THE PERIODIC SOLUTIONS
OF THE SECOND ORDER NONLINEAR
DIFFERENCE EQUATION

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Abstract

Periodic and asymptotically periodic solutions of the nonlinear equation
\( \Delta^2 x_n + a_n f(x_n) = 0, \ n \in \mathbb{N}, \) are studied.

In several recent papers ([2],[3]) the periodicity of solutions of linear difference equations have been investigated. In this paper we examine the periodic solutions of the nonlinear equation

\( (E) \quad \Delta^2 x_n + a_n f(x_n) = 0, \ n \in \mathbb{N}, \)

where \( \mathbb{N} = \{0,1,2,\ldots\}, \ \mathbb{R} \) is the set of real numbers, \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( a, x : \mathbb{N} \rightarrow \mathbb{R} \) are sequences of real numbers.

Throughout the paper we use the following notations. By \( \overline{0,t} \) we denote the set of integers \( \{0,1,2,\ldots,t\}. \) For the function \( y : \mathbb{N} \rightarrow \mathbb{R} \) the forward difference operator \( \Delta^k \) is defined

\[ \Delta^k y_n = y_{n+k} - y_n, \quad \Delta^k y_n = \Delta^{k-1}(\Delta y_n) \quad \text{for} \ k > 1. \]

**Definition 1.** The function \( y \) will be called \( t \)-periodic if \( y_{n+t} = y_n \) for all \( n \in \mathbb{N}. \) (Furthermore we suppose that no \( t_1 \) exists, \( 0 < t_1 < t \) such that \( y_{n+t_1} = y_n \) for all \( n \in \mathbb{N} \) and that \( t > 1). \)

**Definition 2.** The function \( y \) will be called asymptotically \( t \)-periodic \( (t > 1) \) if

\[ y = u + v, \]

where \( u \) is a \( t \)-periodic function and \( \lim_{n \to \infty} v_n = 0. \)

**Definition 3.** We say that the equation \( (E) \) has a \( p_t \)-constant if there exists a constant \( p \in \mathbb{R}, \) such that the equation

\[ (E_1) \quad \Delta^2 x_n + a_n f(x_n) = p \]
has a $t$-periodic solution.

We say that the equation $(E)$ possesses a $p_t^\infty$-constant if there exists a constant $p \in \mathbb{R}$ such that $(E_1)$ has an asymptotically $t$-periodic solution.

**Definition 4.** The equation $(E)$ is said to have a $p_t$-function ($p_t^\infty$-function) if there exists a $t$-periodic function $p : \mathbb{N} \to \mathbb{R}$ such that the equation

$$(E_2) \quad \Delta^2 x_n + a_n f(x_n) = p_n$$

has a $t$-periodic (asymptotically $t$-periodic) solution.

**Remark 1.** Note that if $(E)$ has a $p_t^\infty$-constant (function) then $(E)$ has a $p_t$-constant (function) and if $(E)$ has not a $p_t^\infty$-constant (function) then it has no $p_t$-constant (function).

**Theorem 1.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and $\lim_{n \to \infty} a_n = 0$. Then the equation $(E)$ has not a $p_t^\infty$-constant for any $t > 1$.

**Proof.** We show the proof for simplicity in the case $t = 2$. Similar reasoning can be made for $t > 2$.

Suppose that there exists a $p_t^\infty$-constant $q$ such that the equation

$$(E_3) \quad \Delta^2 x_n + a_n f(x_n) = q$$

has one asymptotically $2$-periodic solution $x$.

Let $x_{2n} \to C_1, x_{2n+1} \to C_2$ as $n \to \infty, C_1 \neq C_2$. Hence

$$\Delta^2 x_{2n} \to 2C_1 - 2C_2$$

$$\Delta^2 x_{2n+1} \to 2C_2 - 2C_1.$$

As a result of the assumption we obtain

$$2C_1 - 2C_2 = q$$

$$2C_2 - 2C_1 = q.$$

The above system has a solution if and only if $q = 0$, but in this case we obtain $C_1 = C_2$, which is a contradiction. \(\blacksquare\)

**Theorem 2.** Let $f \neq 0$ on $\mathbb{R}$. If the equation $(E)$ possesses a $p_t$-constant then $a$ is a $t$-periodic function.

