EVALUATING A $p$-TH ORDER COHOMOLOGY OPERATION

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Abstract

A certain $p$-th order cup product is detected by a $p$-th order cohomology operation. The result is applied to finite $H$-spaces, to show that several properties of compact Lie groups do not hold for arbitrary torsion free finite $H$-spaces.

1. Summary of results

In this paper we give the detailed computations of results announced in [3]. As the computations involve many technical details we try to present the main line of proof separately in Chapter 2—filling in the missing details in Chapters 3–7. In Chapter 8 we bring some applications.

Let $p$ be a prime. Throughout this paper we write $H^*(X) = H^*(X, \mathbb{Z}/p\mathbb{Z})$. Let $\phi_n$ denote an $n + 1$ order mod $p$ cohomology operation associated with the $n+1$ times Toda brackets $\langle \overline{P^a, \ldots, P^{a-1}, P^1} \rangle$ ($a = 1$ if $n$ is even $a = p - 1$ if $n$ is odd). By this we mean a stable cohomology operation universally defined on a spectra

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The editor regrets very much to inform that Professor Zabrodsky died on November 86.

The first author dedicates this paper to his memory.
$E_n$ with a Postnikov system given by:

$$
\begin{align*}
\Omega K_n & \xrightarrow{j_n} E_n \xrightarrow{k_n} K_{n+1} \\
\Omega K_r & \xrightarrow{j_r} E_r \xrightarrow{k_r} K_{r+1} \\
\Omega K_1 & \xrightarrow{j_1} E_1 \xrightarrow{k_1} K_2
\end{align*}
$$

with the properties:

1.0. 

(1) $K_0 = K(Z/pZ, 0)$ - the Eilenberg MacLane Spectrum.

(2) for $r \geq 1$, $K_r = \sum^{\ell_r} K_0$, $t_r = \begin{cases} spq + q - 2s & r = 2s + 1 \\ spq - 2s + 1 & 0 < r < 2s \end{cases}$.

(3) $\Omega K_r \xrightarrow{j_r} E_r \xrightarrow{h_r} E_{r-1}$ and for $0 \leq r < n$, $E_{r+1} \xrightarrow{h_{r+1}} E_r \xrightarrow{K_r} K_{r+1}$ are (co) fibrations.

(4) $[k_r \circ j_r] \in [\Omega K_r, K_{r+1}] = P_0^r \in A(p)$ where

$$a_r = \begin{cases} 1 & \text{for } r \text{ - even} \\ p-1 & \text{for } r \text{ - odd} \end{cases}$$

$\phi_n$ is then given by the universal example (in the sense of [1]) $(x = H^0(h_1 \circ \ldots \circ h_n))_n$, $E_n$, $u = H^{i+n+1}(k_{n+1})(\Sigma^{i+n+1})$.

1.1. Remark. We shall use the notations of D1 to describe a non stable (say $m-1$ connected) representative of $\phi_n$. In that case the maps in (3) are fibrations, $K_0 = K(Z/pZ, m)$, $K_r = K(Z/pZ, m + t_r)$.

The main purpose of this paper is to show:

**The Main Theorem.**

(1) $\phi_{p-1}$ exists and if $n \neq (-1) \mod p$ then $P^i_{2n+1} \in H^{2n+1}(K(Z/pZ, 2n+1))$ is in the domain of $\phi_{p-1}$ and $i_{2n+1} \cdot P^1_{2n+1} \cdot P^2_{2n+1} \cdot \ldots \cdot P^p_{2n+1} \in \phi_{p-1}(P^i_{2n+1})$. 


The principal applications of the main theorem (which generalizes results in [11]) is the construction of counterexamples that the following properties shared by Lie groups, are not valid in general for finite H-spaces:

1.2. Examples.
1.2.1. Let $X$ be a compact connected Lie group. If $H^*(X) \cong \wedge(x_{2m_1+1}, \ldots, x_{2m_n+1})$ with $m_1 \leq m_2 \leq \cdots \leq m_n$ then there exists a map $f: X \to S^{2m_1+1}$ with $x_{2m_n+1} \in \text{Im} H^*(f)$.

1.2.2. It was recently proved by H. Miller ([4]) that for $X = O(n), U(n), Sp(n)$ (and consequently $X$ may belong to a larger family of Lie groups) $X$ splits stably according to the (integral) algebraic weight of its generators, i.e. if $H^*(X)$ is as above $\Sigma^\infty X \cong Y_1 \vee Y_2 \vee \cdots \vee Y_r$ where $H^*(Y_j)$ contains all the monomials $x_{2m_{j_1}+1} \cdot x_{2m_{j_2}+1} \cdots x_{2m_{j_r}+1}$ in $H^*(X)$.

1.2.3. It is also well known (e.g. see [2, theorem 1.1] with $f = \Sigma T$, $T$ - the $\lambda$-th power map) that for any H-space $X$ with $H^*(X)$ as above $\Sigma^\infty X \cong Y_1 \vee Y_2 \vee \cdots \vee Y_{r-1}$ where $H^*(Y_j)$ contains all the monomials in the $x_{2m_{j_1}+1}$’s of length $j \mod p - 1$ (these will be called elements of mod $p - 1$ algebraic weight $j$).

Thus, the following question naturally arises:

1.3. (See e.g. [9, problem 14]) Which of these Lie groups properties are shared by finite H-spaces in general?
1.3.1. If $X$ is a mod $p$ H-space with

$$H^*(X) = \wedge(x_{2m_1+1}, x_{2m_2+1}, \ldots, x_{2m_n+1})$$

does there exist a map $f: X \to S^{2m_1+1}$ with $x_{2m_n+1} \in \text{Im} H^*(f)$?

1.3.2. Is there a mod $p$ stable splitting of a mod $p$ finite H-space according to the (integral) algebraic weight of its cohomology?

1.3.3. (The weakest common property): Let $X$ be as in 1.3.2. Is there a stable map $\tilde{f}: X \to \Omega^\infty \Sigma^\infty S^{2m_1+1}$ with $x_{2m_n+1} \in \text{Im} H^*(\tilde{f})$?

All these questions are negatively answered by the following:

1.4. Example. (See [3]): There exists a mod $p$ H-space ($p \geq 5$) $X$ with $H^*(X) = \wedge(x_{2p+1}, p^1 x_{2p+1}, \ldots, p^p x_{2p+1})$. It follows from the main theorem that there is no stable map $f: X \to \Omega^\infty \Sigma^\infty S^{2p^1+1}$ with $p^p x_{2p+1} \in \text{Im} H^*(f)$.

A consequence of the above example (and similar $\infty$ dimensional examples) is that for an arbitrary stable splitting of $K(\mathbb{Z}/p\mathbb{Z}, 2n + 1)$ ($n \not\equiv -1(p)$) any summand containing $p^\nu x_{2n+1}$ in its cohomology will have to include the product $x_{2n+1} \cdot p^1 x_{2n+1} \cdots p^\nu x_{2n+1}$.

Finally one can deduce a non realizability theorem:
1.5. Proposition. (See Corollary 8.5) If \( n \neq -1 \mod p \) then there is no mod p \( H \)-space (or even a \( \lambda \)-power space) \( X \) with

\[
H^r(X) = \wedge(x_{2n+1}, p^1 x_{2n+1}, \ldots, p^{p-1} x_{2n+1}).
\]

If \( n \) is large enough there is no space with the above cohomology. (Note that \( \wedge(x_{2p-1}, p^1 x_{2p-1}, \ldots, p^{p-1} x_{2p-1}) \) is the mod p cohomology of one of Nishida's factors of \( SU(p^2 - p + 1) \) ([7]). Hence \( n \neq -1 \mod p \) is essential).

2. Outline of proof

Evaluating a \( p \)-th order cohomology operation cannot be expected to be an easy task. Generally it involves many technical details. Still, the core of the computations has a definite straightforward line of thought. In this section we shall try to outline the proof of our main results referring the reader to the auxiliary technical computations in the following sections. The first observation (3.1) is that \( \phi_{p-1} \) exists. (This is obvious for \( p = 2 \) thus we restrict ourselves to \( p \)-odd). The reason is quite simple: Assuming \( \phi_{r-1} \) exists, the obstruction for the existence of \( \phi_r \) is a class in the cohomology of \( E_{r-1} \). It turns out that for small \( r \) the cohomology of \( E_{r-1} \) in the appropriate dimension is zero.

The fact that \( P_{n2n+2} \) is in the domain of \( \phi_{p-1} \) is embedded in the inductive proof of the main theorem but essentially it is a consequence of 4.3.

