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# VALUES IN THE INTERIOR OF THE $L^2$–MINIMAL SOLUTIONS OF THE $\partial\bar{\partial}$–EQUATION IN THE UNIT BALL OF $\mathbb{C}^n$

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*Abstract*

We derive formulas for the values in the interior of the $L^2$–minimal solutions of the $\partial\bar{\partial}$–equation in the unit ball of $\mathbb{C}^n$. These formulas generalizes previously known formulas for the boundary values of the same solutions. We estimate the solutions and obtain a (known) result concerning weighted Nevanlinna classes.

## 0. Introduction

Let $\Theta = \sum \Theta_{jk} d\zeta_j \wedge d\bar{\zeta}_k$ be a smooth closed $(1,1)$–form on the closure of the unit ball $B_n$ of $\mathbb{C}^n$, and consider the equation $i\partial\bar{\partial}u = \Theta$. For $\alpha > 0$ we let $M_{n,\alpha}(\Theta)$ denote the solution which has minimal norm in $L^2((1-|\zeta|^2)^{\alpha-1}d\lambda)$, where $d\lambda$ is Lebesgue measure. For integer values of $\alpha$, we constructed in [1] kernels $L_{n,\alpha}(\zeta, z)$ such that for $z$ on the boundary of $B_n$,

$$M_{n,\alpha}(\Theta)(z) = \int_{B_n} L_{n,\alpha}(\zeta, z) \wedge \Theta(\zeta). \tag{1}$$

The purpose of this note is to describe these kernels in the interior of $B_n$. The idea is the following. Let $\tilde{\Theta}$ be $\Theta$ but considered as a form on $\bar{B}_{n+1}$, not depending on the last variable. Then, see [1] Section 1, $M_{n,\alpha}(\Theta)(z) = M_{n+1,\alpha-1}(\tilde{\Theta})(z, z_{n+1})$ independently of $z_{n+1}$, so in particular

$$M_{n,\alpha}(\Theta)(z) = M_{n+1,\alpha-1}(\tilde{\Theta})(z, \sqrt{1-|z|^2}), \tag{2}$$

and since $(z, \sqrt{1-|z|^2})$ is on the boundary of $B_{n+1}$ we get the interior values of $M_{n,\alpha}(\Theta)$ from the formula for $L_{n+1,\alpha-1}$.

Note that when $n = 1$, the $\partial\bar{\partial}$–equation is essentially the inhomogenous Laplace equation and even in this case the formulas are nontrivial. Though, D. Pascuas has recently obtained these without using the projection of formulas in the unit ball in $\mathbb{C}^2$. In fact, he obtains similar formulas in any plane domain, see [7].

In §1 we carry out some computations and give a rather explicit formula for $L_{n,\alpha}(\zeta,z)\wedge\Theta(\zeta)$ for arbitrary $z$ in $B_n$ (Theorem 1).

It was shown in [1] that (1) makes sense and defines a solution with boundary values in $L^1$ (for $\alpha\geq 1$) if $\Theta$ is just a positive current in $B_n$ which satisfies the Blaschke condition

$$\int_{B_n}(1-|\zeta|^2)\sum\Theta_{jj}d\lambda<\infty.$$

From this result one obtains the Henkin–Skoda theorem in the unit ball, see [5], [6] or [1] Section 3.

In §2 of this note we prove

**Theorem 0.** *Let $\Theta$ be a closed positive $(1,1)$-current in $B_n$ which satisfies the generalized Blaschke condition*

$$\int_{B_n}(1-|\zeta|^2)^{\beta+2}\sum\Theta_{jj}d\lambda<\infty,\quad \beta>-1.$$

*If $\alpha>\beta+1$, then (1) defines a solution to $i\partial\bar\partial u=\Theta$ such that*

$$\int_{B_n}(1-|z|^2)^{\beta}|M_{n,\alpha}(\Theta)(z)|d\lambda(z)<\infty.$$

Theorem 0 provides the essential part of the proof of the Dautov–Henkin theorem, see [3].

