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ON QUOTIENTS OF HOLOMORPHIC FUNCTIONS IN THE DISC WITH BOUNDARY REGULARITY CONDITIONS

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Abstract

In this paper we give characterizations of those holomorphic functions in the unit disc in the complex plane that can be written as a quotient of functions in $A(D)$, $A^\infty(D)$ or $\Lambda_1(D)$ with a nonvanishing denominator in D. As a consequence we prove that if $f \in \Lambda_1(D)$ does not vanish in D, then there exists $g \in \Lambda_1(D)$ which has the same zero set as f in $\overline{D}$ and such that $fg \in A^\infty(D)$.

1. Introduction and statement of results

Let D denote the unit disc in the complex plane and T its boundary. We denote by $A(D)$ the Banach algebra of all continuous functions on $\overline{D}$, holomorphic in D, and by $A^\infty(D)$ the Frechet algebra of all holomorphic functions in D such that all its derivatives extend continuously to $\overline{D}$. We will also consider the Banach algebra $\Lambda_1(D)$ consisting of all holomorphic functions in D satisfying a Lipschitz conditions of order one.

In this paper we are interested in the characterization of those holomorphic functions in D that can be written as a quotient of functions in $A(D)$, $A^\infty(D)$ or $\Lambda_1(D)$, respectively, with a nonvanishing denominator in D.

The corresponding real–variable problems are very simple. For example, every continuous (resp.C^∞) function in an open set V of $\mathbf{R}^n$ is a quotient of two continuous (resp.C^∞) functions in $\mathbf{R}^n$ vanishing exactly on $\mathbf{R}^n \setminus V$ (see [6]). On the other hand, this kind of problems has also been treated in complex analysis, the most well known one being the result of F. and R. Nevanlinna which characterizes the quotients of bounded holomorphic functions in D as the functions in the class N, i. e.

$$\sup_r \int_0^{2\pi} \log^+ |f(re^{it})| dt < \infty.$$

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(see also [**11**] for another problem of this type).

Before stating our main results we will recall some well–known notions and introduce some notations. Each function f in N has an unique factorization $f = BSF$, where B is the Blaschke product with the same zero sequence as f, F is the outer function with boundary absolute value $|F| = |f|$ and S is a singular function

$$S(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right\}$$

for some signed measure μ. The descomposition of $d\mu$ in its positive and negative parts gives $S = S_1/S_2$, with S_1, S_2 singular inner functions. The *Smirnov class* N^+ is the subclass of N defined by the condition $S_2 = 1$. We will use the fact that $f \in N^+$ if and only if the family $\log^+ |f_r|$, where $f_r(e^{i\theta}) = f(re^{i\theta})$, has uniformly absolutely continuous integrals. From this it easily follows that N^+ is a linear space (see [**3**, chapter 2]) as a general reference on N and N^+).

If $f \in N$ is as above we will write

$$\operatorname{sing}(f) = E \cup \operatorname{supp}\mu,$$

where E is the set of accumulation points of the zeros of f and supp μ is the closed support of μ. Then B and S extend analytically to $T \setminus \operatorname{sing}(f)$ and are not zero there (see [**4**, p.68]).

Now we state our first result:

Theorem 1. *An holomorphic function f in D is the quotient of two functions $f_1, f_2 \in A(D)$, $f_2 \neq 0$ in D if and only if the following two conditions hold:*

(a) $f \in N$.

(b) There is a closed set M of Lebesque measure zero such that:

(b.1) f extends continuously to $T \setminus M$.

(b.2) $\operatorname{sing}(S_2) \subset M$, where $f = BFS_1/S_2$ is the factorization of f.

In this case, f_2 can be chosen to vanish exactly on M and outer if and only if $f \in N^+$.

To state our other main theorems we need the notion of *Carleson set*. These are the closed sets $M \subset T$ such that

$$\int_T \log d(e^{i\theta}, M) \quad d\theta > -\infty$$

where $d(e^{i\theta}, M)$ denotes the distance from $e^{i\theta}$ to M (and so, obviously, subsets and finite unions of Carleson sets are Carleson sets, too). These sets are precisely the zero boundary sets of functions in $A^\infty(D)$ and $\Lambda_1(D)$ (see [**8**]). Finally, we say that $f \in A^\infty(D)$ is *flat* at a closed set $E \subset T$ if $f^{(n)}(z) = 0$ for all n and $z \in E$.

