-injective modules. In this paper, we consider the left orthogonal with respect to the Ext functor, in particular over commutative noetherian rings.

These classes have several appealing properties. Specifically, they are known to be closed under direct limits, as well as having good approximation properties, specifically, they are covering classes. In other terminology, these classes are the left constituents of perfect complete cotorsion pairs, a non-trivial consequence of being cogenerated by pure-injective modules, a result which was used to prove the long-standing question of whether flat covers necessarily exist [BEBE01].

Important examples of classes that are Ext -orthogonal to a collection of pure-injective modules are the classes appearing in a Tor -pair. Tor -pairs were introduced as an analogue of cotorsion pairs, that is pairs of classes which are orthogonal with respect to the Tor functor instead of the Ext functor. Via a Tor - Ext duality induced by the hom-tensor adjunction, all classes which appear in a Tor -pair arise as a left-hand class of a complete cotorsion pair cogenerated by the character dual of a module (recall, that these are always pure-injective).

Going back to general cotorsion pairs cogenerated by pure-injective modules, we recall that if there is a bound on the injective dimensions of the pure-injective modules and additionally one imposes the condition that the left-hand class of the cotorsion pairs is definable, that is, in particular closed under arbitrary products, then these cotorsion pairs are exactly the cotilting cotorsion pairs, see Proposition 2.7.

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Such cotorsion pairs over commutative noetherian rings [AHPŠT14], and even over certain general commutative rings, [Hrb16, HŠ20, Baz07, BH21] have been classified in terms of various ring-theoretic data. In particular, the cotilting cotorsion pairs of cofinite type, which are the ones that appear as a Tor -orthogonal class to a family of strongly finitely presented modules of infinite projective dimension, are classified using the spectrum of the ring, or more specifically, Thomason subsets. Over commutative noetherian rings all cotilting classes are of cofinite type [AHPŠT14]. It is also interesting to note that in the characterisations in [AHPŠT14, Hrb16, HŠ20], the Koszul complex with respect to a finitely generated ideal plays a pivotal role.

In this paper, we are principally interested in commutative noetherian rings. In Theorem 4.17, we show that the cotorsion pairs cogenerated by pure-injective modules of finite injective dimension are classified by functions from the spectrum to the non-negative integers, which are bounded by the local depth of the prime. The left-hand constituent of the associated cotorsion pair consists precisely of those modules whose local depths are bounded by the corresponding function. As we detail in Corollary 4.20, our results nicely extend the classification of cotilting cotorsion pairs in the commutative noetherian case, which corresponds to those functions which are in addition order-preserving.

Next, we pay attention to Tor -pairs. It follows that, over any given ring, there is a set of Tor -pairs, and the size of the set has a bound that depends on the cardinality of the ring, we recall this bound in Corollary 2.3. By duality, the Tor -orthogonal classes to a class of modules of finite flat dimension, are also classes that are Ext -orthogonal to a class of pure-injective modules of finite injective dimension. As an outcome of our work, we show that in the commutative noetherian setting, the converse is true as well, so that the Ext -orthogonal classes we consider come from a Tor -pair. So our classification Theorem 4.17 is also a classification of Tor -pairs generated by modules of finite flat dimension. The choice of generators can be made explicit — the function gives rise to a collection of suitable cocycle modules in localized Koszul complexes associated to each prime ideal, see Proposition 4.14.

Characterisation of Tor -pairs over a commutative noetherian ring R was posed as an open problem in the monograph of Göbel and Trlifaj [GT12, §16.5, 3.]. This was motivated by the aforementioned classification of cotilting classes, as well as the fact that if R is hereditary then constituents of Tor -pairs coincide with cotilting classes [GT12, Theorem 16.31]. In §5, we show that if R is regular then our main classification result covers all hereditary Tor -pairs and all hereditary cotorsion pairs cogenerated by pure-injectives. On the other hand, in the singular case there are hereditary Tor -pairs which do not fit into our classification, see Example 4.6, Example 4.8, and Remark 5.6. Leaving the noetherian setting for a short while, we will also supplement this in Example 2.9 by producing a commutative semihereditary ring (in fact, a valuation domain), over which there are Tor -pairs which contain a class which is not a cotilting class.

Additionally, while cotilting classes originate in representation theory, the present work also extends ideas that were studied in commutative algebra and module theory. For example, the large and small restricted flat dimension introduced by Christensen, Frankild and Foxby [CFF02] can be interpreted in terms of suitable Tor -pairs, and their results on when these two dimension theories coincide can be interpreted as a special case of our classification, see §6.

We begin this manuscript with some background the aptly named §2, which begins over an arbitrary ring before specialising to commutative rings and then commutative noetherian rings, introducing the machinery we will need at each stage. Next we introduce some preliminary results in §3 on Tor -pairs, cotorsion pairs and pure-injective modules over commutative noetherian rings. The bulk of

the main results of this paper are in §4, which all take place over a commutative noetherian ring, except for examples. The main classification is stated in Theorem 4.17. Along the way we provide various examples of Tor-pairs which do not fit various hypotheses. We specialise to regular rings in §5 and show that over them our main result gives a classification of all hereditary Tor-pairs. In the final §6, we show that a special case of our classification can be used to characterise almost Cohen–Macaulay rings in terms of small and large restricted flat dimensions being equal, providing a global version of [CFF02, Theorem 3.2]. This last argument also employs the recent study of Gvorov–Lazard theorem in higher dimensions of [HLG24].

2. BACKGROUND

Let R be an associative unital ring. We denote by $\mathbf{Mod}\text{-}R$ the category of all right R -modules and by $\mathbf{mod}\text{-}R$ be the (full, isomorphism-closed) subcategory consisting of right R -modules which admit a resolution by finitely generated projective R -modules, which we refer to as the *strongly finitely presented modules*. For a module $M \in \mathbf{Mod}\text{-}R$, we denote the *character module* of M by $M^+ := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

Fix a class \mathcal{C} . We let $\mathbf{Sub}(\mathcal{C})$ denote the class of modules which are isomorphic to a submodule of \mathcal{C} .

2.1. For $n \geq 0$, let $\mathcal{P}_n(R) = \{M \in \mathbf{Mod}\text{-}R \mid \mathrm{pd}_R M \leq n\}$, $\mathcal{I}_n(R) = \{M \in \mathbf{Mod}\text{-}R \mid \mathrm{id}_R M \leq n\}$, and $\mathcal{F}_n(R) = \{M \in \mathbf{Mod}\text{-}R \mid \mathrm{fd}_R M \leq n\}$ denote the subcategories of $\mathbf{Mod}\text{-}R$ consisting of all modules of projective, injective, and respectively flat dimension bounded above by n . We put $\mathcal{P}_n^{<\aleph_0}(R) = \mathcal{P}_n(R) \cap \mathbf{mod}\text{-}R$. We use the notation $\mathcal{P}(R) = \bigcup_{n \geq 0} \mathcal{P}_n(R)$ for modules of finite projective dimension, similarly we put $\mathcal{F}(R) = \bigcup_{n \geq 0} \mathcal{F}_n(R)$ and $\mathcal{P}^{<\aleph_0}(R) = \bigcup_{n \geq 0} \mathcal{P}_n^{<\aleph_0}(R)$. We often omit the reference to the ring when the ring is clear from the context and write simply \mathcal{P}_n , \mathcal{I}_n , \mathcal{F}_n , \mathcal{P} , \mathcal{F} , or $\mathcal{P}_n^{<\aleph_0}$ and $\mathcal{P}^{<\aleph_0}$.

For an R -module M , $\Omega_i(M)$ is defined to be an i th syzygy if $i > 0$, or, the i th minimal cosyzygy if $i < 0$. By convention, $\Omega_0(M) = M$, and $\Omega_i(M)$ is some fixed chosen representative up to a projective direct summand for $i > 0$. We shall also use the following less usual notion. Let the following be the minimal flat presentation of M .

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{d_0} M \longrightarrow 0,$$

Then the i th yoke, denoted by $\Upsilon_i(M)$, of M is defined to be $\mathrm{Ker}(d_{i-1})$ for $i \geq 1$, and the 0th yoke is M itself.

For a class \mathcal{X} in $\mathbf{Mod}\text{-}R$, we let

$$\mathcal{X}^{\top_1} = \{M \in R\text{-Mod} \mid \mathrm{Tor}_1^R(X, M) = 0, \forall X \in \mathcal{X}\}$$

and

$$\mathcal{X}^{\top_{\geq i}} = \{M \in R\text{-Mod} \mid \mathrm{Tor}_j^R(X, M) = 0 \forall X \in \mathcal{X}, \forall j \geq i\},$$

and for brevity,

$$\mathcal{X}^{\top} := \mathcal{X}^{\top_{\geq 1}}.$$

Analogously, for a class \mathcal{Y} in $R\text{-Mod}$ we can define the classes ${}^{\top_1}\mathcal{Y}$, ${}^{\top_{\geq i}}\mathcal{Y}$ and ${}^{\top}\mathcal{Y}$ in $\mathbf{Mod}\text{-}R$.

For a class \mathcal{X} in $\mathbf{Mod}\text{-}R$, we let

$$\mathcal{X}^{\perp_1} = \{M \in \mathbf{Mod}\text{-}R \mid \mathrm{Ext}_R^1(X, M) = 0, \forall X \in \mathcal{X}\}$$

, and

$$\mathcal{X}^{\perp_{\geq i}} = \{M \in \mathbf{Mod}\text{-}R \mid \mathrm{Ext}_R^j(X, M) = 0 \forall X \in \mathcal{X}, \forall j \geq i\},$$

and for brevity,

$$\mathcal{X}^\perp := \mathcal{X}^{\perp_{\geq 1}}.$$

Analogously, for a class \mathcal{Y} in $\text{Mod-}R$ we can define the classes ${}^{\perp_1}\mathcal{Y}$, ${}^{\perp_{\geq 1}}\mathcal{Y}$ and ${}^{\perp}\mathcal{Y}$ in $\text{Mod-}R$.

2.2. Homological formulae. It is well known that the $\text{Hom-}\otimes$ adjunction yields the following natural isomorphism

$$\text{Ext}_R^i(A, \text{Hom}_S(B, C)) \cong \text{Hom}_S(\text{Tor}_i^R(A, B), C)$$

for any $i \geq 0$, where $M \in \text{Mod-}R$, $B \in R\text{-Mod-}S$ and $C \in \text{Mod-}S$ is injective. If C_S is, in addition, an injective cogenerator this yields that $\text{Ext}_R^n(A, \text{Hom}_S(B, C)) = 0$ if and only if $\text{Tor}_n^R(A, B) = 0$.

An R -module M is *cotorsion* if $\text{Ext}_R^1(F, M)$ vanishes for every $F \in \mathcal{F}_0(R)$.

Let R and S be rings, consider the modules $A \in \text{Mod-}R$, $B \in R\text{-Mod-}S$ which is flat both as an R -module and as an S -module, and $C \in \text{Mod-}S$ a cotorsion S -module. Then, by the derived $\text{Hom-}\otimes$ adjunction, there are the following natural isomorphisms as abelian groups for every $i \geq 0$.

$$(2.1) \quad \text{Ext}_S^i(A \otimes_R B, C) \cong \text{Ext}_R^i(A, \text{Hom}_S(B, C))$$

2.3. Pure-injective modules. A module N is *pure-injective* if $\text{Hom}_R(B, N) \rightarrow \text{Hom}_R(A, N)$ is an epimorphism for any pure embedding $A \rightarrow B$. In other words, the pure-injective modules are the modules which are injective with respect to pure monomorphisms. In particular, all pure-injective modules are cotorsion, and in fact a module is flat if and only if $\text{Ext}_R^1(F, N)$ vanishes for every pure-injective module N . We let \mathcal{PI} denote the class of pure-injective modules.

We will now prove another useful homological formula. For an R -module N and any directed system $\{M_i\}_{i \in I}$, there is the following natural isomorphism of abelian groups.

$$\text{Hom}_R(\varinjlim_I M_i, N) \cong \varprojlim_I \text{Hom}_R(M_i, N)$$

By a well-known result of Auslander, if N is additionally pure-injective, then the same result also holds for higher Ext -groups, see [GT12, Lemma 6.28].

$$\text{Ext}_R^j(\varinjlim_I M_i, N) \cong \varprojlim_I \text{Ext}_R^j(M_i, N)$$

Lemma 2.1. *Let R be a ring, $F \in \text{Mod-}R$ a flat R -module, $Y \in R\text{-Mod-}S$ and $N \in \text{Mod-}S$ a pure-injective module. Then for every $j \geq 0$, there is the following natural isomorphism of abelian groups.*

$$\text{Ext}_S^j(F \otimes_R Y, N) \cong \text{Hom}_R(F, \text{Ext}_R^j(Y, N))$$

Proof. Let $F = \varinjlim_{i \in I} F_i$ where each F_i is a finitely generated free R -module. Then there are the following natural isomorphisms.

$$\begin{aligned} \text{Ext}_S^j(F \otimes_R Y, N) &\cong \text{Ext}_S^j(\varinjlim_{i \in I} F_i \otimes_R Y, N) \\ &\cong \text{Ext}_S^j(\varinjlim_{i \in I} (F_i) \otimes_R Y, N) \\ &\cong \varprojlim_{i \in I} \text{Ext}_S^j(F_i \otimes_R Y, N) \\ &\cong \varprojlim_{i \in I} \text{Hom}_R(F_i, \text{Ext}_R^j(Y, N)) \\ &\cong \text{Hom}_R(F, \text{Ext}_R^j(Y, N)) \end{aligned}$$

□

2.4. Cotorsion pairs. A *cotorsion pair* is a pair of classes $(\mathcal{A}, \mathcal{B})$ in $\text{Mod-}R$, such that $\mathcal{A} = {}^{\perp 1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp 1}$. A cotorsion pair is *hereditary* if moreover $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$, that is, all the higher Ext -groups vanish as well.

Let \mathcal{X} be a class in $\text{Mod-}R$. The cotorsion pair *generated* by \mathcal{X} is the cotorsion pair $({}^{\perp 1}(\mathcal{X}^{\perp 1}), \mathcal{X}^{\perp 1})$. The cotorsion pair *cogenerated* by \mathcal{X} is the cotorsion pair $({}^{\perp 1}\mathcal{X}, ({}^{\perp 1}\mathcal{X})^{\perp 1})$.

The left-hand class in a cotorsion pair is closed under direct sums and extensions, and contains all the projective modules. The right-hand class in a cotorsion pair is closed under products and extensions, and contains all the injective modules. A class \mathcal{X} in $\text{Mod-}R$ is *resolving* if \mathcal{X} is closed under extensions, contains the projectives, and is closed under kernels of epimorphisms. Dually, a class \mathcal{X} in $\text{Mod-}R$ is *coresolving* if \mathcal{X} is closed under extensions, contains the injectives, and is closed under cokernels of monomorphisms. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is hereditary if and only if \mathcal{A} is a resolving class if and only if \mathcal{B} is a coresolving class.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *complete* if \mathcal{A} is special precovering, or equivalently, if \mathcal{B} is special preenveloping, see [GT12, §5 and §6] for details.