It should be emphasized that the main reason for proving Thoerem 0 here is to illustrate how the new formulas can be estimated. For instance, if $\beta$ is an integer then Theorem 0 follows more or less directly from the above mentioned analogous result in [1] because of the following easily verified lemma:

**Lemma 1.** *If $\psi$ is a (continuous) function on the closure of $B_n$, then $\pi\int_{B_n}(1-|\zeta|^2)^{\beta+1}\psi d\lambda=(\beta+1)\int_{B_{n+1}}(1-|\zeta|^2)^{\beta}\psi d\lambda$ if $\beta>-1$, and $\pi\int_{B_n}\psi d\lambda=$ $=\frac{1}{2}\int_{\partial B_{n+1}}\psi dS$, where $dS$ denotes surface measure.*

In fact, if $\Theta$ satisfies the hypotesis of Theorem 0 then, by the lemma, $\tilde\Theta$ satisfies the ordinary Blaschke condition in $B_{n+\beta+1}$. Thus $M_{n+\beta+1,\alpha}(\tilde\Theta)$ has boundary values in $L^1$ (if $\alpha\geq 1$) and is independent of the $\beta+1$ last variables so, by the lemma again and (2), $M_{n,\alpha-\beta-1}(\Theta)$ is in $L^1((1-|\zeta|^2)^{\beta}d\lambda)$ in $B_n$.

We use the following notation. For any $\zeta$ and $z$ in $\mathbb{C}^n$ we put $\zeta\cdot\bar z=\sum\zeta_j\bar z_j$, in particular, $\zeta\cdot\bar\zeta=|\zeta|^2$, and $d\bar z\cdot\zeta=\sum\bar z_j d\zeta_j$ i.e. we only consider differentials of $\zeta$. Moreover, $\beta_k,\mu$ and $\gamma$ denote the positive forms

$$(\frac{i}{2}\partial\bar\partial|\zeta|^2)^k\frac{1}{k!},\ \frac{i}{2}d\bar z\cdot\zeta\wedge dz\cdot\bar\zeta\quad\text{and}\quad\frac{i}{2}\partial|\zeta|^2\wedge\bar\partial|\zeta|^2,$$

respectively. Note that $\beta_n$ is Lebesgue measure in $B_n$.

Finally, the notation $x\lesssim y$ means that $x$ is less than a constant times $y$ and $x\sim y$ is equivalent to $x\lesssim y$ and $y\lesssim x$.

## 1. Values in the interior

In this paragraph $\Theta$ is a fixed closed smooth $(1,1)$–form on the closure of $B_n$. We start by recalling the formula, derived in [1], for $L_{n,\alpha}(\zeta,z)\wedge\Theta(\zeta)$, (cf. [1] above). If $z\in\partial B_n$ and $\alpha\geq -1$ we have

$$(3)\qquad L_{n,\alpha}(\zeta,z)\wedge\Theta(\zeta)=\frac{1}{2\pi^n}\left[\sum_{\substack{j,k\geq 1\\ j+k\leq n+\alpha}} A_{jk}\frac{(1-|\zeta|^2)^{\alpha+1}\Theta\wedge\beta_{n-1}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^k}+\right.$$

$$+\sum_{\substack{j,k\geq 1\\ j+k\leq n+\alpha}} B_{jk}\frac{(1-|\zeta|^2)^{\alpha+2}\Theta\wedge\mu\wedge\beta_{n-2}}{(1-\bar{\zeta}\cdot z)^{j+1}(1-\zeta\cdot\bar{z})^{k+1}}-$$

$$\left.-\sum_{\substack{j,k\geq 1\\ j+k\leq n+\alpha+1}} C_{jk}\frac{(1-|\zeta|^2)^{\alpha+1}\Theta\wedge\mu\wedge\beta_{n-2}}{(1-\bar{\zeta}\cdot z)^j(1-\zeta\cdot\bar{z})^k}\right]$$