The principal idea of the proof of the main theorem is as follows: Let \( K = K(Z/pZ, 2n + 1) \) and let

\[
\Sigma K = B_1 \subset B_2 \subset B_3 \subset \cdots \subset B_\infty = BK = K(Z/pZ, 2n + 2)
\]

be the Milnor–Rothenberg–Steenrod–Milgram filtration ([5], [8]) of the classifying space of \( K \). We shall use the following notations:

2.0.

(i) \( i_{r,s}: B_r \rightarrow B_s \) is the inclusion, with the abbreviation \( i_{r,s+1} = i_r \).

(ii) \( B_r \cup CB_{r-1} = B_r, B_{r-1} \), \( j_r: B_r \rightarrow B_r, B_{r-1} \) denotes the inclusion.

(iii) \( \delta_r: B_r, B_{r-1} \rightarrow \Sigma B_{r-1} \) the third leg of the Puppe sequence.

(iv) \( t = 2n+2 \) denotes all fundamental classes in \( H^{2n+2}(B_r), \ 1 \leq r \leq \infty \).

We evaluate \( \phi_{p-1} \) on \( P_{n2n+2} \in H^{2np+2}(\Sigma K) = H^{2np+2}(B_1) \) using a \( 2np+1 \) connected non stable version of (D1). Thus one reads D1 with \( K_r = K(Z/pZ, 2np + 2 + t_r) \) \( (t_0 = 0) \), \( t_r \) as defined in 0.1 - (2).

We start an inductive process:
\[ \begin{array}{c}
B_p & \xrightarrow{\tilde{u}_p} & E_0 = K(\mathbb{Z}/p\mathbb{Z}, 2np + 2) \\
\downarrow j_p & & \downarrow k_0 = p^1 \\
(D2)_p & & \\
B_p, B_{p-1} & \xrightarrow{\tilde{u}_p = 1 \otimes \cdots \otimes 1} & K_1 \\
\downarrow \delta_p & & \downarrow p^{p-1} \\
\Sigma B_{p-1} & \xrightarrow{\tilde{u}_p} & BK_2 \\
\end{array} \]

\( n \not\equiv -1 \pmod{p} \)

\[ [\tilde{u}_p] = \frac{1}{n+1} P^n t_{2n+2}, \text{ so } P^1 [\tilde{u}_p] t_{2n+2} = P^{n+1} t_{2n+2} = \tau_{2n+2} = [j_p](t_1 \otimes t_2 \otimes \cdots \otimes 1) = \]

\[ = [u_p \circ j_p]. \text{ Note that } H^*(B_r, B_{r-1}) \cong \bigoplus H^*(K) \otimes \bigoplus H^*(K) \otimes \cdots \otimes H^*(K) \]

\text{thus the notation } u_p = 1 \otimes \cdots \otimes 1. \text{ Also note that the map induced by } \Sigma t_{r-1} \circ \delta_r 
\text{from } H^*(B_{r-1}, B_{r-2}) \text{ to } H^*(B_r, B_{r-1}) \text{ corresponds to } d_{r-1}: B_{r-1} \rightarrow B_r \text{ in the cobar construction of } A = H^*(K) \text{ (with the appropriate shift of dimension).}

Any choice of homotopies \( W: k_0 \circ \tilde{u}_p \sim u_p \circ j_p, \ell \circ \sim P^{p-1} \circ k_0 \) induces a map \( \tilde{u}_p: \Sigma B_{p-1} \rightarrow BK_2 \) and its adjoint \( \tilde{u}_p^\#: B_{p-1} \rightarrow K_2 \) factors up to homotopy as \( B_{p-1} \xrightarrow{\tilde{u}_p^\#} E_1 \xrightarrow{k_{p-1}} K_2 \), hence \( \tilde{u}_p^\# \) represents \( \phi_1(P^n t_{2n+2}) \). To continue the inductive step one has to show that \( \tilde{u}_p^\# \) factors (up to homotopy) as

\[ B_{p-1} \xrightarrow{j_{p-1}} B_{p-1}, B_{p-2} \xrightarrow{\tilde{u}_p^\#} K_{2}. \]

Inductively, suppose one has a (homotopy) commutative diagram (for \( 3 \leq r \leq p \))

\[ \begin{array}{c}
B_r & \xrightarrow{\tilde{u}_r} & E_{p-r} \\
\downarrow j_r & & \downarrow k_{p-r} \\
(D2)_r & & \\
B_r, B_{r-1} & \xrightarrow{\tilde{u}_r} & K_{p-r+1} \\
\downarrow \delta_r & & \downarrow p^{p-r+1} \\
\Sigma B_{r-1} & \xrightarrow{\tilde{u}_r} & BK_{p-r+2} \\
\end{array} \]

where \( [k_{p-r} \circ \tilde{u}_r] \) represents \( \phi_{p-r}(P^n t_{2n+2}) \). Any choice of homotopies \( W: k_{p-r} \circ \tilde{u}_r \sim u_r \circ j_r \) and \( \ell \circ \sim P^{p-r+1} \circ k_{p-r} \) (which exists by 3.1) induces a map \( \tilde{u}_r: \Sigma B_{r-1} \rightarrow BK_{p-r+2} \) whose adjoint \( (\tilde{u}_r)^\# \) factors as \( B_{r-1} \xrightarrow{\tilde{u}_r^\#} E_{p-r+1} \xrightarrow{k_{p-r+1}} K_{p-r+2} \), hence, \( (\tilde{u}_r)^\# \) represents \( \phi_{p-r+1}(P^n t_{2n+2}) \).
To proceed inductively one has to show that \( \bar{u}_r \) factors as

\[ \Sigma B_{r-1} \xrightarrow{\Sigma i_{r-1}} \Sigma B_{r-1}, \Sigma B_{r-2} \xrightarrow{(u_{r-1})^*} BK_{p-r+2} \]

or equivalently, \((\bar{u}_r)^*\) factors as

\[ B_{r-1} \xrightarrow{i_{r-1}} B_{r-1}, B_{r-2} \xrightarrow{u_{r-1}^*} K_{p-r+2} \]

This is proved in 7.1 using the following preparatory steps:

**2.1.** Fix \( \lambda \) a primitive root of unity mod \( p \). The \( \lambda \)-th power map on \( K \) is an oo-loop map hence induces self maps \( T_{B_i}, T_{K_i}, \Sigma T_{B_i} = T_{\Sigma B_i} \) on \( B_i, (B_i, B_{r-1}), \Sigma B_{r-1} \) respectively and \( i_{s+1}, j_s, \delta_s \) commute with the T's up to homotopy. Similarly the stable multiplication by \( \lambda \) induces self maps \( T_{E_i}, T_{K_i}, T_{D_i} = \Omega T_{K_i} \) (which are oo-loops maps) on \( E_i, K_i, \Omega K_i \) in (D1).

One can easily see that in (D2) \( \bar{u}_p \) factors with these self maps \( T \). Choosing \( W: \cdot \circ \bar{u}_p \sim u_p \circ j_p, t_s \sim \varphi_{p-1} \circ k_0 \) appropriately (for \( \lambda \) any stable null homotopy will do) one may obtain \( \bar{u}_p \) that commutes with the T's: \( \bar{u}_p \circ T_{E_{r-1}} \sim T_{B_{r-1}} \circ \bar{u}_p \). (see 5.1, 5.2, and 5.3).

Now, using self maps theory (section 5) one can make sure that in all inductive steps (D2) \( \implies (D2)_{r-1} \) all maps obtained commute with the self maps \( T \). This translates to the algebraic fact that the classes \( \{u_r\} \in H^*(B_r, B_{r-1}) \approx \Sigma H^*(K) \otimes \cdots \otimes \Sigma H^*(K) \) have algebraic weight 1 mod \( p-1 \). (i.e: \( u_r = \sum_{i_1, \ldots, i_r} \Sigma z_i \cdot \otimes \Sigma z_i \cdot \otimes \Sigma z_i, z_i \in H^*(K) \) is a monomial of primitive \( z_i = x^{(i)}_1 \cdots x^{(i)}_{r} \), \( x^{(i)}_j \in PH^*(K) \) and \( \sum_{i=1}^r \xi_i \equiv 1 \) (mod \( p-1 \)).

**2.2.** To prove 7.1 one has to show that \( \bar{u}_r \circ \Sigma i_{s+1} \sim \ast \). The first step in the proof uses 4.3 to show that \( W \) could be chosen (without violating the restriction required to obtain the properties discussed in 2.1) so that \( \bar{u}_r \circ \Sigma i_{s+1} \sim \ast \).