2.5. Tor-pairs. A *Tor-pair* is a pair of classes $(\mathcal{E}, \mathcal{C})^{\perp}$ with \mathcal{E} in $\text{Mod-}R$ and \mathcal{C} in $R\text{-Mod}$, such that $\mathcal{E} = {}^{\perp 1}\mathcal{C}$ and $\mathcal{C} = \mathcal{E}^{\perp 1}$. A Tor-pair is *hereditary* if moreover $\mathcal{E} = {}^{\perp}\mathcal{C}$ and $\mathcal{C} = \mathcal{E}^{\perp}$, that is, all the higher Tor-groups vanish as well. In particular, $(\mathcal{F}_n, \mathcal{F}_n^{\perp})^{\perp}$ forms a hereditary Tor-pair for every $n \geq 0$.

Let \mathcal{X} be a class in $\text{Mod-}R$. The Tor-pair *generated* by \mathcal{X} is the Tor-pair of the form $({}^{\perp 1}(\mathcal{X}^{\perp 1}), \mathcal{X}^{\perp 1})^{\perp}$. If \mathcal{X} is closed under yokes, then the Tor-pair it generates is hereditary. Both classes in Tor-pairs are closed under direct limits, pure submodules, pure epimorphisms and arbitrary direct sums. Over commutative rings, it is clear that if $(\mathcal{E}, \mathcal{C})^{\perp}$ is a Tor-pair, then also $(\mathcal{C}, \mathcal{E})^{\perp}$ is a (distinct) Tor-pair.

2.6. The lattice of Tor-pairs and cotorsion pairs. The collection of cotorsion pairs in $\text{Mod-}R$ partially ordered by inclusion in their first component forms a complete lattice, denoted L_{Ext_1} , with the cotorsion pairs $(\mathcal{P}_0, \text{Mod-}R)$ and $(\text{Mod-}R, \mathcal{I}_0)$ forming the smallest and largest element respectively. In general, the lattice L_{Ext_1} is class-sized [GT12, Example 5.16].

Similarly, the collection of Tor-pairs partially ordered by inclusion in their first component form a complete lattice, denoted L_{Tor_1} . The Tor-pairs $(\mathcal{F}_0, R\text{-Mod})^{\perp}$ and $(\text{Mod-}R, \mathcal{F}_0)^{\perp}$ form the smallest and largest element respectively. The lattice L_{Tor_1} can be canonically embedded into L_{Ext_1} via the following assignment.

Lemma 2.2. [GT12, Lemma 2.16(b), Lemma 5.17] *Let $(\mathcal{E}, \mathcal{C})^{\perp}$ be a Tor-pair. Then $(\mathcal{E}, \mathcal{E}^{\perp 1})$ is a cotorsion pair which is cogenerated by $\mathcal{C}^+ := \{C^+ \mid C \in \mathcal{C}\}$. Moreover, if $(\mathcal{E}, \mathcal{C})^{\perp}$ is hereditary, so is $(\mathcal{E}, \mathcal{E}^{\perp 1})$.*

In contrast to the case of cotorsion pairs, the lattice of Tor-pairs forms a set.

Corollary 2.3. [GT12, Corollary 6.24] *Let R be a ring and $\kappa = |R| + \aleph_0$. Then $|L_{\text{Tor}_1}| \leq 2^{2^\kappa}$.*

The question remains of whether one can classify all the Tor-pairs over a particular ring; in case of commutative noetherian rings, this was formulated in [GT12, §16.5, 3.]. The Tor-pairs over a Dedekind domain have been completely characterised in terms of subsets of the spectrum. Moreover, the Tor-pairs over a right hereditary ring have been classified in terms of resolving subcategories of $\text{mod-}R$, see [GT12, Theorem 16.29 and 16.31]. We will show that it is not possible to classify all the Tor-pairs in such a way over an arbitrary commutative noetherian ring, that is, either in terms of the spectrum or in terms of resolving subcategories. Instead, we restrict to a specific type of Tor-pair.

2.7. Hereditary cotorsion pairs cogenerated by pure-injectives. Let $(\mathcal{C}, \mathcal{W})$ be a hereditary cotorsion pair cogenerated by a class \mathcal{P} of pure-injective R -modules, that is, $\mathcal{C} = {}^{\perp}\mathcal{P}$. Then $(\mathcal{C}, \mathcal{W})$ is complete and perfect, the latter meaning that \mathcal{C} is closed under direct limits, see [GT12, §6]. As stated in Lemma 2.2, a hereditary \mathbf{Tor} -pair $(\mathcal{E}, \mathcal{C})^{\top}$ gives rise to a hereditary cotorsion pair $(\mathcal{C}, \mathcal{W})$ cogenerated by pure-injectives via the adjunction formula $\mathcal{C} = {}^{\perp>0}(\mathcal{E}^+)$. This in fact gives a lattice isomorphism between the lattice $L_{\mathbf{Tor}_1}$ of hereditary \mathbf{Tor} -pairs and the sublattice of $L_{\mathbf{Ext}^1}$ consisting of those hereditary cotorsion pairs which are cogenerated by character dual modules, which in turn embeds into the sublattice of $L_{\mathbf{Ext}^1}$ consisting of those hereditary cotorsion pairs which are cogenerated by pure-injective modules.

Remark 2.4. We do not know whether each hereditary cotorsion pair cogenerated by pure-injective modules is induced by a hereditary \mathbf{Tor} -pair in the above sense, or equivalently, is cogenerated by character duals. In §4, we show that this holds for a commutative noetherian ring when restricted to pure-injective modules of finite injective dimension. In the last section, we show that this covers all cotorsion pairs cogenerated by pure-injectives if the ring is regular.

Outside of commutative noetherian rings, one could ask the same question for valuation domains. In this case, each pure-injective module is of injective dimension at most one. Bazzoni proved in [Baz15, Theorem 7.11] that the left-hand class of a cotorsion pair cogenerated by a cotilting module is a class in a \mathbf{Tor} -pair, see §2.9 for results on cotilting modules. However, we do not know if this is necessarily the case for an arbitrary pure-injective, even of finite injective dimension or a cotilting module over more general families of rings.

The question is similar in shape to asking whether each cohomological Bousfield class in the derived module category is a homological Bousfield class. For details, we refer to Stevenson's paper [Ste14], in which he proves in particular that there is a commutative von Neumann regular ring over which a cohomological Bousfield class exists which is not a homological Bousfield class, see [Ste14, Corollary 4.12] in particular.

2.8. Generation of \mathbf{Tor} -pairs. We call a class \mathcal{C} of R -modules *definable* if it is closed under products, direct limits, and pure submodules. It is straightforward to see that the \mathbf{Tor} -orthogonal of a set of finitely presented modules is a definable class. The converse does not hold, however the following theorem describes exactly when this happens. An analogous result for cotorsion pairs is known to hold, see [Š18, Theorem 6.1].

Theorem 2.5. [Her14, Theorem 3.11] *Let \mathcal{X} be a class of right R -modules and let $\mathcal{C} = \mathcal{X}^{\top 1}$. Then \mathcal{C} is definable if and only if there exists a class of countably presented modules \mathcal{S} such that*

- (1) $\mathcal{X} \subseteq \varinjlim \mathcal{S}$,
- (2) $\mathcal{S}^{\top 1} = \mathcal{C}$, and
- (3) the first syzygy of any module in \mathcal{S} is \mathcal{C} -Mittag Leffler.

In particular, if $(\mathcal{E}, \mathcal{C})^{\top}$ is a (not-necessarily hereditary) \mathbf{Tor} -pair such that \mathcal{C} is definable, then there exists a set of countably presented modules \mathcal{S} which generates the \mathbf{Tor} -pair and \mathcal{E} coincides with the direct limit of its countably presented constituents.

2.9. Cotilting classes.

Definition 2.6. A left R -module C is an *n-cotilting module* if the following conditions hold.

- (C1) $\text{id}(C) \leq n$
- (C2) $\text{Ext}_R^i(C^I, C) = 0$ for all $i > 0$ and any set I .

(C3) There exists a $\text{Prod}(C)$ -resolution of the injective cogenerator W of $\text{Mod-}R$. That is, there is an exact sequence of the following form where $C^i \in \text{Prod}(C)$ for each $0 \leq i \leq n$.

$$0 \rightarrow C^n \rightarrow \cdots \rightarrow C^1 \rightarrow C^0 \rightarrow W \rightarrow 0.$$

The class ${}^\perp C$ is called the *cotilting class* associated to C . In particular, a cotilting module is always a pure-injective module of finite injective dimension [Baz03, Što06].

Equivalently, a class of modules \mathcal{C} is a cotilting class if and only if it is resolving, covering, closed under direct products and direct summands, and $\mathcal{C}^\perp \subseteq \mathcal{I}_n(R)$. In particular, cotilting classes are definable classes. See [GT12, §15] for details.

A class \mathcal{X} of left R -modules is of *cofinite type* if there exists a set $\mathcal{S} \subseteq \text{mod-}R \cap \mathcal{P}_n$ of right R -modules such that $\mathcal{S}^\top = \mathcal{X}$. All classes of cofinite type are cotilting classes, but not all cotilting classes are of cofinite type, even over commutative rings, see for example [GT12, Example 15.33]. However, over commutative noetherian rings, all cotilting classes are of cofinite type [AHPŠT14, Theorem 4.2].

By the following proposition, the hereditary Tor-pairs generated by classes of modules of flat dimension at most n where the right-hand class is a definable class are exactly the n -cotilting Tor-pairs for $n \geq 0$.

Proposition 2.7. [AHPŠT14, Proposition 3.14], [AHT04, Theorem 2.3] *Let R be any ring and $(\mathcal{E}, \mathcal{C})^\top$ be a hereditary Tor-pair with $\mathcal{E} \subseteq \mathcal{F}_n$ for some $n \geq 0$. Then \mathcal{C} is definable if and only if \mathcal{C} is an n -cotilting class.*

Moreover, \mathcal{C} is an n -cotilting class of cofinite type if and only if $\mathcal{E} = \varinjlim \mathcal{E}^{<\aleph_0} = {}^\top((\mathcal{E}^{<\aleph_0})^\top)$.

Proof. For the first statement, we comment that the n -th syzygy of a module in $\mathcal{E} \subseteq \mathcal{F}_n$ is flat, so will belong to \mathcal{C} .

The second equivalence holds as an n -cotilting class is of cofinite type if and only if there exists $\mathcal{S} \subseteq \mathcal{P}_n^{<\aleph_0}$ such that $\mathcal{S}^\top = \mathcal{C}$, so in particular $\mathcal{E} = {}^\top((\mathcal{E}^{<\aleph_0})^\top)$. That ${}^\top((\mathcal{E}^{<\aleph_0})^\top) = \varinjlim (\mathcal{E}^{<\aleph_0})$ holds by [AHT04, Theorem 2.3]. \square

Remark 2.8. Bazzoni showed that over a valuation domain, there exist cotilting classes which are not of cofinite type, [Baz07, Proposition 4.5]. Thus, there exist definable classes in hereditary Tor-pairs generated by modules of bounded flat dimension which are not of cofinite type, even over commutative (semihereditary) rings.

Additionally, as mentioned above, by [Baz15, Theorem 6.11] every cotilting class over a valuation domains is a class in a Tor-pair. As valuation domains are of weak global dimension at most one, every Tor-pair is hereditary and generated by modules of flat dimension at most one.

However, does it follow from Theorem 2.5 that every cotilting class over any ring is of cocountable type? This holds for valuation domains and commutative noetherian rings. What is missing is to show that the left-hand class in a cotorsion pair cogenerated by a cotilting module is a class in a Tor-pair, see Remark 2.4.

Let R be a commutative noetherian ring. In view of Proposition 2.7, a *cotilting Tor-pair* is a hereditary Tor-pair $(\mathcal{E}, \mathcal{C})^\top$ generated by a class of modules of finite flat dimension and where the right-hand class a definable class. If additionally a cotilting Tor-pair is generated by a class of modules of flat dimension at most n , then it is an n -cotilting Tor-pair. Given a cotilting Tor-pair $(\mathcal{E}, \mathcal{C})$, Proposition 2.7 yields that $\mathcal{C}_\mathfrak{p}$ is a cotilting class for all $\mathfrak{p} \in \text{Spec}(R)$.

By [GT12, Theorem 16.31], over a right hereditary ring all Tor-pairs are classified in terms of resolving subcategories of $\text{mod-}R$. This implies that every class in a

Tor -pair over a commutative hereditary ring is a cotilting class of cofinite type. We show that this is not necessarily the case over semihereditary commutative rings, in particular over a commutative valuation domain. Thus, in this setting, there exist classes in Tor -pairs which arise as left- Ext -orthogonals of pure-injective modules of finite injective dimension that are not cotilting classes.

Example 2.9. Let R be a (rank one) maximal valuation domain with idempotent maximal ideal \mathfrak{m} , fraction field Q and valuation $v: Q \rightarrow \Gamma$. As there is an idempotent prime (maximal) ideal, there is a subgroup of $v(R)$ order-isomorphic to the rationals \mathbb{Q} . Recall that over a valuation domain, every module is of flat dimension at most one, and all pure-injective modules are of injective dimension at most one. In particular, R being maximal implies that all ideals of R (including the regular module R) are pure-injective R -modules, see [FS01, Theorem XIII.5.2 and Theorem XV.3.2].

Moreover, by [Baz08, Proposition 4.5], in this setting the module $C := Q \oplus R \oplus R/\mathfrak{m}$ is a 1-cotilting module not of cofinite type, and the cotilting class ${}^\perp C$ is exactly the class of modules which are extensions of flat modules by direct sums of copies of the residue field R/\mathfrak{m} , that is

$$\mathcal{C} := {}^\perp C = \{X \in \text{Mod-}R \mid 0 \rightarrow (R/\mathfrak{m})^{(\alpha)} \rightarrow X \rightarrow F \rightarrow 0, F \in \text{Flat-}R\}.$$

In particular, we note that ${}^\perp R/\mathfrak{m} \subsetneq {}^\perp R = {}^\perp C$ since \mathfrak{m} is of injective dimension one and Q is injective, and, for example, $R/s\mathfrak{m} \in {}^\perp R/\mathfrak{m} \setminus {}^\perp R$ where s is a non-zero element of R .

