Publicacions Matemàtiques, Vol 33 (1989), 163–171.

ON RAMIFICATION DIVISORS OF FUNCTIONS IN A PUNCTURED COMPACT RIEMANN SURFACE

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Abstract

Let $\mathcal{V}$ be a compact Riemann surface and $\mathcal{V}'$ be the complement in $\mathcal{V}$ of a nonvoid finite subset. Let $\mathcal{M}(\mathcal{V}')$ be the field of meromorphic functions in $\mathcal{V}'$. In this paper are studied the ramification divisors of the functions in $\mathcal{M}(\mathcal{V}')$ which have exponential singularities of finite degree at the points of $\mathcal{V}-\mathcal{V}'$, and one proves, for instance, that if a function in $\mathcal{M}(\mathcal{V}')$ belongs to the subfield generated by the functions of this type, and has a finite ramification divisor, it also has a finite divisor. It is also proved that for a given finite divisor δ in $\mathcal{V}'$, the ramification divisors (with a fixed degree) of the functions of the said type whose divisor is δ, define a proper analytic subset of a certain symmetric power of $\mathcal{V}'$.

1. Introduction and notations

Let $\mathcal{V}$ be a compact Riemann surface and $\mathcal{V}'$ be the complement in $\mathcal{V}$ of a nonvoid finite subset S. Let $\mathcal{M}(\mathcal{V}')$ be the field of meromorphic functions in $\mathcal{V}'$. Then, it is known (see for instance Cutillas [2]) that given divisors δ, δ' in $\mathcal{V}'$ which are compatible (in an obvious sense), there exists $f \in \mathcal{M}(\mathcal{V}')$ with divisor δ and with ramification divisor (i.e. the divisor of df) δ'. Let now $G_0(\mathcal{V}')$ be the multiplicative group formed by the functions in $\mathcal{M}(\mathcal{V}')$ with exponential singularities of finite degree at the points of S in the sense of Cutillas [1] (i.e. those functions in $\mathcal{M}(\mathcal{V}')$ whose logarithmic differentials are meromorphic in $\mathcal{V}$), and let $K_0(\mathcal{V}')$ be the subfield of $\mathcal{M}(\mathcal{V}')$ generated by $G_0(\mathcal{V}')$. Our aim in this paper is to show that for a general compatible pair $\{\delta, \delta'\}$, a function f as above can not belong to $K_0(\mathcal{V}')$. In fact, we shall prove that if a function in $K_0(\mathcal{V}')$ has a finite ramification divisor, then it also has a finite divisor. We shall also draw out some consequences of this, including that if the genus of $\mathcal{V}$ is > 0 there is no locally univalent holomorphic function in $K_0(\mathcal{V}')$. Finally, we shall study the set of pairs of finite divisors $\{\delta, \delta'\}$ corresponding to functions in $G_0(\mathcal{V}')$ and prove that for a given δ, if we identify in the natural way each set of all compatible pairs $\{\delta, \delta'\}$ verifying that the degree of δ' is fixed, with a certain symmetric power $\mathcal{V}'_k$ of $\mathcal{V}'$, then the pairs corresponding to functions in $G_0(\mathcal{V}')$ form a proper analytic subset of $\mathcal{V}'_k$.

$\mathcal{V}'$, $G_0(\mathcal{V}')$ and $K_0(\mathcal{V}')$ will be as above, $A(\mathcal{V}')$ will be the ring of finite sums of functions in $G_0(\mathcal{V}')$, and $G(\mathcal{V}')$ will be the group of meromorphic functions in $\mathcal{V}'$ with finite divisor. Note that $G_0(\mathcal{V}') \subset G(\mathcal{V}')$.

For every connected open subset U of a Riemann surface, $\mathcal{O}(U)$ will be the ring of holomorphic functions in U, $E(U) = \{e^h : h \in \mathcal{O}(U)\}$, $\mathcal{O}^*(U)$ the group of holomorphic functions without zeroes in U, $\mathcal{M}(U)$ will be the field of meromorphic functions in U, and $\mathcal{M}^*(U)$ the multiplicative group of $\mathcal{M}(U)$.

The divisor of f will be denoted by $\operatorname{div}(f)$ for every $f \in \mathcal{M}(\mathcal{V}')$, and the analogous notation will be used for meromorphic differentials in $\mathcal{V}'$. $\mathcal{D}(\mathcal{V}')$ will be the set of finite divisors in $\mathcal{V}'$ and $\mathcal{P}(\mathcal{V}')$ will be the set of pairs $\{\delta, \delta'\}$ of compatible finite divisors in $\mathcal{V}'$ (such a pair can be defined for instance by the condition that there exists $f \in G(\mathcal{V}')$ with divisor δ and with ramification divisor δ').

