ERGODIC RESULTS FOR CERTAIN CONTRACTIONS
ON ORLICZ SPACES WITH FIXED POINTS

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Abstract

Let \((X, \mathcal{M}, \mu)\) be a \(\sigma\)-finite measure space, \(L_\phi \equiv L_\phi(X, \mathcal{M}, \mu)\) an Orlicz space associated to an \(N\)-function \(\phi\) and let \(T: L_\phi \rightarrow L_\phi\) be a linear operator with a fixed point \(h \neq 0\) a.e., such that

\[\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu \quad (f \in L_\phi)\]

and it is either a \(\| \|_1\)-contraction in \(L_\phi \cap L_2\) or a \(\| \|_\infty\)-contraction in \(L_\phi \cap L_\infty\). The main result of this paper is that for a wide class of \(N\)-functions \(\phi\), the ergodic maximal operator associated to \(T\) is bounded in \(L_\phi\). Moreover, for every \(f \in L_\phi\) we have the almost everywhere convergence and the norm convergence of certain weighted averages which include the Césaro averages.

1. Introduction and preliminaries

Let \((X, \mathcal{M}, \mu)\) be a \(\sigma\)-finite measure space and \(L_\phi \equiv L_\phi(X, \mathcal{M}, \mu)\) and Orlicz space associated to an \(N\)-function \(\phi\) (\(L_\phi\) may be a complex Banach space). In this paper we will consider linear operators \(T\) such that

i) \(\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu, \; f \in L_\phi\)

ii) \(T\) has a fixed point \(h, \; h \neq 0\) a.e.

iii) \(T\) is either a \(\| \|_1\)-contraction in \(L_\phi \cap L_1\) or a \(\| \|_\infty\)-contraction in \(L_\phi \cap L_\infty\).

The main aim of this paper is to prove that, for a wide class of \(N\)-functions \(\phi\), the ergodic maximal operator \(M_T\) defined by

\[
M_Tf = \sup_{n \geq 1} \left| \frac{1}{\tau_n} \sum_{k=0}^{n-1} T^k f \right|
\]

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is bounded in $L_\phi$ (dominated ergodic theorem). Moreover, we shall prove that if $\{b_k\}$ is a bounded Besicovitch sequence, then for every $f \in L_\phi$ there exists $f^* \in L_\phi$ such that

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} b_k T^k f(x) = f^*(x) \quad \text{a.e.,} \quad \lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} b_k T^k f - f^* \right\|_\phi = 0.
$$

A sequence of complex numbers $\{b_k\}$ is called a Besicovitch sequence if for every $\varepsilon > 0$ there exists a trigonometric polynomial $\alpha_\varepsilon$ such that

$$
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |b_k - \alpha_\varepsilon(k)| < \varepsilon.
$$

As a special case we obtain the almost everywhere convergence (individual ergodic theorem) and the norm convergence (mean ergodic theorem) of the Cesàro-averages $n^{-1}(f + T f + \cdots + T^{n-1} f)$.

In the real $L_p$-case, with $1 < p < \infty$, and $(X, \mathcal{M}, \mu)$ a finite measure space the corresponding dominated ergodic theorem is proved by A. de la Torre in [10]. R. Sato proved in [9] that the de la Torre's result may be extended to the case $(X, \mathcal{M}, \mu)$ $\sigma$-finite and a complex $L_p$-space. The ergodic result for an operator which only satisfies conditions i) and iii) is an open problem even in the $L_p$-case, $1 < p < \infty$.

The bounded Besicovitch sequences as weights in the averages were used by J.H. Olsen in [8].

In order to obtain the dominated ergodic theorem we first need some extrapolation theorems which extend the ones given by M.A. Akcoglu and R.V. Chacon in [1] and R. Sato in [9], for $L_p$, $1 < p < \infty$.

Now, we shall present the basic definitions and results concerning to $N$-functions and Orlicz spaces which will be used in this paper. The proofs of most of these results can be found in [5] or in II-13 of [7].

An $N$-function is a continuous and convex function $\phi : [0, \infty) \to \mathbb{R}$ such that $\phi(s) > 0$, $s > 0$, $s^{-1} \phi(s) \to 0$ as $s \to 0$ and $s^{-1} \phi(s) \to \infty$ as $s \to \infty$.

The function $\phi$ is an $N$-function if and only if it has the representation $\phi(s) = \int_0^s \varphi$ where $\varphi : [0, \infty) \to \mathbb{R}$ is continuous from the right, non decreasing such that $\varphi(s) > 0$, $s > 0$, $\varphi(0) = 0$ and $\varphi(s) \to \infty$ as $s \to \infty$. More precisely $\varphi$ is the right derivative of $\phi$ and will be called the density function of $\phi$.

Associated to $\varphi$ we have the function $\rho : [0, \infty) \to \mathbb{R}$ defined by $\rho(t) = \sup\{s : \varphi(s) \leq t\}$ which has the same aforementioned properties of $\varphi$. We will call $\rho$ the generalized inverse of $\varphi$.

The $N$-function $\psi$ defined by $\psi(t) = \int_0^t \rho$ is called the complementary $N$-function of $\phi$. Thus, if $\phi(s) = p^{-1} s^p$, $p > 1$, then $\psi(t) = q^{-1} t^q$ where $pq = p + q$. 
Young's inequality asserts that \( st \leq \phi(s) + \psi(t) \) for \( s, t \geq 0 \), equality holding if and only if \( \phi(s-) \leq t \leq \phi(s) \) or else \( \psi(t-) \leq s \leq \psi(t) \) (See [3]).

