Publicacions Matemàtiques, Vol 32 (1988), 267–274.

NONASSOCIATIVE REAL H^*-ALGEBRAS

M. CABRERA, J. MARTÍNEZ, A. RODRÍGUEZ

Abstract

We prove that, if A denotes a topologically simple real (non-associative) H^*-algebra, then either A is a topologically simple complex H^*-algebra regarded as real H^*-algebra or there is a topologically simple complex H^*-algebra B with *-involution τ such that $A = \{b \in B : \tau(b) = b^*\}$. Using this, we obtain our main result, namely: (algebraically) isomorphic topologically simple real H^*-algebras are actually *-isometrically isomorphic.

0. Introduction

Since the work by Ambrose [1] introducing complex associative H^*-algebras and obtaining for they a complete structure theory, many papers on particular and/or general nonassociative complex H^*-algebras have been published with the aim of finding a similar complete structure theory in their context (see [**4, 5, 6, 12, 14, 15, 17, 18**]).

From a general nonassociative point of view the most remarkable results are the following:

Decomposition Theorem. [6] (See also [8]). *Every non zero (nonassociative) complex H^*-algebra with zero annihilator is the closure of the orthogonal sum of (automatically *-invariant) closed ideals which are topologically simple H^*-algebras.*

Essential Uniqueness of the Topologically simple H^*-Algebra Structure [5]. *Once a complex algebra A has been structured as a topologically simple H^*-algebra, every H^*-algebra structure on A is (up to a positive multiple of the inner product) totally isomorphic to the given one.*

The theory of real H^*-algebras begins with the almost simultaneous works by Balachandran [2] De la Harpe [9] and Unsain [16] on real Lie H^*-algebras. Only very recently the theory of real associative H^*-algebras has been achieved by Balachandran and Swaminathan [3]. Also we must cite the work by Cuenca and Sanchez on real noncommutative Jordan H^*-algebras [7]. In all these particular cases it is remarked that the decomposition theorem for real H^*-algebras with zero annihilator in the class under consideration follows essentially from the same techniques used in the proof of the corresponding decomposition theorem in the complex case. Actually the proof of the general decomposition theorem for nonassociative complex H^*-algebras with zero annihilator given in [**6**] does not depend of the base field, so decomposition theorem is also true for nonassociative real H^*-algebras with zero annihilator.

The situation for the essential uniqueness of the topologically simple H^*-algebra structure is not the same, because the proof given in [**5**] for the complex case depends in the most of the steps of the base field $\mathbb{C}$. Thus the main purpose in this paper is to prove the essential uniqueness of the H^*-algebra structure for topologically simple nonassociative real H^*-algebras. This is achieved in Section 2. The proof uses the results in [**5**] and a Theorem (to which is devoted Section 1), previously known in the case of Lie algebras [**2, 9, 16**], stating that, if A is a topologically simple real H^*-algebra, then either A is a topologically simple complex H^*-algebra regarded as real H^*-algebra or there is a topologically simple complex H^*-algebra B with *-involution τ such that $A = \{b \in B : \tau(b) = b^*\}$.

1. Building topologically simple real H^*-algebras from topologically simple complex H^*-algebras

We recall that an H^*-algebra over $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$ is a nonassociative algebra A over $\mathbb{K}$ with (conjugate-linear) algebra involution $*$, called the H^*-algebra involution of A, which is also a Hilbert space over $\mathbb{K}$ under an inner product $(\ / \ )$ satisfying

$$(xy/z) = (x/zy^*) = (y/x^*z)$$

for all x, y, z in A.

All the results obtained in section 1 of [**6**] for complex H^*-algebras are true also in the real case because the arguments used there for the proofs of these results are not dependent of the base field. In particular the product of any H^*-algebra is continuous for the topology of the Hilbert norm $x \to \|x\| := \sqrt{(x/x)}$, so (by multiplying the inner product by a suitable positive number if necessary) every H^*-algebra is a (complete) normed algebra in the usual sense of the word.