**2.3.** To show that \( \bar{u}_r \circ \Sigma i_{s+1,r-1} \sim \ast \) implies \( \bar{u}_r \circ \Sigma i_{s+1,r-1} \sim \ast \) for \( s < r-2 \) one observes: \( \bar{u}_r \circ \Sigma i_{s+1,r-1} \sim \ast \) is equivalent to the existence of \( \bar{u}'_r : \Sigma B_{s+1}, \Sigma B_{r-1} \rightarrow BK_{p-r+2}, \bar{u}'_r \circ \Sigma j_{s+1} \sim \bar{u}_r \circ i_{s+1,r-1} \), \( [\bar{u}'_r]_1 \) has an algebraic weight 1 mod \( p-1 \) and \( d_{s+1}([\bar{u}'_r]) = 0 \) in the cobar construction. Following computations in the cobar construction (6.2), \( [\bar{u}'_r]_1 = d_s(\bar{u}'_r) \) and \( \bar{u}_r \circ \Sigma i_{s+1,r-1} \sim \ast \).

**2.4.** Now one deduces the existence of (D2) for \( p \geq r \geq 2 \), and \( (\bar{u}_p)^* : B_1 \rightarrow K_p \) represents \( \phi_{2-1}(P^{n_x}) \). As \( \tilde{H}^*(B_1) \approx H^*(B_1, B_0) \) \((B_0 = *)\) one can write \( (\bar{u}_p)_1 = u_1 \) obtaining classes \( u_r \in B_{r+1} = H^*(B_r, B_{r-1}) \), \( 1 \leq r \leq p \) subject to the conditions: \( u_r \) have \( (\text{mod } p-1) \) algebraic weight 1, \( u_{r+1} = 1 \otimes \cdots \otimes 1, P^{n_x} u_{r+1} = d_{r+1} u_{r+1} \). By 6.3 \( \Sigma^{-1} u_1 \in H^*(K) \) has the form \( \varepsilon_1 \cdot P^{1} \cdot \cdots \cdot P^{p-1} \) (mod \( \text{Im } P^1 \)) and the first part of the main theorem follows.

To show that \( 0 \notin \phi_{r-1}(P^{n_x}) \) \((n \neq -1(p))\) suffices to show that for some space \( X \) and some class \( x \in H^{2n+1}(X) \), \( 0 \notin \phi_{r-1}(P^n x) \). \( (P^n x)_{2n+1} \) is obviously in the domain of \( \phi_{2-1} \), being the image of \( P^n x_{2n+1} \). This is done in Chapter 8 where some examples and applications are discussed.
3. The existence of $\phi$

We use the notation of the stable version of (D1).

3.1. Proposition. If $r (r < q)$ satisfies the inequality $|r/2| < \sum_{i=0}^{r-1} \sum_{p_i (q = 2p - 2)}$ then $\phi_r$ exists. In particular $\phi_{p-1}$ exists.

Proof: $\phi_0 = P^1$ and $\phi_1$ obviously exist so one may assume $p > 2, r \geq 1$. Now suppose $\phi_{n-1}$ exists, i.e. there exists a map $k_n: E_{n-1} \rightarrow K_n = \Sigma^{t*} K_0$.

The obstruction for the existence of $\phi_n$ is the composition $E_{n-1} \xrightarrow{k_n} K_n \rightarrow \Sigma K_{n+1}$ ($a_n = 1$ if $n$ is even, $a_n = p - 1$ if $n$ is odd).

Now $\Omega K_{n-1} = \Sigma^{t*} K_{n-1} \rightarrow E_{n-1} \rightarrow K_n$ is given by $P^{p-a_n}$, hence $\ast = P^{p-a_n} \circ k_{n-1} \circ j_{n-1}$ and $P^{p-a_n} \circ k_{n-1}$ factors as $E_{n-1} \rightarrow E_{n-2} \rightarrow \Sigma K_{n+1} = \Sigma^{t*+1} K_0$.

If one can show that $H^{t*+1}(E_{n-2}) = 0$, 3.1 will follow. Using the exact sequences $H^* (E_{j-1}) \rightarrow H^* (E_j) \rightarrow H^* (\Sigma^{-1} K_j)$ and $H^* (E_0) = H^* (K_0)$ suffices to prove that $H^{t*+1}(\Omega K_j) = H^{t*+1}(\Sigma^{t*} K_0) = H^{t*+1}(K_0) = 0$ for $j \geq 0$ ($t_0 = 0$) provided that $[n/2]_p < \sum_{i=0}^{n-1} \sum_{i=0}^{t_i} p^i = \lambda_{n-1}$. Thus, putting $t_n = \nu q + q - s + 1$, $t_{n+1} = t_j + 2 = mq + k$ where $m \leq \nu_{n+1}, k \geq q - n + 1$, $\nu_{n+1} = \lfloor \frac{n}{2} \rfloor - 1$ if $n$ is even.

3.2. Lemma. For $2 < k < q - 1$ and $m < \sum_{i=0}^{k-2} \sum_{i=0}^{t_i} p^i = \lambda_k$, one has $H^{m+t+k}(K_0) = 0$. In particular the inequality holds if $m \leq \nu_{n+1} \leq \lfloor \frac{n}{2} \rfloor p < \lambda_{q-n+1} = \lambda_k$.

Proof: Every element in $H^{m+t+k}(K_0)$ has the form $\alpha \omega$ where $\alpha$ is a sum of admissible monomials in $A(p), |\alpha| = mq + k$ (see [10, p. 77]). An admissible monomial of dim $k \mod q$ has to contain at least $k$ Bocksteins. The lowest dimensional admissible monomial with $k$ Bocksteins is $\beta p^s \cdot \ldots \cdot \beta p^s \cdot \beta$ its dimension is $\lambda_k q + k$.

4. Some computations in the Steenrod algebra

Let $w \in A(p)$ be a linear combination of admissible monomials of excess $\leq 2n + 1$. (See [10]). If $P^1 w$ is a sum of admissible monomials of excess $> 2n + 1$ then (modulo ker $P^1$) $w$ is a linear combination of elements of one of the two forms:

(i) $w = P^m \omega_0, m \neq -1(p), ((\omega_0 | +1)/2) = m - n$.
(ii) $w = ((m + 1)\beta P^m - P^m \beta) \omega_0, m \neq -1(p), ((\omega_0 | /2) = m - n$.

where $\omega_0$ is an admissible monomial.

One has the following:
4.1. Proposition. Let \( K = K(\mathbb{Z}/p\mathbb{Z}, 2n + 1) \). Given \( 0 \neq u = \omega t_{2n+1} \in \text{PH}^t(k/\mathbb{Z}/p\mathbb{Z})(\omega \in A(p)) \) where

\[
t = \begin{cases} 
2np + 1 + kpq - 2k + 1 & 2 \leq 2k \leq p - 1 \\
2np + 1 + kpq + sq - 2k & \begin{cases} 1 \leq s \leq p - 1 \\
2 \leq 2k \leq p - 3 & (\text{the latter is void for } p = 3)
\end{cases}
\end{cases}
\]

\( (\omega \in A(p)) \)

If \( P^1 u = 0 \) then \( P^1 \omega = 0 \) in \( A(p) \).

Proof: We may assume that excess \( \omega \leq 2n + 1 \) and \( P^1 u = 0 \) implies excess \( (P^1 \omega) > 2n + 1 \) hence \( \omega \) has one of the two forms listed above:

(i) \( u = P^m \omega_0 t_{2n+1}, \quad |\omega_0 t_{2n+1}| = 2m \quad \text{or} \quad 2m + 1 \).

(ii) \( u = ((m + 1) \beta P^m - P^m \beta) \omega_0 t_{2n+1}, \quad |\omega_0 t_{2n+1}| = 2m + 1 \quad \text{or} \quad 2m + 2 \)

hence

\[ |u| = 2mp \quad \text{or} \quad 2mp + 1 \quad \text{(case i)}. \]

\[ |u| = 2mp + 2 \quad \text{or} \quad 2mp + 3 \quad \text{(case ii)}. \]

The hypothesis on \( t \) excludes \( |u| = 2mp + 2 \), \( |u| = 2mp + 1 \) can occur only if \( t = 2np + pq \ (k = 1), \quad |\omega_0 t_{2n+1}| = 2n + q \quad \text{and} \quad |\omega_0| = q - 1 \). But there are no elements in \( A(p) \) of dimension \( q - 1 \). \( |u| = 2mp + 1 \) can occur only if \( s + k = p > 3 \) and then \( |\omega_0| = kq + q - 2k \) and again, there are no elements in this dimension in \( A(p) \). Similarly, \( |u| = 2mp + 3 \) occurs only if \( s + k = p - 1 > 2 \). Then \( |\omega_0| = kq + q - 2k - 1 \) and there are no elements in this dimension in \( A(p) \). 

