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ON THE JACOBIAN CRITERION OF FORMAL SMOOTHNESS

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Abstract

We give a short proof of the jacobian criterion of formal smoothness using the Lichtenbaum–Schlessinger cotangent complex.

The aim of this note is to get a proof of the following jacobian criterion of formal smoothness:

Theorem 1. *Let A be a ring, B a noetherian A-algebra, J an ideal of B, and $C = B/J$. Let us consider over A and B the discret and J-adic topology, respectively. Then, the following statements are equivalent*

1) *The A-algebra B is formally smooth*
2) *For every representation $B \simeq R/I$, where R is a smooth A-algebra, the canonical homomorphism*

$$I/I^2 \otimes_B C \longrightarrow \Omega_{R|A} \otimes_R C$$

is left invertible.

This theorem has been obtained by M. André [1, prop. 16.17] using simplicial methods. We shall get a more elementary proof, based on the Lichtenbaum–Schlessinger (co–)homology theory [3], and a counter–example showing that this result is not true for arbitrary B.

First we recall the definition of the Lichtenbaum–Schlessinger (co–)homology functors. Let A be a ring, B an A–algebra and M a B–module. Choose a polynomial algebra R over A such that $B \simeq R/I$, and a free R–module F such that there exists an exact sequence of R–modules

$$0 \longrightarrow U \longrightarrow F \xrightarrow{j} I \longrightarrow 0.$$

Let U_0 be the image of the homomorphism $\phi : F \otimes_R F \to F$, $\phi(x \otimes y) = j(x)y - j(y)x$ and consider the complex of B–modules

$$L_{B|A} : 0 \longrightarrow U/U_0 \longrightarrow F/IF \longrightarrow \Omega_{R|A} \otimes_R B \longrightarrow 0.$$

Then, $T_i(B|A,M) = H_i(L_{B|A} \otimes_B M)$, $T^i(B|A,M) = H^i(\mathrm{Hom}_B(L_{B|A},M))$, $i = 0,1,2$.

There exist isomorphisms: $T_0(B|A,M) \simeq \Omega_{B|A} \otimes_B M$, $T^0(B|A,M) \simeq \mathrm{Der}_A(B,M) \simeq \mathrm{Hom}_B(\Omega_{B|A},M)$, $T^1(B|A,M) \simeq \mathrm{Exalcom}_A(B,M)$ the set of equivalence classes of infinitesimal extensions of B over A by M, and $T^1(B|A,M) \simeq \mathrm{Hom}_B(I/I^2,M)$ if $B \simeq A/I$.

Proposition 1. *Let A be a ring, B an A-algebra, J an ideal of B and $C = B/J$. Assume that $B \simeq R/I$, where R is a smooth A-algebra. Then, the following conditions are equivalent*

1) *The canonical homomorphism $I/I^2 \otimes_B C \to \Omega_{R|A} \otimes_R C$ is left invertible*
2) *$T_1(B|A,C) = 0$ and $\Omega_{B|A} \otimes_B C$ is a projective C-module*
3) *$T_1(B|A,M) = 0$ for all C-module M.*

Proof: Since R is a smooth A-algebra, we have $T_1(R|A,-) = 0 = T^1(R|A,-)$ [**3**, prop. 3.1.3]. Hence there exist exact sequences [**3**, 2.3.5]

$$0 \longrightarrow T_1(B|A,C) \longrightarrow I/I^2 \otimes_B C \longrightarrow \Omega_{R|A} \otimes_R C \longrightarrow \Omega_{B|A} \otimes_B C \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}_C(\Omega_{B|A} \otimes_B C, M) \longrightarrow \mathrm{Hom}_C(\Omega_{R|A} \otimes_R C, M) \longrightarrow \mathrm{Hom}_C(I/I^2 \otimes_B C, M) \longrightarrow T^1(B|A,M) \longrightarrow 0,$$

where M is a C-module.

The result follows from this sequences having into account that $\Omega_{R|A} \otimes_R C$ is a projective C-module [**3**, prop. 3.1.3].

For every C-module M, the homomorphisms $A \to B \to B_n = B/J^n$ induce an exact sequence

$$T^1(B_n|B,M) \longrightarrow T^1(B_n|A,M) \longrightarrow T^1(B|A,M) \longrightarrow T^2(B_n|B,M)$$

and, therefore, an exact sequence

$$\varinjlim T^1(B_n|B,M) \longrightarrow \varinjlim T^1(B_n|A,M) \longrightarrow T^1(B|A,M) \longrightarrow \varinjlim T^2(B_n|B,M).$$

The formal smoothness of B over A is equivalent, by [**2**, prop. 19.4.4], to the vanishing of $\varinjlim T^1(B_n|A,M)$ for all C-module M. Then, Theorem 1 is a consequence of Proposition 1 and

Proposition 2. *Let A be a noetherian ring, I an ideal of A, $A_n = A/I^n$, and M an A/I- module. Then*

i) $\varinjlim T^1(A_n|A,M) = 0$.
ii) $\varinjlim T^2(A_n|A,M) = 0$.

Part i) *is easy:* $\varinjlim T^1(A_n|A,M) \simeq \varinjlim \operatorname{Hom}_{A/I^n}(I^n/I^{2n},M) \simeq \varinjlim \operatorname{Hom}_{A/I}(I^n/I^{n+1},M)=0$.