Since R/\mathfrak{m} is simple, ${}^\perp R/\mathfrak{m} = R/\mathfrak{m}^\perp$, so we conclude that $(\mathcal{C}, {}^\perp R/\mathfrak{m})^\perp$ is a hereditary Tor -pair by the structure of the cotilting class \mathcal{C} described above, and that all module have flat dimension at most one. Moreover, we claim that R/\mathfrak{m}^\perp is not closed under products, so is not a cotilting class. That is, consider the modules $\{R/s\mathfrak{m}\}_{s \in R \setminus \{0\}}$. Then the R -module $\text{Tor}_1^R(R/s\mathfrak{m}, R/\mathfrak{m}) \cong s\mathfrak{m} \otimes_R (R/\mathfrak{m}) \cong (s\mathfrak{m})/(s\mathfrak{m}^2)$ necessarily vanishes as \mathfrak{m} is idempotent, where the first isomorphism is the connecting homomorphism of $(-\otimes_R R/\mathfrak{m})$ applied to $0 \rightarrow s\mathfrak{m} \rightarrow R \rightarrow R/s\mathfrak{m} \rightarrow 0$. On the other hand, $\text{Tor}_1^R(R/sR, R/\mathfrak{m}) \cong R/\mathfrak{m}$ for every non-zero element s of R .

Now fix a non-zero element $s \in \mathfrak{m}$. Then we claim that $sR = \bigcap_{v(t) < v(s)} t\mathfrak{m} = \bigcap_{v(t) < v(s)} tR$. The inclusion “ \subseteq ” is clear. For “ \supseteq ”, we show the contrapositive. Take $r \notin sR$. Then $sR \subsetneq rR$, or $v(s) > v(r)$. As the value group $\Gamma = \Gamma(R)$ contains the rationals by assumption, there exists an element t of R such that $v(s) > v(t) > v(r)$, so in particular $r \notin tR$ so $r \notin \bigcap_{v(t) > v(s)} tR$.

From the previous paragraph, we can conclude that there is a monomorphism $0 \rightarrow R/sR \rightarrow \prod_{v(t) < v(s)} R/t\mathfrak{m}$. Thus in particular since $\text{Tor}_1^R(R/sR, R/\mathfrak{m}) \neq 0$, $\text{Tor}_1^R(\prod_{v(t) < v(s)} R/t\mathfrak{m}, R/\mathfrak{m}) \neq 0$, so ${}^\perp R/\mathfrak{m} = {}^\perp R/\mathfrak{m}^\perp$ is not closed under products.

We will return to the discussion of cotilting classes over commutative rings and commutative noetherian rings in §2.11.

2.10. Commutative rings: grade, depth and Koszul complexes. We define the grade with respect to an ideal for a general commutative ring R . We follow the convention of using grade for general commutative rings and ideals, and reserving depth for local rings and their maximal ideal. The following definition makes this precise.

Definition 2.10. Let R be a commutative ring. For a fixed ideal J of R , we let the *grade* of M relative to J , denoted $\text{grade}_R(J; M)$, be the non-negative integer $\inf\{i \mid \text{Ext}_R^i(R/J, M) \neq 0\}$. If $\text{Ext}_R^i(R/J, M) \neq 0$ for all $i \geq 0$, then we set $\text{grade}_R(J; M) = \infty$.

If R is local with maximal ideal \mathfrak{m} , then the *depth* of a module M is the grade of M with respect to the maximal ideal. That is, $\text{depth}_R(M) := \text{grade}_R(\mathfrak{m}; M)$.

The Koszul complex is grade-sensitive. We introduce the concepts and results we need to explain that following [BH98], [Bou07, §1] and [HŠ20].

Let $x \in R$. Then we let $K_\bullet(x)$ denote the complex

$$0 \longrightarrow R \xrightarrow{(-)\cdot x} R \longrightarrow 0,$$

with the leftmost R in degree 1 and the rightmost R in degree 0. Let $\mathbf{x} := x_1, \dots, x_\ell$, $\ell > 0$ a sequence of elements in R . Then we let $K_\bullet(\mathbf{x})$ denote the tensor product of the complexes $K_\bullet(x_i)$, that is

$$K_\bullet(\mathbf{x}) := \bigotimes_{1 \leq i \leq \ell} K_\bullet(x_i).$$

All entries in this complex are finitely generated free modules and are nonzero only in homological degrees $0, 1, \dots, \ell$.

Let M be an R -module. The *Koszul (chain) complex* of \mathbf{x} with coefficients in M is the chain complex $K_\bullet(\mathbf{x}) \otimes_R M$, denoted by $K_\bullet(\mathbf{x}; M)$. The i th *Koszul homology* of \mathbf{x} with coefficients in M is the i th homology of the chain complex $K_\bullet(\mathbf{x}; M)$, and is denoted by $H_i(\mathbf{x}; M)$. We denote by $K_\bullet(\mathbf{x})$ the chain complex $K_\bullet(\mathbf{x})$, and by $H_i(\mathbf{x})$ the homology of $K_\bullet(\mathbf{x})$.

The *Koszul cochain complex* of \mathbf{x} with coefficients in M is the chain cocomplex $\text{Hom}_R(K_\bullet(\mathbf{x}), M)$, denoted by $K^\bullet(\mathbf{x}; M)$, and is nonzero only in cohomological degrees $0, 1, \dots, \ell$. The i th *Koszul cohomology* of \mathbf{x} with coefficients in M is the i th cohomology of the cochain complex $K^\bullet(\mathbf{x}; M)$, and is denoted by $H^i(\mathbf{x}; M)$. Similarly as in the dual, we denote by $K^\bullet(\mathbf{x})$ the cochain complex $K^\bullet(\mathbf{x})$, and by $H^i(\mathbf{x})$ the homology of $K^\bullet(\mathbf{x})$.

By [BH98, Proposition 1.6.10], the complexes $K_\bullet(\mathbf{x}; M)$ and $K^\bullet(\mathbf{x}; M)$ are isomorphic, up to a relabelling of the cohomological degrees and a shift by ℓ . In particular, $H_i(\mathbf{x}; M) \cong H^{\ell-i}(\mathbf{x}; M)$ for all i .

Let $\mathbf{x} := x_1, \dots, x_\ell$ and $\mathbf{y} := y_1, \dots, y_m$ with $m, \ell > 0$ be two sequences which generate an ideal I of R . Then $H^i(\mathbf{x}; M) = 0$ for all $i \leq n$ if and only if $H^i(\mathbf{y}; M) = 0$ for all $i \leq n$, see [HŠ20, Proposition 3.4] which uses [BH98, Corollary 1.6.2 and Corollary 1.6.10(d)]. Since we are only interested in when the cohomology vanishes, we shall abuse the notation and denote $H^i(\mathbf{x}; M)$ simply by $H^i(I; M)$.

The following Proposition will be essential for our purposes. It is implicit in the proof of in [Bou07, p. 5, Théorème 1], and moreover allows J to be an infinitely generated ideal. The proposition is stated and proved using derived categories in [HŠ20, Corollary 3.10].

Proposition 2.11. *If R is a commutative ring, then*

$$\bigcap_{i=0}^{n-1} \text{Ker } \text{Ext}_R^i(R/J, -) = \bigcap_{i=0}^{n-1} \text{Ker } H^i(J; -)$$

as subcategories of $\text{Mod-}R$ for every $n > 0$.

Proof. We comment on how the statement follows from [Bou07]. This follows by the proof of [Bou07, p. 5, Théorème 1]. Explicitly, they first show that $\bigcap_{i=0}^{n-1} \text{Ker } \text{Ext}_R^i(R/J, -) \subseteq \bigcap_{i=0}^{n-1} \text{Ker } H^i(J; -)$ using a cohomology version of [Bou07, p. 4, Corollaire 1] and finally show by contradiction that there is no module M such that $M \in \bigcap_{i=0}^n \text{Ker } H^i(J; -)$ and $\text{Ext}_R^n(R/J, M) \neq 0$. \square

Remark 2.12. Let I be a finitely generated ideal. Let the following denote the Koszul cochain complex $K^\bullet(I; M)$ defined using an arbitrarily chosen set of generators $\mathbf{x} := x_1, \dots, x_\ell$ of I ,

$$0 \longrightarrow M_0 \xrightarrow{d^0} M_1 \xrightarrow{d^1} \dots \xrightarrow{d^{\ell-1}} M_\ell \longrightarrow 0,$$

where each M_i denotes a finite direct sum of copies of M .

Let $S_k(I; M)$ for $0 < k < \ell$ denote the module $\text{Coker}(d^{k-1})$. If

$$M \in \bigcap_{i=0}^{k-1} \text{Ker} H^i(I; -) = \bigcap_{i=0}^{k-1} \text{Ker} \text{Ext}_R^i(R/I, -),$$

then the complex provides a finite resolution of length k of $S_k(I; M)$ by finite direct sums of copies of M .

In the special case that $M = S$ for a flat ring morphism $R \rightarrow S$, then $H_i(I; S)$ is exactly the i th Koszul homology of IS with coefficients in S in $\text{Mod-}S$, see [BH98, Proposition 1.6.7].

Particular instances of the Koszul-type modules $S_k(I; M)$ are going to be our prototype of modules with finite flat dimension, and its character duals (or also their Matlis duals) are going to be the prototype of pure-injective modules with finite injective dimension, cf. Proposition 2.15.

2.11. Cotilting classes over commutative rings. Over a commutative ring, cotilting classes of cofinite type are classified in terms of certain finite sequences of faithful hereditary torsion pairs of finite type, or equivalently, finite sequences of certain faithful finitely generated Gabriel topologies, see [HŠ20, §2] for details.

Theorem 2.13. [HŠ20, Theorem 6.2],[AHPŠT14, Theorem 4.2, Corollary 4.4] *Let R be a commutative ring and $n \geq 0$. Then there are one-to-one correspondences between the following collections.*

(i) *Descending sequences of torsion-free classes of finite type and closed under injective envelopes*

$$(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{n-1}),$$

such that $\Omega_{-i}(R) \in \mathcal{E}_i$ for $0 \leq i < n$,

(ii) *Descending sequences of finitely generated Gabriel topologies*

$$(\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_{n-1}),$$

such that $\text{Ext}_R^i(R/J, R) = 0$ for every $J \in \mathcal{G}_i$ and $0 \leq i < n$,

(iii) *n -cotilting classes of cofinite type in $\text{Mod-}R$,*

(iv) *resolving subcategories of $\text{mod-}R$ of modules of projective dimension at most n .*

The correspondences (using the notation of §2.10) are given as follows.

(i) \rightarrow (ii)

$$\mathcal{G}_i := \{J \leq R \mid \text{Hom}_R(R/J, F) = 0 \text{ for every } E \in \mathcal{E}_i\}$$

(ii) \rightarrow (iii)

$$\mathcal{C} := \bigcap_{i=0}^{n-1} \bigcap_{J \in \mathcal{G}_i^f} (S_i(J; R)^\top)$$

where \mathcal{G}_i^f are the finitely generated ideals in \mathcal{G}_i .

If R is moreover noetherian, then we can omit the “of cofinite type” in (iii) and for every ideal J in \mathcal{G}_i , $\text{depth}(I; R) > i$, or $i < \inf\{j \mid \text{Ext}_R^j(R/J, R) \neq 0\}$.

Corollary 2.14. *Let R be a commutative noetherian ring. If $(\mathcal{E}, \mathcal{C})^\top$ is a cotilting Tor-pair (that is, a hereditary Tor-pair generated by a class of modules of finite flat dimension such that \mathcal{C} is definable), then for every prime $\mathfrak{p} \in \text{Spec}(R)$, $\mathcal{C}_\mathfrak{p}$ is a cotilting class of dimension $\text{depth}(R_\mathfrak{p})$.*

Proof. By Lemma 3.4 $(\mathcal{E}_\mathfrak{p}, \mathcal{C}_\mathfrak{p})$ is a hereditary Tor-pair and $\mathcal{C}_\mathfrak{p}$ is a definable class in $\text{Mod-}R_\mathfrak{p}$. Recall that since $R_\mathfrak{p}$ is local, $\text{depth}(R_\mathfrak{p}) = \text{findim}(R_\mathfrak{p})$ is finite (see discussion in §2.12) so in particular every cotilting class is of dimension at most $\text{findim}(R_\mathfrak{p})$, see Theorem 2.13. \square

2.12. Homological properties of commutative noetherian rings. For any prime $\mathfrak{p} \in \text{Spec}(R)$, we let $k(\mathfrak{p})$ denote the field of fractions of the residue field R/\mathfrak{p} .

In Proposition 2.15 we introduce the prototype of module of finite flat dimension that we will use to prove our classification results which are a particular instance of the Koszul cosyzygies, cf. Remark 2.12.

The local dual of such modules are pure injective modules of finite injective dimension, and also are their Matlis dual.

Proposition 2.15. *Let R be a commutative noetherian ring, and fix a prime $\mathfrak{p} \in \text{Spec}(R)$ and an integer k such that $\text{depth}R_\mathfrak{p} \geq k > 0$. Then the following statements hold for $S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})$,*

- (1) $S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p}) = S_k(\mathfrak{p}; R_\mathfrak{p})$ is an R -module of flat dimension k and an $R_\mathfrak{p}$ -module of projective dimension k ;
- (2) its character dual $S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})^+$ as well as its Matlis dual

$$\text{Hom}_{R_\mathfrak{p}}(S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p}), E(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})) = S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})^\nu$$

are both pure-injective modules of finite injective dimension k and

$$S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})^\top = {}^\perp(S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})^+) = {}^\perp(S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})^\nu).$$

Moreover, the following are equivalent for an R -module M .

- (i) $M \in S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})^\top$,
- (ii) $\text{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq k$,
- (iii) $\mathfrak{p} \notin \text{Ass}(\bigoplus_{i=0}^{k-1} \Omega_{-i}(M))$.

Proof. (1) By Proposition 2.11, if $\text{depth}R_\mathfrak{p} \geq k > 0$ then the Koszul cochain complex $K^\bullet(\mathfrak{p}; R_\mathfrak{p}) = K^\bullet(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})$ is exact up to degree $k-1$ and all its terms are finite direct sums of copies of $R_\mathfrak{p}$, cf. Remark 2.12. Therefore $S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})$ is an R -module of flat dimension k , as claimed in the statement.

(2) Taking duals with respect to an injective module takes any module to a pure injective module, and flat modules to injective modules. Moreover, both the character module and the Matlis dual are duals with respect to an injective cogenerator, hence they are not only exact functors but also preserve dimensions. That is, the dual of the Koszul resolution of $S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})$ witnessing that the module has projective dimension k , yields an injective resolution witnessing that the dual module has injective dimension k .

For the equalities between the Tor-orthogonals and the left Ext-orthogonals of the local dual of $S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})$ we appeal to Lemma 2.2 which holds because of the homological formulae given in §2.2. As these homological formulas also hold for the case of the Matlis dual, we also deduce that $S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})^\top = {}^\perp(S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})^\nu)$.

Now we prove the second part of the statement describes the Tor-orthogonal of $S_k(\mathfrak{p}R_\mathfrak{p}; R_\mathfrak{p})$.