4.2. Proposition. If \( 1 \leq a \leq p - 1 \) then the following sequence is exact

\[
\cdots \rightarrow A(P) \xrightarrow{P^a \times} A(P) \xrightarrow{P^{a+1} \times} A(P) \xrightarrow{P^a \times} A(P) \rightarrow \cdots
\]

\( (P^a \times \text{means the endomorphism of left multiplication by } P^a) \).

Proof: We assume \( p \)-odd. The Milnor representation of \( A(p) \), \([6]\), yields an isomorphism \( A(p) \cong P \otimes Q \). This is an isomorphism of left \( P \) modules where \( P \) is the subalgebra of \( A(p) \) generated by \( P^p \). Thus, suffices to prove the exactness of the sequence where \( P \) replacing \( A(p) \). For \( P \) one can use again the basis of admissible monomials in the \( P^i \)'s. Now, if \( a < p \) and \( \omega \in P \) is an admissible monomial then \( P^a \omega \) is (a multiple of) an admissible monomial. If \( \omega_1 \neq \omega_2 \) are admissible monomials then \( P^a \omega_1 \neq P^a \omega_2 \) (unless both are zero). It follows that if \( u = \Sigma k_i w_i \in A(p), \quad k_i \in \mathbb{Z}/p\mathbb{Z}, \quad \omega_i, \) admissible monomials \( P^a u = 0 \) if and only if \( P^a \omega_i = 0 \) for all \( i \) and suffices to prove that \( P^a \omega = 0 \) if and only if \( \omega = P^{-a} \hat{P} \omega \) for an admissible monomial \( \omega \). Now \( \omega = P^m \omega_0 = \hat{k_0} P^p P^{a+b} \omega_0 \) for some admissible monomial \( \omega_0 \), \( 0 \leq \hat{m}, \quad 0 \leq b < p, \quad P^a \omega = 0 \) if and only if \( a + b \geq p \) and then \( \omega = P^{a-b} \hat{k_0} P^{a+b} \omega_0 \). 

\]
4.3. Corollary. Let \( K = K(\mathbb{Z}/p\mathbb{Z}, 2n + 1) \) and let \( \omega_{2n+1} = u \in \Phi^{2n+1}(K) \), \( \omega \in A(p) \) (as in 1.0-(2)). If \( \Phi^\alpha u = 0 \) then \( u = \Phi^{\alpha+1}u_0 \) (as in 1.0-(4)).

Proof: If \( r \) is even \( t_r = t_2k = kpq - 2k + 1 \), then \( u = t \) as in 4.1 and 0 = \( \Phi^\alpha u = \Phi^1u \) implies \( \Phi^1\omega = 0 \). \( \omega = \Phi^{p-1}\omega_0 \) and \( u = \Phi^{p-1}\omega_{2n+1} = \Phi^{n+1}u_0 \). If \( r \) is odd \( t_r = t_{2k+1} = kpq + q - 2k \). If 0 = \( \Phi^\alpha u = \Phi^{p-1}u \) then for some \( s, 1 \leq s \leq p - 2, \Phi^{s-1}u \neq 0 \) and \( \Phi^1(\Phi^{s-1}u) = 0 \). The proof \( u = \Phi^1u_0 \) is by induction on \( s \). If \( \Phi^{s-1}u = t \) as in 4.1 and \( \Phi^1(\Phi^{s-1}u) = 0 \) implies \( \Phi^1(\Phi^{s-1}\omega) = 0 \) (where \( u = \omega_{2n+1}, \omega \in A(p) \)). By 4.2 \( \Phi^{p-1}\omega = \Phi^{p-1}\omega_0 \). If \( s = 1 \), \( \omega = -\Phi^1(\Phi^{p-2}\omega_0) \) and \( u = \Phi^1(\Phi^{p-2}\omega_0)_{t_{2n+1}} \). Otherwise one has for some \( \varepsilon \neq 0, \Phi^{p-1}(\omega - \varepsilon\Phi^{p-2}\omega_0)_{t_{2n+1}} = 0 \), by induction \( u - \varepsilon\Phi^{p-2}\omega_0_{t_{2n+1}} = \Phi^1u_0 \) and \( u = \Phi^1(u_0 + \varepsilon\Phi^{p-2}\omega_0_{t_{2n+1}}) \).}

5. Review of homotopy theory of self maps

In this section we extract some notations and statements from [12], needed in our computations. Assume all spaces and maps are pointed. Consider pairs \( X, TX \) of spaces \( X \) with a self map \( TX : X \rightarrow X \) and “morphisms” \( f, V_f : TX \rightarrow Y, TY \) where \( f : X \rightarrow Y \) and \( V_f : X \rightarrow PY = \text{map}(I, Y) \) is a homotopy \( V_f : f \circ TX \sim TY \circ f \). Our main object here is to show that some standard homotopy theoretic constructions extend to our “category”.

Now, given pairs \( E, TE \), \( B, TB \), \( B_0, TB_0 \), morphims \( f_0, V_{f_0} : E, TE \rightarrow B, TB \), \( f_1, V_{f_1} : B, TB \rightarrow B_0, TB_0 \) and a homotopy \( \ell : * \sim f_1 \circ f_0 \) (\( \ell : E \rightarrow LB_0 = \text{map}(I, 0; B_0, *) \)). We denote by \( \alpha(\ell) = \alpha(\ell, V_{f_0}, V_{f_1}) \) the class of \( \ell \circ T_B + Pf_1 \circ V_{f_1} \circ f_0 \rightarrow LB_0 \circ \ell \in \pi_1(\text{map}(E, B_0), *) \).

For convenience assume that \( B_0 = K(G, m) \) where \( G \) is an elementary abelian \( p \)-group. Then one has:

5.1. Proposition (3.9 of [12]). One can choose \( \ell \) so that after replacing \( T_B, TB, TB_0 \) by their \( p' \)-th iteration \( \alpha(\ell) = 0 \). If \( \ell \) is altered by \( \omega : E \rightarrow \Omega B_0 \) and \( \omega \circ T_E \sim \Omega T_B \circ \omega \) then \( \alpha(\omega + \ell) = \alpha(\ell) \).

Similarly, suppose in addition that one is given pairs \( L, TL, K, TK \) and maps \( \sigma, V_\sigma : L, TL \rightarrow E, TE \), \( \tau, V_\tau : K, TK \rightarrow B, TB \), \( h, V_h : L, TL \rightarrow K, TK \) and a homotopy \( W : f_0 \circ \sigma \sim \tau \circ h \), to obtain a diagram:

\[
\begin{array}{ccc}
L, TL & \xrightarrow{\sigma} & E, TE \\
\downarrow \alpha & & \downarrow f_0 \\
K, TK & \xrightarrow{\tau} & B, TB \\
& \downarrow f_1 & \\
& B_0, TB_0 & \\
\end{array}
\]
One can define an element \( \alpha(W) = \alpha(W, V_o, V_f, V_h, V_r) \)\( \in \pi_1(\text{map}_*(L, B), f_0 \circ \alpha T_L) \) as the class of \( Pf_0 \circ V_o + V_f \circ \alpha + PT_B \circ W - V_h \circ \rho - P \circ V_h - W \circ T_L \). If \( B \) is an \( H \)-space one can translate \( \alpha(W) \) to \( \pi_1(\text{map}_*(L, B), \alpha) \) and then it coincides with \( \alpha \) of the first type (replacing the square by maps \( L \to E \times K \to B \)).

In this case 5.1 yields:

5.2. Proposition. \( W \) can be chosen so that after replacing \( T_L, T_K, T_E, T_B \) by their \( p \)-th iteration \( \alpha(W) = 0 \). If \( W \) is altered by \( w : L \to \Omega B \) and \( w \circ T_L \sim T_B \circ w \) then \( \alpha(w \circ W) = \alpha(W) \) (where \( * \) denote the pointwise multiplication of paths in the function space).

The classes \( \alpha(\ell), \alpha(W) \) represent obstructions to the following problems:

Given \( f_1, V_f : B, T_B \to B_0, T_{B_0} \) one naturally obtains a self map \( T_{F_1} \) of the homotopy fiber \( F_{f_1} \) of \( f_1 \) so that the "inclusion of the fiber map" \( j_{f_1} : F_{f_1} \to B \) strictly commutes with the self maps: \( j_{f_1} \circ T_{F_1} = T_B \circ j_{f_1} \). If \( \ell : * \sim f_1 \circ f_o, \ f_o, V_f, E, T_B \to B, T_{B_0} \), then \( \ell \) induces a lifting \( f_1 : E \to V_f \) of \( f_1 : j_{f_1} \circ \alpha(\ell) = \alpha(\ell, V_o, V_f) \) is the obstruction to lift \( V_f \) to a homotopy \( V_{f_1} : f_1 \circ T_B \sim T_{V_{f_1}} \circ f_1 \).