$A_1, \ldots, A_g, B_1, \ldots, B_g$ (g being the genus of $\mathcal{V}$) will denote, if $g > 0$, closed piecewise analytic curves in $\mathcal{V}$ defining a canonical system of generators of the fundamental group of $\mathcal{V}$ (following the usual terminology explained for instance in Gunning [3]). We shall suppose without loss of generality that no point of S is contained in any of these curves.

The following known theorem will be useful in the sequel.

Theorem 1.1. *If A is a subset of $G(\mathcal{V}')$ which is independent over $\mathcal{M}^*(\mathcal{V})$ (in the sense that no product $f_1^{n_1} \ldots f_r^{n_r}$, with $f_1, \ldots, f_r \in A$ and $n_1, \ldots, n_r \in \mathbb{Z} - \{0\}$, belongs to $\mathcal{M}^*(\mathcal{V})$), then it is also algebraically independent over $\mathcal{M}(\mathcal{V})$.*

Proof: It is a consequence of Theorem 7.9 in Cutillas [1]. ∎

2. Functions in $K_0(\mathcal{V}')$ with finite ramification divisor

Before stating the fundamental theorem of this section, we prove an auxiliary result which is a consequence of Theorem 1.1.

Lemma 2.1. *If $f_1, \ldots, f_r \in A(\mathcal{V}')$, then there exists $\beta_1, \ldots, \beta_s \in G_0(\mathcal{V}')$, algebraically independent over $\mathcal{M}(\mathcal{V})$, such that $f_1, \ldots, f_r$ are rational functions in $\beta_1, \ldots, \beta_s$ with coefficients in $\mathcal{M}(\mathcal{V})$ and with monomials as denominators.*

Proof: Since $G_0(\mathcal{V}')/\mathcal{M}^*(\mathcal{V})$ is a divisible group without torsion, it can be considered as a Q-vector space. Note that a set of functions $\{\beta_i\}_{i \in I}$ is a maximal independent subset of $G_0(\mathcal{V}')$ over $\mathcal{M}^*(\mathcal{V})$ if and only if the set $\{\bar{\beta}_i\}_{i \in I}$, where $\bar{\beta}_i$ is the natural image of β_i in $G_0(\mathcal{V}')/\mathcal{M}^*(\mathcal{V})$ for every $i \in I$, is a basis of this vector space. Let now, for every $n \in \mathbb{N}$ and $i \in I$, $\beta_{i,n} \in G_0(\mathcal{V}')$ and $h_{i,n} \in \mathcal{M}^*(\mathcal{V})$, be such that $\beta_i = \beta_{i,n}^n h_{i,n}$. Then $\{\beta_{i,n}\}_{i \in I}$ is also a maximal independent subset of $G_0(\mathcal{V}')$ over $\mathcal{M}^*(\mathcal{V})$ for every $n \in \mathbb{N}$, and so, if A_n is the subring of $A(\mathcal{V}')$ generated over $\mathcal{M}(\mathcal{V})$ by the functions $\{\beta_{i,n}\}_{i \in I}$ and their inverses (which coincides with the localization of the ring of polinomials in the

$\{\beta_{i,n}\}_{i\in I}$ with coefficients in $\mathcal{M}(\mathcal{V})$ with respect to the multiplicative system formed by the products of powers of the $\{\beta_{i,n}\}_{i\in I}$ with nonnegative exponents), then $A_m \supset A_n$ if $m,n \in \mathbb{N}$ and n divides m, and $A(\mathcal{V}') = \cup_{n=1}^{\infty} A_n$; whereupon it is easy to obtain from Theorem 1.1 the conclusion of the statement. ■

Theorem 2.2. *If $f \in K_0(\mathcal{V}')$ and df has a finite divisor, then $f \in G_0(\mathcal{V}')$.*