(i) \Leftrightarrow (ii) Consider the following equivalences.

$$(2.2) \quad M \in S_k(\mathfrak{p}R_{\mathfrak{p}}; R_{\mathfrak{p}})^{\top} \Leftrightarrow \text{Tor}_i^R(S_k(\mathfrak{p}R_{\mathfrak{p}}; R_{\mathfrak{p}}), M) = 0 \text{ for } 1 \leq i \leq k$$

$$(2.3) \quad \Leftrightarrow \text{Tor}_i^{R_{\mathfrak{p}}}(S_k(\mathfrak{p}R_{\mathfrak{p}}; R_{\mathfrak{p}}), M_{\mathfrak{p}}) = 0 \text{ for } 1 \leq i \leq k$$

$$(2.4) \quad \Leftrightarrow \text{Ext}_{R_{\mathfrak{p}}}^i(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0 \text{ for } 0 \leq i \leq k-1$$

The equivalence (2.3) holds as $R \rightarrow R_{\mathfrak{p}}$ is a flat ring epimorphism. The equivalence (2.4) holds by [HŠ20, Proposition 5.12], which is shown via the equivalence of the Koszul homology and cohomology, that is that $H_i(I; -) \cong H^{n-i}(I; -)$ for an ideal I . Finally, by definition, $\text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geq k$ if and only if $\text{Ext}_{R_{\mathfrak{p}}}^i(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$ for $0 \leq i \leq k-1$.

(ii) \Leftrightarrow (iii) If $k = 1$, then $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq 1$ if and only if $\text{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$ if and only if $\mathfrak{p} \notin \text{Ass}(M)$, as $\mathfrak{p} \in \text{Ass}(M)$ if and only if $\mathfrak{p}R_{\mathfrak{p}} \in \text{Ass}(M)_{\mathfrak{p}}$.

For the inductive step, we apply the functor $\text{Hom}_R(R/\mathfrak{p}, -)$ to the minimal injective resolution of M . Then, as $\text{Ass}(M) = \text{Ass}(E(M))$ and $E(M_{\mathfrak{p}}) \cong E(M)_{\mathfrak{p}}$, we find an isomorphism $\text{Hom}_R(R/\mathfrak{p}, \Omega_{-i-1}(M)) \cong \text{Ext}_R^1(R/\mathfrak{p}, \Omega_{-i}(M))$. By dimension shifting, $\text{Ext}_{R_{\mathfrak{p}}}^i(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}}) \cong \text{Ext}_R^1(R/\mathfrak{p}, \Omega_{-i}(M))$ for $i > 0$, so the vanishing of the $\text{Ext}_{R_{\mathfrak{p}}}^i(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}})$ for $0 \leq i < k$ coincides with the vanishing of $\text{Hom}_R(R/\mathfrak{p}, \Omega_{-i-1}(M))$, or in other words that $\mathfrak{p} \notin \text{Ass}(\Omega_{-i-1}(M))$. \square

For further quoting, we introduce some of the finitistic dimensions of a ring. These are quite well understood for commutative noetherian rings.

Let us denote by $\text{Findim}(R) = \sup\{\text{pd}M \mid M \in \mathcal{P}\}$, $\text{findim}(R) = \sup\{\text{pd}M \mid M \in \mathcal{P}^{<\aleph_0}\}$, and $\text{W.Findim}(R) = \sup\{\text{fd}M \mid M \in \mathcal{F}\}$, respectively, the *finitistic*, *small finitistic*, and *weak finitistic dimension* of R .

If R is a commutative noetherian ring, by classical results of Bass [Bas62] and Raynaud-Gruson [RG71], $\text{Findim}(R)$ coincides with $\dim R$, the Krull dimension of R , [RG71, Théorème 3.2.6]. Assume now that $\dim(R) < \infty$. Then any flat R -module belongs to \mathcal{P} [Jen70], [RG71, Corollaire 3.2.7], and therefore \mathcal{P} coincides with the class \mathcal{F} of all modules of finite flat dimension. It follows that \mathcal{P} can be described as the class of all modules of flat dimension bounded by $\dim(R)$. In symbols, we have $\mathcal{P}_{\dim(R)} = \mathcal{P} = \mathcal{F} = \mathcal{F}_{\dim(R)}$.

If R is local with maximal ideal \mathfrak{m} then $\text{findim}(R) = \text{depth}(R)$ by the Auslander-Buchsbaum formula. Finally, for any R we have the identity $\text{W.Findim}(R) = \sup\{\text{depth}(R_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Spec}(R)\}$, [Bas62, Proposition 5.1].

3. PRELIMINARY RESULTS

3.1. Hereditary cotorsion pairs cogenerated by pure-injectives. Let $(\mathcal{C}, \mathcal{W})$ be a hereditary cotorsion pair cogenerated by a class \mathcal{P} of pure-injective R -modules. We define the classes $\mathcal{C}_{(m)} = {}^{\perp_{>m}}\mathcal{W} = {}^{\perp_{>m}}\mathcal{P}$. This way we obtain for any $m \geq 0$ a hereditary cotorsion pair $(\mathcal{C}_{(m)}, \mathcal{W}_{(m)})$. If $(\mathcal{C}, \mathcal{W})$ is cogenerated by a class \mathcal{P} of pure-injectives then $(\mathcal{C}_{(m)}, \mathcal{W}_{(m)})$ is cogenerated by $\Omega_{-k}\mathcal{P}$, the class of k -th minimal cosyzygies of modules from \mathcal{P} . Since any such minimal cosyzygy is again pure-injective [GT12, Lemma 6.20], this hereditary cotorsion pair is again cogenerated by pure-injectives.

Lemma 3.1. *Let $(\mathcal{C}, \mathcal{W})$ be a hereditary cotorsion pair and $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of modules in $R\text{-Mod}$. Then the following hold.*

- (i) *If $L \in \mathcal{C}$ and $M \in \mathcal{C}_{(i)}$ for $i > 0$, then $N \in \mathcal{C}_{(i)}$.*
- (ii) *If $M \in \mathcal{C}_{(i)}$ and $N \in \mathcal{C}_{(i+1)}$ for $i \geq 0$, then $L \in \mathcal{C}_{(i)}$.*

Proof. For (i), we must show that $\text{Ext}_R^j(N, W) = 0$ for every $j > i$ and $W \in \mathcal{W}$. Then $\text{Ext}_R^j(N, W) = 0$ for every $j > i > 0$ by the exact sequences

$$\text{Ext}_R^j(L, W) \rightarrow \text{Ext}_R^{j+1}(N, W) \rightarrow \text{Ext}_R^{j+1}(M, W).$$

For (ii) we must show that $\text{Ext}_R^j(L, W) = 0$ for every $j > i$ and $W \in \mathcal{W}$. There is an exact sequence $\text{Ext}_R^j(M, W) \rightarrow \text{Ext}_R^{j+1}(L, W) \rightarrow \text{Ext}_R^{j+1}(N, W)$, where the outer two modules vanish for all $j > i$ by assumption. \square

3.2. Hereditary Tor-pairs. A hereditary Tor-pair $(\mathcal{E}, \mathcal{C})^\top$ induces a class $\mathcal{E}^{\top_{>m}}$ for each integer $m > 0$. The class $\mathcal{E}^{\top_{>m}}$ forms the right-hand class of a hereditary Tor-pair, as the associated Tor-pair is generated by $\Upsilon_m(\mathcal{E})$.

We will let $\mathcal{C}_{(m)}$ denote the right hand class of the induced cotorsion pair, that is $\mathcal{C}_{(m)} := \mathcal{E}^{\top_{>m}} = (\Upsilon_m(\mathcal{E}))^\top$ and the hereditary Tor-pair by $(\mathcal{E}_{(m)}, \mathcal{C}_{(m)})^\top$. In particular, $\mathcal{C}_{(0)} = \mathcal{C} \subseteq \mathcal{C}_{(1)} \subseteq \dots \subseteq \mathcal{C}_{(i)} \subseteq \dots$ and $\mathcal{E}_{(0)} = \mathcal{E} \supseteq \mathcal{E}_{(1)} \supseteq \dots \supseteq \mathcal{E}_{(i)} \supseteq \dots$. If moreover $\mathcal{E} \subseteq \mathcal{F}_n(R)$, so all the modules in the left-hand class have flat dimension at most n , then $(\mathcal{E}_{(m)}, \mathcal{C}_{(m)})^\top = (\mathcal{F}_0(R), R\text{-Mod})^\top$ for all $m \geq n$, and, $\mathcal{E}_{(m)} \subseteq \mathcal{F}_{n-m}(R)$ for $0 \leq m \leq n$.

Lemma 3.2. *Let $(\mathcal{E}, \mathcal{C})^\top$ be a hereditary Tor-pair and $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of modules in $R\text{-Mod}$. Then the following hold.*

- (i) *If $L \in \mathcal{C}$ and $M \in \mathcal{C}_{(i)}$ for $i > 0$, then $N \in \mathcal{C}_{(i)}$.*
- (ii) *If $M \in \mathcal{C}_{(i)}$ and $N \in \mathcal{C}_{(i+1)}$ for $i \geq 0$, then $L \in \mathcal{C}_{(i)}$.*

Proof. Analogous to the proof of Lemma 3.1. \square

3.3. Cotorsion pairs cogenerated by pure-injectives and colocalisation over commutative rings. Let R be a commutative ring and S a multiplicative subset of R , and R_S the localisation of R at S . Localisation over an arbitrary commutative ring preserves flat dimension, and interacts well with the Tor_R -functor. However, in general, localisation does not preserve the injective dimension of a module. Instead, the *colocalisation* of an R -module M , which is the module $\text{Hom}_R(R_S, M)$, when M is a pure-injective module has many desirable properties. We will denote the colocalisation of a class \mathcal{X} with respect to a multiplicative subset S by $\mathcal{X}^S := \{X^S \mid X \in \mathcal{X}\}$, where $X^S = \text{Hom}_R(R_S, X) \in \text{Mod-}R_S$.

More generally, let $R \rightarrow R'$ be any ring homomorphism. Then for an R -module N we have following basic properties:

- $\text{id}_R(N) = n \implies \text{id}_{R'} \text{Hom}_R(R', N) \leq n$ provided that $\text{Ext}_R^j(R', \Omega_{-i}N) = 0$ for all $i \geq 0$ and all $j > 0$.
- N pure-injective $\implies \text{Hom}_R(R', N)$ pure-injective as an R' -module.
- If N is an R' -module and $\text{Tor}_j^R(M, R') = 0$ for every j , then $\text{Ext}_R^j(M, N) \cong \text{Ext}_{R'}^j(M \otimes_R R', N)$.

Suppose R is a commutative ring, Y is any module, F is a flat R -module and N is a pure-injective module. Uniting the homological identity in Lemma 2.1 for pure-injective modules with the derived tensor-Hom adjunction §2.2 we have the following isomorphisms.

$$(3.1) \quad \text{Hom}_R(F, \text{Ext}_R^i(M, N)) \cong \text{Ext}_R^i(F \otimes_R M, N) \cong \text{Ext}_R^i(M, \text{Hom}_R(F, N))$$

In particular, the equation (3.1) shows that colocalisation of pure-injective modules commutes with the Ext -functor in the second variable. Thus if F is the localisation of the ring R_S , combining the above information we have the following isomorphisms for a pure-injective module N , M any R -module, and $i \geq 0$.

$$(3.2) \quad \mathrm{Ext}_R^i(M, N)^S \cong \mathrm{Ext}_R^i(M_S, N) \cong \mathrm{Ext}_R^i(M, N^S) \cong \mathrm{Ext}_{R_S}^i(M_S, N^S)$$

Lemma 3.3. *Let R be a commutative ring and S a multiplicative subset of R . Consider a hereditary cotorsion pair $(\mathcal{C}, \mathcal{W})$ cogenerated by a set \mathcal{P} of pure-injectives. Then there is a hereditary cotorsion pair $(\mathcal{C}_S, \mathcal{V})$ in $\mathrm{Mod}-R_S$ cogenerated by the set \mathcal{P}^S of pure-injectives, and moreover $\mathcal{C}_S = \mathcal{C} \cap \mathrm{Mod}-R_S$.*

In addition $(\mathcal{C}_S)_{(i)}$ coincides with $((\mathcal{C}_{(i)})_S$ for $i \geq 0$.

Proof. First note that $\mathcal{C}_S = \mathcal{C} \cap \mathrm{Mod}-R_S$ as \mathcal{C} is closed under direct limits.

Take $P \in \mathrm{Mod}-R$ pure-injective and $M \in \mathrm{Mod}-R$. Then for any $i > 0$:

$$\mathrm{Ext}_R^i(C_S, P) \cong \mathrm{Ext}_{R_S}^i(C_S, P^S) \cong \mathrm{Ext}_R^i(C, P^S)$$

by (3.2). Then $M_S \in \mathcal{C}_S$ if and only if $M_S \in {}^{\perp > 0} \mathcal{P}^S$. \square

3.4. Localisations and Tor-pairs. Let R be a commutative ring and S a multiplicative subset of R , and \mathcal{X} a class in $\mathrm{Mod}-R$. Then we let $\mathcal{X}_S := \{X_S \mid X \in \mathcal{X}\}$, a class in $\mathrm{Mod}-R_S$.

Lemma 3.4. *Let R be a commutative ring and S a multiplicative subset of R . Consider a Tor-pair $(\mathcal{E}, \mathcal{C})^\top$. Then there is a Tor-pair $(\mathcal{E}_S, \mathcal{C}_S)^\top$ in $\mathrm{Mod}-R_S$, and moreover $\mathcal{E}_S = \mathcal{E} \cap \mathrm{Mod}-R_S$ and $\mathcal{C}_S = \mathcal{C} \cap \mathrm{Mod}-R_S$.*

If in addition the Tor-pair is hereditary, then $((\mathcal{E}_S)_{(i)}, (\mathcal{C}_S)_{(i)})^\top$ coincides with $((\mathcal{E}_{(i)})_S, (\mathcal{C}_{(i)})_S)^\top$ as Tor-pairs in $\mathrm{Mod}-R_S$ for $i \geq 0$.

Proof. First note that $\mathcal{E}_S = \mathcal{E} \cap \mathrm{Mod}-R_S$ and $\mathcal{C}_S = \mathcal{C} \cap \mathrm{Mod}-R_S$ as classes in a Tor-pair are always closed under direct limits.

Take $M \in \mathrm{Mod}-R_S$ and $E_S \in \mathcal{E}_S$. Then $\mathrm{Tor}_1^{R_S}(E_S, M) \cong \mathrm{Tor}_1^R(E_S, M)$ as $R \rightarrow R_S$ is a flat ring morphism. The latter module vanishes if and only if $M \in \mathcal{C}$, so the right Tor-orthogonal of \mathcal{E}_S in $\mathrm{Mod}-R_S$ is exactly \mathcal{C}_S , and a symmetric argument shows that the left Tor-orthogonal of \mathcal{C}_S in $\mathrm{Mod}-R_S$. \square

3.5. Pure-injective modules over commutative noetherian rings. When R is a commutative noetherian ring, it is known that every flat cotorsion module is pure-injective. Moreover, by work of Enochs [Eno84], or [EJ11, Section 5.3, Theorem 5.3.28], every flat cotorsion module is isomorphic to a product $\prod_{\mathfrak{p} \in \mathrm{Spec}(R)} T_{\mathfrak{p}}$, where $T_{\mathfrak{p}}$ is the $\mathfrak{p}R_{\mathfrak{p}}$ -completion of a free $R_{\mathfrak{p}}$ -module.