If \( \phi_1 \) and \( \phi_2 \) are \( N \)-functions with complementary \( N \)-functions given by \( \psi_1 \) and \( \psi_2 \) respectively, then, the inequality for complementary functions asserts that if \( \phi_1(s) \leq \phi_2(s) \) for \( s \geq s_0 \), then \( \psi_2(t) \leq \psi_1(t) \) for \( t \geq \phi_2(s_0) \), where \( \phi_2 \) is the density function of \( \phi_2 \).

An \( N \)-function \( \phi \) is said to satisfy the \( \Delta_2 \)-condition in \([s_0, \infty)\), \( s_0 \geq 0 \), if there exists a constant \( \alpha \) such that \( \phi(2s) \leq \alpha \phi(s) \) for every \( s \geq s_0 \).

If \( \phi \) is the density function of \( \phi \), then \( \phi \) satisfies \( \Delta_2 \) in \([s_0, \infty)\) if and only if there exists a constant \( \alpha > 1 \) such that \( s \phi(s) \leq \alpha \phi(s) \), \( s \geq s_0 \).

The \( \Delta_2 \)-condition for \( \phi \) does not transfer necessarily to the complementary \( N \)-function.

If \((X, \mathcal{M}, \mu)\) is a \( \sigma \)-finite measure space we denote by \( M = M(X, \mathcal{M}, \mu) \) the space of \( \mathcal{M} \)-measurable and \( \mu \)-a.e. finite functions from \( X \) to \( \mathbb{R} \) or to \( \mathbb{C} \). If \( \phi \) is an \( N \)-function we consider the Orlicz spaces \( L_\phi \equiv L_\phi(X, \mathcal{M}, \mu) \) and \( L_\phi^* \equiv L_\phi^*(X, \mathcal{M}, \mu) \) defined by \( L_\phi = \{ f \in M : \int_X \phi(|f|)d\mu < \infty \} \) and \( L_\phi^* = \{ f \in M : f g \in L_1 \text{ for all } g \in L_\psi \} \) where \( \psi \) is the complementary \( N \)-function of \( \phi \). We have \( L_\phi \subset L_\phi^* \) and if \( \phi \) satisfies \( \Delta_2 \) then \( L_\phi = L_\phi^* \).

We have that \( L_\phi^* \) is a linear space with the usual operations on which we may define the norms \( \| f \|_\phi = \sup \{ \int_X |f g|d\mu : g \in S_\psi \} \), where \( S_\psi = \{ g \in L_\psi : \int_X \psi(|g|)d\mu \leq 1 \} \), and \( \| f \|_\phi = \inf \{ \lambda > 0 : \int_X \phi(\lambda^{-1}|f|)d\mu \leq 1 \} \) which are called Orlicz norm and Luxemburg norm respectively. Both norms are equivalent.

Holder's inequality asserts that for every \( f \in L_\phi^* \) and every \( g \in L_\psi \), we have \( \| fg \|_1 \leq \| f \|_\phi \| g \|_\psi \) where \( \phi \) and \( \psi \) are complementary \( N \)-functions.

If \( \phi(s) = s^p \) with \( p > 1 \) then \( L_\phi^* = L_{\phi^p} \), \( \| f \|_\phi = \| f \|_{\phi^p} \) and \( \| g \|_\psi = \| g \|_\psi \) where \( pq = p + q \).

The convergence \( f_n \to f \) in \( [L_\phi^*] \| \cdot \|_\phi \) implies the mean convergence \( \lim_{n \to \infty} \int_X (|f_n - f|)d\mu = 0 \) but, in general, mean convergence only implies norm convergence when \( \phi \) satisfies \( \Delta_2 \). Then the set \( \mathcal{S} \) of simple functions (with support of finite measure) is dense in \( [L_\phi, \| \cdot \|_\phi] \) if \( \phi \) satisfies \( \Delta_2 \).

If \( \phi \) verifies \( \Delta_2 \), then for every continuous linear functional \( F \) over \( [L_\phi, \| \cdot \|_\phi] \) there exists an unique function \( g \in L_\psi^* \) such that \( F(f) = \int_X f g d\mu, f \in L_\phi \), and moreover \( \| F \|_\phi = \| g \|_\psi \), where \( \psi \) is the complementary \( N \)-function of \( \phi \), but if \( \phi \) does not satisfy \( \Delta_2 \) then there exist linear functionals on \( L_\phi^* \) which are not represented by functions of \( L_\psi^* \).

If \( \phi \) and \( \psi \) satisfy \( \Delta_2 \) then \( [L_\phi, \| \cdot \|_\phi] \) is reflexive.