Similarly, \( f_0, V_{f_0} \) induces a self map \( T_{G_{f_0}} \) on the mapping cone \( C_{f_0} \) of \( f_0 \) so that \( i_{f_0} : B \to C_{f_0} \) strictly commutes with the self maps: \( i_{f_0} \circ T_B = T_{G_{f_0}} \circ \alpha i_{f_0} \). \( \ell : * \sim f_1 \circ f_0 \) induces a map \( \hat{f}_E : C_{f_0} \to B_0 \) extending \( f_1 : f_1 \circ i_{f_0} = \alpha(\ell) = \alpha(\ell, V_o, V_f) \) is also the obstruction to extend \( V_{f_1} \) to a homotopy \( V_{f_1} : \hat{f}_E \circ T_{G_{f_0}} \sim T_{B_0} \circ f_{f_0} \). Analogously, \( T_K, T_E, T_{B_0}, V_f, V_r \) in D3 induce a self map \( T_{U(f_0, \tau)} \) on the the homotopy pull back \( U(f_0, \tau) \) of \( f_0 \) and \( \tau \). \( W \) induces a lifting \( h_w : L \to U(f_0, \tau) \) of

\[
(\sigma \times h) \circ \Delta : L \to E \times K.
\]

\( \alpha(W) = \alpha(W, V_o, V_f, V_h, V_r) \) is the obstruction to lift \( (V_o \times V_h) \circ \Delta \) to a homotopy \( V_{h_w} : h_w \circ T_L \sim T_{U(f_0, \tau)} \circ h_w \). Now, if in D3 \( \sigma, \tau, W, \ell \) are given, they induce a map \( \rho : C_{f_0} \to B_0 \).

5.3. Proposition. Suppose in (D3) \( B, B_0 \) are \( K(G, n) \) and \( K(G_0, m) \) respectively where \( G, G_0 \) are elementary abelian \( p \)-groups. If \( W : f_1 \circ \sigma \sim \tau \circ oh, \ \ell : * \sim f_1 \circ f_0 \) are chosen so that \( \alpha(W) = 0, \ \alpha(\ell) = 0 \) then the natural map \( \rho \) admits a homotopy \( V_\rho : \rho \circ T_{C_{f_0}} \sim T_{B_0} \circ \rho \).

Proof: Writing down the formulas for the various maps involved one gets

\[
\rho(x, t) = \begin{cases} 
\ell_o(x) \lfloor 2t \\ Pf_1 \circ W(x) \lfloor 2t - 1 \end{cases}, \quad T_{C_{f_0}}(z, t) = \begin{cases} T_L(z) \lfloor 2t \\ V_h(x) \lfloor 2t - 1 \end{cases}
\]

\[
\rho(y) = f_1 \circ \tau(y)
\]

\( T_{C_{f_0}}(y) = T_K(y) \)
The obstruction to extend the homotopy

\[ P f_1 \circ W \circ T_L (x) [4t - 1] \]

\[ P (f_1 \circ \tau) \circ V_h (x) [2t - 1] \]

\[ \rho T_{C_h} (y) = f_1 \circ \tau \circ T_k (y) \]

\[ T_{B_0} \circ \rho (x, t) = \left\{ \begin{array}{ll} LT_{B_0} \circ \ell \circ \sigma (x) [2t] \\
PT_{B_0} \circ P f_1 \circ W (x) [2t - 1] \end{array} \right. \]

\[ T_{B_0} \circ \rho (y) = T_{B_0} \circ f_1 \circ \tau (y) \]

The obstruction to extend the homotopy

\[ Pf_1 \circ V_r + V_{f_1} \circ \tau : f_1 \circ \tau \circ T_k \sim T_{B_0} \circ f_1 \circ \tau \]

is the class in \( \pi_1 (\text{map}_* (L, B_0), +) \) of the loop

\[ \gamma = \ell \circ \sigma \circ T_L + Pf_1 \circ W \circ T_L + Pf_1 \circ P \tau \circ V_h + \\
+ Pf_1 \circ V_r \circ h + V_{f_1} \circ \tau \circ h - PT_{B_0} \circ P f_1 \circ W - LT_{B_0} \circ \ell \circ \sigma \]

which can be illustrated as follows as a loop in \( \text{map}_* (L, B_0) \):

Thus \( [\gamma] = -(\ell \circ \sigma \circ T_L) \# \pi_1 (f_1 \circ *) \alpha (W) + \pi_1 (\sigma \circ *) \alpha (\sigma) \) where

\[ f_1 \circ \text{map}_* (L, B) \longrightarrow \text{map}_* (L, B_0) \]

\[ \sigma \circ \text{map} (E, B_0) \longrightarrow \text{map}_* (L, B_0) \]

are the left and right compositions with \( f_1 \) and \( \sigma \) respectively.

\[ (\ell \circ \sigma \circ T_L) \# : \pi_1 (\text{map}_* (L, B_0), f_1 \circ f_0 \circ \sigma \circ T_L) \longrightarrow \pi_1 (\text{map}_* (L, B_0), +) \]

is the isomorphism induced by the path \( \ell \circ \sigma \circ T_L \). Hence, \( \alpha (W) = 0, \quad \alpha (\ell) = 0 \)

implies \( [\gamma] = 0. \)
6. Derivations in the co-bar construction

Let \( A \) be a graded connected primitively generated Hopf algebra of finite type over \( \mathbb{Z}/p\mathbb{Z} \) \( p \)-odd. Then the elements of \( A \) are sums of monomials in the primitives. One has a mod \( p-1 \) algebraic weight function \( w \) defined on homogeneous polynomials: \( w(a) \in \mathbb{Z}/(p-1)\mathbb{Z} \), satisfying: \( w(1) = 0 \), \( W(PA) = k \), \( w(\text{Im}(PA \otimes \cdots \otimes PA \rightarrow A)) = k \) (\( \Delta \) - the multiplication in \( A \)). Thus, together with the grading this gives a bigrading of \( A \), \( A_{s,t} \), \( s, t \in \mathbb{Z}^+ \).

Let \( B = \bigoplus_{r=0}^{\infty} B_r \) be the reduced cobar construction on \( A; B_r = A \otimes \cdots \otimes A \) with a bigrading \( B_{r,t} = \bigoplus_{m,t} A_m \otimes \cdots \otimes A_m \). Then \( w \) extends to \( B_r \) by \( \omega(a_1 \otimes \cdots \otimes a_r) = \sum \omega(a_i) \) for monomials \( a_i \) in \( A \). Let \( d_r: B_r \rightarrow B_{r+1} \) be the derivation \( d_r(a_1 \otimes \cdots \otimes a_r) = \sum_{i=1}^r (-1)^{i+1} a_1 \otimes \cdots \otimes \tilde{a}_i \otimes \cdots \otimes a_r \) (\( \tilde{a} \) - the reduced coproduct in \( A \)). Then \( d_r \) respects the weight function thus extending it to the homology of \( d_r \), namely to \( \text{Ext}^*_{\mathbb{A}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \). The latter is a free algebra generated by \( \text{Ext}^*_{\mathbb{A}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \) and by some elements in \( \bigoplus_{n} \text{Ext}^{2kp}_{\mathbb{A}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \). All generators in \( \text{Ext}^*_{\mathbb{A}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \) are of weight one. \( \text{Ext}^{1,m}_{\mathbb{A}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \approx PA_m \) and the generators in \( \text{Ext}^{2,kp}_{\mathbb{A}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \) are represented by \( \sum_{a=-1}^{p-1} \frac{1}{p}(v^a \otimes v^{-a}, v \in PA_{2k}) \). Thus we have:

6.1. Lemma. \( \text{Ext}^{s,r}_{\mathbb{A}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \) contains no elements of weight 1 for \( 2 < s < p \) and for \( s = 2 \), \( r \neq 0 \) (mod \( 2p \)).

6.2. Corollary. If \( u \in \ker d_r \subseteq B_r \), \( w(u) = 1 \), \( 2 < r < p \) or if \( r = 2 \) and \( |u| \neq 2 \) (mod \( 2p \)) then \( u \in \text{Im} d_{r-1} \) (\( |u| \) - the total degree \( s+r \) in \( \text{Ext}^{s,r} \)).

If \( A \) above is a non stable module over \( \mathbb{Z}/p\mathbb{Z} \) \( \mathbb{P}^i/(\mathbb{P}^i)^p \subset \mathbb{A}(p) \) (i.e., \( \mathbb{P}^i u = u^p \) for \( u \in \mathbb{A}_2 \), \( \mathbb{P}^1 A_1 = 0 \) and \( \mathbb{P}^1 \) acts as a derivation on \( a \otimes b \) and \( a \otimes \otimes b \). \( \mathbb{P}^1(PA) \subset PA \). Then \( \mathbb{P}^i = \frac{1}{i}(\mathbb{P}^1)^i \) \( (i < p) \) respects the mod \( p-1 \) algebraic weight and \( \mathbb{P}^i \) commuted with \( d_r \) in \( A \otimes \cdots \otimes A \approx B_r \).