Lemma 3.5. *Let R be a commutative noetherian ring and $C \in \mathrm{Mod}-R$ a cotorsion module. If $\mathrm{Hom}_R(R_{\mathfrak{m}}, C) = 0$ for every maximal ideal $\mathfrak{m} \in \mathrm{Spec}(R)$, then $C = 0$.*

Proof. Let $F \xrightarrow{\phi} C \rightarrow 0$ be a flat cover of C . Using [EJ11, Theorem 5.3.28], we can explicitly write F as $\prod_{\mathfrak{p} \in \mathrm{Spec}(R)} T_{\mathfrak{p}}$ as described above.

First note that if \mathfrak{m} is a maximal ideal of R and $\mathrm{Hom}_R(R_{\mathfrak{m}}, C) = 0$, then necessarily $\mathrm{Hom}_R(M_{\mathfrak{m}}, C) = 0$ for every $R_{\mathfrak{m}}$ -module $M_{\mathfrak{m}}$. Therefore, as $T_{\mathfrak{p}}$ is an $R_{\mathfrak{m}}$ -module for every maximal ideal \mathfrak{m} containing \mathfrak{p} , $\mathrm{Hom}_R(T_{\mathfrak{p}}, C) = 0$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$. As the contravariant Hom-functor commutes with coproducts, $\mathrm{Hom}_R(\bigoplus_{\mathfrak{p} \in \mathrm{Spec}(R)} T_{\mathfrak{p}}, C) \cong \prod_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathrm{Hom}_R(T_{\mathfrak{p}}, C) = 0$. Let

$$0 \longrightarrow \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(R)} T_{\mathfrak{p}} \xrightarrow{\tau} \prod_{\mathfrak{p} \in \mathrm{Spec}(R)} T_{\mathfrak{p}} \xrightarrow{\pi} \mathrm{Coker}(\tau) \longrightarrow 0$$

denote the short exact sequence where τ is the natural inclusion of a coproduct into a product. As $\phi\tau = 0$, ϕ factors through $\mathrm{Coker}(\tau)$, that is, there exists a homomorphism $\beta: \mathrm{Coker}(\tau) \rightarrow C$ such that $\beta\pi = \phi$.

As τ is a pure monomorphism, it follows that $\mathrm{Coker}(\tau)$ is a flat R -module. Thus by the flat-precover property of ϕ , there exists a morphism $\alpha: \mathrm{Coker}(\tau) \rightarrow$

$\prod_{\mathfrak{p} \in \text{Spec}(R)} T_{\mathfrak{p}}$ such that $\phi\alpha = \beta$. Therefore, $\phi = \beta\pi = \phi\alpha\pi$, and by the minimality property of the flat cover ϕ , $\alpha\pi$ is an automorphism. We conclude that π is an isomorphism, and therefore $F \cong \prod_{\mathfrak{p} \in \text{Spec}(R)} T_{\mathfrak{p}} = 0$, so also $C = 0$, as desired. \square

Proposition 3.6. *Let R be a commutative noetherian ring. Let N be a pure-injective module and $i \geq 0$ an integer. Then, for an R -module M , $\text{Ext}_R^i(M, N) = 0$ if and only if $\text{Ext}_R^i(M, N^{\mathfrak{m}}) = 0$ for every maximal ideal $\mathfrak{m} \in \text{Spec}(R)$.*

Proof. Let N be a pure-injective module. By (3.2), there is an isomorphism

$$\text{Ext}_R^i(M, N^{\mathfrak{m}}) \cong \text{Ext}_R^i(M, N)^{\mathfrak{m}}.$$

Thus it is sufficient to show that $\text{Ext}_R^i(M, N)^{\mathfrak{m}}$ vanishes for every maximal ideal $\mathfrak{m} \in \text{Spec}(R)$ if and only if $\text{Ext}_R^i(M, N)$ vanishes.

The if statement is trivial. For the only if statement, recall that $\text{Ext}_R^i(M, N)$ is a pure-injective R -module by [War69, Proposition 7], and apply Lemma 3.5. \square

We have the following corollary, the special case of N being a cotilting module appears in [T§14, Theorem 2.4].

Corollary 3.7. *Let R be a commutative noetherian ring. Let N be a pure-injective module. Then, ${}^{\perp}N = {}^{\perp}(\prod_{\mathfrak{m}} N^{\mathfrak{m}})$. Moreover, ${}^{\perp}N$ is a definable class in $\text{Mod-}R$ then ${}^{\perp}(N^{\mathfrak{m}})$ is a definable class in $\text{Mod-}R_{\mathfrak{m}}$ for each maximal ideal \mathfrak{m} .*

Proof. The first statement follows directly from Proposition 3.6. The moreover clause follows from the same identities in the proof of Proposition 3.6 and because ${}^{\perp}N^{\mathfrak{m}} = ({}^{\perp}N)_{\mathfrak{m}}$ by Lemma 3.3. Explicitly, $\text{Ext}_R^i(M, N)$ vanishes if and only if $\text{Ext}_R^i(M, N)^{\mathfrak{m}}$ vanishes for every maximal ideal \mathfrak{m} if and only if $\text{Ext}_{R_{\mathfrak{m}}}^i(M_{\mathfrak{m}}, N)^{\mathfrak{m}}$ for every maximal ideal \mathfrak{m} . \square

4. THE MAIN CLASSIFICATION

Throughout this section, R will denote a commutative noetherian ring. Recall \mathcal{I} the class of all R -module of finite injective dimension and \mathcal{PI}_0 is the class of all pure-injective R -modules. The goal of this section is to prove our main classification result Theorem 4.17. We start by showing that the left constituents of the hereditary cotorsion pairs cogenerated by modules from $\mathcal{I} \cap \mathcal{PI}_0$ are closed under taking injective envelopes.

Lemma 4.1. *Let R be a local ring with maximal ideal \mathfrak{m} , and take a short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$, such that L is \mathfrak{m} -torsion, or equivalently, L is filtered by copies of R/\mathfrak{m} . Then $\text{Ass}(M) = \text{Ass}(L) \cup \text{Ass}(N)$*

Proof. Suppose that $L \neq 0$. It is sufficient to show that if $\mathfrak{p} \in \text{Ass}(N) \setminus \{\mathfrak{m}\}$, then $\mathfrak{p} \in \text{Ass}(M)$ for $\mathfrak{p} \in \text{Spec}(R)$. Fix $\mathfrak{p} \in \text{Ass}(N)$ and localise the short exact sequence at \mathfrak{p} . Then $L_{\mathfrak{p}} = 0$ as $(R/\mathfrak{m})_{\mathfrak{p}} = 0$, so as the sequence remains exact $M_{\mathfrak{p}} \cong N_{\mathfrak{p}}$, so $\mathfrak{p}R_{\mathfrak{p}} \in \text{Ass}(N_{\mathfrak{p}})$ implies that $\mathfrak{p} \in \text{Ass}(M)$, as required. \square

Lemma 4.2. *Let $(\mathcal{C}, \mathcal{W})$ be a hereditary cotorsion pair cogenerated by a class $\mathcal{P} \subseteq \mathcal{PI}_0$. If $k(\mathfrak{p}) \in \mathcal{C}$ for some $\mathfrak{p} \in \text{Spec}(R)$, then the modules $E(k(\mathfrak{p})), \Omega_{-1}(k(\mathfrak{p})) \in \mathcal{C}$. More generally, for an R -module M , if $\{k(\mathfrak{p}) \mid \mathfrak{p} \in \text{Ass}(M)\} \subseteq \mathcal{C}$, then $E(M) \in \mathcal{C}$.*

Proof. The first statement holds since $E(k(\mathfrak{p}))$ and $\Omega_{-1}(k(\mathfrak{p}))$ are filtered by direct sums copies of the module $k(\mathfrak{p})$, and since \mathcal{C} is closed under direct limits. The second statement holds since for a module M , $E(M) \cong \bigoplus_{\mathfrak{p} \in \text{Ass}(M)} E(k(\mathfrak{p}))^{(\alpha_{\mathfrak{p}})}$ for cardinals $\alpha_{\mathfrak{p}}$. So $E(M) \in \mathcal{C}$ by the first statement. \square

Lemma 4.3. *Let $(\mathcal{C}, \mathcal{W})$ be a hereditary cotorsion pair cogenerated by a class $\mathcal{P} \subseteq \mathcal{I} \cap \mathcal{PI}_0$. For any $M \in \mathcal{C}$, if $\mathfrak{p} \in \text{Ass}(M)$, then $k(\mathfrak{p}) \in \mathcal{C}$.*

Proof. It is enough to prove the statement for $\mathcal{P} = \{N\}$ where N is pure-injective of injective dimension $n > 0$, we proceed by induction on n . For $n = 1$, \mathcal{C} is closed under submodules. So if $\mathfrak{p} \in \text{Ass}(M)$, then $R/\mathfrak{p} \in \mathcal{C}$. Moreover, \mathcal{C} is closed under direct limits, and $k(\mathfrak{p})$ is an direct limit of copies of R/\mathfrak{p} in $\text{Mod-}R$, so it follows that $k(\mathfrak{p}) \in \mathcal{C}$.

Fix a $n > 1$ and suppose that the statement holds for all $0 < k < n$. In particular, this means that the statement holds for the cotorsion pairs $(\mathcal{C}_{(n-k)}, \mathcal{W}_{(n-k)})$. Take $M \in \mathcal{C}$ and $\mathfrak{p} \in \text{Ass}(M)$. We claim that we can assume without loss of generality that $M = M_{\mathfrak{p}}$, so \mathfrak{p} is maximal in $\text{Ass}(M)$. This is because \mathcal{C} is closed under direct limits, so if $M \in \mathcal{C}$, then $M_{\mathfrak{p}} \in \mathcal{C}$, and $\mathfrak{p} \in \text{Ass}(M)$ if and only if $\mathfrak{p}R_{\mathfrak{p}} \in \text{Ass}(M)_{\mathfrak{p}} \subseteq \text{Spec}(R_{\mathfrak{p}})$.

Suppose L is a \mathfrak{p} -torsion $R_{\mathfrak{p}}$ -submodule of $M \cong M_{\mathfrak{p}}$. By the induction hypothesis, we can assume that $k(\mathfrak{q}) \in \mathcal{C}_{(1)}$ for every $\mathfrak{q} \in \text{Ass}(M)$. As L is necessarily filtered by $k(\mathfrak{p})$, we can assume that $L \in \mathcal{C}_{(1)}$, and similarly, for any module X such that $\text{Ass}(X) \subseteq \text{Ass}(M)$, its injective envelope $E(X)$, is in $\mathcal{C}_{(1)}$. Consider the following commutative diagram with exact rows and columns formed by the horseshoe lemma.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 (4.1) \quad 0 & \longrightarrow & E(L) & \longrightarrow & E(L) \oplus E(N) & \longrightarrow & E(N) \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \Omega_{-1}(L) & \longrightarrow & Y & \longrightarrow & \Omega_{-1}(N) \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

By the above explanation, $E(L), E(N) \in \mathcal{C}_{(1)}$, so also their direct sum is in $\mathcal{C}_{(1)}$. Considering the middle vertical column and applying Lemma 3.2(i), $Y \in \mathcal{C}_{(1)}$. As also $\Omega_{-1}(L)$ is filtered by copies of $k(\mathfrak{p})$, $\Omega_{-1}(L) \in \mathcal{C}_{(1)}$, so it follows that $\Omega_{-1}(N) \in \mathcal{C}_{(2)}$ by considering the last row. Therefore, by applying Lemma 3.2(ii) to the right-most column, we find that $N \in \mathcal{C}_{(1)}$. In particular, $L \in \mathcal{C}$ by applying Lemma 3.2(ii) to the first row.

As this proof holds for any \mathfrak{p} -torsion module L , it holds for $k(\mathfrak{p})$, so we conclude that $k(\mathfrak{p}) \in \mathcal{C}$, as required. \square

Corollary 4.4. *Let $(\mathcal{C}, \mathcal{W})$ be a hereditary cotorsion pair cogenerated by a class $\mathcal{P} \subseteq \mathcal{I} \cap \mathcal{P}\mathcal{J}_0$. Then \mathcal{C} is closed under injective envelopes.*

The embedding of L_{Tor_1} into L_{Ext^1} of §2.6 takes hereditary Tor -pairs generated by a class of modules in \mathcal{F} to hereditary cotorsion pairs cogenerated by modules from $\mathcal{I} \cap \mathcal{P}\mathcal{J}_0$. Thus, we also get:

Corollary 4.5. *Let $(\mathcal{E}, \mathcal{C})^{\top}$ be a hereditary Tor -pair generated by a class $\mathcal{G} \subseteq \mathcal{F}$. Then \mathcal{C} is closed under injective envelopes.*

Before proceeding, we show that constituents of a Tor -pair not satisfying the assumptions of Corollary 4.5 may fail to be closed under injective envelopes. We take the opportunity to exhibit more examples.

Example 4.6. Let (R, \mathfrak{m}, k) be a local noetherian which is not self-injective and with $\mathfrak{m} \in \text{Ass}(R)$. Consider the Tor -pair generated by k . Then $R \in k^{\top}$, however we claim that $E(R) \notin k^{\top}$. It suffices to show that $E(k) \notin k^{\top}$ as $E(k)$

is a direct summand of $E(R)$ by the assumption that $\mathfrak{m} \in \text{Ass}(R)$. Recall that $\text{Hom}_R(E(k), E(k)) \cong \hat{R}$, the \mathfrak{m} -adic completion of R .

Suppose $d = \dim R > 0$, so in particular by Grothendieck's vanishing theorem, the local cohomologies $H_{\mathfrak{m}}^d(\hat{R}) \cong H_{\mathfrak{m}}^d(R) \neq 0$, see [BH98, Proposition 3.5.4 (d), Theorem 3.5.7]. Assume for the contradiction that $\text{Tor}_d^R(E(k), k) = 0$. Then, also $\text{Ext}_R^d(k, \hat{R})$ vanishes via the isomorphism

$$\text{Ext}_R^d(k, \hat{R}) \cong \text{Hom}_R(\text{Tor}_d^R(E(k), k), E(k)) = 0.$$

So by [BH98, Remark 3.5.3(c)]

$$H_{\mathfrak{m}}^d(\hat{R}) \cong \varinjlim_i \text{Ext}_{\hat{R}}^d(\hat{R}/\mathfrak{m}^i, \hat{R}) \cong \varinjlim_i \text{Ext}_R^d(R/\mathfrak{m}^i, \hat{R}) = 0,$$

a contradiction.

The following remark observes that over commutative rings, the classes of a Tor-pair are interchangeable, but not symmetric. Thus a Tor-pair generated by a module of infinite flat dimension may give rise to a hereditary Tor-pair (co)generated by modules of finite flat dimension. See §5 for when hereditary Tor-pairs are both generated and cogenerated by modules of bounded flat dimension.