In the following, we shall always assume that \((X, \mathcal{M}, \mu)\) is a \( \sigma \)-finite measure space and \( \phi \), together with its complementary \( N \)-function \( \psi \), satisfy the \( \Delta_2 \)-condition in \([0, \infty)\). The \( \Delta_2 \)-condition for \( \phi \) is a very important condition that plays fundamental roles in many questions and the best known Orlicz spaces are associated to functions which satisfy \( \Delta_2 \). The \( \Delta_2 \)-condition for \( \psi \) may seem
to be a restrictive assumption. Some know Orlicz spaces as, for example, the 
Zygmund Orlicz space $L \log L$ and the $L \log^k L$ spaces, $k > 0$, are associated 
to $N$-functions which satisfy $\Delta_2$ but their complementary $N$-functions do not; 
but the above spaces do not satisfy our dominated ergodic result. In fact the $\Delta_2$-condition 
for the complementary $N$-function is necessary for such result.

Precisely, let $([0,1],\mathcal{B},\lambda)$ be the Lebesgue-space and let $\tau$ an invertible $\lambda$-
measure preserving transformation from $[0,1]$ into itself. In [2] B. Bru and 
H. Heinich characterize the Orlicz spaces, associated to Young's functions, for 
which the ergodic maximal operator associated to the operator $T$, defined by 
$Tf = f \circ \tau^{-1}$, is bounded in $L_{\phi}$ (classical dominated ergodic theorem) (the 
Young's functions in [2] are our $N$-functions). The characterizing condition 
given in [2] is the condition of comoderation on $\phi$.

The function $\phi$ is said to be comoderated if there exist $s_0$, $a$ and $b > 1$ 
such that $\varphi(as) \geq b \varphi(s)$ for $s \geq s_0$, where $\varphi$ is the density function of $\phi$ or, 
equivalently, if there exist $s_0$, $a$ and $b > 1$ such that $\varphi(as) \geq ab \varphi(s)$ for $s \geq s_0$ 
(in [2] a function continuous from the left is taken as density function of $\phi$ 
whereas our density function is right continuous).

The paper [2] does not establish the equivalence between the comoderation 
of $\phi$ and the moderation ($\Delta_2$-condition in some $[t_0,\infty)$) of the complementary 
$N$-function $\psi$ unless $\varphi$ be continuous. However, we observe that the comoderation 
of $\phi$ is equivalent to the moderation of $\psi$. At the same time, we shall 
prove another characterization of the moderation of $\psi$, which is used in this 
paper, and which appear in [2], [5] and in the rest of the literature with more 
restrictive hypothesis. Exactly:

**Proposition 1.2.** Let $\phi$ be an $N$-function and $\psi$ the complementary $N$-
function of $\phi$. The following conditions are equivalent:

a) $\phi$ is comoderated.

b) $\psi$ is moderated.

c) There exist $s_0$ and $\beta > 1$ such that $b \psi(s) \leq \alpha \varphi(s)$ for $s \geq s_0$.

**Proof:** a) $\implies$ b). If $\phi$ is comoderated then $\phi(s) \leq \phi_1(s)$ for $s \geq s_0$ where $\phi_1$ 
is the $N$-function given by $\phi_1(s) = (ab)^{-1} \varphi(s)$. The complementary function 
of $\phi_1$ is given by $\psi_1(t) = (ab)^{-1} \psi(bt)$. Taking into account the inequality for 
complementary $N$-functions we obtain that $\psi(bt) \leq ab \psi(t)$ for $t \geq t_0 = \varphi_1(s_0)$, 
where $b > 1$, which equivalces to condition $\Delta_2$ of $\psi$ for $t \geq t_0$.

b) $\implies$ c). Let $\rho$ be the generalized-inverse of $\varphi$. Since $\psi$ is moderated there 
exist $t_0$ and $\alpha > 1$ such that $t \rho(t) \leq \alpha \psi(t)$ for every $t \geq t_0$. On the other 
hand, it follows from the equality cases in Young’s inequality that $t \rho(t) = 
\phi(\rho(t)) + \psi(t)$ and therefore 
$$\phi(\rho(t)) \leq \alpha^{-1}(\alpha - 1)t \rho(t), \quad t \geq t_0.$$ 
Then, since $\rho(\varphi(s)) \geq s$ and the function $u \mapsto u^{-1} \phi(u)$ increases for $u > 0$ we 
obtain 
$$s^{-1} \phi(s) \leq \phi(\rho(\varphi(s)))/\rho(\varphi(s)) \leq \alpha^{-1}(\alpha - 1)\varphi(s), \quad s \geq \rho(t_0)$$
and thus we obtain c) with \( s_0 = \rho(t_0) \) and \( \beta = \alpha(\alpha - 1)^{-1} > 1 \).

\( c) \Rightarrow a). \) Condition c) implies that there exist \( s_0 \) and \( \beta > 1 \) such that the function \( s \mapsto s^{-\beta} \phi(s) \) increases for \( s \geq s_0 \) (or for \( s > s_0 \) if \( s_0 = 0 \)). Then, if \( \alpha > 1 \) is such that \( \alpha^{\beta-1} \geq 2 \) we have \( \phi(as) \geq \alpha^\beta \phi(s) \geq 2a\phi(s) \) for \( s \geq s_0 \) and thus we obtain the moderation of \( \phi \).

Note. Since \( \varphi(0) = \rho(0) = 0 \), if some of the conditions of Proposition 1.2 is satisfied for every \( s \geq 0 \), then the others two conditions are also valids for every \( s \geq 0 \).