where

$$A_{jk}=\frac{(n+\alpha)(n+\alpha-j-1)!(n+\alpha-k-1)!}{(\alpha+1)!(n+\alpha-j-k)!},$$

$$B_{jk}=\frac{jk(n+\alpha-j-1)!(n+\alpha-k-1)!}{(\alpha+2)!(n+\alpha-j-k)!}$$

and

$$C_{jk}=\frac{(n+\alpha-j)!(n+\alpha-k)!}{(\alpha+1)!(n+\alpha-j-k+1)!}.$$

When $n=1$ terms involving $\Theta\wedge\mu\wedge\beta_{n-2}$ are interpreted as zero.

Let

$$\sigma(\zeta,z)=|1-\bar{\zeta}\cdot z|^2-(1-|\zeta|^2)(1-|z|^2).$$

It is shown in §2 that $\sigma(\zeta,z)\geq 0$ and that equality occurs if and only if $\zeta=z$. Now we state our main result.

**Theorem 1.** *For any $z$ (and $\zeta$) on the closure of $B_n$ we have*

$$(4)\qquad L_{n,\alpha}(\zeta,z)\wedge\Theta(\zeta)=$$

when $\sum \Theta_{jj} = 0$. This follows from the fact that the restriction of the corresponding orthogonal projection $\Pi_\mu$ (cf. the proof above) to the $\mathbb{R}^{2n}$-harmonic functions on $\bar{B}_n$ is nothing but the natural projection

$$\bigoplus_{p,q\geq o} H_{p,q} \to \bigoplus_{p\geq 0} H_{0,p} \oplus H_{p,0} \ ,$$

where $H_{p,q}$ denotes the homogenous harmonic polynomials of bidegree $(p,q)$.

In particular, if $v$ is the solution to $\sum \frac{\partial^2 v}{\partial \zeta_j \partial \bar{\zeta}_j} = \sum \Theta_{jj}$ which vanish on the boundary then $M_\mu(\Theta - i\partial\bar{\partial}v) = M_{n,0}(\Theta)$, i.e. the solution which is $L^2$-minimal on the boundary, since (cf. (16)) the projection $\Pi_{n,0}$ only depends on the boundary values, see [1].

## 2. Estimates of the solutions

**Proposition 2.** *If $\alpha \geq 0$, then*

(21) $$|L_{n,\alpha}(\zeta,z) \wedge \Theta(\zeta)| \lesssim$$

$$\lesssim (1-|\zeta|^2)^{\alpha+1} \left[\frac{|1-\bar{\zeta}\cdot z|^2}{\sigma(\zeta,z)}\right]^{n-1} \left[\frac{\Theta \wedge \beta_{n-1}}{|1-\bar{\zeta}\cdot z|^{n+\alpha}} + \frac{\Theta \wedge \gamma \wedge \beta_{n-2}}{|1-\bar{\zeta}\cdot z|^{n+\alpha+1}}\right]$$

*if $(\cdot)^{n-1}$ is interpreted as $1+\log(\cdot)$ when $n=1$.*

*Proof:* We need the estimate

(22) $$\Theta \wedge \mu \wedge \beta_{n-2} \lesssim \Theta \wedge \gamma \wedge \beta_{n-2} + |\zeta - z|^2 \Theta \wedge \beta_{n-1} \ ,$$

see [1], Section 3 formula (9), and the inequalities

(23) $$|\zeta - z|^2 \lesssim 2|1-\bar{\zeta}\cdot z| \ ,$$

$\sigma(\zeta,z) \leq |1-\bar{\zeta}\cdot z|^2$ and $(1-|z|^2) \leq 2|1-\bar{\zeta}\cdot z|$. The last two ones are trivial whereas (23) comes out from an easy computation. Now Proposition 2 follows from Theorem 1. ∎

We will also use some properties of $\sigma(\zeta,z)$ and we collect them in a lemma.