If B is a complex H^*-algebra the underlying real algebra $B_{\mathbb{R}}$, obtained by restriction of the base field, can and will be considered as a real H^*-algebra (called the *realization* of B) under the same H^*-algebra involution that of B and the real inner product $Re(\ / \ )$. Clearly closed *-subalgebras of the real H^*- algebra $B_{\mathbb{R}}$ are new examples of real H^*-algebras. But we are interested in a very particular case of this last situation. Assume that the H^*-algebra involution of B is continuous and let τ be any continuous "*-involution" of B (linear antiautomorphism of period two such that $\tau(b^*) = [\tau(b)]^*$ for all b in B). Then the set $\{b \in B : \tau(b) = b^*\}$ is a closed *-subalgebra of $B_{\mathbb{R}}$ so a real H^*-algebra. We remark that the assumption of continuity for the H^*-algebra involution of B and for the *-involution τ is superfluous if B has zero annihilator (where the *annihilator* of B is defined as the closed ideal $\{b \in B : bB = Bb =$

$= \{0\}\})$ because of the uniqueness of the complete algebra norm topology of B in this case [**13**, Remark 2.8.(i)]. Actually if B has zero annihilator its H^*-algebra involution is isometric [**6**, Proposition 2. (ix)] and if in particular B is "topologically simple" (non zero product and with no non zero proper closed ideals) also the *-involution τ is isometric, [**5**, Corollary 3.5.] (see also our Lemma 2 below), so if b, c are in B with $\tau(b) = b^*$ and $\tau(c) = c^*$ then the inner product (b/c) is a real number and the passing to $Re(\ \ /\ \ )$ in the real H^*-algebra $\{b \in B : \tau(b) = b^*\}$ is unnecessary.

Now we are in position to give the canonical method for obtaining topologically simple real H^*-algebras from the complex ones.

Proposition 1. *If B is a topologically simple complex H^*-algebra, $B_{\mathbb{R}}$ is a topologically simple real H^*-algebra. Moreover, if τ is any *-involution of B, then $\{b \in B : \tau(b) = b^*\}$ is also a topologically simple real H^*-algebra.*

Proof: Let M be a closed ideal of $B_{\mathbb{R}}$. Then $M \cap iM$ is a closed ideal of B so, by the topological simplicity of B, we have either $M \cap iM = B$, in which case $M = B_{\mathbb{R}}$, or $M \cap iM = 0$. From this last equality we deduce $MB = M(iB) = i(MB) \subset M \cap iM = 0$ and, analogously, $BM = 0$ so $M \subset Ann(B) = 0$ and so $B_{\mathbb{R}}$ is topologically simple. Now let τ be a *-involution of B, let A denote the real H^*-algebra $\{b \in B : \tau(b) = b^*\}$ and let M be a closed ideal of A. The fact that for any b in B $b + \tau(b^*)$ and $i(\tau(b^*) - b)$ lie in A yields to the equality $B = A \oplus iA$ as topological direct sum of real subspaces. Now it is clear that $M + iM$ is a closed ideal of B so either $M + iM = B$ (and so $M = A$) or $M + iM = 0$ (and so $M = 0$). Therefore A is topologically simple. ∎

Our next result will show that the topologically simple real H^*-algebras obtained in the above proposition are the only topologically simple real H^*-algebras. For the proof we recall the concept of "complexification" of a real algebra A. The complexification of A is the complex algebra $A_{\mathbb{C}} := A \oplus iA$ with addition, multiplication by complex numbers and product defined by

$$(x + iy) + (z + it) = (x + z) + i(y + t)$$
$$(\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\alpha y + \beta x)$$
$$(x + iy)(z + it) = (xz - yt) + i(xt + yz)$$

for all x, y, z, t in A and α, β real numbers. If A is actually a real H^*-algebra, then $A_{\mathbb{C}}$ can and will be considered in a natural way as a complex H^*-algebra with H^*-algebra involution the extension by conjugate-linearity of the one of A and inner product the extension by sesquilinearity of the one of A. Moreover the extension by linearity of the H^*-algebra involution of A gives a *-involution of $A_{\mathbb{C}}$ which will be called the *canonical *-involution* of the complex H^*-algebra $A_{\mathbb{C}}$.

Theorem 1. *Let A be a topologically simple real H^*-algebra. Then either A is the realization of a topologically simple complex H^*-algebra or there exist a topologically simple complex H^*-algebra B with $*$-involution τ such that*