6.3. Proposition. Let \( A \) be a primitively generated graded connected Hopf algebra and a module over \( \mathbb{Z}/p\mathbb{Z} \) \( \mathbb{P}^i/(\mathbb{P}^i)^p \). Suppose \( PA_m = 0 \) for \( m < 2n+1 \) and let \( x \in PA_{2n+1} \). Then there exist elements \( u_s \in A \otimes \cdots \otimes A \) of mod \( p-1 \) algebraic weight 1, \( 1 \leq s \leq p \), so that:

\[
(i) \quad u_p = x \otimes x \otimes \cdots \otimes x \\
(ii) \quad \mathbb{P}^{a+s} u_s = d_{a-1} u_{s-1} \quad \text{for } s > 1.
\]
(\(u_{s-1} \in A \otimes \cdots \otimes A\) considered as an element in \(B_{s-1}\) \(a_{s+1}\) as in 1.0 (4).

For any choice of \(u_s\) of weight \(1 \pmod{p-1}\) satisfying (i) and (ii) one has 
\[u_1 = \epsilon p \cdot P^1 x \cdot P^2 x \cdots P^{p-1} x \pmod{\text{Im} P^1}\]
\[\epsilon_p = (-2)^{p-\frac{1}{2}}.\]

**Proof:** Suppose by induction \(u_{p-r}\) of weight one were found for \(0 \leq r \leq r_0\) so that, \(u_p = x \otimes \cdots \otimes x\) and \(P^{p-r} u_{p-r+1} = d_{p-r} u_{p-r}\), \(|u_s| = \text{odd}.\) Further assume:

\[(\text{for } r = 2s)\quad u_{p-2s} = (P^{p-1}(P^1 x \cdot x))^{s} \cdot x \otimes x \otimes \cdots \otimes x + \sum_i w_1^i \otimes w_2^i\]
\[(\text{for } r = 2s+1)\quad u_{p-2s-1} = (P^{p-1}(P^1 x \cdot x))^{s} (P^1 x \cdot x) \otimes x \otimes \cdots \otimes x + \sum_i v_1^i \otimes v_2^i\]

where \(w_1^i, w_2^i \in A \otimes \cdots \otimes A\), \(|w_1^i| > (2n+1)(p-2s-1)\)
\[v_2^i \in A \otimes \cdots \otimes A, \quad |v_2^i| > (2n+1)(p-2s-2)\).

If \(r_0 = 0\), \(P^1 u_p = P^1(x \otimes \cdots \otimes x) = \sum_{i=1}^p x \otimes \cdots \otimes P^i x \otimes \cdots \otimes x\)
\[= d_{p-1} \sum_{i=1}^{p-1} ((-1)^{i+1} \cdot x) \otimes \cdots \otimes P^i x \otimes \cdots \otimes x\]
\[\text{and } u_{p-1}\text{ exists having the desired form. If } r_0 = 2s_0 + 1, s_0 \geq 0, P^1 u_{p-2 s_0} = d_{p-2 s_0-1} u_{p-2 s_0-1} \text{ imply}\]
\[d_{p-2 s_0-1} P^{p-1} u_{p-2 s_0-1} = 0. \text{ As } \omega(P^{p-1} u_{p-2 s_0-1}) = 1 \quad \text{for } p > p - 2s_0 - 1 > 1, \text{ hence } P^{p-1} u_{p-2 s_0-1} \in \text{Im } d_{p-2 s_0-1}.\]

Now \(P^{p-1} u_{p-2 s_0-1} = (P^{p-1}(P^1 x \cdot x))^{s+1} \otimes x \otimes x \otimes \cdots \otimes x + \sum_i \tilde{v}_1^i \otimes \tilde{v}_2^i, \quad |\tilde{v}_1^i| \geq (2n+1)(p-2s-2)\). One can easily argue that if \(P^{p-1} u_{p-2 s_0-1} = d_{p-2 s_0-2} u_{p-2 s_0-2}\), then \(u_{p-2 s_0-2}\) has the desired leading term \((\text{mod } \ker d_{p-2 s_0-2})\). If \(r_0 = 2s_0, 0 < s_0 < \frac{p-1}{2}, P^{p-1} u_{p-2 s_0+1} = d_{p-2 s_0+1} u_{p-2 s_0+1}\) and \(d_{p-2 s_0} u_{p-2 s_0} = 0. \text{ As } p > p - 2s_0 > 1, |P^1 u_{p-2 s_0}| = \text{odd } \not\equiv 2 (\text{mod } 2p), \text{ again } P^1 u_{p-2 s_0} \in \text{Im } d_{p-2 s_0-1}. P^{p-1} u_{p-2 s_0} = \]
\[(P^{p-1}(P^1 x \cdot x))^s P^1 x \otimes x \otimes \cdots \otimes x + \sum_i \tilde{w}_1^i \otimes \tilde{w}_2^i, \quad |\tilde{w}_2^i| \geq (2n+1)(p-2s-1)\).

Again one can see that \(P^1 u_{p-2 s_0} = d_{p-2 s_0-1} u_{p-2 s_0-1}, u_{p-2 s_0-1}\) has the desired leading term \((\text{mod } \text{Im } d_{p-2 s_0-1})\).

Now, one can see by induction that \(u_{p-r}\) are uniquely determined \(\text{mod } (\ker d_{p-r} = \text{Im } d_{p-r-1}) + \text{Im } P^{p-r+1}\). Thus \(u_1\) is determined up to image \(P^{p_0} = P^1\) and \(u_1 = [P^{p-1}(P^1 x \cdot x)]^{p-\frac{1}{2}} x = \epsilon_p x P^1 x \cdots P^{p-1} x \pmod{\text{Im } P^1}.\)
7. Proof of the main theorem, part (1)

Fix $p$ odd and $n \not\equiv -1 \pmod{p}$. We follow the notations of 2.0 (i)-(iv) and recall the observation that $H^*(B_r, B_{r-1}) \approx B_r A$ where $A = H^*(K)$, $B_r A \approx \Sigma A \otimes \cdots \otimes \Sigma A$ ($\Sigma A = \tilde{H}^*(\Sigma K)$). The composition $B_{r+1}, B_r \overset{d_r}{\to} \Sigma B_r \overset{e_{f_r}}{\to} \Sigma B_r, B_{r-1}$ induces a morphism $H^*(B_r, B_{r-1}) \to H^*(B_{r+1}, B_r)$ corresponding to the derivation in the cobar construction

$$d_r: A \otimes A \otimes \cdots \otimes A \to \tilde{A} \otimes \cdots \otimes \tilde{A}, \quad d_r(a_1 \otimes \cdots \otimes a_r) = \sum_{j=1}^r (-1)^j a_1 \otimes \cdots \otimes \tilde{a}_j \otimes \cdots \otimes a_r$$

where $\tilde{\mu}^*: \tilde{A} \to \tilde{A} \otimes \tilde{A}$ is the reduced coproduct (note that $\tilde{A} \approx \Sigma A$ with a shift of dimensions).

Now, starting of with (D2), one observes that all maps (except possibly $\tilde{\alpha}_r$) commute with the self maps $T$ induced by the $\lambda$-th power maps in $K$ (and consequently in $\Sigma K, B_r(B_r, B_{r-1})$) and by those in $E_r, K$. One can see that if $\ell: \kappa \sim P^{p-1} \circ k_0, V_k: k_0 \circ T_{B_0} \circ T_{K_0} \circ k_0, V_{p^{r-1}}: P^{p-1} \circ T_{K_r} \sim T_{B_{K_r}} \circ P^{p-1}$ are chosen to be the stable homotopies then $\alpha(\ell, V_k, V_{p^{r-1}}) = 0$.

By 5.2, 5.3 one can choose $W: k_0 \circ \tilde{\alpha}_r \sim \alpha_r \circ j_r$ so that $\tilde{\alpha}_r: \Sigma B_{p-1} \to BK_2$ will satisfy $\tilde{\alpha}_r \circ T_{B_{p-r-1}} \sim T_{B_{K_r}} \circ \tilde{\alpha}_r$ (where the $T$'s may have to be replaced by their $p'$-th iteration).