To prove part ii) *we need the following lemmas.*

Lemma 1. *Let A be a ring, I an ideal of A, $B=A/I$, and M a B-module. Then, there exists a natural monomorphism of B-modules $T^2(B|A,M)\to \operatorname{Ext}^1_A(I,M)$.*

Proof: Let F be a free A-module such that there exists an exact sequence of A-modules $0\to U\to F\xrightarrow{j} I\to 0$. Let U_0 be the image of the homomorphism $\phi: F\otimes_A F\to F$, $\phi(x\otimes y)=j(x)y-j(y)x$. Then, $T^2(B|A,M)=\operatorname{Coker}(\operatorname{Hom}_A(F,M)\to \operatorname{Hom}_A(U/U_0,M))$. The result follows from the diagram of exact sequences

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
\operatorname{Hom}_A(F,M) & \longrightarrow & \operatorname{Hom}_A(U/U_0,M) & \longrightarrow & T^2(B|A,M) & \longrightarrow & 0\\
\| & & \downarrow & & \downarrow & & \\
\operatorname{Hom}_A(F,M) & \longrightarrow & \operatorname{Hom}_A(U,M) & \longrightarrow & \operatorname{Ext}^1_A(I,M) & \longrightarrow & 0
\end{array}$$

Lemma 2. *Let A be a noetherian ring, I an ideal of A and $n>0$ an integer number. Then, there exists $s\geq n$ such that the canonical homomorphism*

$$\operatorname{Tor}_1^A(I^s,A/I)\longrightarrow \operatorname{Tor}_1^A(I^n,A/I)$$

is trivial.

Proof: We have $\operatorname{Tor}_1^A(I^n,A/I)\simeq \operatorname{Tor}_2^A(A/I^n,A/I)\simeq \operatorname{Tor}_1^A(A/I^n,I)$. Let $0\to U\to F\to I\to 0$ be an exact sequence of A-modules where F is free and of finite type. Then

$$\operatorname{Tor}_1^A(A/I^n,I)\simeq (U\cap I^nF)/I^nU.$$

By Artin-Rees lemma [4, th. 15] there exists $r>0$ such that $I^tF\cap U=I^{t-r}(I^rF\cap U)$ for $t>r$. Taking $s=n+r$ we obtain $I^sF\cap U\subseteq I^nU$ and therefore

$$\operatorname{Tor}_1^A(I^s,A/I)\longrightarrow \operatorname{Tor}_1^A(I^n,A/I)$$

is trivial.

We now prove part ii) of Proposition 2. By lemma 1 there exists a monomorhism

$$\varinjlim T^2(A_n|A,M)\longrightarrow \varinjlim \operatorname{Ext}^1_A(I^n,M).$$

On the other hand, for each n we have an exact sequence

$$0 \to \operatorname{Ext}^1_{A/I}(I^n/I^{n+1}, M) \to \operatorname{Ext}^1_A(I^n, M) \to \operatorname{Hom}_{A/I}(\operatorname{Tor}_1^A(I^n, A/I), M)$$

which is deduced, for instance, from the change-rings spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{A/I}(\operatorname{Tor}_q^A(I^n, A/I), M) \Rightarrow \operatorname{Ext}^{p+q}_A(I^n, M).$$

Since $\varinjlim \operatorname{Ext}^1_{A/I}(I^n/I^{n+1}, M) = 0$ and $\varinjlim \operatorname{Hom}_{A/I}(\operatorname{Tor}_1^A(I^n, A/I), M) = 0$, by lemma 2, we obtain $\varinjlim \operatorname{Ext}^1_A(I^n, M) = 0$. Therefore, $\varinjlim T^2(A_n|A, M) = 0$.

Theorem 1 is not true for arbitrary B. To exhibit a counter-example, we need the following result.

Lemma 3. *Let A be a ring and $I \subseteq T$ two ideals of A such that $T^2 = T$ and $IT \neq I$. Let $B = A/I$, $J = T/I$ and $C = B/J$. Then, the A-algebra B is formally smooth for the J-adic topology, but there exists a C-module M such that $T^1(B|A, M) \neq 0$.*

Proof: We have $J^2 = (T^2 + I)/I = (T + I)/I = T/I = J$. Hence, for each $C = A/T$-module M

$$\varinjlim T^1(B_n|A, M) \simeq T^1(C|A, M) \simeq \operatorname{Hom}_{A/T}(T/T^2, M) = 0,$$

where $B_n = B/J^n$. Therefore, B is formally smooth.

On the other hand $T^1(B|A, M) \simeq \operatorname{Hom}_B(I/I^2, M) \simeq \operatorname{Hom}_{A/T}(I/IT, M)$. Since $I/IT \neq 0$, we obtain $T^1(B|A, I/IT) \neq 0$.

Counter-example. $A = C(R, R)$ the ring of all real-valued continous functions on R, I = principal ideal of A generated by the identity function, and T = maximal ideal of A containing I (see [5], Ch. 2, § 2, Ex. 15).

This counter-example solves negatively a question of A. Brezuleanu [6, Remark 1.3 (i)].

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