$$A = \{b \in B : \tau(b) = b^*\}.$$

Proof: If the complex H^*-algebra $A_{\mathbb{C}}$ is topologically simple, call it B and let τ denote its canonical $*$-involution. Clearly $A = \{b \in B : \tau(b) = b^*\}$ and we are in the last situation of the statement. Otherwise $A_{\mathbb{C}}$ has a non zero proper closed ideal M and such an ideal can not be invariant under τ (use the topological simplicity of A and the fact that closed ideals of $A_{\mathbb{C}}$ are $*$-invariant [**6**, Proposition 2.(v)]) Therefore $M \cap \tau(M) = 0$, so M and $\tau(M)$ are mutually orthogonal [**6**, Proposition 2.(vii)], and so $A_{\mathbb{C}} = M \oplus \tau(M)$ because $M \oplus \tau(M)$ is a non zero closed τ-invariant ideal of $A_{\mathbb{C}}$. Now from the fact that M is an arbitrary non zero proper closed ideal of $A_{\mathbb{C}}$ it follows easily that M and $\tau(M)$ are the only non zero proper closed ideals of $A_{\mathbb{C}}$ and, as a consequence, that M is a topologically simple (complex)H^*-algebra. Now routine verification shows that the mapping $m \to m + \tau(m^*)$ is a total isomorphism (isometric $*$-isomorphism) from the realization of the topologically simple complex H^*-algebra M (once its inner product has been multiplied previously by two) onto A. ■

The proof of the above theorem shows that the two posibilities in the statement for the topologically simple real H^*-algebra A are mutually exclusive because they depend on wether or not $A_{\mathbb{C}}$ is topologically simple. The following corollary exhibits more explicitly this fact. We recall that the *centroid* $C(A)$ of a nonassociative algebra A is defined as the set of mappings T from A into A such that $T(xy) = T(x)y = xT(y)$ for all x, y in A. As in [11,Theorem 2.1], if A is a (nonassociative) complete normed algebra with zero annihilator, then $C(A)$ is a closed commutative subalgebra of $BL(A)$ (the Banach algebra of all continuous linear operators on A) containing the identity operator.

Corollary 1. *Let A be a topologically simple real H^*-algebra. Then the centroid of A is isomorphic to $\mathbb{C}$ or $\mathbb{R}$ according to A is the realization of a topologically simple complex H^*-algebra or there is a topologically simple complex H^*-algebra B with $*$-involution τ such that*

$$A = \{b \in B : \tau(b) = b^*\}$$

Proof: If A is the realization of a topologically simple complex H^*-algebra we have $C(A) = \mathbb{C}$ since the centroid is not dependent of the base field and so [**5**, Theorem 1.2] can be applied. Otherwise, by Theorem 1, $A = \{b \in B : \tau(b) = b^*\}$ for suitable topologically simple complex H^*-algebra B and $*$-involution τ of B, Since $B = A \oplus iA$ (recall the proof of Proposition 1), every element in $C(A)$ can be exented by linearity to an element in $C(B)$. Therefore $C(A) = \mathbb{R}$ because $C(B) = \mathbb{C}$ [**5**, Theorem 1.2]. ■

2. Isomorphisms of real H^*-algebras

Our aim in this section is to prove that algebraically isomorphic topologically simple real H^*-algebras are actually totally isomorphic. The first step to this end is the following theorem on decomposition of isomorphisms between real H^*-algebras with zero annihilator, from which it follows that isomorphic real H^*-algebras with zero annihilator are $*$-isomorphic. The points for the proof are the analogous result for complex H^*-algebras [5] and the fact that the complexification of a real H^*-algebra is a complex H^*-algebra which has zero annihilator if the same is true for the given one.

If T is any linear mapping from an H^*-algebra A into another one B, we denote by T^* the (linear) mapping from A into B defined by

$$T^*(a) = (T(a^*))^* \text{ for all } a \text{ in } A.$$

Theorem 2. *Let A and B be real H^*-algebras with zero annihilator and F be an isomorphism from A onto B. Then F can be written in a unique way as $F = Gexp(D)$ with G a $*$-isomorphism from A onto B and D a continuous derivation of A such that $D^* = -D$.*

Proof: The mapping $F_{\mathbb{C}}$ from $A_{\mathbb{C}}$ onto $B_{\mathbb{C}}$ defined by $F_{\mathbb{C}}(x+iy) = F(x) + iF(y)$ is an isomorphism so, by [5, Theorem 3.3 and Corollary 2.3], $F_{\mathbb{C}}$ can be written in a unique way as $F_{\mathbb{C}} = G_0 exp(D_0)$ where G_0 is a $*$-isomorphism from $A_{\mathbb{C}}$ onto $B_{\mathbb{C}}$ and D_0 a continuous derivation of $A_{\mathbb{C}}$ with $D_0^* = -D_0$.