In this way, the moderation of \( \psi \) is necessary for the classical dominated ergodic result and, therefore, for our dominated ergodic result since that the operator \( T \), defined by \( Tf = f o\tau^{-1} \) satisfies conditions i), ii) and iii), whatever the \( N \)-function \( \phi \) may be. On the other hand, the space \(([0,1],B,\lambda)\) is of finite measure and our spaces can be of infinite measure. For this reason we shall assume the \( \Delta_2 \)-condition in \([0,\infty)\), but un the case \( \mu(X) < \infty \) the argument which we shall use can be adapted if only we suppose the \( \Delta_2 \)-condition in some \([s_0,\infty)\).

Our results are valid, for example, for the known \( L^p \text{Log}^k L \) spaces, with \( p > 1 \) and \( k > 0 \) since the \( N \)-functions of the form \( \phi(s) = s^p \text{log}^k(1+s) \) satisfy that \( 1 < p < \phi(s)/s\varphi(s) \leq p + b \) for every \( s > 0 \) and certain constant \( b \).

2. Extrapolation Theorems

We first observe that the convexity theorem for positive operators given by M.A. Akcoglu and R.V. Chacon in [1] can be easily extended to Orlicz spaces, following the same type of arguments, as follows

**Proposition 2.1.** Let \( \phi \) be an \( N \)-function strictly convex in some interval and let \( T \) be a conservative positive contraction in \( L_1 \) such that

\[
\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu, \quad (f \in L_1 \cap L_\phi).
\]

Then, \( \|Tf\|_\infty \leq \|f\|_\infty \) for every \( f \in L_1 \cap L_\infty \).

**Proof:** The operator \( T \) is said to be conservative when \( \mu(D) = 0 \), where \( D \) is the dissipative part of \( X \) with respect to \( T \).

First assume that \( \mu(X) < \infty \). It is enough to prove that \( Tc \leq c \) almost everywhere for some constant \( c \neq 0 \).

We have that \( \varphi \) increases strictly in some interval \( I \), where \( \varphi \) is the density function of \( \phi \). Let \( c \in I \) with \( c \neq 0 \). Then, we get that

\[
\phi(c + s) > \phi(c) + s\varphi(c) \quad (0 \neq s \geq -c).
\]

Since \( T \) is conservative we have \( \int_X Tf d\mu = \int_X f d\mu \) for every \( f \in L_1 \).
Let $Tc(x) = c + g(x)$; then $\int_X g d\mu = 0$ and therefore if $\mu \{x \in X : g(x) > 0\} > 0$ we have

$$\int_X \phi(|Tc|) d\mu > \int_X \phi(c) d\mu,$$

which contradicts (2.2). This proves that $Tc \leq c$.

The general case follows from the preceding by a method similar to the one given in [1] using the following result:

**Lemma 2.4.** Let $\phi$ be an $N$-function and $T$ a positive contraction in $L_1$ satisfying (2.2). Then, for every $A \in \mathcal{M}$ there exists a linear operator $T_A : L_1(A) \rightarrow L_1(A)$ such that

a) $T_A$ is a positive contraction in $L_1(A)$ and

$$\int_X \phi(|T_A f|) d\mu \leq \int_X \phi(|f|) d\mu, \quad (f \in L_1(A) \cap L_{\phi}(A)).$$

b) For every $f \in L_1^+(A)$ and every $n \geq 1$

$$\sum_{k=0}^{n} T^k f(x) \leq \sum_{k=0}^{n} T_A^k f(x) \quad \text{a.e. in } A.$$

The proof of Lemma 2.4 can be obtained easily following the arguments of [1].

**Remarks.**

1. The conservative condition of $T$ cannot be eliminated from the hypothesis of Proposition 2.1 since in $\mathbb{R}$ with Lebesgue-measure if $Tf(x) = \sqrt{2}f(2x)$ then $T$ is a positive contraction in $L_1$, an isometry in $L_2$ but $\|Tf\|_{\infty} = \sqrt{2}\|f\|_{\infty}$.

2. There exist $N$-function which are strictly convex over no interval. An example is the following. We consider the dyadic intervals $I_n = [2^{n-1}, 2^n)$ and $J_n = [2^{-n}, 2^{-n+1})$ where $n$ is a positive integer and let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be defined by $\varphi(0) = 0$, $\varphi(t) = 2^{-n}$ if $t \in J_n$ and $\varphi(t) = 2^{n-1}$ if $t \in I_n$. Then $\phi$ defined by $\phi(s) = \int_0^s \varphi$ is an $N$-function. Since $\phi(2s) = 4\phi(s)$ we have that $\phi$, as well as its complementary $N$-function, satisfy the $\Delta_2$-condition. However $\phi$ is not strictly convex over any interval. Furthermore there is no constant $c \neq 0$ such that (2.3) holds.

However most of $N$-functions are strictly convex in some interval.

In the following results the operators are not necessarily positive but they have a fixed point $h$ with $h \neq 0$ a.e.
Theorem 2.5. Let $\phi$ be an $N$-function, strictly convex in some interval and let $T: L_\phi \to L_\phi$ be a linear operator such that

i) $\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu$, \quad ($f \in L_\phi$).

ii) $\|Tf\|_1 \leq \|f\|_1$, \quad ($f \in L_1 \cap L_\phi$).

iii) There exists $h \in L_\phi$, $h \neq 0$ a.e., such that $Th = h$.

Then, $\|Tf\|_\infty \leq \|f\|_\infty$ for every $f \in L_1 \cap L_\infty$ and consequently for every $f \in L_\phi \cap L_\infty$.