**Lemma 2.** *Suppose $\zeta, z \in \bar{B}_n$. Then*

*a) $|\zeta - z|^2 \geq \sigma(\zeta,z) \geq 0$ and $\sigma(\zeta,z) = 0$ if and only if $\zeta = z$.*

*b) There is a constant $c > 0$ such that for fix $\zeta$, $|\{z \in B_n; \sigma(\zeta,z) < r^2\}| \leq \leq cr^{2n}(1-|\zeta|^2)^{1-n}$; ($|\cdot|$ is volume measure).*

*c) If either* $1-|z|^2 \geq 8(1-|\zeta|^2)$, $(1-|\zeta|^2) \geq 8(1-|z|^2)$ *or* $|1-\bar\zeta\cdot z| \geq$ $\geq 4(1-|\zeta|^2)$, *then* $\sigma(\zeta,z) \geq 1/2|1-\bar\zeta\cdot z|^2$.

*Proof:* Since $\sigma(\zeta-z) = |\zeta-z|^2 - |\zeta|^2|z|^2 + |\zeta\cdot z|^2$ the first inequality in a) is obvious. We can assume that $\zeta = (\zeta_1, 0, \ldots, 0)$ and $z = (z_1, z_2, \ldots, z_n) =$ $= (z_1, z')$. Then

$$\sigma = |\zeta_1 - z_1|^2 + (1-|\zeta_1|^2)|z'|^2\,, \tag{24}$$

hence $\sigma \geq 0$ and equality holds only if $z_1 = \zeta_1$ and $z' = 0$. This proves a). Now suppose $\sigma < r^2$. Then, by (24),

$$|\zeta_1 - z_1|^2 < r^2 \quad \text{and} \quad (1-|\zeta_1|^2)|z'|^2 < r^2$$

so that

$$|\{\sigma < r^2\}| \leq cr^2\left[\frac{r^2}{1-|\zeta_1|^2}\right]^{n-1} = cr^{2n}(1-|\zeta|^2)^{1-n}\,,$$

which proves b). To see c), suppose for instance that $1-|\zeta|^2 \geq 8(1-|z|^2)$. Since $\sigma(\zeta,z) = |1-\bar\zeta\cdot z|^2 - (1-|\zeta|^2)(1-|z|^2)$ and $|1-\bar\zeta\cdot z| \geq 1/2(1-|\zeta|^2)$ we find that

$$\sigma(\zeta,z) \geq \frac{1}{2}|1-\bar\zeta\cdot z|^2 + \frac{1}{8}(1-|\zeta|^2)^2 - (1-|\zeta|^2)(1-|z|^2)$$

so that, by assumption,

$$\sigma(\zeta,z) \geq \frac{1}{2}|1-\bar\zeta\cdot z|^2 + (1-|\zeta|^2)(1-|z|^2) - (1-|\zeta|^2)(1-|z|^2) = \frac{1}{2}|1-\bar\zeta\cdot z|^2.$$

The other cases in c) are handled similarly, and thus the lemma is verified. ∎

*Proof of Theorem 0:* First assume that $\Theta$ is a closed positive smooth form on $\bar B_n$. Our objective is to obtain the estimate

$$\int_{B_n} |M_{n,\alpha}(\Theta)(z)|(1-|z|^2)^\beta d\lambda(z) \lesssim \int_{B_n} (1-|\zeta|^2)^{\beta+2}\Theta\wedge\beta_{n-1} \quad \text{if} \quad \alpha > \beta+1. \tag{25}$$

Once this is done, Theorem 0 follows by regularization, e.g. as in [1] Section 3, but in this case it is much simpler since we will have no problem with regularity at the boundary, and we omit the details here.