The same observation will hold in all the following inductive steps: If (D2), is given so that all maps (with the possible exception of $\tilde{\alpha}_r$) commute up to homotopy with the self maps $T$ then one chooses the stable homotopies $\ell: \kappa \sim P^{p^{r-1}} \circ k_0, V_{k_r}: k_0 \circ T_{B_{r-1}} \circ T_{K_{r-1}} \circ k_0, V_{p^{r-1}}: P^{p^{r-1}} \circ T_{K_r} \sim T_{B_{K_r}} \circ P^{p^{r-1}} \circ T_{K_{r-1}} \sim T_{B_{K_{r-1}}} \circ \beta_{p^{r-1}}$ and a suitable $W: k_0 \circ \tilde{\alpha}_r \sim \alpha_r \circ j_r$ so that $\tilde{\alpha}_r$ induced by $\ell$ and $W$ will satisfy $\tilde{\alpha}_r \circ T_{B_{p-r-1}} \sim T_{B_{K_{p-r-2}}} \circ \tilde{\alpha}_r$. Thus if one assumes inductively (induction on $p-r$) that (D2), exists for all $p \geq r \geq p_0 \geq 3$ so that all maps commute up to homotopy with the self maps $T$ one has to prove the following proposition in order to continue the inductive process:

7.1. Proposition. $\tilde{\alpha}_r \circ \Sigma i_{p-r-2} \sim \lambda_r$.

If 7.1 is proved then $\tilde{\alpha}_r$ may be factored as

$$\Sigma B_{p-r-2} \xrightarrow{\Sigma i_{p-r-1}} \Sigma B_{p-r-2}, \Sigma B_{p-r-2} \xrightarrow{(u_{r-1})^*} BK_{p-r-2}$$

and $(u_{r-1})^*$ commutes with the $T$'s (5.1). The same holds for the adjoint map $u_{r-1}: B_{p-r-2}, B_{p-r-2} \to K_{p-r-2}$. 

Theorem 7.2. The $\alpha_i$'s are stable homotopies.
By induction \([k_{p-r_0} \circ \tilde{u}_{r_0}]\) represents \(\Phi_{p-r_0}(P^n_{2n+2})\) and consequently \(\Phi_{p-r_0+1}(P^n_{2n+2})\) is represented by \([\tilde{u}_{r_0}]\in [B_{r_0-1}, K_{p-r_0+2}]\), in particular \((\tilde{u}_{r_0})_\ast\) factors as

\[
\begin{align*}
B_{r_0-1} & \xrightarrow{\tilde{u}_{r_0-1}} E_{p-r_0+1} \\
& \xrightarrow{k_{p-r_0+1}} K_{p-r_0+2} : \tilde{u}_{r_0-1}
\end{align*}
\]

is the lifting of \(\tilde{u}_{r_0} \circ i_{r_0-1}\) induced by \(L_{u_{r_0}} \circ \ell_0 + W \circ i_{r_0-1} : \ast \sim k_{p-r_0} \circ \tilde{u}_{r_0} \circ i_{r_0-1}\) where \(\ell_0\) is the standard homotopy \(\ell_0 : i_{r_0} \circ j_{r_0} \sim \ast\). One can easily see that \(\alpha(\ell_0) = 0\) and \(W\) was chosen to have \(\alpha(W) = 0\), hence, by a dual of 5.3 \(\tilde{u}_{r_0-1}\) commutes with the self maps. Applying 7.1 one obtains \((D2), r_0-1\) as required. This inductive process yields diagrams \((D2)_r\) for \(2 <_r p\). If one denotes the adjoint of \(\hat{u}_r : \Sigma B_1 \rightarrow B K_p\) by \(u_r = B_1 \rightarrow K_p\) and (as all maps are pointed \(u_1 = B_1, B_0 \rightarrow K_p\) where \(B_0 = \ast\)) then \([u_r]_\ast\) represents \(\Phi_{p-1}(P^n_{2n+2})\). In this way one obtains a sequence of maps \([u_r]_\ast \in [B_r, B_{r-1} : K_{p-r+1}] = H^{2n+2+P_{r-1}}(B_r, B_{r-1})\) satisfying:

(i) \(u_p = t_{2n+2} \ast \cdots \ast t_{2n+2}\),

(ii) \(P^{r_{r-1}}[u_r]_\ast = [u_{r-1} \circ i_{r-1} \circ \delta_r]_\ast\).

One shifts dimensions and considers \(\Sigma^{-r}u_r \in \hat{A} \otimes \cdots \otimes \hat{A} = B_r(\hat{A} = \overline{H}^*(K))\) then (i), (ii) are shifted to satisfy the hypothesis of 6.3 \((u_r\) of mod \(p-1\) algebraic weight \(1\) is equivalent to the property that \(u_r\) commutes with the self maps). Hence, \(\Sigma^{-r}u_r = \varepsilon_{p}t_{2n+2} \ast P^1t_{2n+2} \ast \cdots \ast P^{p-1}t_{2n+2}\) (mod \(\text{Im } P^1\)). As \(\text{Im } P^1\) is part of the indeterminacy of \(\Phi_{p-1}\) part (1) of the main theorem follows.

**Proof of 7.1:** First consider the composition:

\[
\Sigma B_1 \xrightarrow{\Sigma i_{r_0-1}} \Sigma B_{r_0-1} \xrightarrow{\tilde{u}_{r_0}} B K_{p-r_0+2} \quad (r_0 \geq 3).
\]

Now

\[
\hat{H}^*(\Sigma B_1) \approx H^*(\Sigma B_1, \Sigma B_0) \approx \Sigma^2 \hat{H}(K) \approx \Sigma^2 \hat{A}
\]

and the image of \(H^*(\Sigma i_{r_0-1})\) is \(\Sigma^2 PA\). Now, \((\tilde{u}_{r_0})_\ast : B_{r_0-1} \rightarrow K_{p-r_0+2}\) represents \(\Phi_{p-r_0+1}(P^n_{2n+2})\), hence \(\ast \sim P^{r_{r-1}} \circ \tilde{u}_{r_0}\) and \(P^{r_{r-1}}[\tilde{u}_{r_0} \circ \Sigma i_{r_0-1}] = 0\), \(\tilde{u}_{r_0} \circ \Sigma i_{r_0-1} \in \Sigma^2 PA\). Let \(u = \Sigma^{-2}[\tilde{u}_{r_0} \circ \Sigma i_{r_0-1}] \in PA\) then \(u\) has weight \(1\) (mod \(p-1\)) and its dimension is \(2np + 1 + t_{p-r_0}\). By 4.3 \(u = \hat{P}^{r_{r-1}} \circ v, v \in PA\) Now the map \(H^*(BK) = H^*(B\omega_0) \rightarrow H^*(B_1)\) induces a surjection \(PH^*(BK) \rightarrow PH^*(K)\) and \(\Sigma^2 v\) belongs to \(\text{Im } H^*(\Sigma B_{r_0}) \rightarrow H^*(\Sigma B_1)\), say \(\Sigma^2 v = H^*(\tilde{u}_{r_0})\). Thus one can alter \(W : k_{p-r_0} \circ \tilde{u}_{r_0} \sim u_{r_0} \circ j_{r_0}\) by \(v \in \Sigma B_{r_0}, K_{p-r_0-1}\) (as \(v\) is of weight \(1\) it commutes with the self maps, by 5.2 \(a((-v) \ast W) = 0\) and \(\tilde{u}_{r_0}\) is thus altered to obtain \(\tilde{u}_{r_0} \circ \Sigma i_{r_0-1} \sim \ast\).

Suppose inductively that \(\tilde{u}_{r_0} \circ \Sigma i_{r_0-1} \sim \ast\) for some \(r_1 \leq r < r_0 - 2\). Then the composition \(\Sigma B_r \rightarrow \Sigma B_{r+1} \rightarrow \Sigma B_{r_0-1} \rightarrow B K_{p-r_0+2}\) is null homotopic. We also assume that all maps commute with the self maps \(T\).
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By 5.1 (and by replacing the self maps by their \( p \) iterates, if necessary) \( u_{r_0} \circ o \Sigma_{r_{r+1}, r_{r-1}} \) extends to a map \( \Sigma B_{r+1}, \Sigma B_r \to BK_{p-r_{r+2}} \) which homotopically commutes with the self maps. The \( \lambda \) power map induces a multiplication by \( \lambda \) on the fundamental class of \( BK_{p-r_{r+2}} \), the fact that \( u' \) commutes with the self maps is equivalent to the fact that \( [u'] \in H^* (\Sigma B_{r+1}, \Sigma B_r) \) is a \( \lambda \) characteristic vector of \( H^* (\Sigma B_{r+1}, \Sigma B_r) \), hence it is a class of mod \( p - 1 \) algebraic weight.