Proof: In this proof we follow the idea given by Sato in [9].

Let $k$ be such that $\phi(s) < s$ for $0 < s < k$. Given $f \in L_1 \cap L_\infty$ let $B = \{x \in X: |f(x)| \geq k\};$ then $\mu(B) < \infty$ and therefore $\int_X \phi(|f|)d\mu \leq \|f\|_1 + \mu(B)\phi(|f|) < \infty$. Consequently $L_1 \cap L_\infty \subset L_\phi$.

Let $\hat{T}: L_1 \to L_1$ be the linear extension of $T: [L_1 \cap L_\phi, \|\cdot\|_1] \to L_1$ and $P$ the linear modulus of $\hat{T}$. (See Theorem 4.1.1 in [6]). We shall prove that $P$ satisfies the hypotheses of Proposition 2.1 and therefore $\|Pf\|_\infty \leq \|f\|_\infty$ for $f \in L_1 \cap L_\infty$; in this way, since $|\hat{T}f| \leq |Pf|, \quad f \in L_1$, and $L_1 \cap L_\infty \subset L_1 \cap L_\phi$ we obtain that $\|Tf\|_\infty \leq \|f\|_\infty$, $f \in L_1 \cap L_\infty$, and consequently for every $f \in L_\phi \cap L_\infty$ since $L_1 \cap L_\infty$ is dense in $L_\phi \cap L_\infty$ with the $L_\infty$-norm.

Now, we show that $P$ satisfies the conditions of Proposition 2.1. The $\Delta_2$-condition implies that $L_1 \cap L_\phi$ is dense in $[L_\phi, \|\cdot\|_\phi]$. On the other hand, it follows from i) that $\|Tf\|_\phi \leq \|f\|_\phi$, $f \in L_\phi$, and consequently given $\epsilon > 0$ there is $f_\epsilon \in L_1 \cap L_\phi$ such that for every $n \geq 1$

$$\|h - \frac{1}{n} \sum_{k=0}^{n-1} T^k f_\epsilon\|_\phi \leq \epsilon/2.$$  \hfill (2.6)

If $T$ is a power bounded linear operator in a reflexive Banach space $V$, that is, the powers $T^k, \quad k \geq 0$, are uniformly bounded in $V$, then the Césaro-averages

$$R_n f = \frac{1}{n} \sum_{k=0}^{n-1} T^k f$$

converge in norm to a $T$-invariant limit for all $f \in V$ (See Theorem 2.1.2 in [6]).

Let $f_\epsilon^*$ be the limit in $[L_\phi, \|\cdot\|_\phi]$ of $R_n f_\epsilon$. It follows from (2.6) that for $0 < \epsilon < 1$ we have $\|h - f_\epsilon^*\|_\phi < \epsilon$ and consequently

$$\int_X \phi(|h - f_\epsilon^*|)d\mu < \epsilon.$$  \hfill (2.7)

On the other hand, $f_\epsilon^*(x) = 0$ for a.e. $x \in D$, where $D$ is the dissipative part of $X$ with respect to $P$, since (Theorem 3.1.6 in [6]) $\sum_{k=0}^{\infty} P^k f(x) < \infty$.
on \( D \) for all \( f \in L_1^+ \). Since \( \phi(|h|) > 0 \) a.e. (2.7) shows that \( \mu(D) = 0 \) and thus \( P \) is conservative.

Now, in order to prove that \( P \) satisfies condition (2.2) we consider the Akcoglu and Brunel's theorem related with the structure of \( \hat{T} \) on the conservative part \( C \) of \( X \) with respect to \( P \) (see Theorem 4.1.10 in [6]). Let \( \mathcal{F} \) be the family of \( P \)-absorbing subsets of \( C \); there exists a set \( \Gamma \in \mathcal{F} \) and a function \( s \in L_\infty(\Gamma) \), with \(|s| = 1 \) on \( \Gamma \), such that \( \hat{T}f = \hat{s}P(sf) \) for any \( f \in L_1(\Gamma) \), where \( \hat{s} \) is the complex conjugate of \( s \), and if \( \Delta = C - \Gamma \) then \((I-T)L_1(\Delta)\) is dense in \( L_1(\Delta) \).

We have that \( \text{supp} \, T(\chi_{\Delta}h) \subset \Gamma \) and \( \text{supp} \, T(\chi_{\Delta}h) \subset \Delta \); therefore \( Tg = g \) where \( g = \chi_{\Delta}h \). Carrying out a similar reasoning to the used for \( h \) we have that for every \( \varepsilon > 0 \) there exist \( f_\varepsilon \in L_1(\Delta) \cap L_\infty(\Delta) \) and \( f_\varepsilon^* \in L_\phi(\Delta) \) such that \( \|g - f_\varepsilon^*\|_\phi < \varepsilon \) and \( \lim_{n \to \infty} \|R_n f_\varepsilon - f_\varepsilon^*\|_\phi = 0 \).