**Proof.** Let $x$ be a $t$-periodic solution of $(E_3)$. Then $\Delta^2 x$ is $t$-periodic. By virtue of the assumption $f \neq 0$ and we get

$$\frac{\Delta^2 x_n - q}{f(x_n)} = -a_n.$$
The left hand side of the above equality is a $t$-periodic function so the right hand side must also be $t$-periodic.

Remark 2. We can prove analogously that if $f \neq 0$ on $\mathbb{R}$, then $t$-periodicity of $a$ is the necessary condition for the existence of a $p_t$-function $q$ for the equation $(E)$. However in this case we do not require for $t$ to be the basic period. Eventually $a$ can be a constant function. It is easy to see that if $f(C_i) = 0$ then the equation $(E)$ has $p_t$-constant $q = 0$. Then a $t$-periodic solution takes the form $x = C_t$.

By $i_R$ we denote the identity function on $\mathbb{R}$.

**Theorem 3.** Let $a : \mathbb{N} \rightarrow \mathbb{R}$, let $f$ be a continuous function on $\mathbb{R}$, $f \neq 0$ such that the functions

\begin{equation}
    i_R + a_n f : \mathbb{R} \rightarrow \mathbb{R}
\end{equation}

are surjections for every $n \in \mathbb{N}$. If

\begin{equation}
    \sum_{j=1}^{\infty} j |a_j| < \infty
\end{equation}

then the equation $(E)$ has a $p_t^{\infty}$-function for arbitrary $t \geq 1$.

**Proof:** Choose $t \geq 1$. By assumption there exist constants $C_r, r = 1, 2, \ldots, t$, $C_i \neq C_j, i \neq j$, such that

\[ f(C_r) \neq 0. \]

The case

\begin{equation}
    f(C_r) > 0, \ r = 1, 2, \ldots, t
\end{equation}

will be considered. The proof for the other cases $f(C_i) > 0, f(C_j) < 0$ is similar.

By virtue of the continuity of the function $f$ there exist intervals

\begin{equation}
    I_r = [C_{r+1} - \delta, C_{r+1} + \delta], \ r = 0, 1, \ldots, t - 1
\end{equation}

such that

\begin{equation}
    f(u) > 0 \text{ for } u \in I_r, \ r = 0, 1, \ldots, t - 1.
\end{equation}

From (2) it follows that

\begin{equation}
    \lim_{n \rightarrow \infty} \sum_{j=n}^{\infty} j |a_j| = 0.
\end{equation}
Let us denote

\[(7) \quad D = \max_{0 \leq r \leq t-1} \{ \max_{u \in I} f(u) \} \]

and

\[n_1 = \min \{ n \in \mathbb{N} : n = tk + t - 1, D \sum_{j=n}^{\infty} j|a_j| \leq \delta \}. \]

In the space \(l^\infty\) of bounded sequences with the norm

\[\|x\| = \sup_{n \geq 0} |x_n| \]

we define the set \(T\) in the following way:

\[x = \{x_i\}_{i=0}^{\infty} \in T\]

if

\[x_r = x_{t+r} = x_{2t+r} = \cdots = x_{n_1-t+r+1} = C_{r+1}, \quad x_{tk+r} \in I_{tk+r} :=\]

\[\quad = |C_{r+1} - D \sum_{j=tk+r}^{\infty} j|a_j|; C_{r+1} + D \sum_{j=tk+r}^{\infty} j|a_j||,\]

\[r = 0, 1, \ldots, t-1; \quad k \in \mathbb{N}; \quad k > \frac{1}{t}(n_1 + 1 - t).\]