Proof: Let $\alpha_1, \ldots, \alpha_q, f_1, \ldots, f_p \in G_0(\mathcal{V}')$ be such that $\bar{f}_1, \ldots, \bar{f}_p$ are different, $\bar{\alpha}_1, \ldots, \bar{\alpha}_q$ are different (where the bar over a function means its natural image in $G(\mathcal{V}')/\mathcal{M}^*(\mathcal{V})$), and such that $f = \frac{f_1+\cdots+f_p}{\alpha_1+\cdots+\alpha_q}$. Let h be a function in $\mathcal{M}^*(\mathcal{V})$. Then

$$\frac{df}{dh} = \frac{(\frac{df_1}{dh} + \cdots + \frac{df_p}{dh})(\alpha_1 + \cdots + \alpha_q) - (f_1 + \cdots + f_p)(\frac{d\alpha_1}{dh} + \cdots + \frac{d\alpha_q}{dh})}{(\alpha_1 + \cdots + \alpha_q)^2}$$

where, of course, $\frac{df}{dh}$ represents the unique meromorphic function F in $\mathcal{V}'$ such that $df = F\, dh$ in $\mathcal{V}'$, and the same is valid for the other functions $f_1, \ldots, f_p, \alpha_1, \ldots, \alpha_q$.

Note that $\frac{df_i}{dh}, \frac{d\alpha_j}{dh} (i = 1, \ldots, p; j = 1, \ldots, q)$ belong also to $G_0(\mathcal{V}')$ and therefore the same must be true for $\frac{df}{dh}$ (reason for instance as in Theorem 7.11 of Cutillas [1]). Observe also that by the above equality, $\alpha_1 + \cdots + \alpha_q$ divides the product $(f_1 + \cdots + f_p)(\frac{d\alpha_1}{dh} + \cdots + \frac{d\alpha_q}{dh})$ in $A(\mathcal{V}')$. Let now $\beta_1, \ldots, \beta_s \in G_0(\mathcal{V}')$ be algebraically independent over $\mathcal{M}(\mathcal{V})$ and such that the functions $f_1 + \cdots + f_p, \alpha_1 + \cdots + \alpha_q$ and $\frac{d\alpha_1}{dh} + \cdots + \frac{d\alpha_q}{dh}$ can be expressed respectively as rational functions $\frac{P(\beta_1,\ldots,\beta_s)}{M(\beta_1,\ldots,\beta_s)}, \frac{Q(\beta_1,\ldots,\beta_s)}{N(\beta_1,\ldots,\beta_s)}$ and $\frac{Q_1(\beta_1,\ldots,\beta_s)}{N_1(\beta_1,\ldots,\beta_s)}$ in $\beta_1, \ldots, \beta_s$, with coefficients in $\mathcal{M}(\mathcal{V})$, whose denominators are monomials. We can also suppose that $P(\beta_1, \ldots, \beta_s)$, $Q(\beta_1, \ldots, \beta_s)$ have no common irreducible factor in $\mathcal{M}(\mathcal{V})[\beta_1, \ldots, \beta_s]$ (the ring of polynomials in $\beta_1, \ldots, \beta_s$ with coefficients in $\mathcal{M}(\mathcal{V})$). Hence, since, up to multiplication by monomials, $Q(\beta_1, \ldots, \beta_s)$ must divide the product $P(\beta_1, \ldots, \beta_s)Q_1(\beta_1, \ldots, \beta_s)$ in $\mathcal{M}(\mathcal{V})[\beta_1, \ldots, \beta_s]$, we deduce that every factor of $Q(\beta_1, \ldots, \beta_s)$ which is not a monomial divides also $Q_1(\beta_1, \ldots, \beta_s)$. But this implies that $\alpha_1 + \cdots + \alpha_q$ must divide $\frac{d\alpha_1}{dh} + \cdots + \frac{d\alpha_q}{dh}$ in $A(\mathcal{V}')$, whence we conclude that $d \log (\alpha_1 + \cdots + \alpha_q)$ has finitely many poles in $\mathcal{V}'$, that is, $\alpha_1 + \cdots + \alpha_q \in G_0(\mathcal{V}')$; and then by Theorem 7.9 in Cutillas [1], it must be $q = 1$.

Finally, let $\gamma_i = \frac{f_i}{\alpha_1}$ for $i = 1, \ldots, p$; then $\gamma_i \in G_0(\mathcal{V}')$ and since $\frac{d\gamma_1}{dh} + \cdots + \frac{d\gamma_p}{dh}$ would have infinitely many zeroes in $\mathcal{V}'$ if $p > 1$ (again by Theorem 7.9 in Cutillas [1]), we deduce from the hypotheses that p must be 1, that is, $f \in G_0(\mathcal{V}')$. ■

As we have remarked in the Introduction, given compatible divisors δ, δ' in $\mathcal{V}'$, there exists $f \in \mathcal{M}(\mathcal{V}')$ with divisor δ and with ramification divisor δ'. However, if we assume further that $\delta \in \mathcal{D}(\mathcal{V}')$, then it is easy to see that there is at least one more condition that the pair $\{\delta, \delta'\}$ must verify in order that a corresponding f can belong to $K_0(\mathcal{V}')$, as shows the following theorem. Before stating it, it will be useful to introduce some more notation.