If τ (resp.: τ') denotes the canonical $*$-involution of $A_{\mathbb{C}}$ (resp.: $B_{\mathbb{C}}$), clearly we have

$$F_{\mathbb{C}} = \tau'(F_{\mathbb{C}})^*\tau$$

so

$$\begin{aligned} F_{\mathbb{C}} &= \tau'(G_0 \exp(D_0))^*\tau = \tau' G_0 \exp(-D_0)\tau = \\ &= \tau' G_0 \tau\tau \exp(-D_0)\tau = \tau' G_0 \tau \exp(-\tau D_0 \tau). \end{aligned}$$

Since $\tau' G_0 \tau$ is a $*$-isomorphism from $A_{\mathbb{C}}$ onto $B_{\mathbb{C}}$ and $-\tau D_0 \tau$ is a continuous derivation of $A_{\mathbb{C}}$ with $(-\tau D_0 \tau)^* = -(-\tau D_0 \tau)$, by the uniqueness of the decomposition for $F_{\mathbb{C}}$, we have $\tau' G_0 \tau = G_0$ and $-\tau D_0 \tau = D_0$. Therefore $G_0(A) = B$ and $D_0(A) \subset A$. Now, if we consider the mapping G (from A onto B) and D (from A into A) defined by $G(x) = G_0(x)$ and $D(x) = D_0(x)$ for all x in A, these satisfy the requirements in the statement of the theorem. The uniqueness of the decomposition for F follows easily from the uniqueness of the decomposition for $F_{\mathbb{C}}$. ■

Our next purpose is to show that $*$-isomorphisms between given topologically simple real H^*-algebras are constant positive multiples of isometries. The proof depends on Theorem 1 and the following two lemmas. Actually Lemma 2 is a little improvement of [5, Corollary 3.4].

Lemma 1. *Let A and B be topologically simple complex H^*-algebras. Then any isomorphism from $A_{\mathbf{R}}$ onto $B_{\mathbf{R}}$ is, as mapping from A to B, either linear or conjugate linear.*

Proof: Let F be an isomorphism from $A_{\mathbf{R}}$ onto $B_{\mathbf{R}}$. As in [10, Theorem x.5] F induces a ring–isomorphism u from $C(A)$ onto $C(B)$ (both equal $\mathbb{C}$ [5, Theorem 1.2]) such that $F(\lambda x) = u(\lambda)F(x)$ for all x in A and all λ in $\mathbb{C}$. The proof is concluded by observing that in our case u is real–linear. ■

Lemma 2. *Let A and B be topologically simple complex H^*-algebras. Then there is a positive number K ($K = 1$ if $A = B$) such that for every isomorphism or antiisomorphism F from A onto B we have that $F^{\cdot} = K(F^*)^{-1}$.*

Proof: We recall that F is automatically continuous and that $F^{\cdot}$ is then defined as the unique continuous linear mapping from B into A satisfying $(F(a)/(b) = (a/F^{\cdot}(b))$ for all a in A and b in B. By [5, Corollary 3.4] there is a positive number $K(K = 1$ if $A = B)$ such that $F^{\cdot} = K(F^*)^{-1}$ for every isomorphism F from A onto B and also, by regarding each antiisomorphism from A onto B as an isomorphism from A onto the H^*-algebra obtained by reversion of the product of B, there is K' such that $G^{\cdot} = K'(G^*)^{-1}$ for every antiisomorphism G from A onto B. Therefore we must show that if A and B are at the same time isomorphic and antiisomorphic, then $K = K'$. First assume $A = B$ and let F be an antiautomorphism of A. Then, since F^2 is an automorphism, we have $(F^2)^{\cdot} = (F^{2*})^{-1}$ and also clearly $(F^2)^{\cdot} = K'^2(F^{2*})^{-1}$ so $K' = 1$ in this particular case. In general let G be an antiisomorphism from A onto B and H be an antiautomorphism of A. Then GH is an isomorphism from A onto B, so we have

$$K((GH)^*)^{-1} = (GH)^{\cdot} = H^{\cdot}G^{\cdot} = K'(H^*)^{-1}(G^*)^{-1} = K'((GH^*)^{-1}.$$

and so $K = K'$ as required. ■

Theorem 3. *Let A and B be topologically simple real H^*-algebras. Then there is a positive number $K(K = 1$ if $A = B)$ such that for every isomorphism F from A onto B we have that $F^{\cdot} = K(F^*)^{-1}$.*