Remark 4.7. Given a generating set of a Tor-pair, it is not immediately clear whether it will be generated or cogenerated by modules of bounded flat dimension. For example, let (R, \mathfrak{m}, k) be a local noetherian self-injective ring which is not a field, and consider again the Tor-pair generated by k . Then $k^{\top_1} = k^{\top} = \mathcal{P}_0 = \mathcal{J}_0 \subsetneq \text{Mod-}R$, so the Tor-pair generated by k is $(\text{Mod-}R, \mathcal{P}_0)^{\top}$. The class \mathcal{P}_0 is closed under injective envelopes, but the flat dimension of k is not bounded as R is not a field. However, clearly this Tor-pair is cogenerated by R .

There exist hereditary Tor-pairs which are not generated (or cogenerated) by classes of modules of finite flat dimension, as seen in the following example.

Example 4.8. Let R be a non-Gorenstein commutative noetherian ring with a Gorenstein flat module which is not flat (the commutative noetherian rings which do not have an analogue of the Govorov-Lazard Theorem for the Gorenstein flat modules provide examples of such rings, see [HJ11, Theorem 2.7 and Example 2.8]). Let \mathcal{GF}_0 denote the Gorenstein flat modules in $\text{Mod-}R$. Then, by [SS20, Corollary 5.7], $(\mathcal{GF}_0, \mathcal{GF}_0^{\top})^{\top}$ forms a hereditary Tor-pair. By assumption, $\mathcal{F}_0 \subsetneq \mathcal{GF}_0$, therefore there exists a module in \mathcal{GF}_0 which is of infinite flat dimension by [Hol04, Theorem 3.19]. Moreover, $\mathcal{J} \subseteq \mathcal{GF}_0^{\top}$ where \mathcal{J} denotes the class of modules of finite injective dimension by [Hol04, Theorem 3.14]. By the assumption that R is not Gorenstein, there exists an injective module of infinite flat dimension by [Iwa80, Proposition 1]. Thus $(\mathcal{GF}_0, \mathcal{GF}_0^{\top})^{\top}$ is a hereditary Tor-pair which is not generated by modules of bounded flat dimension on either side. Moreover, \mathcal{GF}_0^{\top} is closed under injective envelopes as it contains all injective modules.

Let us continue again in the direction of the promised classification. The closure under injective envelopes yields the following reduction important for one of the main inductive arguments in Theorem 4.17.

Lemma 4.9. *Let $(\mathcal{C}, \mathcal{W})$ be a hereditary cotorsion pair cogenerated by a class $\mathcal{P} \subseteq \mathcal{I} \cap \mathcal{P}\mathcal{J}_0$. Then the following hold for any R -module M .*

- (i) $M \in \text{Sub}(\mathcal{C})$ if and only if $E(M) \in \mathcal{C}$.
- (ii) $M \in \mathcal{C}$ if and only if $M \in \text{Sub}(\mathcal{C})$ and $\Omega_{-1}(M) \in \mathcal{C}_{(1)}$.

Proof. For (i), if $M \in \text{Sub}(\mathcal{C})$, then $E(M)$ is a direct summand of a module in $\mathcal{J}_0(R) \cap \mathcal{C}$ by properties of injective envelopes. If $E(M) \in \mathcal{C}$, then trivially $M \in \text{Sub}(\mathcal{C})$.

For (ii), suppose $M \in \mathcal{C}$. Then it follows trivially that $M \in \text{Sub}(\mathcal{C})$, and $\Omega_{-1}(M) \in \mathcal{C}_{(1)}$ follows since $E(M) \in \mathcal{C}$ by Corollary 4.4 and applying Lemma 3.1(i).

Now suppose that $M \in \text{Sub}(\mathcal{C})$ and $\Omega_{-1}(M) \in \mathcal{C}_{(1)}$. Then $E(M) \in \mathcal{C}$ by (i), so by Lemma 3.1(ii), $M \in \mathcal{C}$. \square

The following lemmas will be used in the proof of Proposition 4.13.

Lemma 4.10. *Let (R, \mathfrak{m}, k) be a local commutative noetherian ring with $\text{depth}(R) = 0$ and N a pure-injective module of finite injective dimension. Then $\text{Ext}_R^i(k, N) = 0$ for all $i > 0$.*

Proof. Let (R, \mathfrak{m}, k) be a local ring of zero depth and N a module of finite injective dimension. We will prove the statement by induction on the injective dimension of N . For the base step, suppose that $\text{id}(N) = 1$. Then, as $\text{depth}(R) = 0$, k is a submodule of R , so there is a short exact sequence $0 \rightarrow k \rightarrow R \rightarrow R/I \rightarrow 0$. Applying $\text{Hom}_R(-, N)$, we find the exact sequence

$$0 \rightarrow \text{Ext}_R^1(R, N) \rightarrow \text{Ext}_R^1(k, N) \rightarrow \text{Ext}_R^2(R/I, N) \rightarrow 0,$$

where the two outer modules vanish as $\text{id}(N) = 1$ and R is projective. Therefore we conclude that $\text{Ext}_R^1(k, N)$.

For the induction step, suppose that $k \in \mathcal{C}' := {}^\perp(\mathcal{I}_{n-1}(R) \cap \mathcal{P}\mathcal{I}_0)$, and we will show that $k \in \mathcal{C} = {}^\perp(\mathcal{I}_n(R) \cap \mathcal{P}\mathcal{I}_0)$. We will show that k satisfies the conditions in Lemma 4.9(ii). As k is a submodule of R , $k \in \text{Sub}(\mathcal{C})$, as clearly $R \in \mathcal{C}$. Moreover, $\Omega_{-1}(k)$ is filtered by copies of k which by the induction hypothesis is in \mathcal{C}' , and we note additionally that $\mathcal{C}' \subset \mathcal{C}_{(1)}$. Therefore, we conclude that $k \in \mathcal{C}$, as required. \square

Lemma 4.11. *Let (R, \mathfrak{m}, k) be a local commutative noetherian ring and N a pure-injective module of finite injective dimension. Then $\text{Ext}_R^j(k(\mathfrak{p}), N) = 0$ for every $j > \text{depth}(R_{\mathfrak{p}})$.*

Proof. In view of the isomorphism $\text{Ext}_R^j(k(\mathfrak{p}), N) \cong \text{Ext}_{R_{\mathfrak{p}}}^j(k(\mathfrak{p}), N^{\mathfrak{p}})$, it is sufficient to prove the statement for when \mathfrak{p} is the maximal ideal.

Let x_1, \dots, x_m denote a maximal regular sequence, so $m = \text{depth}(R)$, and let $J := \langle x_1, \dots, x_m \rangle$. Fix a module N of finite injective dimension over R . Then, for $j > \text{depth}(R)$, $\text{Ext}_R^j(R/J, N) = 0$ as $\text{pd}(R/J) = \text{depth}(R)$, so in particular, $\text{Ext}_R^i(R/J, \Omega_{-\text{depth}(R)} N) = 0$ for all $i > 0$.

Using this and (2.1), we have the following isomorphisms for all $i > 0$.

$$\text{Ext}_R^{i+\text{depth}(R)}(k, N) \cong \text{Ext}_R^i(k, \Omega_{-\text{depth}(R)} N) \cong \text{Ext}_{R/J}^i(k, \text{Hom}_R(R/J, \Omega_{-\text{depth}(R)} N))$$

Moreover, $\text{Hom}_R(R/J, \Omega_{-\text{depth}(R)} N)$ is pure-injective of finite injective dimension as an R/J -module.

As J was generated by a maximal regular sequence, R/J is a local ring with zero depth, so it remains only to apply Lemma 4.10. \square

The following two propositions provide the two main assignments of the classification Theorem 4.17. First, we introduce the following function will play an essential role in our classification results.

Definition 4.12. We let the *depth function* be the function $\text{depth}: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ assigning to each prime ideal \mathfrak{p} , the depth of the ring localised at \mathfrak{p} , $\text{depth}(R_{\mathfrak{p}})$.

We consider functions on the spectrum $\text{Spec}(R)$ with values in non-negative integers $\mathbb{Z}_{\geq 0}$. For a general function $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$, we will write $\phi \leq \text{depth}$ as a shorthand notation for $\phi(\mathfrak{p}) \leq \text{depth}(\mathfrak{p})$ for all $\mathfrak{p} \in \text{Spec}(R)$.

Proposition 4.13. *Let $(\mathcal{C}, \mathcal{W})$ be a hereditary cotorsion pair cogenerated by a class $\mathcal{P} \subseteq \mathcal{I} \cap \mathcal{P}\mathcal{I}_0$, and define a map $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ by $\mathfrak{p} \mapsto \inf\{i \in \mathbb{Z}_{\geq 0} \mid k(\mathfrak{p}) \in \mathcal{C}_{(i)}\}$. Then $\phi \leq \text{depth}$.*

Proof. We have for any $\mathfrak{p} \in \text{Spec}(R)$

$$\text{Ext}_R^i(k(\mathfrak{p}), N) \cong \text{Ext}_{R_{\mathfrak{p}}}^i(k(\mathfrak{p}), N^{\mathfrak{p}})$$

for any pure-injective R -module N and $i > 0$. By Lemma 4.11, we know that $\text{Ext}_{R_{\mathfrak{p}}}^j(k(\mathfrak{p}), N^{\mathfrak{p}}) = 0$ for every pure-injective N of finite injective dimension and every $j > \text{depth}(\mathfrak{p})$. It follows that $k(\mathfrak{p}) \in {}^{\perp > \text{depth}(\mathfrak{p})}\mathcal{P}$, which by definition means that $k(\mathfrak{p}) \in \mathcal{C}_{(\text{depth}(\mathfrak{p}))}$ so $\phi(\mathfrak{p}) \leq \text{depth}(\mathfrak{p})$, as required. \square

Proposition 4.14. *Let $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ be a function satisfying $\phi \leq \text{depth}$. Then there is a hereditary Tor-pair $(\mathcal{E}, \mathcal{C})^{\top}$ generated by a set of modules of finite flat dimension such that $\mathcal{C} = \{M \in \text{Mod-}R \mid \text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \phi(\mathfrak{p}) \ \forall \mathfrak{p} \in \text{Spec}(R)\}$.*

Moreover, $\mathcal{C}_{(i)} = \{M \in \text{Mod-}R \mid \text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq (\phi(\mathfrak{p}) - i) \ \forall \mathfrak{p} \in \text{Spec}(R)\}$ for any $i \geq 0$. In particular, $k(\mathfrak{p}) \in \mathcal{C}_{(i)}$ if and only if $\phi(\mathfrak{p}) \leq i$.

Proof. We construct a generating set \mathcal{S}_{ϕ} for the Tor-pair, consisting of modules of finite flat dimension in $\text{Mod-}R$. For each $\mathfrak{p} \in \text{Spec}(R)$ we have $\phi(\mathfrak{p}) \leq \text{depth}(\mathfrak{p})$, so Proposition 2.15 yields that the module $S_{\phi(\mathfrak{p})}(\mathfrak{p}R_{\mathfrak{p}}; R_{\mathfrak{p}})$ constructed in §2.10 is of finite flat dimension over R for each $\mathfrak{p} \in \text{Spec}(R)$. We let

$$\mathcal{S}_{\phi} := \{S_{\phi(\mathfrak{p})}(\mathfrak{p}R_{\mathfrak{p}}; R_{\mathfrak{p}})\}_{\mathfrak{p} \in \text{Spec}(R)},$$

and let $(\mathcal{E}, \mathcal{C})^{\top}$ be the Tor-pair generated by \mathcal{S}_{ϕ} . Then again by Proposition 2.15, we conclude that the modules in $\mathcal{C}_{\phi} := \mathcal{S}_{\phi}^{\top}$ are of the desired form.

For the moreover statement, let $M \in \text{Mod-}R$ and consider a short exact sequence $0 \rightarrow \Omega_1 M \rightarrow P \rightarrow M \rightarrow 0$ with P projective. Then $\text{depth}_{R_{\mathfrak{p}}}((\Omega_1 M)_{\mathfrak{p}}) = \min(\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + 1, \text{depth}(\mathfrak{p}))$. Since $P \in \mathcal{C}$, we get $M \in \mathcal{C}_{(1)} \iff \Omega_1 M \in \mathcal{C}$ by Lemma 3.1, which yields the desired description of $\mathcal{C}_{(1)}$, then we proceed by induction. The final claim then follows directly as $\text{depth}_{R_{\mathfrak{p}}}(k(\mathfrak{p})) = 0$. \square

Remark 4.15. In the generating set \mathcal{S}_{ϕ} constructed above, for $n > 0$ we note that

$$\mathcal{S}_{\phi} \cap \mathcal{F}_n(R) = \{S_{\phi(\mathfrak{p})}(\mathfrak{p}R_{\mathfrak{p}}; R_{\mathfrak{p}}) \mid \phi(\mathfrak{p}) \leq n\}.$$

However, for $i \geq 0$, the induced Tor-pairs are generated by the following set of R -modules of finite flat dimension,

$$(\mathcal{E}_{\phi})^{\top > i} = (\mathcal{C}_{\phi})_{(i)} = \{S_{\phi(\mathfrak{p})-i}(\mathfrak{p}R_{\mathfrak{p}}; R_{\mathfrak{p}}) \mid \phi(\mathfrak{p}) > i\}^{\top}.$$

In case the flat dimensions of the generators are bounded by 1, the class \mathcal{C} is closed under submodules. In this case, we can readily capture these classes in terms of subsets of $\text{Spec}(R)$; this will serve as the base step of the coming induction. For a subset X of $\text{Spec}(R)$, we will denote by \mathcal{C}_X the class of R -modules $\{M \in \text{Mod-}R \mid \text{Ass}(M) \subseteq X\}$

Lemma 4.16. *Let \mathcal{C} be a class of R -modules closed under submodules, extensions, and direct limits. Then there is a subset X of $\text{Spec}(R)$ such that $\mathcal{C} = \{M \in \text{Mod-}R \mid \text{Ass}(M) \subseteq X\}$.*

Proof. Given $M \in \mathcal{C}$, we have $k(\mathfrak{p}) \in \mathcal{C}$ for any $\mathfrak{p} \in \text{Ass}(M)$ as \mathcal{C} is closed under submodules. Then also $E(k(\mathfrak{p})) \in \mathcal{C}$, because $E(k(\mathfrak{p}))$ admits a filtration by coproducts of copies of $k(\mathfrak{p})$ by Matlis [Mat58]. Put $X = \bigcup_{M \in \mathcal{C}} \text{Ass}(M)$, then clearly $\mathcal{C} \subseteq \mathcal{C}_X$. To prove the converse, let $N \in \text{Mod-}R$ such that $\text{Ass}(N) \subseteq X$. Then the injective envelope $E(N)$ of N is a coproduct of copies of $E(k(\mathfrak{p}))$ with $\mathfrak{p} \in \text{Ass}(N)$, again by [Mat58]. Then $E(N) \in \mathcal{C}$, so that also $N \in \mathcal{C}$. \square

Finally, we are ready to state and prove our main theorem.