Given \( \eta > 0 \) there is \( u_\eta \in L_1(\Delta) \) such that \( \|u_\eta - Tu_\eta - f_\varepsilon\|_1 < \eta/2 \) and therefore for every \( n \geq 1 \) we have \( \|\eta^{-1}(u_\eta - T^n u_\eta) - R_n f_\varepsilon\|_1 = \|R_n(u_\eta - Tu_\eta - f_\varepsilon)\|_1 < \eta/2 \), which proves that \( \lim_{n \to \infty} \|R_n f_\varepsilon\|_1 = 0 \) and so \( f_\varepsilon^*(x) = 0 \) a.e. This shows that \( \mu(\Delta) = 0 \). Then, we have \( \hat{T}f = \hat{s}P(sf) \) for every \( f \in L_1 \) and therefore it follows from i) that \( \int_X \phi(|Pf|)d\mu = \int_X \phi(|\hat{s}T(\hat{s}f)|)d\mu \leq \int_X \phi(|f|)d\mu \) for every \( f \in L_1 \cap L_\phi \) and this finishes the proof.

Now, our aim is to prove that the roles of \( L_1 \) and \( L_\infty \) in Theorem 2.5 can be interchanged. For this we shall considerer the adjoint operator of \( T \).

Let \( T : L_\psi \to L_\phi \) be a bounded linear operator; more precisely, we suppose that there is a constant \( C \) such that \( \|Tf\|_\phi \leq C\|f\|_\psi \), \( f \in L_\phi \). Then, if \( g \in L_\psi^* \), where \( \psi \) is the complementary \( N \)-function of \( \phi \), the linear functional \( F \) over \([L_\phi, \|[\cdot]\|_\phi]\) defined by \( F(f) = \int_X gTfd\mu \) is continuous since by Holder's inequality we have \( \|F(f)\| \leq C\|g\|_\psi\|f\|_\phi \) and therefore, since \( \phi \) satisfies \( \Delta_2 \), there exists an unique function \( g^* \in L_\psi^* \) such that \( \int_X gTfd\mu = \int_X f g^* d\mu \), \( f \in L_\phi \). Then, we can define the bounded linear operator \( T^* : L_\psi^* \to L_\psi^* \), \( g \to T^*g \), where \( T^*g \) is the function in \( L_\psi^* \) such that

\[
\int_X gTfd\mu = \int_X f T^*gd\mu, \quad f \in L_\phi.
\]

We shall call \( T^* \) the adjoint operator of \( T \). \( T^* \) satisfies \( \|T^*g\|_\psi \leq C\|g\|_\psi \). In our case we have

**Lemma 2.8.** Let \( T : L_\phi \to L_\phi \) be a linear operator such that

\[
\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu \quad (f \in L_\phi).
\]

Then, its adjoint operator \( T^* \) satisfies

\[
\int_X \psi(|T^*g|)d\mu \leq \int_X \psi(|g|)d\mu \quad (g \in L_\psi)
\]

(2.9)
and moreover, if $T$ admits an invariant function $h$ with $h \neq 0$ a.e., then there exists $g \in L_\varphi$ with $g \neq 0$ a.e., such that $T^* g = g$.

Proof. We write $\text{sig } z$ for $z/|z|$ and by $\bar{u}$ we denote the complex conjugate of $u$. For $g \in L_\varphi^+$ we have

$$\int_X f |T^* g| d\mu = |\int_X f(\text{sig } T^* g) T^* g d\mu| \leq \int_X |T(f \text{ sig } T^* g)||g| d\mu \leq \int_X \phi(|f|) d\mu + \int_X \psi(|g|) d\mu.$$  

Let $\varphi$ be the density function of $\varphi$ and $\rho$ the generalized inverse of $\varphi$. Since $\psi$ satisfies $\Delta_2$ there exists $\alpha > 1$ such that $s\rho(s) \leq \alpha \psi(s)$ and therefore $\phi(\rho(s)) = s\rho(s) - \psi(s) \leq (\alpha - 1)\psi(s)$. Therefore, for every $g \in L_\varphi$ the function $\rho(|T^* g|)$ belongs to $L_\varphi^+$ and so (2.9) follows from (2.10) for $f = \rho(|T^* g|)$.

Now, let us assume that $Th = h$ with $h \neq 0$ a.e. If $\varphi$ is not continuous then there exists an at most countable set of positive reals $s_1, s_2, \ldots, s_n$ where $\varphi$ is not continuous; in this situation, since $h \in L_\varphi$, it is easy to see that $\{c > 0 : \mu\{x \in X : |s_i^{-1} h(x)| = c\} > 0\}$ is at most countable and therefore there exists $\lambda > 0$ such that for every $s_i$ we have

$$\mu\{x \in X : |\lambda^{-1} h(x)| = s_i\} = 0.$$  

In the case $\varphi$ continuous (2.11) holds trivially with $\lambda = 1$.

Let $u = \lambda^{-1} h$ and $g = \varphi(|u|) \text{ sig } \bar{u}$. We have that $g \neq 0$ a.e. and $g \in L_\varphi$ since $\varphi$ satisfies $\Delta_2$. It follows from (2.9) that

$$\int_X |T^* g| d\mu = \int_X \varphi(|u|) d\mu + \int_X \psi(|T^* g|) d\mu = \int_X |T^* g| d\mu = \int_X \varphi(|u|) d\mu + \int_X \psi(|T^* g|) d\mu,$$

and therefore

$$\int_X |u||T^* g| d\mu = \int_X (\varphi(|u|) + \psi(|T^* g|)) d\mu.$$  

Then, Young's inequality shows that

$$|uT^* g| = \varphi(|u|) + \psi(|T^* g|) \text{ a.e.}$$  

It follows from (2.11) and (2.13) that $|T^* g| = \varphi(|u|)$ a.e. On the other hand we obtain from (2.12) that $(\text{sig } \bar{u}) \text{ sig } T^* u = 1$ and therefore $T^* g = g$ which finishes the proof of the Lemma.