Theorem 2. *For an holomorphic function f in D to be the quotient of two functions f_1, $f_2 \in A^\infty(D)$, $f_2 \neq 0$ in D it is necessary and sufficient that:*

(a) $f^{(n)} \in N$ for all n, and if $f = BFS_1/S_2$ is its factorization then $(S_2 f)^{(n)} \in N^+$ for all n.

(b) There is a Carleson set M such that:

(b.1) There are a function g, satisfying a Lipschitz condition of order 1 in T, with $\log g \in L^1(T)$, positive outside M, and for each n an integer q_n such that

$$|f^{(n)}(z)|^{1/q_n} = 0(g(z)^{-1}), z \in T \setminus M.$$

(b.2) $sing(S_2) \subset M$.

In this case, f_2 can be chosen flat exactly on M and outer if and only if $f \in N^+$.

Theorem 3. *An holomorphic function f in D is the quotient of two functions $f_1, f_2 \in \Lambda_1(D), f_2 \neq 0$ in D if and only if the conditions (a) and (b) of Theorem 2 hold for $n = 0$ and $n = 1$. In this case, f_2 can always be chosen in $A^\infty(D)$ and flat exactly on M and outer if and only $f \in N^+$.*

In condition (b) of Theorem 2 and 3, $f^{(n)}$ is to be understood as the almost everywhere defined boundary value of the function $f^{(n)} \in N$. It is not difficult to see directly that conditions (a) and (b) imply in fact that $f^{(n)}$ has a continuous extension to $T \setminus M$ (for all n in Theorem 2 and $n = 0$ in Theorem 3). We point out however that, contrary to what one might expect in analogy with Theorem 1, condition (b.1) cannot be replaced by the weaker assumption that $f^{(n)}$ extends continuously to $T \setminus M$ for all n (see Remark 2 in section 3).

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2. Proof of Theorem 1

Lemma 1. *Assume $f \in N$ and that there is a closed set M such that f extends continuously to $T \setminus M$. Let $K = \{z \in T \setminus M : f(z) = 0\}$. Then $sing(f) \subset M \cup K$ and the outer part F of f also extends continuously to $T \setminus M$.*

Proof: This is well known when $M = \phi$ (see [4,p.69]) and the same proof applies in this general case. ■

Lemma 2. *Let φ be a non-negative continuous function in (a, b) such that $\log\varphi$ is integrable. Then there exists a C^∞ function ψ in (a, b) such that $\psi \geq 1$, $\psi \geq \varphi$ and*

$$\int_a^b \log\psi(t)dt \leq 4\int_a^b \log^+ \varphi(t)dt.$$

Proof: We consider a partition of (a,b) in intervals

$$[a+\frac{b-a}{2^{n+1}},a+\frac{b-a}{2^n}),[b-\frac{b-a}{2^n},b-\frac{b-a}{2^{n+1}}), \quad n\geq 1.$$

Let $[c,d)$ be one of these intervals. Let s be a step function in $[c,d)$ such that

$$\log^+\varphi \leq s\,, \int_c^d s \leq 2\int_c^d \log^+\varphi.$$

Doing the same in every interval we obtain a subdivision of (a,b) in intervals $[\lambda_n,\lambda_{n+1})$ and a function g , equal to some constant c_n on $[\lambda_n,\lambda_{n+1})$ such that

$$\log^+\varphi \leq g\,, \int_a^b g \leq 2\int_a^b \log^+\varphi.$$

We can also assume that $c_n\geq 0$ and $c_n\neq c_{n+1}$. Let $[\lambda_n,\lambda_{n+1})$ and $[\lambda_{n+1},\lambda_{n+2})$ be two contiguous intervals and let μ_n and μ_{n+1} their respective middle points. It is easy now to construct a function h on $[\mu_n,\mu_{n+1}]$ which verifies:

1) $h(x)\geq c_n$ for $x\in[\mu_n,\lambda_{n+1}]$ and $h(x)\geq c_{n+1}$ for $x\in[\lambda_{n+1},\mu_{n+1}]$.
2) $h(\mu_i)=2c_i \quad , i=n,n+1$.
3) h is C^∞ and all its derivatives vanish at μ_n and μ_{n+1}.
4) $\int_{\mu_n}^{\mu_{n+1}} h \leq 2\int_{\mu_n}^{\mu_{n+1}} g$.