Since $\Theta\wedge\beta_{n-1} = (n-1)\Theta\wedge\beta_{n-2}\wedge\frac{i}{2}\partial\bar\partial|\zeta|^2$, Stokes' theorem gives that the right hand side of (25) is equal to $(\beta+2)(n-1)\int_{B_n}(1-|\zeta|^2)^{\beta+1}\Theta\wedge\gamma\wedge\beta_{n-2}$. Hence (25) follows from Fubinis theorem and the estimate

$$\begin{aligned}\int_{B_n} |L_{n,\alpha}(\zeta,z)\wedge\Theta(\zeta)|(1-|z|^2)^\beta d\lambda(z) &\lesssim \\ &\lesssim (1-|\zeta|^2)^{\beta+2}\Theta\wedge\beta_{n-1} + (1-|\zeta|^2)^{\beta+1}\Theta\wedge\gamma\wedge\beta_{n-2}.\end{aligned} \tag{26}$$

To show (26), fix $\zeta$ in $B_n$, put

$$I = (1-|\zeta|^2)^{\beta+2}\Theta\wedge\beta_{n-1} + (1-|\zeta|^2)^{\beta+1}\Theta\wedge\gamma\wedge\beta_{n-2} \tag{27}$$

and let $E$ be the set of $z$ in $B_n$ such that $(1-|z|^2)\leq 8(1-|\zeta|^2)$, $(1-|\zeta|^2)\leq$ $\leq 8(1-|z|^2)$, $|1-\bar{\zeta}\cdot z|\leq 4(1-|\zeta|^2)$ and $\sigma(\zeta,z)\leq\frac{1}{2}|1-\bar{\zeta}\cdot z|^2$.

From part c) of Lemma 2, it turns out that $|1-\bar{\zeta}\cdot z|^2\leq 2\sigma(\zeta,z)$ in $B_n\backslash E$ and thus, by Proposition 2,

$$|L_{n,\alpha}(\zeta,z)\wedge\Theta(\zeta)| \lesssim (1-|\zeta|^2)^{\alpha+1}\left[\frac{\Theta\wedge\beta_{n-1}}{|1-\bar{\zeta}\cdot z|^{n+\alpha}} + \frac{\Theta\wedge\gamma\wedge\beta_{n-2}}{|1-\bar{\zeta}\cdot z|^{n+\alpha+1}}\right].$$

Since

$$\int_{B_n}\frac{(1-|z|^2)^\beta d\lambda(z)}{|1-\bar{\zeta}\cdot z|^{n+\alpha}} \lesssim \left[\frac{1}{1-|\zeta|^2}\right]^{\alpha-\beta-1} \text{ if } \alpha-\beta-1>0,$$

(Proposition 1.4.10 in [4]) we find that

$$\int_{B_n\backslash E}|L_{n,\alpha}(\zeta,z)\wedge\Theta(\zeta)|(1-|z|^2)^\beta d\lambda(z)\lesssim I. \tag{28}$$

Note that $|1-\bar{\zeta}\cdot z|\sim 1-|\zeta|^2\sim 1-|z|^2$ and $\sigma(\zeta,z)\leq 8(1-|\zeta|^2)^2$ in $E$. Thus, using Proposition 2 again and putting $1-|\zeta|^2=\sqrt{8}\delta$ we find that

$$\begin{aligned}&\int_E|L_{n,\alpha}(\zeta,z)\wedge\Theta(\zeta)|(1-|z|^2)^\beta \lesssim\\ &\lesssim\left[\delta^{\beta-1+n}\Theta\wedge\beta_{n-1}+\delta^{\beta-2+n}\Theta\wedge\gamma\wedge\beta_{n-2}\right]\int_{\sigma\leq\delta^2}\frac{d\lambda(z)}{\sigma^{n-1}}.\end{aligned}$$

From part b) of Lemma 2 it turns out that

$$\int_{\sigma\leq\delta^2}\frac{d\lambda(z)}{\sigma^{n-1}}\lesssim\delta^{3-n},$$

(the case $n=1$, requires a slightly different argument because of the logarithm in Proposition 2) and hence

$$\int_E|L_{n,\alpha}(\zeta,z)\wedge\Theta(\zeta)|(1-|z|^2)^\beta d\lambda(z)\lesssim I. \tag{29}$$

Now, (27), (28), and (29) imply (26), and this completes the proof. ∎

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