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ON A CHARACTERIZATION OF AZUMAYA ALGEBRAS

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Abstract

A direct proof of Braun's characterization of Azumaya algebras is given.

Let A be a ring (associative, with 1), Z its centre, and $U = A \otimes_Z A^{op}$. There is an obvious left U-action on A, which we shall denote by $U \times A \to A$, $(u, a) \mapsto u * a$. Let J denote the kernel of the map $U \to A$, $u \mapsto u * 1$, so J is just the left annihilator of $1 \in A$. Notice J contains $a \otimes 1 - 1 \otimes a$ for all $a \in A$.

It is well known that the following properties are equivalent, cf.[2, p.52].

(1) ${}_UA$ is a projective generator; that is, there is some $e \in U$ such that $e * 1 = 1$, $Je = 0$, $UeU = U$.

(2) A_Z is a projective generator and the natural ring homomorphism $U \to End_Z(A)$ is an isomorphism.

(3) ${}_UA$ is projective; that is, there is some $e \in U$ such that $e * 1 = 1$ and $Je = 0$.

A ring satisfying these equivalent conditions is called an *Azumaya algebra*. In [1, Theorem 4.1], Braun gives the further characterization

(4) There is some $e \in U$ such that $e * 1 = 1$, $e * A \subseteq Z$.

In fact (1) $\Leftrightarrow$ (2) by Morita equivalence; (1) $\Rightarrow$ (3) is trivial; and (3) $\Rightarrow$ (4) since $J * (e * A) = (Je) * A = 0 * A = 0$ so $e * A \subseteq Z$. The purpose of this note is to give a direct proof that (4) $\Rightarrow$ (1).

Thus, suppose there is some $e \in U$ such that $e * 1 = 1$, $e * A \subseteq Z$. We show first that $UeU = U$. Let $\mathfrak{a} = \{a \in A | a \otimes 1 \in UeU\}$. This is clearly a (two-sided) ideal of A; if $\mathfrak{a} = A$ then $UeU = U$ as desired, so we may assume there is a maximal ideal $\mathfrak{m}$ of A containing $\mathfrak{a}$.

Let us give U two left U-module structures $U \times U \to U$, $(u, v) \mapsto u *_1 v$, $u *_2 v$ by setting $u *_1 (a \otimes b) = (u * a) \otimes b$, $u *_2 (a \otimes b) = a \otimes (u * b_2)$. It is easy to see these are well-defined. Notice $e *_2 (a \otimes b) = a \otimes (e * b) = a(e * b) \otimes 1$; notice also that $U *_2 (UeU) \subseteq UeU$. Hence every element of $e *_2 (UeU)$ is of the form $a \otimes 1$ for some $a \in \mathfrak{a} \subseteq \mathfrak{m}$.

Let $\bar{A} = A/\mathfrak{m}$, $\bar{Z} = Z/(Z \cap \mathfrak{m})$, $\bar{U} = \bar{A} \otimes_{\bar{Z}} \bar{A}^{op}$ and let the maps $A \to \bar{A}$, $U \to \bar{U}$ be denoted $x \mapsto \bar{x}$. Then $\bar{e} * \bar{1} = \bar{1}$, $\bar{e} * \bar{A} \subseteq \bar{Z}$. But $\bar{Z}$ lies in the centre of $\bar{A}$, which in turn is acted on trivially by $\bar{e}$ so $\bar{Z}$ is the centre of $\bar{A}$. Since $\bar{A}$ is simple, it follows that $\bar{U}$ is simple. Also $\bar{e} * \bar{1} = \bar{1}$ so $\bar{e} \neq 0$ and therefore $\bar{U}\bar{e}\bar{U} = \bar{U}$. But $\overline{1 \otimes 1} = \bar{e} *_2 (\overline{1 \otimes 1}) \in \bar{e} *_2 \bar{U}\bar{e}\bar{U} = \overline{e *_2 UeU} = 0$, a contradiction. This completes the proof that $UeU = U$.

It remains to verify that $Je = 0$.

We claim that for any $u \in U$, $(UeU) *_2 u \subseteq (u *_1 U)U$. Indeed

$$vew *_2 (\sum a_i \otimes b_i) = \sum a_i \otimes vew * b_i) = \sum a_i \otimes (v * 1)(ew * b_i)$$
$$= \sum a_i(ew * b_i) \otimes (v * 1) = \sum_j [(\sum_i a_i \otimes b_i) *_1 (c_j \otimes (v * 1))](d_j \otimes 1)$$

where $ew = \sum c_j \otimes d_j$. Now $Je * A = J * (e * A) \subseteq J * Z = 0$ so $Je * A = 0$. But $Je \subseteq U *_2 (Je) = (UeU) *_2 (Je) \subseteq ((Je) *_1 U)U$ which is 0 since $Je * A = 0$, so $Je = 0$ and (1) holds.

Remarks. (i) The above proof of (3)$\Rightarrow$ (1) is more direct than the one given in **[2]**.

(ii) It is clear from (4) that $A \to Z$, $a \mapsto e * a$, is a retraction of A to Z; thus in (2) one can strengthen the condition that A_Z be a projective generator to $(A/Z)_Z$ being finitely generated projective.

References

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2. F. DEMAYER, E. INGRAHAM, Separable algebras over commutative rings, *Lecture Notes in Mathematics* **181** (1971), Springer-Verlag, New York.

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