Doing the same with all intervals we obtain a C^∞ function in (a,b), which we continue denoting by h, such that $h\geq g$ and

$$\int_a^b h \leq 2\int_a^b g.$$

Then $\psi = e^h$ satisfies all required conditions. ■

Proof of Theorem 1: The necessity of the conditions is clear in all the cases, using Lemma 1 for part (b.2). Assume now first that $f\in N^+$ and extends continuously to $T\setminus M$. Let $f=BSF$. Applying Lemma 2 to $\varphi=|f|$ in each complementary interval of $T\setminus M$ we obtain a C^∞ function φ_1 in $T\setminus M$, $\varphi_1\geq 1$, $\varphi_1\geq\varphi$ with $\log\varphi_1\in L^1(T)$. Now we consider the outer functions F_1,F_2 with boundary values $|F_1|=\varphi/\varphi_1$, $|F_2|=1/\varphi_1$, respectively, so that $|F_1(z)|$, $|F_2(z)|\leq 1$ and $F_1=FF_2$. The function F_2 extends continuously to $T\setminus M$ because $\log\varphi_1$ is C^∞ off M by Lemma 1 and hence $F_1=FF_2$ also extends continuously to $T\setminus M$. Let now G be an outer function in $A(D)$ vanishing exactly on M. Then F_1G, F_2G are outer functions in $A(D)$, $F_2G=0$ exactly on M and $F_1G=0$ on $M\cup K$ where K is as in Lemma 1. Now

$$f=\frac{BSF_1G}{F_2G}.$$

Since by Lemma 1, $\operatorname{sing}(BS) \subset M \cup K$ and F_1G vanishes there, it follows that $f_1 = BSF_1G \in A(D)$. This proves the theorem when $F \in N^+$. If f is just in N and $f = BFS_1/S_2$ with $\operatorname{sing}(S_2) \subset M$ repeating the proof we end up with

$$f = \frac{BS_1F_1G}{S_2F_2G}$$

and also $S_2F_2G \in A(D)$ because $\operatorname{sing}(S_2) \subset M$ and F_2G vanishes on M. ∎

3. Proof of Theorem 2

In order to prove Theorems 2 and 3 we will follow the method used by B.A. Taylor and D.L. Williams in [11] to obtain such a representation for certain Blaschke products. As in [11], we say that a function defined in (a, b) satisfies a *weak Lipschitz condition* if

$$|\varphi(t+\Delta t) - \varphi(t)| \leq 2|\Delta t||\varphi(t)|^2$$

for $|\Delta t| \leq 1/2|\varphi(t)|^{-1}$. We will use the following three lemmas (see [11] for Lemmas 3 and 4 and [10] for Lemma 5):

Lemma 3. *Let φ be a real-valued in (a, b) which satisfies a weak Lipschitz condition, $\varphi \geq 2$, and*

$$|\varphi(t)| \geq (min(|t-a|, |t-b|))^{-1}.$$

Then there exists a real-valued C^∞ function h in (a, b) such that $h \geq 2$, $\varphi - 2 \leq h \leq \varphi + 2$ and $|h^{(n)}| \leq c_n\varphi^{3n}$ for some constant c_n and all n.

Lemma 4. *Let g be a non-negative integrable function on $[-\pi, \pi]$. There exists a non-negative C^∞ function $w(x)$, defined for $x \geq 0$, which verifies:*

(a) $x^{-1}w(x) \to +\infty$ as $x \to +\infty$.