The points of S will be denoted by $a_1, \ldots, a_m$. If $f \in G_0(\mathcal{V}')$, let $p_{i,f}$ be for $i = 1, \ldots, m$ a polynomial in one variable with coefficients in C such that $fe^{p_{i,f}(z_i^{-1})}$ is meromorphic in some neighbourhood of a_i for some holomorphic coordinate z_i in a neighbourhood of a_i such that $z_i(a_i) = 0$, and let $r_i(f)$ be either the degree of $p_{i,f}$ plus 1, or 0, depending on whether $d \log f$ has a pole at a_i or not. Note that $r_i(f)$ coincides with the multiplicity, changed of sign, of $d \log f$ at a_i. The element $(r_1(f), \ldots, r_m(f))$ of $(\mathbb{Z}^+)^m$ will be denoted by $r(f)$, and the sum of its components by $\mid r(f) \mid$. Finally, given f as above, Z_f will be its number of zeroes (counting multiplicities) in $\mathcal{V}'$, Z_{df} will be the analogue for df, and N_f^- will be the number of points of $\mathcal{V}'$ at which f has a pole (without counting multiplicities).

Theorem 2.3. *If $f \in G_0(\mathcal{V}')$, then $Z_{df} - Z_f - N_f^- = 2g - 2 + \mid r(f) \mid$.*

Proof: Note that the number of poles of df in $\mathcal{V}'$ minus the analogous number for f (both counting multiplicities) is N_f^-. Since $d \log f$ is a meromorphic differential in $\mathcal{V}$, the degree of its divisor is $2g-2$. From this and from the definition of $r(f)$ follows easily the equality of the statement. ■

Let now δ be a finite divisor in $\mathcal{V}'$. Let Z_δ be the sum of the positive multiplicities of δ at the points of $\mathcal{V}'$, and N_δ^- be the number of points of $\mathcal{V}'$ at which δ has negative multiplicity. Then, a clear consequence of Theorems 2.2 and 2.3 is the following:

Corollary 2.4. *If $\{\delta, \delta'\} \in \mathcal{P}(\mathcal{V}')$, then a function with divisor δ and with ramification divisor δ' can belong to $K_0(\mathcal{V}')$ only if $D_{\delta,\delta'} \equiv Z_{\delta'} - Z_\delta - N_\delta^- - 2g + 2 \geqq 0$. Furthermore, such a function can belong to $G_0(\mathcal{V}') - \mathcal{M}^*(\mathcal{V})$ only if $D_{\delta,\delta'} \geqq 2$.*

Corollary 2.5. *If a nonconstant holomorphic function φ in $\mathcal{V}'$ with finite ramification divisor belongs to $K_0(\mathcal{V}')$, then $d\varphi$ has at least $2g$ zeroes (counting multiplicities) in $\mathcal{V}'$. In particular, if $g > 0$, there is no locally univalent holomorphic function in $K_0(\mathcal{V}')$.*

Proof: If $\varphi \in \mathcal{M}(\mathcal{V})$, it must have at least a pole in $\mathcal{V}$. Hence, $Z_{d\varphi} \geqq Z_\varphi + 2g - 2 + \mid r(\varphi) \mid \geqq 2g$. If $\varphi \in K_0(\mathcal{V}') - \mathcal{M}^*(\mathcal{V})$, the conclusion follows from Theorem 2.2 and Corollary 2.4. ■

3. Pairs in $\mathcal{P}(\mathcal{V}')$ corresponding to functions of $G_0(\mathcal{V}')$

Let $\{\delta, \delta'\}$ be a pair of compatible finite divisors in $\mathcal{V}'$ such that (with the notation explained in Corollary 2.4) $D_{\delta,\delta'} \geqq 0$. Then, from which we have seen in Section 2, the following question arises: is there a nonconstant function in $G_0(\mathcal{V}')$ with divisor δ and with ramification divisor δ'?.

The answer, as we shall see, is in general negative. Before proving this we explain more notation whose use will be convenient.

