Universitat Autònoma de Barcelona

Topology of manifolds

2020/2021

Code: 100114
ECTS Credits: 6

Degree	Type	Year	Semester
2500149 Mathematics	OT	4	0

The proposed teaching and assessment methodology that appear in the guide may be subject to changes as a result of the restrictions to face-to-face class attendance imposed by the health authorities.

Contact

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Use of Languages

Principal working language: spanish (spa)

Some groups entirely in English: No

Some groups entirely in Catalan: No

Some groups entirely in Spanish: Yes

Teachers

Roberto Rubio Nuñez

Prerequisites

It is better to have succeded in the course "Diferential Calculus".

Objectives and Contextualisation

Ever since the concept of homeomorphism was clearly defined, the "ultimate" problem in topology has been to classify topological spaces "up to homeomorphism". That this was a hopeless undertaking was very soon apparent, the subspaces of the plane R2 being an obvious example. From this impossibility were born algebraic and differential topology, by a shift of emphasis which consisted in associating "invariant" objects to some types of spaces (objects are the same for two homeomorphic spaces). At first these objects were integers, but it was soon realized that much more information could be extracted from invariant algebraic structures such as groups and rings.

(Jean Dieudonné, A history of algebraic and differential topology 1900--1960)

The objective of this course is twofold. On the one hand, we will introduce one of the most important and most studied classes of topological spaces: differentiable manifolds. These spaces, very common in both mathematics and physics, have the remarkable feature of admitting generalizations of concepts previously studied in Multivariable Calculus and Differential Geometry.

On the other hand, we will introduce the cohomological methods in topology. De Rahm's cohomology is an extremely useful technique to understand the "shape" of manifolds: it consists of transforming (part of) the geometric information that supports a manifold into algebraic objects, in this case, a sequence of vector spaces. This makes it possible to try the following strategywhen solving a problem on a manifold: translate it into an algebraic problem, calculate the solution at the algebraic level and geometrically reinterpret the result.

In particular, these vector spaces nicely encode several properties of the manifold: its dimension, its orientation, superior orientation properties (spin structures, etc.). In addition to introducing these cohomology groups, we will present some of the tools used to extract the relevant information from these spaces.

This course is a first look at a theory that has been developing since the end of the 19th century and that continues to be active. Among his great achievements are: the classification of surfaces, the demonstration of the Poincaré conjecture in dimensions greater than 5, the "Kervaire invariant 1" problem and more recently the development of topological techniques for data analysis.

Competences

- Actively demonstrate high concern for quality when defending or presenting the conclusions of ones work.
- Apply critical spirit and thoroughness to validate or reject both ones own arguments and those of others.
- Demonstrate a high capacity for abstraction.
- Develop critical thinking and reasoning and know how to communicate it effectively, both in ones own languages and in a third language.
- Effectively use bibliographies and electronic resources to obtain information.
- Formulate hypotheses and devise strategies to confirm or reject them.
- Generate innovative and competitive proposals for research and professional activities.
- Students must be capable of collecting and interpreting relevant data (usually within their area of study) in order to make statements that reflect social, scientific or ethical relevant issues.
- Students must be capable of communicating information, ideas, problems and solutions to both specialised and non-specialised audiences.
- Students must develop the necessary learning skills to undertake further training with a high degree of autonomy.
- Students must have and understand knowledge of an area of study built on the basis of general secondary education, and while it relies on some advanced textbooks it also includes some aspects coming from the forefront of its field of study.

Learning Outcomes

1. Actively demonstrate high concern for quality when defending or presenting the conclusions of ones work.
2. Apply critical spirit and thoroughness to validate or reject both ones own arguments and those of others.
3. Develop critical thinking and reasoning and know how to communicate it effectively, both in ones own languages and in a third language.
4. Devise demonstrations of mathematical results in the field of geometry and topology.
5. Effectively use bibliographies and electronic resources to obtain information.
6. Generate innovative and competitive proposals for research and professional activities.
7. Students must be capable of collecting and interpreting relevant data (usually within their area of study) in order to make statements that reflect social, scientific or ethical relevant issues.
8. Students must be capable of communicating information, ideas, problems and solutions to both specialised and non-specialised audiences.
9. Students must develop the necessary learning skills to undertake further training with a high degree of autonomy.
10. Students must have and understand knowledge of an area of study built on the basis of general secondary education, and while it relies on some advanced textbooks it also includes some aspects coming from the forefront of its field of study.
11. Understand abstract language and in-depth demonstrations of some advanced theorems of geometry and topology.

Content

The course will cover the following topics:

1) Definition of smooth manifold:

- Notion of atlas.

- Smooth functions.

- Submanifolds.

2) Tangent space:

- Tangent vector.

- Tangent map.

- Immersions, submersions.

- Transversality

3) Vector fields:

- Definition.

- Flow of a vectorfield.

- Lie calculus.

- Vector bundles.

4) Differential forms:

- Definition.

- Exterior product and differential.

- De Rahm's complex.

- Notion of homology.

5) Cohomology of de Rahm:

- Formal properties.

- Homotopy invariance.

- Mayer-Vietoris sequence.

- Künneth's theorem.

6) Orientability:

- Cup product.

- Integration of differential forms.

- Stokes' formula.

- Poincaré duality.