(b) $\int_{-\pi}^{\pi} w(g(t))dt < +\infty$.

(c) For each n there exists a constant c_n such that

$$|w^{(n)}(x)| \leq c_n(1+x^2).$$

Lemma 5. *Let $f \in A^\infty(D)$, let $f = BSF$ and let $K = \{z \in T : f^{(n)}(z) = 0$ for all $n\}$. Then $sing(f) \subset K$, (so that $sing(f)$ is a Carleson set), $F \in A^\infty(D)$ and F is flat on K. More generally, if $f \in A^\infty(D)$ and S_0 is a singular inner function dividing f (i. e. S/S_0 is bounded), then $f/S_0 \in A^\infty(D)$. Conversely, if S is a singular inner function and $F \in A^\infty(D)$ is flat on $sing(S)$ then $FS \in A^\infty(D)$.*

Proof of Theorem 2: Let $f = f_1/f_2$, with $f_1, f_2 \in A^\infty(D)$, $f_2 \neq 0$ in D, and let M be the set of zeros of f_2, a Carleson set in T. It is obvious that

$f^{(n)} \in N$ for all n and that $f^{(n)} f_2^{n+1} \in A^\infty(D)$, and hence (b.1) holds with $q_n = n+1$ and $g = |f_2|$. It remains to prove that $\operatorname{sing}(S_2) \subset M$ and that $(S_2 f)^{(n)} \in N^+$. Let $d\nu_1, d\nu_2$ be the measures corresponding to the singular parts of f_1, f_2 respectively and $d\mu_1, d\mu_2$ the ones corresponding to S_1 and S_2. Then $\mu_1 = (\nu_1 - \nu_2)_+ \le \nu_1$, $\mu_1 - \mu_2 = \nu_1 - \nu_2$, that is, $\nu_i = \mu_i + \nu$, $i = 1,2$ for some positive measure ν. Therefore, $\operatorname{sing}(S_2) \subset \operatorname{supp} \mu_2 \subset M$, by Lemma 5. Also, the singular inner function S_0 corresponding to $d\nu$ divides both f_1, f_2 and so by Lemma 5, f_1/S_0 and f_2/S_0 are again in $A^\infty(D)$. This means that replacing f_1 by f_1/S_0 and f_2 by f_2/S_0 we can assume that S_2 is the singular inner part of f_2. Then $S_2 f = f_1/F_2$, where F_2 is the outer part of f_2, which is in $A^\infty(D)$ by Lemma 5, and now it is clear that $(S_2 f)^{(n)} \in N^+$ for all n.

Assume now that $f \in N^+$ satisfies (a), i. e. $f^{(n)} \in N^+$, for every n, and (b). Without loss of generality we can assume that the Lipschitz constant for g is 1, that $|g| \le 1/2$ and, multiplying g by a function in $A^\infty(D)$ vanishing on M, that g is zero on M. Then the hypothesis of Lemma 3 hold for $\varphi = 1/g$ in each complementary interval of M in T (if $|\Delta t| \le 1/2\varphi(t)^{-1} = |g(t)|/2$ then $g(t+\Delta t) \ge g(t)/2$ and hence

$$|\varphi(t+\Delta t) - \varphi(t)| = \frac{|g(t+\Delta t) - g(t)|}{|g(t)||g(t+\Delta t)|} \le \frac{|\Delta t|}{\frac{1}{2}|g(t)|^2} = 2|\Delta t||\varphi(t)|^2).$$

If h is the function given by Lemma 3, then $h_1 = h + 2$ is a real valued C^∞ function in $T \setminus M$ such that $h_1 \ge 4$, $1/g \le h_1 \le 3/g$ and for every n there exists c_n such that

$$|h_1^{(n)}| \le c_n g^{-3n}.$$

In particular h_1 has an integrable logarithm. Let now w be a function as in Lemma 4, with $g = \log h_1$. We consider the outer function $F = \exp G$ with

$$G(z) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} w(\log h_1(t))dt.$$

Notice that $F \in C^\infty(\overline{D} \setminus M)$ and it is not zero on $\overline{D} \setminus M$.

