A REPRESENTATION THEOREM FOR CERTAIN BOOLEAN LATTICES

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Abstract

Let $R$ be an associative ring with 1 and $R$-tors the complete brouwerian lattice of all hereditary torsion theories on the category of left $R$-modules. A well known result asserts that $R$ is a left semiartinian ring iff $R$-tors is a complete atomic boolean lattice. In this note, we prove that if $\mathcal{L}$ is a complete atomic boolean lattice then, there exists a left semiartinian ring $R$ such that $\mathcal{L}$ is lattice-isomorphic to $R$-tors.

1. Background. Throughout the following, $R$ will denote an associative ring with unit element and $R$-tors will denote the complete brouwerian lattice of all hereditary torsion theories on the category $R$-mod of unitary left $R$-modules. Notation and terminology concerning $R$-tors will follow [1].

If $M \in R$-mod, let us denote by $\xi(M)$ the smallest element of $R$-tors respect to which $M$ is torsion; the unique minimal element of $R$-tors is then $\xi = \xi(0)$. We will denote by $R$-simp a complete set of representatives of isomorphism classes of simple left $R$-modules.

A lattice $\mathcal{L}$ is said to be complete if arbitrary meets and joins are defined in $\mathcal{L}$. In particular, in a complete lattice there are a minimal element and a maximal element, which will be denoted by $\xi$ and $\chi$ respectively. An element $a$ of a complete lattice $\mathcal{L}$ is an atom if $a \neq \xi$ and $b < a$ implies $b = \xi$; we will denote by $\mathfrak{A}(\mathcal{L})$, the set of atoms of $\mathcal{L}$. $\mathcal{L}$ is said to be atomic if for every $x \in \mathcal{L}$, there exists $a \in \mathfrak{A}(\mathcal{L})$ such that $a \leq x$. $\mathcal{L}$ is locally atomic if every element of $\mathcal{L}$ is a join of atoms. A complemented distributive lattice is called a boolean lattice; in a boolean lattice, complements are unique.

For any ring $R$, the atoms of $R$-tors are the torsion theories of the form $\xi(S)$ with $S \in R$-simp. As a result, $R$-tors is an atomic lattice and $R$ is left semiartinian iff $R$-tors is a complete atomic boolean lattice or iff $R$-tors is locally atomic. See III of [1] for details.

2. Complete Atomic Boolean Lattices. In [2], professor Golan asks if for every boolean lattice $\mathcal{L}$, there exists a left semiartinian ring $R$ such that $\mathcal{L}$ is lattice-isomorphic to $R$-tors. Inasmuch as for every ring $R$, $R$-tors is complete atomic lattice, in this note, we will consider complete atomic boolean lattices.
Lemma 1. If $\mathcal{L}$ is a complete atomic boolean lattice, then

a) $\mathcal{L}$ is locally atomic

b) The decomposition of every element $x \in \mathcal{L}$, $x \neq \xi$ as join of atoms is unique.

Proof: The proof follows easily by using proposition 4.5 of Ch. III of [3]. $lacksquare$

Corollary 2. Let $\mathcal{L}$ and $\mathcal{L}'$ be complete atomic boolean lattices; if $\varphi : \mathfrak{A}(\mathcal{L}) \rightarrow \mathfrak{A}(\mathcal{L}')$ is a bijection, then $\varphi$ can be extend to a lattice isomorphism $\overline{\varphi} : \mathcal{L} \rightarrow \mathcal{L}'$.

Theorem 3. Let $\mathcal{L}$ be a complete atomic boolean lattice; then there exists a left semiartinian ring $R$ such that $\mathcal{L}$ is lattice-isomorphic to $R$-tors.

Proof: Let $\mathcal{L}$ be a complete atomic boolean lattice; by Corollary 2, it is enough to show a left semiartinian ring $R$ and a bijective correspondence from $\mathfrak{A}(\mathcal{L})$ to $R$-simp, then we will have that $\mathcal{L}$ is isomorphic to $R$-tors.

Case 1: $\mathfrak{A}(\mathcal{L})$ is a finite set.

Let us supose $|\mathfrak{A}(\mathcal{L})| = n < \infty$ and let $R = \mathbb{Z}_2^n$ be the ring direct product of $n$ copies of $\mathbb{Z}_2$, the integers modulo 2. Then $|R$-simp$| = n$ and inasmuch as $R$ is semiartinian ring, we are done.

Case 2: $\mathfrak{A}(\mathcal{L})$ is an infinite set.

Let us denote $\mathfrak{A}(\mathcal{L}) = \alpha$ and let $A = \mathbb{Z}_2^\alpha$ the ring direct product of $\alpha$ copies of $\mathbb{Z}_2$. Now, let $R$ be the subring of $A$ generated by the sum $I = \mathbb{Z}_2^{(\alpha)}$ and the unit element of $A$. In $R$, the projective simple modules are, except by isomorphisms, the simple direct sumands in the decomposition $I = \mathbb{Z}_2^{(\alpha)}$. On the other hand, the singular simple modules are all isomorphic to $R/I$. Then $|R$-simp$| = \alpha$ and so, there exists a bijection between $\mathfrak{A}(\mathcal{L})$ and $R$-simp.

Finally, it is not difficult to prove that $R$ is a semiartinian ring, and so we complete the proof.

References


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