We shall go on utilizing the notation introduced in Section 2. Moreover, for $k \in \mathbb{N}$, $\mathcal{V}'_k$ will be the $k-th$ symmetric power of $\mathcal{V}'$ endowed with its natural k–dimensional complex analytic manifold structure (see, for instance, Gunning [3]). An arbitrary $(r_1, \ldots, r_m) \in (\mathbb{Z}^+)^m$ will be briefly represented as r, and the sum of its components by $| r |$. If δ is a finite divisor in $\mathcal{V}'$ and $r \in (\mathbb{Z}^+)^m$, N_δ will be $N_\delta^- + N_{-\delta}^-$ (i.e., the number of points of $\mathcal{V}'$ having nonzero coefficient in δ), $N(r)$ will be the number of components of r which are $\geqq 2$, and $N(\delta, r)$ will be N_δ plus the number of components of r which are $\geqq 1$. Observe that if $f \in G_0(\mathcal{V}')$ has divisor δ and $r = r(f)$, then $N(r)$ is the number of points in $\mathcal{V}$ at which $d \log f$ has a pole of order $\geqq 2$, and $N(\delta, r)$ is the number of points in $\mathcal{V}$ at which $d \log f$ has a pole.

Lemma 3.1. *If for some $\delta \in \mathcal{D}(\mathcal{V}')$, some $r \in (\mathbb{Z}^+)^m$ with every component $\leqq 1$, and every $\delta' \in \mathcal{D}(\mathcal{V}')$ such that $\{\delta, \delta'\} \in \mathcal{P}(\mathcal{V}')$ and such that $D_{\delta,\delta'} = | r |$, there exists $f \in G_0(\mathcal{V}')$ with divisor δ and with ramification divisor δ', then $g = 0$ and $N(\delta, r) = N_\delta + | r | = 2$.*

Proof: If $\delta = \Sigma_{i=1}^s n_i b_i$, with all b_i different in $\mathcal{V}'$ and $n_i \in \mathbb{Z} - \{0\}$, then every δ' as in the statement must be of the form $\Sigma_{i=1}^s (n_i - 1) b_i + \delta^+$ for some nonnegative $\delta^+ \in \mathcal{D}(\mathcal{V}')$ of degree $2g - 2 + D_{\delta,\delta'} + N_\delta \geqq 2g$. If g were > 0, δ^+ could be replaced by another nonnegative $\delta_1^+ \in \mathcal{D}(\mathcal{V}')$, different from δ^+, in order to obtain $\delta'' = \delta_1^+ + \Sigma_{i=1}^s (n_i - 1) b_i$ such that $\{\delta, \delta''\} \in \mathcal{P}(\mathcal{V}')$ and such that $D_{\delta,\delta''} = | r |$. But this is impossible since the hypotheses imply that there exists $h \in \mathcal{M}^*(\mathcal{V})$ such that δ and δ' are respectively the divisor and the ramification divisor of h in $\mathcal{V}'$, and therefore there can be no such a function for δ and δ''. Hence, it must be $g = 0$. Finally, taking into account that a similar argument also proves that $-2 + D_{\delta,\delta'} + N_\delta$ can not be > 0, we deduce that it must also be $D_{\delta,\delta'} + N_\delta = 2$, whereupon up to easy details the lemma is proved. ■

Lemma 3.2. *Let $F \in \mathcal{M}(\mathcal{V}')$ and $H_1, \ldots, H_n \in \mathcal{O}(\mathcal{V}')$ be such that for every $\lambda = (\lambda_1, \ldots, \lambda_n)$ in an open subset U of $\mathbb{C}^n$, the function $F_\lambda = F + \Sigma_{j=1}^n \lambda_j H_j$ has exactly k zeroes in $\mathcal{V}'$ (counting multiplicities), whereas its number of zeroes in $\mathcal{V}'$ is less than k if $\lambda \in \mathbb{C}^n - U$. Then, the mapping τ from U into $\mathcal{V}'_k$ which assigns to each $\lambda \in U$ the divisor of zeroes (in $\mathcal{V}'$) of F_λ, is holomorphic and proper.*