We will prove now that all the derivates of F and fF are bounded and tend to zero at any point of M, which will finish the proof of the theorem in case $f \in N^+$.

First we estimate $|F|$ on $T \setminus M$. Since $w(x)/x \to +\infty$ as $x \to +\infty$ we can write

$$|F(z)| = h_1(z)^{-\mu(z)} \le (3/g(z))^{-\mu(z)} \tag{1}$$

with $\mu(z) \to +\infty$ as $z \in T$ approaches $z_o \in M$. We estimate now $G^{(n)}$ following a method similar to the one in Carleson [**2**] in the form expressed by the following Lemma (see Lemma 2.3 in [**1**]):

Lemma 6. *Let Ψ be a function of class C^{n+1} in some arc $J = [a,b]$ of T, let c be the middle point of J and let $A(z)$ be defined by*

$$A(z) = \int_a^b \frac{e^{it}}{(e^{it}-z)^{n+1}} \Psi(e^{it})dt \quad , \quad z \in D.$$

If $\Psi^0 = \Psi$, $\Psi_k(e^{it}) = e^{-it}\frac{d}{dt}\Psi_{k-1}(e^{it})$, $k = 1,\dots, n+1$ and $M_k = max\ \{|\Psi_k(e^{it})|, a \le t \le b\}$, then for $z = re^{ic}$, $0 \le r < 1$

$$|A(z)| \le const(\Sigma_{k=0}^{n-1} \frac{M_k}{|z-e^{ia}|^{n-k}} + M_n + |J|\, M_{n+1}).$$

Let $z \in T \setminus M$. In

$$|G^{(n)}(rz)| = \frac{n!}{\pi} |\int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it}-rz)^{n+1}} w(\log\ h_1(t))dt|$$

we break the integral into two parts corresponding to the interval of center z and radious $\frac{1}{2}|g(z)|$, which is less than $\frac{1}{2}d(z,M)$, and its complementary. In the second integral, $|e^{it}-rz| \ge c|g(z)|$ and so this integral is bounded by const $|g(z)|^{-n-1}$. In the first integral we use the bounds for h_1 and its derivatives, and those of w, obtaining with the notations of Lemma 6,

$$M_n \le c \max\{|g(e^{it})|^{-p_n}, a \le t \le b\}$$

for some integer p_n.

Using the fact that $g(z+\Delta z) \ge \frac{1}{2}g(z)$ for $\Delta z < \frac{1}{2}g(z)$ we see that $M_n \le$ const $g(z)^{-p_n}$ and so we conclude that

$$|G^{(n)}(rz)| \le \text{const}\, g(z)^{-p_n} \tag{2}$$

for some integers p_n. From (1) and (2) it follows that

$$|F^{(n)}(z)| \le \text{const}(\frac{3}{g(z)})^{-\mu(z)} g(z)^{-q_n} \quad , z \in T \setminus M,$$

for some integers q_n. Now, by the hypothesis (b.1) on f, $(fF)^{(n)}$ will also satisfy this bound. This implies that $(fF)^{(n)}$ and $F^{(n)}$ are bounded and tend to zero at any point of M along $T \setminus M$, because $\mu(z) \to +\infty$.

The proof will be finished if we show that $(fF)^{(n)}$ and $F^{(n)}$ belong to N^+ for any n. Since $f^{(n)} \in N^+$ for all n by hypothesis, it is enough to prove that any derivate of G belongs to N^+. But this follows from (2), because $\log^+|G_r^{(n)}|$ is uniformly integrable being dominated by $\log^+ 1/g$.