1. \( r < r_0 - 2 \) implies that the composition \( B_{r+1}, B_r \xrightarrow{\delta^{r+1}} \Sigma B_{r+1} \to B_{r-1} \) is null homotopic, hence, \( \hat{u}' \circ \Sigma j_{r+1} \circ \delta_{r+1} \sim \hat{u}_{r_0} \circ \Sigma \delta_{r+1} \circ \delta_{r+1} \sim * \), \( \delta_{r+1} [\Sigma^{-1} \hat{u}'] = 0 \) in the cobar construction. As \( p - 1 \geq r + 1 \geq 2 \) and \( |\Sigma^{-1} \hat{u}'| = 2np + 2 + t_{p-r_{r+2}} \not\equiv 2 \mod 2p \) by 6.2 \( \Sigma^{-1} \hat{u}' \in \text{Im} \, d_r, \ \hat{u}' \sim \sim \hat{u}' \circ \Sigma^2 j_r \circ \Sigma \delta_{r+1}, \Sigma B_{r+1}, \Sigma B_r \xrightarrow{\delta_r} BK_{p-r_{r+2}} \)

hence, \( \hat{u}_{r_0} \circ \Sigma i_{r_{r+1}, r_{r-1}} \sim \hat{u}' \circ \Sigma j_{r+1} \sim \hat{u}' \circ \Sigma^2 j_r \circ \Sigma \delta_{r+1} \circ \Sigma j_{r+1} \sim * \). By induction \( \hat{u}_{r_0} \circ \Sigma i_{r_{r+1}, r_{r-1}} \sim * \) for all \( r \leq r_0 - 2 \), \( \hat{u}_{r_0} \circ \Sigma i_{r_{r-2}, r_{r-1}} = \hat{u}_{r_0} \circ \Sigma i_{r_{r-2}} \sim \sim \sim \sim * \) and 7.1 follows. \( \blacksquare \)

8. The main theorem (part (2)) and applications

The proof of the main theorem (2) (see 8.3) is a consequence of some applications of the main theorem part (1).

8.1. Proposition. Let \( \lambda \) be a primitive root of unity mod \( p \). Let \( X, T \) be a \( Q - \lambda \) power space, i.e. \( T \circ X \to X \) induces multiplication by \( \lambda \) on the quotient module \( QH^* (X) \) of algebra generators. If for some \( n, \ n \not\equiv -1 \mod p \), \( QH^* (X) = 0 \) for all \( i \not\equiv 2n + 1 \mod p - 1 \) and if \( x \in H^{2n+1} (X) \) is an indecomposable \( \lambda \) characteristic vector of \( H^* (T) \) then \( \Phi_{p-1} (P^n x_{2n+1}) = \varepsilon_p x \cdot P^1 x \ldots P^{p-1} x + \text{Im} \, P^1 + \bigoplus_i W_i \), where \( W_i \subseteq H^* (X) \) is the \( \lambda^i \) eigenspace, \( 1 \leq i \leq p - 1 \). (One may assume that \( H^* (T) \) has a diagonal form for every \( n \).)

Proof: \( \Sigma T \) induces a mod \( p \) splitting of \( \Sigma X \), \( \Sigma X \approx \coprod_{i \in \mathbb{Z}_2} Y_i \), \( H^* (Y_i) = \Sigma W_i \). (\( Y_i \) is the mapping telescope of \( \prod_{i \in \mathbb{Z}_2} (\Sigma T - \lambda^i) \)). \( P^n x \) is obviously in the domain of \( \Phi_{p-1} \). A representative of \( \Phi_{p-1} (P^n x) \) could be obtained by \( X \xrightarrow{\varepsilon_p x \cdot P^1 x \ldots P^{p-1} x} K(Z/pZ, 2n + 1) \xrightarrow{(u_1)_*} E_{p-1} \xrightarrow{k_p} K_p \), hence, by the main theorem (1) \( \varepsilon_p x \cdot P^1 x \ldots P^{p-1} x \in \Phi_{p-1} (P^n x) \). Let \( : X \to E_{p-1} \xrightarrow{k_{p-1}} K_{p} \) be any other representative of \( \Phi_{p-1} (P^n x) \), hence the composition \( X \to E_{p-1} \xrightarrow{k_{p-1}} K(Z/pZ, 2np + 1) \) is \( P^n x \). Denote \( f = (u_1)_* \circ x \) then
\[ f_\# - f_\#: \Sigma X \to BE_{p-1} \text{satisfies } B(h_1 \circ \cdots \circ h_{p-1}) \circ (f_\# - f_\#) \sim * \text{ and } \tilde{f}_\# - f_\# \text{ lifts to } \Sigma X \to \tilde{B}E_{p-1}. \] 

On the other hand, \( W_1 \subset \bigoplus_i H^{2n+1+(p-1)i}(X) \) (as \( W_1 \) is spanned by monomials in the indecomposable \( \lambda \)-eigenvectors of length \( \equiv 1 \mod p-1 \)). It follows that the map \( \tilde{j}_{p-1}: \Omega K_{p-1} \to \tilde{E}_{p-1} \) (the lifting of \( j_{p-1} \) in (D1)) induces a surjection

\[ [Y_1, K_{p-1}] \to [Y_1, \tilde{B}E_{p-1}]. \]

Hence, \( Y_1 \to \Sigma X \xrightarrow{\tilde{f}_\# - f_\#} BE_{p-1} \xrightarrow{Bh_\#} BK_p \) factors as \( Y_1 \to K_{p-1} \xrightarrow{p^1} BK_p \), and \( [k_p \circ \tilde{f}] - [k_p \circ f] = p^1(\Sigma^{-1} \omega) + \bigoplus_i W_i. \) Hence \( [k_p \circ \tilde{f}] = \varepsilon_n x \cdot P^1 x \ldots P^{p-1} x + \Im P^1 + \bigoplus_i W_i. \]

**8.2. Example.** Let \( X = B(2n+1) \) be the \( 2n \) connected Wilson's irreducible factor of the \( 2n \) connected element of the \( \Omega \)-spectrum of \( BP \). If \( n \neq -1 \mod p \) then for the fundamental class \( x \in H^{2n+1}(X) \) one has \( \varepsilon_n x \cdot P^1 x \ldots P^{p-1} x \notin \Im P^1 + \bigoplus_i W_i \), hence, by 8.1 \( 0 \notin \Phi_{p-1}(P^n x). \)

**8.3. Proof of the main theorem (part 2):** \( 0 \in \Phi_{p-1}(P^n x_{2n+1}) \) will contradict 8.2. ■

**8.4. Corollary.** Let \( X, T, n, x \) be as in 8.1. If \( 0 \notin \Phi_{p-1}(P^n x_{2n+1}) \) (in particular if \( P^n x_{2n+1} = 0 \)) then \( x \cdot P^1 x \ldots P^{p-1} x \in \Im P^1. \)

**Proof:** By 8.1, \( 0 \notin \Phi_{p-1}(P^n x) = \varepsilon_n x \cdot P^1 x \ldots P^{p-1} x + \Im P^1 + \bigoplus_i W_i. \)

**8.5. Corollary.** There is no \( Q-\lambda \) power space \( X \) having \( H^*(X) = \wedge(x, P^1 x, \ldots, P^{p-1} x) \), \( \dim X = 2n + 1, n \neq -1 \mod p). \)

**Proof:** \( P^n x = 0 \) by hypothesis but there are no elements \( w \) with \( P^1 w = x \cdot P^1 x \ldots P^{p-1} x \) to satisfy the conclusions of 8.4 ■

**8.5.1. Remark.** If \( n \) is large enough \( (n > \frac{1}{2}(p^2 - p - 1)(p-1)) \) then there is no space (not necessarily a power space) with \( H^*(X) = \wedge(x, P^1 x, P^2 x, \ldots, P^{p-1} x) \). This follows from the fact that \( H^m(X) = 0 \) for \( m \neq 2n + 1 \mod p - 1 \) in the relevant range (as in the proof of 8.1), hence, \( \Im P^1 \) is the total indeterminacy. In [3] a mod \( p \) \( (p \geq 5) \) \( H \)-space is constructed with \( H^*(X) = \wedge(x, P^1 x, \ldots, x, P^p x), |x| = 2p + 1. \)

**8.6. Corollary.** There is no stable map \( f: X \to \Omega^\infty \Sigma^\infty S^{2p+1} \) with \( P^p x \in \Im H^*(f). \)

**Proof:** As \( \Phi_{p-1} \) is stable \( 0 \in \Phi_{p-1}(w) \) where \( 0 \neq w \in H^{2p+1}(\Omega^\infty \Sigma^\infty S^{2p+1}) \).

A map \( f \) with \( H^*(f)w = P^p x \) will yield (by 8.4) \( x \cdot P^1 x \ldots P^{p-1} x \in \Im P^1 \) which is impossible. ■
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