Proof: The continuity of τ can be proved as in Theorem 4A of Appendix V in Whitney [4], and will be used in order to show that it is holomorphic in a neighbourhood of each $\mu = (\mu_1, \dots, \mu_n) \in U$. Let $\Sigma_{j=1}^s n_j p_j$, with $n_j \in \mathbb{N}$ and the p_j different, be the divisor of zeroes of F_μ. Let V be a neighbourhood of $p_1, \dots, p_s$ in $\mathcal{V}'$ with a holomorphic coordinate z (we can suppose for instance that none of the points p_j is in any of the curves $A_1, \dots, A_g$, $B_1, \dots, B_g$, if $g > 0$, and take as V the intersection of $\mathcal{V}'$ with the complement in $\mathcal{V}$ of the union of these curves). Let now z_j be, for $j = 1, \dots, k$, the function in the power V^k defined for each $q = (q_1, \dots, q_k) \in V^k$ by $z_j(q) = z(q_j)$. Then the sums $\sigma_1 = z_1 + \dots + z_k, \dots, \sigma_k = z_1^k + \dots + z_k^k$ form a holomorphic system of coordinates in the $k-th$ symmetric power $V^{(k)}$ considered as an open subset of $\mathcal{V}'_k$ (see for instance Gunning [**3**]). Let $C_1, \dots, C_s$ be the positively oriented boundaries of *disks* (with respect to the coordinate z) $D_1, \dots, D_s$, centered at $p_1, \dots, p_s$ respectively and such that their closures are pairwise disjoint and contained in V, and let $\varepsilon > 0$ be such that $\mid \lambda_j - \mu_j \mid < \varepsilon$, for $j = 1, \dots, n$, implies that all zeroes in $\mathcal{V}'$ of F_λ are in $\cup_{j=1}^s D_j$. Then, if δ_λ is the divisor of zeroes of F_λ, the coordinates in $V^{(k)}$ of δ_λ are

$$\sigma_1(\delta_\lambda) = \frac{1}{2\pi i}\Sigma_{j=1}^s \int_{C_j} z d \log F_\lambda, \dots, \sigma_k(\delta_\lambda) = \frac{1}{2\pi i}\Sigma_{j=1}^s \int_{C_j} z^k d \log F_\lambda;$$

and, since one can derivate these integrals with respect to $\lambda_1, \dots, \lambda_n$, we obtain the desired conclusion.

Let us now prove that τ is proper. It will suffice to see that if $(\lambda^{(m)})$ is a sequence in U which converges to a point λ_0 in the boundary of U and if K is a compact bordered submanifold of $\mathcal{V}'$ with analytic boundary, then the interior $\mathring{K}$ of K can not contain all the zeroes of all the functions $F_{\lambda^{(m)}}$. If this were not so, we could suppose without loss of generality that the boundary ∂K of K contains no pole of F and no zero of F_{λ_0}, whereupon if k' is the number of poles of F in $\mathring{K}$ (counting multiplicities) we would have:

$$\frac{1}{2\pi i}\int_{\partial K} d \log F_{\lambda^{(m)}} = k - k', \text{ for every } m \in \mathbb{N},$$

whereas by the hypotheses, $\frac{1}{2\pi i}\int_{\partial K} d \log F_{\lambda_0} < k - k'$. But, on the other hand, since $(\lambda^{(m)})$ converges to λ_0, $(\int_{\partial K} d \log F_{\lambda^{(m)}})$ must converge to $\int_{\partial K} d \log F_{\lambda_0}$, and so we obtain a contradiction which proves that τ is proper. ■

Lemma 3.3. *Given $\delta \in \mathcal{D}(\mathcal{V}')$ and $r \in (\mathbb{Z}^+)^m$, then except for the case $g = 0$ and $N(\delta, r) \leqq 2$, the divisors of zeroes of the logarithmic differentials of the functions $f \in G_0(\mathcal{V}')$ such that $\delta =$ div (f) and such that $r = r(f)$ form a proper analytic subset of $\mathcal{V}'_k$, with $k = 2g - 2 + \mid r \mid + N_\delta > 0$.*

Proof: Let f be as in the statement (if such an f does not exist, the thesis of the lemma is of course true) and let $H(f)$ be the subset of $\mathcal{O}(\mathcal{V}') \cap \mathcal{M}(\mathcal{V})$ formed by the functions h such that $r(e^h f) = r$. By excluding the case $N(r) = 0$