Theorem 2.5 and Lemma 2.8 imply easy
Theorem 2.14. Let \( \phi \) be an \( N \)-function whose complementary \( N \)-function is strictly convex in some interval and let \( T : L_\phi \rightarrow L_\phi \) be a linear operator such that

1) \( \int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu \), \( (f \in L_\phi) \).

2) \( \|Tf\|_\infty \leq \|f\|_\infty \), \( (f \in L_\infty \cap L_\phi) \).

3) There exists \( h \in L_\phi \), \( h \neq 0 \) a.e., such that \( Th = h \).

Then, \( \|Tf\|_1 \leq \|f\|_1 \) for every \( f \in L_1 \cap L_\phi \).

Proof: Let \( \psi \) be the complementary \( N \)-function of \( \phi \), \( T^* \) the adjoint operator of \( T \) and let \( \{A_n \} \) be an increasing sequence of measurable sets with \( \mu(A_n) < \infty \) and \( X = \bigcup A_n \). Then, for every \( g \in L_1 \cap L_\psi \) we have

\[
\int_X |T^*g|d\mu = \lim_{n \to \infty} \int_X gT(xA_n \, \operatorname{sgn} T^*g)d\mu \leq \|g\|_1.
\]

Consequently, \( \|T^*g\|_\infty \leq \|g\|_\infty \) for every \( g \in L_\psi \cap L_\infty \) and therefore for any \( f \in L_1 \cap L_\phi \) and \( n \geq 1 \) we get \( \|\int_X fT^*(xA_n \, \operatorname{sgn} Tf)d\mu\| \leq \|f\|_1 \) and thus \( \|Tf\|_1 \leq \|f\|_1 \).

3. Ergodic results

Theorem 3.1. (Dominated, individual and mean weighted ergodic theorem).

Let \( \phi \) and \( T \) satisfy the hypotheses of the extrapolation theorem 2.5 or 2.14. Then

a) The ergodic maximal operator \( M_T \)-defined by (1.1) is bounded in \( [L_\phi, \|\|_\phi]\).

b) If \( \{b_k\} \) is a bounded Besicovitch sequence, then for every \( f \in L_\phi \) there exists \( f^* \in L_\phi \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} b_k T^k f(x) = f^*(x) \ \text{a.e.}, \quad \lim_{n \to \infty} \|\frac{1}{n} \sum_{k=0}^{n-1} b_k T^k f - f^*\|_\phi = 0.
\]

Proof: Since \( L_1 \cap L_\infty \subset L_\phi \) it follows from Theorem 2.5 or 2.14 that \( T : L_1 \cap L_\phi \rightarrow L_1 \) admits an unique extension \( \hat{T} : L_1 \rightarrow L_1 \) which is a Dunford-Schwartz operator, that is, \( \|\hat{T}f\|_1 \leq \|f\|_1 \), \( f \in L_1 \), and \( \|\hat{T}f\|_\infty \leq \|f\|_\infty \), \( f \in L_1 \cap L_\infty \). Therefore the linear modulus \( P \) of \( \hat{T} \) is also a Dunfort-Schwartz operator.

Consequently, for every \( f \in L_1 \) and \( \lambda > 0 \) we have (see Theorem 2.3.2 in [4])

\[
\mu\{x \in X : M_P f(x) > \lambda\} \leq \lambda^{-1} \int_X |f|d\mu,
\]

where \( M_P \) is the maximal operator associated to \( P \). Moreover, trivially, \( \|M_P f\|_\infty \leq \|f\|_\infty \) for \( f \in L_1 \cap L_\infty \).
For $f \in L_1 \cap L_\phi$ set $f_\lambda = f \chi_{A(\lambda)}$ and $f^\lambda = f - f_\lambda$ where $A(\lambda) = \{ x \in X : |f(x)| > \lambda/2 \}$. We have $f_\lambda \in L_1$, $f^\lambda \in L_1 \cap L_\infty$ and therefore

\begin{align}
(3.2) \quad \int_X \phi(M_P f) d\mu &= \int_0^\infty \varphi(\lambda) \mu \{ x \in X : M_P f(x) > \lambda \} d\lambda \\
&\leq 2 \int_0^\infty \lambda^{-1} \varphi(\lambda) \left( \int_X |f^\lambda| d\mu \right) d\lambda = 2 \int_X |f(x)| \left( \int_0^{2|f(x)|} \lambda^{-1} \varphi(\lambda) d\lambda \right) d\mu(x),
\end{align}

where $\varphi$ is the density function of $\phi$.