The set \(T\) is closed, convex and bounded. Furthermore, by \(\text{diam } S\) we mean

\[\text{diam } S = \sup\{\|x - y\|; \quad x \in S, y \in S\}. \]

So

\[(8) \quad \text{diam } I_{tk+r} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.\]

It is easy to find a finite \(\varepsilon\)-net for every \(\varepsilon > 0\). Therefore by Hausdorff’s Theorem the set \(T\) is compact. Let us define an operator \(A\) for \(x \in T\) as follows:

\[Ax = y = \{y_i\}_{i=0}^{\infty}\]

where

\[y_r = y_{t+r} = \cdots = y_{n_1+r+1-t} = C_{r+1}; \quad r = 0, 1, \ldots, t-1, \]

\[y_{tk+1} = C_{r+1} - \sum_{j=tk+r}^{\infty} (j + 1 - tk - r)a_j f(x_j)\]

for \(k \in \mathbb{N}, k > \frac{1}{t}(n_1 + 1 - t), \quad r = 0, 1, \ldots, t-1.\)

Let us observe that

\[I_{tk+r} \subset I_r, \quad r = 0, 1, \ldots, t-1, \quad k > \frac{1}{t}(n_1 + 1 - t).\]
Hence

\[ (9) \quad | \sum_{j=\tau k+r}^{\infty} (j+1-tk-r)a_j f(x_j) | \leq \sum_{j=\tau k+r}^{\infty} j|a_j| |f(x_j)| \leq D \sum_{j=\tau k+r}^{\infty} j|a_j|. \]

Therefore \( y_{\tau k+r} \in I_{\tau k+r} \), \( r = 0, 1, \ldots, t-1 \), \( k \in \mathbb{N} \), \( k > (n_1 + 1 - t)/t \) and this means that \( A : T \to T \). Let us take an arbitrary sequence \( \{x^m\}_{m=1}^{\infty} \) of elements of \( T \) convergent to some \( x^0 \in T \) i.e.

\[ \|x^m - x^0\| \to 0. \]

Hence we have

\[ (10) \quad \sup_{n \geq 0} |x^m_n - x^0_n| \to 0 \]

as \( m \to \infty \). Let \( \varepsilon_1 \) be an arbitrarily taken positive real number. By the uniform continuity of \( f \) on the sets \( I_r \) we have

\[ |u_1 - u_2| < \delta \text{ implies } |f(u_1) - f(u_2)| < \varepsilon_1. \]

From (10) it follows that

\[ (11) \quad \sup_{n \geq 0} |x^m_n - x^0_n| < \delta \]

for \( m \geq M(\delta) \). Let \( y^m = Ax^m \), \( m \in \mathbb{N} \); then

\[ (12) \quad \|Ax^m - Ax^0\| = \sup_{n \geq n_1} | \sum_{j=n}^{\infty} (j+1-n)a_j f(x^m_j) - \sum_{j=n}^{\infty} (j+1-n)a_j f(x^0_j) |. \]

By (9) the series

\[ \sum_{j=n}^{\infty} (j+1-n)a_j f(x^m_j), m \in \mathbb{N} \]

are absolutely convergent. Hence, by (11) and (12)

\[ \|Ax^m - Ax^0\| \leq \varepsilon_1 \sum_{j=n_1}^{\infty} j|a_j| \]

so that the operator \( A \) is continuous on \( T \). By Schauder's Theorem there exists \( z \in T \) such that \( z = Az \). By definition of \( A \) this element \( z = \{z_i\}_{i=0}^{\infty} \) satisfies

\[ (13) \quad z_{r} = z_{r+t} = \cdots = z_{n_{1}+r+t-1} = c_{r+1} \]
Applying the operator $\Delta$ to $z$ we obtain

$$\Delta z_{tk+r} = z_{tk+r+1} - z_{tk+r} =$$

$$= C_{r+2} - C_{r+1} - \sum_{j=tk+r+1}^{\infty} (j + tk - r) a_j f(z_j) + \sum_{j=tk+r}^{\infty} (j + 1 - tk - r) a_j f(z_j) =$$