This ends the proof of the sufficiency part of Theorem 2 in case $f \in N^+$. Assume now that f satisfies (a) and (b) and $\operatorname{sing}(S_2) \subset M$. This later fact clearly implies that for some constanst p_n

$$|S_2^{(n)}(z)| \leq \operatorname{const} d(z,N)^{-p_n} \leq \operatorname{const} g(z)^{-p_n}$$

(recall that we can assume $g(z) \leq d(z,N)$). Hence fS_2 is a N^+ function to which we can apply what we have already proved: $S_2 f = S_2 f F/F$ with F flat on M. By the last part of Lemma 5, $S_2 F \in A^\infty(D)$ and the proof is finished. ■

Remark 1. A particular case of a function which verifies the condition of Theorem 3 was considered by B.A. Taylor and D.L. Williams ([**1**]). It is a Blaschke product such that if E is its set of zeros then

$$\int \log d(e^{i\theta}, E) d\theta > -\infty.$$

In this case , the function $d(e^{it}, E)$ plays the role of the function g in condition (b.1) and $M = \overline{E} \cap T$.

Remark 2. As already said, condition (b.1) in Theorem 2 cannot be replaced by the weaker assumption that $f^{(n)}$ extends continuously to $T \setminus M$ for all n. Consider for instance a Blaschke product whose zero set

$$E = \{r_n e^{i\theta_n}\}_{n \geq 1}$$

satisfies:

(a) $\sum_{n=1}^{\infty} (1 - r_n)^\alpha < +\infty$ for every $\alpha > 0$.
(b) E has $\{1\}$ as unique acumulation point.
(c) The set $\{1\} \cup \{e^{i\theta_n}\}_{n \geq 1}$ is not a Carleson set.

Every Blaschke product that satisfies (a) has derivatives $B^{(n)} \in H^p$ for some $p = p(n)$ (see [**5**]), and then $B^{(n)} \in N^+$. Trivially B extends analitically to $T - \{1\}$, yet B cannot be expresed as a quotient f_1/f_2 with $f_1, f_2 \in A^\infty(B)$, because E, that verifies (b) and (c), is not the zero set of any function in $A^\infty(D)$ (see [**7**]).

4. Proof of Theorem 3

For the proof of Theorem 3 we have to use instead of Lemma 5 the following Lemma (see [**9**]):

Lemma 7. *Let $f = BSF \in \Lambda_1(D)$ and $K = \{z \in T : f(z) = 0\}$. Then $\operatorname{sing}(f) \subset K$, so that $\operatorname{sing}(f)$ is a Carleson set, and $F \in \Lambda_1(D)$. More generally, if S_0 is a singular inner function dividing f (i. e. S/S_0 is bounded),then f/S_0 is in $\Lambda_1(D)$.*

Proof of Theorem 3: Let $f = f_1/f_2$, $f_1, f_2 \in \Lambda_1(D)$, $f_2 \neq 0$ in D and let M be the set of zeros of f_2 wich is a Carleson set in T. It is clear that, f, $f' \in N$

and, as before, condition (b.1) is satisfied with $g = |f_2|$, $q_0 = 1, q_1 = 2$. With the same notations as in the proof of Theorem 2, in this case we have now, by Lemma 7, that f_1/S_0 and f_2/S_0 are again in $\Lambda_1(D)$. Hence we can also write $f = g_1/g_2$ with g_1, g_2 in $\Lambda_1(D)$ and now S_2 is also the singular inner part of g_2. Then $S_2 f = g_1/F_2$ with $F_2 \in \Lambda_1(D)$. Now it is clear that $(S_2 f)' \in N^+$.

Assume now that $f, f' \in N^+$ and that (b.1) holds. We repeat the construction of Theorem 2, thus obtaining an $F \in A^\infty(D)$ flat on M. We must check now that $fF \in \Lambda_1(D)$, that is, $(fF)' \in H^\infty$. This is, as before, consequence of the fact that $(fF)' = f'F + fF'$ belongs to N^+ and that it is bounded on T/M. If $f \in N$ satisfies (a) and (b) then the above case applies to $S_2 f$, so that $S_2 f = f_1/f_2$ with $f_2 \in A^\infty(D)$ and flat on M. Now $S_2 f_2 \in A^\infty(D)$ by Lemma 5 and we are done. ■

Corollary. *If $f \in \Lambda_1(D)$ and $f \neq 0$ in D, there exists $g \in \Lambda_1(D)$ which has the same zero set as f in $\overline{D}$ and such that $fg \in A^\infty(D)$.*

Proof: Apply the theorem to $1/f$. ■

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