Integrating by parts, we obtain

\begin{align}
(3.3) \quad \int_0^s \lambda^{-1} \varphi(\lambda) d\lambda &= s^{-1} \phi(s) + \int_0^s \lambda^{-2} \varphi(\lambda) d\lambda, \quad (s > 0).
\end{align}

Since the $N$-function complementary of $\phi$ satisfies $\Delta_2$ there exists a constant $\beta > 1$ such that $\beta \phi(s) \leq s \varphi(s)$, $s \geq 0$; then, if $0 < \lambda < 1$ we have that $\lambda^{-2} \varphi(\lambda) \leq \phi(1) \lambda^{\beta - 2}$ and therefore $\int_{[0,\lambda]} \lambda^{-2} \varphi(\lambda) d\lambda < \infty$. Then, (3.3) shows that

$$
\int_0^s \lambda^{-1} \varphi(\lambda) d\lambda \leq \beta (\beta - 1)^{-1} s^{-1} \phi(s), \quad (s > 0).
$$

Hence, it follows from (3.2) that

\begin{align}
(3.4) \quad \int_X \phi(M_P f) d\mu &\leq \alpha \beta (\beta - 1)^{-1} \int_X \phi(|f|) d\mu \quad (f \in L_1 \cap L_\phi),
\end{align}

where $\alpha$ is a constant in the $\Delta_2$-condition for $\phi$.

Since $|T_f| \leq P|f|$ for $f \in L_1$, (3.4) shows that there exists a constant $C_1 > 0$ such that $\int_X \phi(M_T f) d\mu \leq C_1 \int_X \phi(|f|) d\mu$, $f \in L_1 \cap L_\phi$, which proves that $\|M_T f\|_{(\phi)} \leq C \|f\|_{(\phi)}$, $f \in L_1 \cap L_\phi$, where $C = \max(1, C_1)$. Since $L_1 \cap L_\phi$ is a dense linear subspace of $[L_\phi, \| \|_{(\phi)}]$ it follows that $\|M_T f\|_{(\phi)} \leq C \|f\|_{(\phi)}$ for every $f \in L_\phi$, which proves a).

Now, let $\{ b_k \}$ be a bounded Besicovitch sequence; then a) and the Banach principle show that for almost everywhere convergence it is enough to prove that the weighted averages

$$
T_n f = \frac{1}{n} \sum_{k=0}^{n-1} b_k T^k f
$$

converges a.e. for all $f$ in a dense subset of $[L_\phi, \| \|_{(\phi)}]$.

Let $m \in \mathbb{N}$ and $S : L_\phi \rightarrow L_\phi$ defined by $S f = e^{imT} f$. Since $L_\phi$ is reflexive and the powers $S^k$, $k \geq 0$, are uniformly bounded, exactly $\|S^k f\|_{(\phi)} \leq \|f\|_{(\phi)}$ for every $f \in L_\phi$ and $k \geq 0$, then, the Césàro averages $R_n f = n^{-1} (f + S f + \ldots + S^{n-1})$ converge in norm for every $f \in L_\phi$. Therefore $L_\phi$ is the closure of
the direct sum of the set of fixed points of $S$ and the space $(I - S)L_\phi$ (see 2.1 in [6]).

On the other hand, given $\beta > 1$ such that $\beta \phi(s) \leq s \varphi(s), s \geq 0$, the function $s \rightarrow s^{-\beta} \phi(s)$ increases for $s > 0$ and consequently $\phi(st) \leq s^\beta \phi(t)$ for $0 \leq s \leq 1$ and $t \geq 0$. Therefore, if $g \in L_\phi$ we have

$$\int_X \sum_{n=1}^\infty \phi(|n^{-1} S^n g|) d\mu \leq \sum_{n=1}^\infty n^{-\beta} \int_X \phi(|S^n g|) d\mu \leq \int_X \phi(|g|) d\mu \sum_{n=1}^\infty n^{-\beta} < \infty.$$ 

Hence $n^{-1} S^n g(x) \rightarrow 0$ a.e. as $n \rightarrow \infty$ and thus $R_n f \rightarrow 0$ a.e. if $f = g - Sg$.

Since the maximal operator $M_S$ is bounded in $[L_\phi, \| \cdot \|_{(\phi)}]$ we obtain that, for any $f \in L_\phi$, $n^{-1} \sum_{k=0}^{n-1} e^{i m k} T_k f$ converges a.e. and therefore for every trigonometric polynomial $\alpha$ and $f \in L_\phi$ we have that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \alpha(k) T_k f(x)$$

exists and is finite a.e.

Then, for every $f \in L_\phi \cap L_\infty$, $T_n f$ converges a.e. since for every $\varepsilon > 0$ there exists a trigonometric polynomial $\alpha_\varepsilon$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} |b_k - \alpha_\varepsilon(k)| < \varepsilon$$

and consequently

$$\limsup_{n \rightarrow \infty} |T_n f(x) - \frac{1}{n} \sum_{k=0}^{n-1} \alpha_\varepsilon(k) T_k f(x)| < \varepsilon \|f\|_{\infty} \text{ a.e.}$$

In this way, since $L_\phi \cap L_\infty$ is dense in $L_\phi$, we conclude that $T_n f$ converges almost everywhere for every $f \in L_\phi$.

Finally, let $f^*(x) = \lim_{n \rightarrow \infty} T_n f(x)$. It follows from a) that $f^* \in L_\phi$ and $\phi(|T_n f - f^*|)$ is dominated by $\phi(M_T f) \in L_1$; thus, taking into account the Lebesgue's dominated theorem, we get that $\lim_{n \rightarrow \infty} \int_X \phi(|T_n f - f^*|) d\mu = 0$ which proves that $\lim_{n \rightarrow \infty} \|T_n f - f^*\|_{(\phi)} = 0$.

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