Theorem 4.17. *Let R be a commutative noetherian ring. There is a bijective correspondence between the following collections.*

- (i) Hereditary Tor -pairs $(\mathcal{E}, \mathcal{C})^\top$ generated by classes of modules of finite flat dimension,
- (ii) Hereditary cotorsion pairs $(\mathcal{C}, \mathcal{W})$ cogenerated by classes of pure-injective modules of finite injective dimension,
- (iii) Functions $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ such that $\phi \leq \text{depth}$,
- (iv) Sequences of subsets $X_0 \subseteq \dots \subseteq X_n \subseteq \dots \subseteq \text{Spec}(R)$ such that $\text{Ass}(\Omega_{-i}(R)) \subseteq X_i$.

The assignments are as follows:

- (i) \rightarrow (ii): See Lemma 2.2.
- (iii) \rightarrow (i) : $\phi \mapsto (\mathcal{E}_\phi, \mathcal{C}_\phi)$, where

$$\mathcal{C}_\phi := \{M \in \text{Mod-}R \mid \text{depth}_{R_p}(M_p) \geq \phi(p) \ \forall p \in \text{Spec}(R)\}.$$

- (ii) \rightarrow (iii) : $\mathcal{C} \mapsto \phi_{\mathcal{C}}$, where

$$\phi_{\mathcal{C}}(p) := \inf\{i \in \mathbb{Z}_{\geq 0} \mid k(p) \in \mathcal{C}_{(i)}\}.$$

- (ii) \rightarrow (iv) : $\mathcal{C} \mapsto X_0 \subseteq \dots \subseteq X_n \subseteq \dots \subseteq \text{Spec}(R)$, where

$$X_n := \text{Ass}(\mathcal{C}_{(n)}).$$

Proof. The assignment (i) \rightarrow (ii) of Lemma 2.2 is clearly injective. The assignments (iii) \rightarrow (i) and (ii) \rightarrow (iii) are well-defined by Proposition 4.14 and Proposition 4.13. There is a straightforward bijection (iii) \rightarrow (iv) which assigns $X_n := \bigcup_{0 \leq i \leq n} \phi^{-1}(i)$ for each $n \geq 0$. In this way, (ii) \rightarrow (iv) is identified with (ii) \rightarrow (iii) via Lemma 4.3.

The composition (iii) \rightarrow (i) \hookrightarrow (ii) \rightarrow (iii) is an identity by Proposition 4.14. To show that all the involved assignments are bijections, it remains to show that (ii) \rightarrow (iii) is injective. Let $(\mathcal{C}, \mathcal{W})$ be a cotorsion pair of the collection (ii) and ϕ the function $\text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ associated via (ii) \rightarrow (iii). We shall show that $\mathcal{C} = \mathcal{C}_\phi$.

Using Corollary 3.7, we can reduce to the case of R local. Then $\mathcal{I} = \mathcal{I}_{\dim(R)}$, which implies $\mathcal{C}_{(\dim(R))} = \text{Mod-}R$. Since \mathcal{C} is closed under injective envelopes by Corollary 4.4, we have $\text{Sub}(\mathcal{C}) = \text{Sub}(\mathcal{C} \cap \mathcal{I}_0)$, where \mathcal{I}_0 is the class of injective modules. Then it is easy to see that $\text{Sub}(\mathcal{C})$ is closed under extensions, and by the argument of [HS20, Lemma 5.7], it is also closed under direct limits. It follows by Lemma 4.16 that $\text{Sub}(\mathcal{C}) = \mathcal{C}_{X_0}$, where again $X_i = \text{Ass}(\mathcal{C}_{(i)})$ for each $i \geq 0$. Using Lemma 4.9, we get the relation $M \in \mathcal{C}_{(i)}$ if and only if $\text{Ass}(M) \subseteq X_i$ and $\Omega_{-1}(M) \in \mathcal{C}_{(i+1)}$. Since $\mathcal{C}_{(\dim(R))} = \text{Mod-}R$, we get by induction that $M \in \mathcal{C}$ if and only if $\text{Ass}(\Omega_{-i}(M)) \subseteq X_i$ for all $i \geq 0$. Then $\mathcal{C} = \mathcal{C}_\phi$ by Proposition 2.15. \square

Remark 4.18. It follows that the character duals of the modules in \mathcal{S}_ϕ defined in Proposition 4.14 are cogenerators of the cotorsion pair $(\mathcal{C}_\phi, \mathcal{W}_\phi)$ in Theorem 4.17. However, it also follows that all hereditary cotorsion pairs cogenerated by pure-injectives of finite injective dimension can be cogenerated instead by modules which are better-understood: Matlis duals.

By Proposition 2.15, $\{S_{\phi(p)}(\mathfrak{p}R_p; R_p)^{v_p} \mid S_{\phi(p)}(\mathfrak{p}R_p; R_p) \in \mathcal{S}_\phi\}$ also cogenerate the desired cotorsion pair, where

$$S_{\phi(p)}(\mathfrak{p}R_p; R_p)^{v_p} = \text{Hom}_{R_p}(S_{\phi(p)}(\mathfrak{p}R_p; R_p), E(R_p/\mathfrak{p}R_p)).$$

The following is a restriction of our main theorem to cotilting Tor -pairs. This provides a generalization of the classification of n-cotilting classes of [AHPŠT14] to

the case when the corresponding resolving subcategory has no uniform bound on projective dimension, which is relevant in the case $\dim(R) = \infty$, see also [HNŠ24, Remark 5.7] and references therein for a discussion about ∞ -cotilting classes relevant to our setting. Note that the bijection $(iii) \leftrightarrow (v)$ is precisely [DT15, Theorem 1.2] of Dao and Takahashi.

Remark 4.19. Here, we call a function $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ *order-preserving* if $\phi(\mathfrak{q}) \leq \phi(\mathfrak{p})$ whenever $\mathfrak{q} \subseteq \mathfrak{p}$. The *grade function* is the function $\text{grade}: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ defined by the rule $\mathfrak{p} \mapsto \text{grade}_R(\mathfrak{p}, R) = \inf\{i \in \mathbb{Z}_{\geq 0} \mid \text{Ext}_R^i(R/\mathfrak{p}, R) \neq 0\}$, cf. §2.10. The relation with the *depth* function used in Theorem 4.17 is given by $\text{grade}(\mathfrak{p}) = \inf\{\text{depth}(\mathfrak{q}) \mid \mathfrak{q} \in V(\mathfrak{p})\}$, see [BH98, Proposition 1.2.10]. In particular, *grade* is an order-preserving function such that $\text{grade} \leq \text{depth}$, and we deduce that for any order-preserving function $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ that $\phi \leq \text{depth}$ if and only if $\phi \leq \text{grade}$.

Corollary 4.20. *Let R be a commutative noetherian ring. The correspondence of Theorem 4.17 restricts to another bijective correspondence between the following collections.*

- (i) *Hereditary Tor-pairs $(\mathcal{E}, \mathcal{C})^\top$ generated by classes of finitely generated modules of finite projective dimension,*
- (ii) *Cotilting Tor-pairs $(\mathcal{E}, \mathcal{C})^\top$,*
- (iii) *Order-preserving functions $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ such that $\phi \leq \text{depth}$,*
- (iv) *Sequences $X_0 \subseteq \dots \subseteq X_n \subseteq \dots \subset \text{Spec}(R)$ of generalization closed subsets such that $\text{Ass}(\Omega_{-i}(R)) \subseteq X_i$.*
- (v) *resolving subcategories of $\text{mod-}R$ consisting of modules of finite projective dimension.*

Proof. The equivalence between (iii) and (iv) follows directly from definitions. To a function $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ as in (iii), we assign the set

$$\mathcal{S}'_\phi = \{S_{\phi(\mathfrak{p})}(\mathfrak{p}; R) \mid \mathfrak{p} \in \text{Spec}(R)\},$$

note that \mathcal{S}'_ϕ consists of finitely generated modules of finite projective dimension (see Remark 2.12), because $\phi \leq \text{grade}$ by Remark 4.19. Let $(\mathcal{E}, \mathcal{C})^\top$ be the hereditary Tor-pair such that $\mathcal{C} = (\mathcal{S}'_\phi)^\top$. We have

$$\mathcal{C} = \{M \in \text{Mod-}R \mid \text{grade}_R(\mathfrak{p}, M) \geq \phi(\mathfrak{p}), \mathfrak{p} \in \text{Spec}(R)\}$$

by [HŠ20, Proposition 5.12]. We claim that this coincides with the Tor-pair associated to ϕ via Theorem 4.17. Indeed, similarly to the discussion above, this follows directly from [BH98, Proposition 1.2.10] and the order-preservation of ϕ . Since \mathcal{S}'_ϕ consists of finitely generated modules, \mathcal{C} is definable, and so $(\mathcal{E}, \mathcal{C})^\top$ is a cotilting Tor-pair. The assignment from (iii) to (v) is by taking the smallest resolving subcategory of $\text{mod-}R$ generated by \mathcal{S}'_ϕ .

If $\dim(R) < \infty$ then we can see that these assignments are bijections directly by Theorem 2.13. It remains to explain the case of infinite Krull dimension. Let $(\mathcal{E}, \mathcal{C})^\top$ be a cotilting Tor-pair. For each maximal ideal \mathfrak{m} , $\mathcal{C}_\mathfrak{m}$ is a cotilting class in $\text{Mod-}R_\mathfrak{m}$ by Proposition 2.7. Let $\phi_\mathfrak{m}: \text{Spec}(R_\mathfrak{m}) \rightarrow \mathbb{Z}_{\geq 0}$ be the associated order-preserving function to each $(\mathcal{E}_\mathfrak{m}, \mathcal{C}_\mathfrak{m})^\top$ and let $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ be the function associated to $(\mathcal{E}, \mathcal{C})^\top$. Since $\Omega_{-i}k(\mathfrak{p}) \in \mathcal{C}$ if and only if $\Omega_{-i}k(\mathfrak{p}) \in \mathcal{C}_\mathfrak{m}$ for each maximal ideal $\mathfrak{p} \subseteq \mathfrak{m}$, we see that $\phi_\mathfrak{m}(\mathfrak{p}R_\mathfrak{m}) = \phi(\mathfrak{p})$ for all inclusions $\mathfrak{p} \subseteq \mathfrak{m}$. Then it is straightforward to show that $\mathcal{C} = (\mathcal{S}'_\phi)^\top$. This establishes the bijection between items (i), (ii), (iii). Finally, items (i) and (v) are in bijection by an argument similar to [GT12, Theorem 13.49]. \square

It is natural to ask to what extent is our Theorem 4.17 broader than the (essentially) known Corollary 4.20. In view of the parametrizing functions, this is to ask

how strong of an assumption for a function $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ bounded by depth is it to be order-preserving. Using a classical result of Bass, we get the following dichotomy.

Proposition 4.21. *Let R be a commutative noetherian ring. Then the following are equivalent:*

- (i) *every function $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ with $\phi \leq \text{depth}$ is order-preserving.*
- (ii) $\dim(R) \leq 1$.

Proof. (i) \implies (ii) : Towards contradiction, assume that $\dim(R) > 1$. Let $\mathfrak{q} \in \text{Spec}(R)$ be a prime of height 2. Then a theorem of Bass [Bas62, Proposition 5.2, Corollary 5.3] provides a non-maximal prime ideal $\mathfrak{p} \subseteq \mathfrak{q}$ such that $\text{depth}(\mathfrak{p}) = 1$. Then the function ϕ assigning 1 to \mathfrak{p} and 0 to all other primes in $\text{Spec}(R)$ is bounded by depth and not order-preserving.

(ii) \implies (i) : We have $\phi(\mathfrak{p}) \leq \text{depth}(\mathfrak{p}) \leq \text{height}(\mathfrak{p})$ for every $\mathfrak{p} \in \text{Spec}(R)$. Then $\phi(\mathfrak{p}) = 0$ if \mathfrak{p} is a minimal prime and $\phi(\mathfrak{p}) \in \{0, 1\}$ if \mathfrak{p} is a maximal prime. Then clearly ϕ is order-preserving. \square

Remark 4.22. The depth function is the largest choice of a parametrizing function in Theorem 4.17, and as such corresponds to the largest Tor-pair $(\mathcal{E}_{\text{depth}}, \mathcal{C}_{\text{depth}})^\top$ in the lattice which is subject to our classification. In fact, $\mathcal{C}_{\text{depth}}$ consists precisely of modules of restricted flat dimension zero, see §6. If R is Cohen–Macaulay then $\text{depth} = \text{grade} = \text{height}$ is itself an order-preserving function and thus the induced Tor-pair is cotilting. In general though, the depth function may fail to be order-preserving. In fact, depth is order-preserving if and only if $\text{depth} = \text{grade}$ if and only if R is almost Cohen–Macaulay, see §6 again.

For example, consider a noetherian local ring R with maximal ideal \mathfrak{m} such that $\text{depth}(\mathfrak{m}) = 0$. Then the only function associated to a cotilting Tor-pair by Corollary 4.20 is the constant zero. However, if $\dim(R) > 1$ then Proposition 4.21 yields more depth-bounded functions and thus more hereditary Tor-pairs subject to Theorem 4.17, including the one corresponding to the depth function itself.

5. THE REGULAR CASE

Let R be a regular ring, that is, every localisation of R at a prime ideal has finite global dimension. We show that Theorem 4.17 characterises *all* hereditary Tor-pairs over R , and in fact, all hereditary cotorsion pair cogenerated by pure-injectives. This follows readily if R is local, or even of finite Krull dimension, but to obtain the general case we need to make the following observation first. Let $\text{mSpec}(R)$ denote the set of maximal ideals of R .

Proposition 5.1. *Let R be a commutative noetherian ring. Any hereditary cotorsion pair cogenerated by a class \mathcal{P} of pure-injective R -modules is cogenerated by the class $\bigcup_{\mathfrak{m} \in \text{mSpec}(R)} \mathcal{P}^{\mathfrak{m}}$, where $\mathcal{P}^{\mathfrak{m}} = \{N^{\mathfrak{m}} \mid N \in \mathcal{P}\}$ for each maximal ideal \mathfrak{m} .*

Proof. Follows directly from Proposition 3.6. \square

Corollary 5.2. *Let R be a regular ring. Any hereditary cotorsion pair cogenerated by pure-injective modules is cogenerated by pure-injective modules of finite injective dimension.*

Proof. This follows directly from Proposition 5.1 as $N^{\mathfrak{m}}$ is a pure-injective R -module whose injective dimension over both $R_{\mathfrak{m}}$ and R cannot exceed $\dim(R_{\mathfrak{m}})$. \square

Note that if R is regular then every hereditary Tor-pair is generated by modules of finite flat dimension. Indeed, if a hereditary Tor-pair $(\mathcal{E}, \mathcal{C})^\top$ is generated by a class \mathcal{G} of modules of finite flat dimension, then it is also generated by the

class $\bigcup_{\mathfrak{m} \in \mathfrak{m}\text{Spec}(R)} \mathcal{G}_{\mathfrak{m}}$, where $\mathcal{G}_{\mathfrak{m}} = \{G_{\mathfrak{m}} \mid G \in \mathcal{G}\}$. Unlike Proposition 5.1, this follows easily by noting that $\text{Tor}_i^R(M, G) = 0$ if and only if $0 = \text{Tor}_i^R(M, G)_{\mathfrak{m}} \cong \text{Tor}_i^{R_{\mathfrak{m}}}(M, G_{\mathfrak{m}})$ for each $\mathfrak{m} \in \mathfrak{m}\text{Spec}(R)$.