$$= C_{r+2} - C_{r+1} + \sum_{j=tk+r}^{\infty} a_j f(z_j),$$

and consequently

$$\Delta^2 z_{tk+r} = \Delta z_{tk+r+1} - \Delta z_{tk+r} =$$

$$= C_{r+3} - 2C_{r+2} + C_{r+1} + \sum_{j=tk+r+1}^{\infty} a_j f(z_j) - \sum_{j=tk+r}^{\infty} a_j f(z_j) =$$

$$= C_{r+3} - 2C_{r+2} + C_{r+1} - a_{tl+k+r} f(z_{tk+r}),$$

$$r = 0, 1, \cdots, t - 1, k > \frac{1}{t} (n_1 + 1 - t)$$

where

$$C_{t+1} = C_1, \ C_{t+2} = C_2.$$

Denoting

$$q_{tk+r} = C_{r+3} - 2C_{r+2} + C_{r+1}, \ r = 0, 1, \cdots, t - 1,$$

we obtain the equation

$$\Delta^2 x_n + a_n f(x_n) = q_n$$

which has an asymptotically $t$-periodic solution defined for $n > n_1$. This follows from (8) and $z_{tk+r} \in I_{tk+r}$, i.e. $z_{tk+r} \to C_{r+1}$ as $k \to \infty$.

It suffices to show that there exist a solution of (15) which coincides with (13) for $n > n_1$.

For this we observe that the equation (15) can be rewritten in equivalent form

$$x_n + a_n f(x_n) = q_n - x_{n+2} + 2x_{n+1}. $$
Taking \( n = n_1 \), \( x_{n+1} = x_{n_1+1} \), \( x_{n+2} = x_{n_1+2} \) we find \( x_{n_1} \), which by the assumptions exists (probably more than one). Repeating this reasoning we find \( x_i \) for \( i = 0, 1, \ldots, n_1 - 1 \). This function \( x \) is of course a solution of (15) which coincides with \( z \) for \( n > n_1 \) and therefore has the desired asymptotic behaviour.

**Remark 3.** If the functions \( i_R + a_n f \) are one-to-one mappings of \( R \) onto \( R \) then the solution obtained in the Theorem 3 is unique. The case \( t = 1 \), i.e. the solutions having the asymptotic property \( \lim_{n \to \infty} x_n = C \), was considered in the paper [1].

Let us observe that by Theorem 3 if we want to have some solutions which have a given asymptotically \( t \)-periodic solution, then it suffices to add to equation (E) the periodic perturbation \( q \) which can be easily found by (14).

**Example.** As an example we consider the difference equation of the form

\[
\triangle^2 x_n + \frac{(-1)^{n+1}}{4[2^n + (-1)^n]} x_n = 0, \quad n = 1, 2, \ldots
\]

It is evident by d'Alembert criterion that the series

\[
\sum_{j=1}^{\infty} \frac{j(-1)^{j+1}}{4[2^j + (-1)^j]}
\]

is absolutely convergent. Furthermore the functions

\[
x + \frac{(-1)^{n+1}}{4[2^n + (-1)^n]} x
\]

are surjections from \( R \) onto \( R \) for all \( n \). Therefore the assumptions of the Theorem 3 hold. We show that this equation has a \( p^\infty_2 \)-function and find a 2-periodic solution of the form

\[
x_n = (-1)^n + y_n.
\]

Applying the proof of the Theorem 3 we see that the \( p^\infty_2 \)-function \( q \) takes the form

\[
q_n = 4(-1)^n.
\]

Considering the equation

\[
\triangle^2 x_n + \frac{(-1)^{n+1}}{4[2^n + (-1)^n]} x_n = 4(-1)^n
\]

we can observe that this equation has the solution

\[
x_n = (-1)^n + \frac{1}{2^n}
\]

which is of the desired form.
References


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