studied in Lemma 3.1, we can suppose that $N(r) > 0$. Let us suppose for simplicity and without loss of generality, that $r_1 \geqq 2, \ldots, r_{N(r)} \geqq 2$, and note that $H(f)$ is contained in the finite dimensional $\mathbb{C}$–vector space E formed by the functions $h \in \mathcal{M}(\mathcal{V})$ such that $\operatorname{div}(h) + \Sigma_{j=1}^{N(r)}(r_j - 1)a_j$ is a nonnegative divisor in $\mathcal{V}$. Moreover, it is not difficult to see that $H(f)$ is in open subset of E which by choosing a basis in E, containing the function 1, can be identified with an open subset $U(f)$ of $\mathbb{C}^l$ where l is, by the Riemann–Roch theorem, $| r | - N(\delta, r) + N_\delta + 1$, if $g = 0$, and $\leqq | r | - N(\delta, r) + N_\delta$ if $g \geqq 1$. Let now k be as in the lemma, and observe that except for the case indicated in the statement, it is $l - 1 < k$. Consider the mapping τ from $U(f)$ into $\mathcal{V}'_k$ defined by assigning to every $\lambda \in U(f)$, with corresponding element $h \in H(f)$ by the identification of above, the divisor of zeroes (in $\mathcal{V}'$) of $d \log (e^h f)$. Note that by Lemma 3.2 this mapping is holomorphic, and that if E' is a supplementary vector subspace of $\mathbb{C}$ in E, if $H'(f)$ is $H(f) \cap E'$, and if $U'(f)$ corresponds to $H'(f)$ via the identification of $H(f)$ with $U(f)$, then $\tau(U'(f)) = \tau(U(f))$. Hence, τ can also be considered as a holomorphic map from an open subset of $\mathbb{C}^{l-1}$ into $\mathcal{V}'_k$, from which by using again Lemma 3.2 and the proper mapping theorem (see for instance Whitney [4]), one deduces that the image of $\mathcal{V}$ is an analytic subset of $\mathcal{V}'_k$ of dimension $\leqq l - 1$.

Consider now an element α of $2\pi \mathbb{Z}^{2g+m}$ whose components coincide respectively with the integrals of $d \log f$ over the curves $A_1, \ldots, A_g, B_1, \ldots, B_g$ and the residues of $d \log f$ at $a_1, \ldots, a_m$ for some f as above (of course, it can be supposed without loss of generality that no point with nonzero coefficient in δ is in any of these curves), and let us call H_α the set $\{e^h f : h \in H(f)\}$ for such an f. Since the union of all the sets H_α which can be obtained in this way is precisely the subset of $G_0(\mathcal{V}')$ formed by the functions f such that $\delta = \operatorname{div}(f)$ and $r = r(f)$, we deduce that the subset of $\mathcal{V}'_k$ formed by the divisors of zeroes of the logarithmic differentials of these functions coincide with the union of a countable family $\{\mathcal{W}_\alpha\}$ of proper analytic subsets of $\mathcal{V}'_k$, being each $\mathcal{W}_\alpha$ the image of a mapping corresponding to α analogous to the τ of above. Hence, in order to conclude the proof, it will suffice to show that the family $\{\mathcal{W}_\alpha\}$ is locally finite. To see this, let $\mathcal{F}$ be the subgroup of $\mathcal{O}^*(\mathcal{V}')$ formed by the functions φ such that $r(\varphi f) \leqq r$ for some fixed f as above, and let $\bar{\mathcal{F}}$ be the quotient group of $\mathcal{F}$ by the subgroup $\mathcal{F}'$ formed by the exponentials of the functions in E. Then, since $\bar{\mathcal{F}}$ can be considered as a subgroup of $\mathcal{O}^*(\mathcal{V}')/E(\mathcal{V}')$ (which is isomorphic to $\mathbb{Z}^{2g+m}$), it is finitely generated. Let $\varphi_1, \ldots, \varphi_s \in \mathcal{F}$ be such that their natural images in $\bar{\mathcal{F}}$ generate $\bar{\mathcal{F}}$. Then, every $\varphi \in \mathcal{F}$ such that $r = r(\varphi f)$ can be expressed as a product $\exp(\Sigma_{j=1}^{l} t_j h_j)\Pi_{j=1}^{s}\varphi_j^{n_j}$, where $\{h_1, \ldots, h_l\}$ is a basis of E, $t_j \in \mathbb{C}$ and $n_j \in \mathbb{Z}$. Let U_1 be the open subset of $\mathbb{C}^{l+s}$ formed by the $t = (t_1, \ldots t_{l+s})$ such that $\omega_t = d \log f + \Sigma_{j=1}^{l} t_j h_j + \Sigma_{j=1}^{s} t_{l+j} d \log \varphi_j$ has multiplicities at the points of S which coincide with those of $d \log f$, and note that if $\varphi \in \mathcal{F}$ and $r = r(\varphi f)$, and if $\varphi = \exp(\Sigma_{j=1}^{l} t_j h_j)\Pi_{j=1}^{s}\varphi_j^{n_j}$, with $n_j \in \mathbb{Z}$ and $t_j \in \mathbb{C}$, then $(t_1, \ldots, t_l, n_1, \ldots, n_s) \in U_1$. Consider now the mapping