In the regular case, our Theorem 4.17 now allows the following formulation. Recall that over a regular (or more generally, Cohen–Macaulay) ring R , we have $\text{height}(\mathfrak{p}) = \text{grade}(\mathfrak{p}) = \text{depth}(\mathfrak{p})$ for all $\mathfrak{p} \in \text{Spec}(R)$.

Theorem 5.3. *Let R be a regular commutative noetherian ring. There is a bijective correspondence between the following collections.*

- (i) Hereditary Tor-pairs $(\mathcal{E}, \mathcal{C})^{\top}$,
- (ii) Hereditary cotorsion pairs $(\mathcal{C}, \mathcal{W})$ cogenerated by classes of pure-injective modules,
- (iii) Functions $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ such that $\phi \leq \text{depth}$,
- (iv) Sequences $X_0 \subseteq \dots \subseteq X_n \subseteq \dots \subseteq \text{Spec}(R)$ such that $\text{Ass}(\Omega_{-i}(R)) \subseteq X_i$.

Remark 5.4. Note that, even in the regular case above, the theory of pure-injective modules can be far from transparent. In fact, outside of the hereditary case of dimension at most one, such rings always admit a superdecomposable pure-injective module, see [JL89, Summary 8.71].

The collection (i) of Theorem 5.3 has an obvious symmetry: Given a hereditary Tor-pair $(\mathcal{E}, \mathcal{C})^{\top}$, also $(\mathcal{C}, \mathcal{E})^{\top}$ is a hereditary Tor-pair. This is reflected by the corresponding functions in the following way.

Proposition 5.5. *If ϕ is the function $\text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ associated to $(\mathcal{E}, \mathcal{C})^{\top}$ in Theorem 5.3, then we claim that the function associated to $(\mathcal{C}, \mathcal{E})^{\top}$ is exactly $\psi(\mathfrak{p}) := \text{height}(\mathfrak{p}) - \phi(\mathfrak{p})$.*

Proof. To see the claim, first note that as \mathcal{E} is hereditary, so is \mathcal{C} , and by construction $S_{\phi(\mathfrak{p})}(\mathfrak{p}R_{\mathfrak{p}}; R_{\mathfrak{p}}) \in \mathcal{E}$. The Koszul complex $K_{\bullet}(\mathfrak{p}R_{\mathfrak{p}}; R_{\mathfrak{p}})$ is a projective resolution of $R_{\mathfrak{p}}$ as $R_{\mathfrak{p}}$ is a regular ring by assumption, so $S_j(\mathfrak{p}R_{\mathfrak{p}}; R_{\mathfrak{p}})$ is a $(\text{height}(\mathfrak{p}) - j)$ -syzygy of $k(\mathfrak{p})$ for every $0 \leq j \leq \text{height}(\mathfrak{p})$. In particular, $\text{depth}_{R_{\mathfrak{p}}} S_j(\mathfrak{p}R_{\mathfrak{p}}; R_{\mathfrak{p}}) = \text{height}(\mathfrak{p}) - j$, so it follows from the characterisation in Proposition 2.15 that $S_j(\mathfrak{p}R_{\mathfrak{p}}; R_{\mathfrak{p}}) \in \mathcal{C}$ for $j \leq \text{height}(\mathfrak{p}) - \phi(\mathfrak{p})$. So the associated function to the hereditary bounded Tor-pair $(\mathcal{C}, \mathcal{E})^{\top}$ is exactly $\text{height}(\mathfrak{p}) - \phi(\mathfrak{p})$, as claimed. \square

Remark 5.6. Let R be a commutative noetherian ring and $(\mathcal{E}, \mathcal{C})^{\top}$ a hereditary Tor-pair generated by a class of modules of finite flat dimension. If R is singular then the opposite Tor-pair $(\mathcal{C}, \mathcal{E})^{\top}$ is never subject to the classification Theorem 4.17. Indeed, towards contradiction assume that both $(\mathcal{C}, \mathcal{E})^{\top}$ and $(\mathcal{E}, \mathcal{C})^{\top}$ are generated by modules of finite flat dimension. Passing to a localisation, we can assume that R is a local singular ring. Then $\mathcal{E}, \mathcal{C} \subseteq \mathcal{F}_d$, where $d = \dim R$. This is a contradiction, as the singularity ensures that there is a module M such that $M \in \mathcal{F}^{\top}$ but $M \notin \mathcal{F}$. Indeed, we can choose M as a cocycle module in an acyclic, but not contractible, complex of projective R -modules.

Proposition 5.7. *Let R be a regular commutative noetherian ring. The bijection of Theorem 5.3 restricts to another bijection between:*

- (i) Tor-pairs $(\mathcal{E}, \mathcal{C})^{\top}$ such that both \mathcal{E} and \mathcal{C} are definable,
- (ii) functions $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ satisfying:
 - $\phi(\mathfrak{q}) = 0$ for any minimal prime \mathfrak{q} , and
 - $0 \leq \phi(\mathfrak{p}) - \phi(\mathfrak{q}) \leq 1$ for all immediate inclusions $\mathfrak{q} \subsetneq \mathfrak{p}$.

Proof. First, notice that any function ϕ as in (ii) satisfies $0 \leq \phi(\mathfrak{p}) \leq \text{height}(\mathfrak{p})$, and so corresponds to a hereditary Tor-pair via Theorem 5.3. Indeed, given any $\mathfrak{p} \in \text{Spec}(R)$, consider a chain of immediate inclusions $\mathfrak{q} = \mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \dots \subsetneq \mathfrak{q}_n = \mathfrak{p}$,

where \mathfrak{q} is some minimal prime ideal. Then the assumption on ϕ yields $0 = \phi(\mathfrak{q}) \leq \phi(\mathfrak{p}) \leq n - \phi(\mathfrak{q}) = n$, and $n = \text{height}(\mathfrak{p})$ as R is catenary.

In view of Corollary 4.20, it remains to check that the functions of (ii) are precisely those functions $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ of Theorem 5.3 such that both ϕ and $\psi = \text{height} - \phi$ are order-preserving. Assume that ϕ is order-preserving. Then ψ is order preserving precisely if for each immediate inclusion $\mathfrak{q} \subsetneq \mathfrak{p}$ of primes we have $\psi(\mathfrak{q}) - \psi(\mathfrak{p}) \geq 0$. We have

$$\psi(\mathfrak{q}) - \psi(\mathfrak{p}) = (\text{height}(\mathfrak{q}) - \text{height}(\mathfrak{p})) - (\phi(\mathfrak{q}) - \phi(\mathfrak{p})) = 1 - (\phi(\mathfrak{q}) - \phi(\mathfrak{p})),$$

which shows that indeed $\psi(\mathfrak{q}) - \psi(\mathfrak{p}) \geq 0$ if and only if $\phi(\mathfrak{p}) - \phi(\mathfrak{q}) \leq 1$. \square

Corollary 5.8. [GT12, Theorem 16.31] *Let R be a regular commutative noetherian ring of Krull dimension 1 (= a hereditary commutative noetherian ring). Then given any Tor-pair $(\mathcal{E}, \mathcal{C})^\top$, both \mathcal{E} and \mathcal{C} are definable, and thus 1-cotilting classes.*

Proof. Let $\phi: \text{Spec}(R) \rightarrow \mathbb{Z}_{\geq 0}$ be a function satisfying $0 \leq \phi(\mathfrak{p}) \leq \text{height}(\mathfrak{p})$. Then the function is determined by values on maximal ideals \mathfrak{m} , with the only possible values being 0 or 1. Such a function automatically satisfies the condition (ii) of Proposition 5.7. \square

6. THE RESTRICTED FLAT DIMENSIONS

Let R be a commutative noetherian ring. Christensen, Foxby and Frankild introduced the restricted homological dimensions over a commutative noetherian ring in various works, see [CFF02] and [Chr00, Chapter 5]. Instead of considering a Tor-pair, their approach involved introducing the *large and small restricted flat dimensions* of a module, denoted Rfd and rfd respectively (also called the large and small restricted Tor-dimensions). They are defined as follows for an R -module M over a commutative noetherian ring R .

$$\text{Rfd}_R(M) := \sup\{i \geq 0 \mid \text{Tor}_i^R(F, M) \neq 0 \text{ for some } F \in \mathcal{F}\}$$

$$\text{rfd}_R(M) := \sup\{i \geq 0 \mid \text{Tor}_i^R(F, M) \neq 0 \text{ for some } F \in \mathcal{F}^{<\aleph_0}\}$$

We will let \mathcal{RF}_n denote the modules of large restricted flat dimension at most n and also let $(\mathcal{F}_{(n)}, \mathcal{RF}_n)^\top$ be the hereditary Tor-pair, which is generated by n -th yokes of modules belonging to \mathcal{F} . Note that this Tor pair corresponds to the function $\mathfrak{p} \mapsto \sup\{0, \text{depth}(\mathfrak{p}) - n\}$ on $\text{Spec}(R)$ via Theorem 4.17, see Proposition 4.14.

In [CFF02], the authors succeed in providing a kind of analogue of the Auslander–Buchsbaum formula for the restricted flat dimensions, see [CFF02, Theorem 2.4(b)], which we recover in our main Theorem 4.17 by considering $\phi := \text{depth}$.

It is straightforward to see that $\text{rfd}_R(M) \leq \text{Rfd}_R(M) \leq \text{fd}_R(M)$ for any R -module M . However, when each of the equalities hold is less straightforward, see [CFF02] for more details. In particular, they show that if R is a local ring, then $\text{rfd}_R(M) = \text{Rfd}_R(M)$ for every (finitely generated) R -module M if and only if R is *almost Cohen–Macaulay*, that is if $\dim R_{\mathfrak{p}} - \text{depth}(R_{\mathfrak{p}}) \leq 1$ for every prime ideal \mathfrak{p} , see [CFF02, Theorem 3.2]. Note that R is almost Cohen–Macaulay if and only if $\text{grade} = \text{depth}$ by [CFF02, Lemma 3.1].

Combining some results with Proposition 2.7 of [AHT04, Theorem 2.3], one can extend this to all noetherian rings. The equivalence of (iii) and (iv) is generalised in our Corollary 4.20.

Proposition 6.1. *Let R be a commutative noetherian ring. Then the following are equivalent.*

- (i) R is almost Cohen–Macaulay
- (ii) $\text{rfd}_R(M) = \text{Rfd}_R(M)$ for every (finitely generated) R -module M .

- (iii) \mathcal{RF}_0 is a definable class, that is, \mathcal{RF}_0 forms the right-hand class of a cotilting Tor-pair generated by $\mathcal{P}^{<\aleph_0}(R)$.
- (iv) $\mathcal{F}(R) \subseteq \varinjlim \mathcal{P}^{<\aleph_0}(R)$, where the equality holds if and only if R is of finite dimension.
- (v) $\mathcal{F}_n(R) = \varinjlim \mathcal{P}_n^{<\aleph_0}(R)$ for every $n > 0$.
- (vi) $\mathcal{F}_n(R_{\mathfrak{p}}) = \varinjlim \mathcal{P}_n^{<\aleph_0}(R_{\mathfrak{p}})$ for every $n > 0$ for every $\mathfrak{p} \in \text{Spec}(R)$.

Proof. (i) \Leftrightarrow (v) \Leftrightarrow (vi) is by [HLG24, Theorem B] and as being almost Cohen–Macaulay is a local condition by definition.

(i) \Rightarrow (ii): This holds by [CFF02, Corollary 3.3]. Alternatively, the equivalence (i) \Leftrightarrow (ii) follows from Theorem 4.17, Corollary 4.20, and the above mentioned fact that (i) amounts to $\text{grade} = \text{depth}$. Indeed, $\text{rfd}_R = \text{Rfd}_R$ is equivalent to \mathcal{RF}_n being definable by Corollary 4.20, which by the same result amounts to depth being order-preserving, and thus equal to grade .

(ii) \Rightarrow (iii): By [CFF02, Theorem 3.2] we can suppose that $\text{rfd}_R(M) = \text{Rfd}_R(M)$ for every R -module M . In particular, \mathcal{RF}_0 is closed under products as it is generated as a Tor-orthogonal to a set of finitely presented modules, and such classes are closed under products.

(iii) \Rightarrow (vi): It follows that for every prime ideal \mathfrak{p} , $\mathcal{RF}_0(R_{\mathfrak{p}}) = \mathcal{RF}_0(R) \cap \text{Mod-}R_{\mathfrak{p}}$ as Tor commutes with localisations, so $\mathcal{RF}_0(R_{\mathfrak{p}})$ is a definable class for each prime $\mathfrak{p} \in \text{Spec}(R)$. By Proposition 2.7, we conclude that $\mathcal{F}_n(R_{\mathfrak{p}}) = \varinjlim \mathcal{P}_n^{<\aleph_0}(R_{\mathfrak{p}})$ for every prime \mathfrak{p} where $n := \dim R_{\mathfrak{p}}$. To see that the statement holds for every n , apply [HLG24, Theorem B].

(v) \Rightarrow (iv): This is clear.

(iv) \Rightarrow (vi): Take $F \in \mathcal{F}_n(R_{\mathfrak{p}}) \subseteq \mathcal{F}_n(R)$ where $n := \dim R_{\mathfrak{p}}$. By assumption, $F = \varinjlim M_i$, where each M_i is a finitely presented R -module of finite projective dimension. As F is an $R_{\mathfrak{p}}$ -module by assumption, and direct limits commute with localisation, $F \cong F \otimes_R R_{\mathfrak{p}} = \varinjlim (M_i \otimes_R R_{\mathfrak{p}})$, where the right-hand side is a direct limit in $\text{Mod-}R_{\mathfrak{p}}$ of finitely presented $R_{\mathfrak{p}}$ -modules of projective dimension at most n . That the statement holds for all n then follows by [HLG24, Theorem B and Subsection 2.13]. \square

For the weaker condition that $\mathcal{F}_n(R) = \varinjlim \mathcal{P}_n^{<\aleph_0}(R)$ for some $n > 0$, see [HLG24, Theorem B].

Remark 6.2. The modules with vanishing restricted flat dimension were also introduced in [Xu96] with the nomenclature *strongly torsion free* modules over an arbitrary ring. Additionally, we recover Xu’s characterisation of the strongly torsion free modules over Gorenstein rings in our Theorem 4.17, see [Xu96, Theorem 5.4.8].

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