Proof: We may assume obviously that A and B are isomorphic. If A and B are complex H^*-algebras regarded as real H^*-algebras, then the equality $H^{\cdot} = K(H^*)^{-1}$ is true for suitable positive number K and every complex isomorphism or antiisomorphism H from A onto B (Lemma 2). But if F denotes any real isomorphism from A onto B, by Lemma 1, F is either a linear complex isomorphism or a conjugate-linear complex isomorphism in which case the mapping $x \to F(x)^*$ is a linear complex antiisomorphism. In both cases, taking into account for the second one that the H^*-algebra involution of B is isometric, we have clearly $F^{\cdot} = K(F^*)^{-1}$. In view of Corollary 1 and Theorem

1 only remains to consider the case of existence of topologically simple complex H^*-algebras C and D with $*$-involutions τ and τ', respectively, such that

$$A = \{c \in C : \tau(c) = c^*\} \text{ and } B = \{d \in D : \tau'(d) = d^*\}.$$

Now, since $C = A \oplus iA$ and $D = B \oplus iB$, each isomorphism from A onto B extends in a unique way to an isomorphism from C onto D so in this case the statement of our theorem follows directely from the analogous one for complex algebras (Lemma 2). ■

Corollary 2. *Let A and B be topologically simple real H^*-algebras. Then there is a positive number L such that, for every $*$-isomorphism F from A onto B, LF is an isometric mapping.*

With the above corollary our purpose in this paper is achieved. If A and B are isomorphic topologically simple real H^*-algebras, they are $*$-isomorphic (Theorem 2) and $*$-isomorphisms from A onto B are constant positive multiples of isometries. This is the essential uniqueness of the H^*-algebra structure for topologically simple real H^*-algebras.

References

1. W. AMBROSE, Structure theorems for a special class of Banach algebras, *Trans. Amer. Math. Soc.* **57** (1945), 364–386.
2. V.K. BALACHANDRAN, Real L^*-algebras, *J. Pure Appl. Math.* **3** (1972), 1224–1246.
3. V.K. BALACHANDRAN AND N. SWAMINATHAN, Real H^*-algebras, *J. Functional Analysis* **65** (1986), 64–75.
4. M. CABRERA, J. MARTÍNEZ AND A. RODRÍGUEZ, Malcev H^*-algebras, *Math. Proc. Camb. Phil. Soc.* (to appear); *Research announcement: Extracta Math.* **1** (1987), 130–132.
5. J.A. CUENCA AND A. RODRÍGUEZ, Isomorphism of H^*-algebras, *Math. Proc. Camb. Phil. Soc.* **97** (1985), 93–99.
6. J.A. CUENCA AND A. RODRÍGUEZ, Structure theory for noncommutative Jordan H^*-algebras, *J. Algebra* **106** (1987), 1–14.
7. J.A. CUENCA AND A. SANCHEZ, Structure theory for real noncommutative Jordan H^*-algebras, *Preprint* (1987), Universidad de Málaga.
8. A. FERNÁNDEZ AND A. RODRÍGUEZ, A Wedderburn theorem for nonassociative complete normed algebras, *J. London Math. Soc.* **37** (1986), 328--338.
9. P. DE LA HARPE, Classification des L^*-algèbres semi-simples réelles séparables, *C.R. Acad. Sci.* **Ser. A 272** (1971), 1559-1561, Paris.

10. N. JACOBSON, "*Lie algebras*," Dover Publication Inc., New York, 1962.
11. C.N. KELLOG, Centralizers and H^*-algebras, *Pacific J. Math.* **17** (1966), 121–129.
12. I. PÉREZ DE GUZMAN, Structure theorem for alternative H^*-algebras, *Math. Proc. Camb. Phil. Soc.* **94** (1983), 437–446.
13. A. RODRÍGUEZ, The uniqueness of the complete norm topology in complete normed nonassociative algebras, *J. Functional Analysis* **60** (1985), 1–15.
14. J.R. SCHUE, Hilbert space methods in the theory of Lie algebras, *Trans. Amer. Math. Soc.* **95** (1960), 69–80.
15. J.R. SCHUE, Cartan decomposition for L^*-algebras, *Trans. Amer. Math. Soc.* **98** (1961), 334–349.
16. I. UNSAIN, Classification of the simple separable real L^*-algebras, *J. Differential Geometry* **7** (1972), 423–451.
17. C. VIOLA DEVAPAKKIAN, Hilbert space methods in the theory of Jordan algebras, *Math. Proc. Camb. Phil. Soc.* **78** (1975), 293–300.
18. C. VIOLA DEVAPAKKIAN AND P.S. REMA, Hilbert space methods in the theory of Jordan algebras II, *Math. Proc. Camb. Phil. Soc.* **79** (1976), 307–319.

Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071-Granada, SPAIN.

Rebut el 14 d'Abril de 1988