$\tau_1 : U_1 \to \mathcal{V}'_k$ which assigns to each $t \in U_1$ the divisor of zeroes in $\mathcal{V}'$ of ω_t, and note also that the above considered family $\{\mathcal{W}_\alpha\}$ coincide with the family of images by τ_1 of all sets of the type $U_{n_1,\ldots,n_s} = \{t \in U_1 : t_{l+j} = n_j, j = 1, \ldots, s\}$, with $n_1, \ldots, n_s$ fixed in $\mathbf{Z}$ and such that $(0, \ldots, 0, n_1, \ldots, n_s) \in U_1$. Let finally P be a point in $\mathcal{V}'_k$ and let V be an open neighbourhood of P with compact closure in $\mathcal{V}'_k$. Then, since one can prove as in Lemma 3.2 that τ_1 is proper, $\tau_1^{-1}(\bar{V})$ is also compact and so it can not contain points of an infinity of different sets of the type $U_{n_1,\ldots,n_s}$, whence we deduce that $\bar{V}$ can not meet an infinity of different sets of the family $\{\mathcal{W}_\alpha\}$ and hence this family of subsets of $\mathcal{V}'_k$ is locally finite. ∎

Remark. Given $\delta \in \mathcal{D}(\mathcal{V}')$, there is an evident biunivocal correspondence between the set of divisores of zeroes of the logarithmic differentials of the functions $f \in G_0(\mathcal{V}')$ such that $\operatorname{div}(f) = \delta$, and the set of $\delta' \in \mathcal{D}(\mathcal{V}')$ such that $\delta = \operatorname{div}(f)$ and $\delta' = \operatorname{div}(df)$ for some $f \in G_0(\mathcal{V}')$. Hence, Lemma 3.3 provide us information about divisors of δ' type.

The proceding lemmas have covered all possible cases which may occur when we consider the problem of finding a function with a given finite divisor and with a certain behaviour at the points of S. It is now easy to deduce from them the following:

Theorem 3.4. *Given $\delta \in \mathcal{D}(\mathcal{V}')$ and $n \in \mathbf{Z}^+$, the following two possibilities occur*

1) If $g = 0$ and $n + N_\delta \leqq 2$ or if $g = 0$, $n \geqq 2$ and $N_\delta \leqq 1$, then for every $\delta' \in \mathcal{D}(\mathcal{V}')$ such that $\{\delta, \delta'\} \in \mathcal{P}(\mathcal{V}')$ and such that $D_{\delta,\delta'} = n$, there exists $f \in G_0(\mathcal{V}')$ with divisor δ and with ramification divisor δ' (and hence such that $| r(f) | = n$).

2) In any other case, the divisors of zeroes of the logarithmic differentials of the functions $f \in G_0(\mathcal{V}')$ such that $\operatorname{div}(f) = \delta$ and such that $| r(f) | = n$ form a proper analytic subset of $\mathcal{V}'_k$, with $k = n + 2g - 2 + N_\delta > 0$.

Proof: If $g = 0$ and $n + N_\delta \leqq 2$ or if $g = 0$, $n \geqq 2$ and $N_\delta \leqq 1$, it is a nondifficult exercise to obtain the conclusion of case 1). In any other case, since the set of $r \in (\mathbf{Z}^+)^m$ such that $| r | = n$ is finite and since for any of them the hypotheses of Lemma 3.3 are valid, one obtains easily from this lemma the desired conclusion. ∎

Final remarks.

1) Note that given $\{\delta, \delta'\} \in \mathcal{P}(\mathcal{V}')$, then we can consider a function in $G_0(\mathcal{V}')$ with divisor δ, and multiply it by the exponential of a suitable function in $\mathcal{O}(\mathcal{V}') \cap \mathcal{M}(\mathcal{V})$, in order to obtain a function $f \in G_0(\mathcal{V}')$ such that $\operatorname{div}(f) = \delta$ and such that $\operatorname{div}(df) \geqq \delta'$.

2) Theorem 3.4 shows that for every finite divisor δ in $\mathcal{V}'$ with $N_\delta \geqq 2$ (or in case that $g \geqq 1$), the question cited at the beginning of this section has in general a negative answer, and even that *almost* every divisor $\delta' \in \mathcal{D}(\mathcal{V}')$ such

that $\{\delta, \delta'\} \in \mathcal{P}(\mathcal{V}')$ verifies that no function exists in $G_0(\mathcal{V}')$ with divisor δ and with ramification divisor δ'.

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Rebut el 4 de novembre de 1988