Methodology

In this course, we will follow the "flipped classroom" approach. Each week students will be provided with a few pages of reading and problems to be thoroughly studied before arriving to class. These pages will be accompanied by a reading guide and questions intended to stimulate the personal reflection of the students. The lecture will be used to understand the key concepts (solving any doubts that may exist), put them into practice through problem solving and analyze their importance or contextualize them. Students are expected to be the main participants in the discussion, while the role of the teachers is to stimulate this discussion, contribute their experience and knowledge, and suggest possible directions. These discussions are expected to take place in small groups in the classroom (following the distancing measures recommended at the time) and then the progress made will be shared with the whole class. This group dynamic will also be the one followed online.

The seminar hours will be used to make presentations by the students on topics previously discussed and researched in groups.

A more detailed guide on the methodology used in this course will be provided on the virtual campus.

Activities

Title	Hours	ECTS	Learning Outcomes
Type: Directed			
Problem Sessions	14	0.56	2, 1, 3, 6, 10, 8, 7
Theory classes	30	1.2	2, 1, 3, 6, 10, 8, 7
Type: Supervised			
Seminars	6	0.24	2, 1, 3, 6, 10, 8, 7
Type: Autonomous			
Assimilations of theoretical results	45	1.8	6, 10, 8, 5
Homework	15	0.6	2, 11, 3, 10, 8, 5
Solving problems	30	1.2	11, 4, 10, 8

Assessment

The assessment will be carried out in two modalities: an important part of continuous assessment together with a final exam. There will be no partial exam.

Each week on the sheet with the theoretical contents to be studied before class you will find three types of exercises:

Type A): Questions about the concepts and some very basic exercise.

Type B): Warm-up exercises to understand the concepts.

Type C): Advanced problems (not necessarily every week).

Each student is expected to submit their responses to exercises A) on Sunday. These are evaluated out of 15, and only four grades are possible: 0, 5, 10 and 15. Only the effort of having tried to answer, and not the correctness of the answer, is assessed. The average of the grades obtained divided by 10 results into a grade A, which goes from 0 to 1.5.

Exercises B) must be submitted at the beginning of the session on Tuesday. They are also evaluated out of 15, with four possible grades: 0, 5, 10 and 15. Likewise, the main part of the assessment is the effort that has been put into answering and the quality of the writing. The mark 15 means having proposed, to a large extent, mostly correct solutions. The average of the grades obtained divided by 10 results into a grade B, which goes from 0 to 1.5.

Exercises C) in general will not be solved in class. Students will be asked to work on some of them and submit their solutions. The goal for these problems is to do them perfectly, so teachers may ask to redo some of them if necessary. The obtained grades are averaged into a grade C, between 0 and 10.

In small groups, students will be asked to study some topics outside the course syllabus, but within its limits. Each group must submit a synopsis of the topic discussed, as well as a plan for their oral presentation. Each group will present their topic orally to the teachers and their classmates. All members of the group will have to present part of the work orally. This results in a grade P, in which both the small written contribution and the quality of the oral presentation are assessed. The grade P is between 0 and 10.

The final exam is assessed with a grade between 0 and 10.

THERE IS NO PARTIAL EXAM.

The final grade will be: A + B + 0.2C + 0.2P + 0.3F

A document in the virtual campus will recall the assessment for this course and give some more details.

The reevaluation exam replaces parts C and F. This exam cannot be used to "raise your grade".

A student who does not take the final exam is considered "No assessable".

Assessment Activities

Title	Weighting	Hours	ECTS	Learning Outcomes
Final exam	30%	3	0.12	11, 1, 4, 9
Homework assignments, type C.	20%	0	0	11, 3, 4, 10, 9, 5
Oral presentation, type P.	20%	1	0.04	11, 1, 3, 6, 9, 8, 7, 5
Reading assignment type B.	15%	0	0	2, 11, 1, 3, 4, 10, 8, 5
Weekly work mod. A	20%	3	0.12	11, 1, 3, 10, 5
Written reevaluation	50%	3	0.12	11, 1, 3, 4, 9, 7, 5

Bibliography

Basic:

J. M. Lee, Introduction to Smooth Manifolds, Graduate Texts in Mathematics 218, Springer-Verlag, New York. xvii+ 631 pp. (https://link-springer-com.are.uab.cat/book/10.1007%2F978-1-4419-9982-5)

L. W. Tu, An introduction to manifolds. Universitext. Springer, New York, second edition, 2011. 384 pp. (https://link-springer-com.are.uab.cat/book/10.1007%2F978-1-4419-7400-6)

Advanced:

A. Hatcher, Algebraic topology. Cambridge University Press, Cambridge, 2002. xii+544 pp. (http://www.math.cornell.edu/~hatcher/AT/ATpage.html)

R. Bott and L.W. Tu, Differential forms in algebraic topology. Graduate Texts in Mathematics, 82. Springer-Verlag, New York-Berlin, 1982. xiv+331 pp.

M. W. Hirsch, Differential Topology. Graduate Texts in Mathematics 33, Springer-Verlag New York Heidelberg Berlin 1976. x + 222pp.

P. Michor, Topics in Differential Geometry, Graduate Studies in Mathematics 93. American Mathematical Society, Providence, RI, 2008. xii+494 pp. (https://www.mat.univie.ac.at/~michor/dgbook.pdf)