AN ELEMENTARY PROOF OF A LIMA’S THEOREM FOR SURFACES

F. J. Turiel

Abstract

An elementary proof of the following theorem is given:

THEOREM. Let \( M \) be a compact connected surface without boundary. Consider a \( C^\infty \) action of \( \mathbb{R}^n \) on \( M \). Then, if the Euler–Poincaré characteristic of \( M \) is not zero there exists a fixed point.

The proof given here adapts for dimension two the ideas used by P. Molino and the author in [2] and [3]. Moreover we show that the theorem remains true if \( \mathbb{R}^n \) is replaced by a connected nilpotent Lie group \( G \).

In the slightly more general case, dealt with by E. L. Lima, of a surface with boundary, it is sufficient gluing together two copies of this surface in order to obtain a surface without boundary.

1. Actions of \( \mathbb{R}^n \)

Let \( V \) be the Lie algebra of \( \mathbb{R}^n \). The action of \( \mathbb{R}^n \) induces a Lie algebra homomorphism \( v \in V \to X_v \in \mathcal{X}(M) \) called infinitesimal action. We recall that the infinitesimal isotropy of a point \( p \) is the set \( I(p) = \{ v \in V | X_v(p) = 0 \} \). As \( V \) is abelian \( I(p) \) depends only on the orbit.

Denote by \( \Sigma_k \) the set of points \( p \) of \( M \) whose orbit is \( k \)-dimensional, i.e. \( \text{codim} I(p) = k \).

Suppose \( \Sigma_0 \) empty. We will gradually arrive to a contradiction.

1) Set \( C_2 = \{ v \in V | X_v(p) = 0 \text{ for some } p \in \Sigma_2 \} \). As there are at most countably many 2-orbits because they are open sets, \( C_2 \) is at most countable union of \( (n-2) \)-planes of \( V \).

2) The map on the grassmanian of \( (n-1) \)-planes \( k : p \in \Sigma_1 \to I(p) \in g_{n-1}(V) \) is differentiable, i.e. it can be locally extended to a differentiable map.

Indeed, consider \( p \in \Sigma_1 \) and \( u \in V \) such that \( X_u(p) \neq 0 \). We can find a coordinate system \( (A, x) \), \( p \in A \), such that \( X_{x_1} = \frac{\partial}{\partial x_1} \) and that the image of \( A \) on \( \mathbb{R}^2 \) is a rectangle.
Let \( \{v_1, \ldots, v_{n-1}\} \) a basis of \( I(p) \). Set \( X_{v_j} = f_j \frac{\partial}{\partial x_1} + g_j \frac{\partial}{\partial x_2} \). We define the map

\[
\tilde{h}: A \longrightarrow g_{n-1}(V)
\]

\[
x \longrightarrow \mathbb{R}\{v_1 - f_1 u, \ldots, v_{n-1} - f_{n-1} u\}
\]

whose differentiability is clear.

Note that \( w \in \tilde{h}(x) \) if and only if \( X_w(x) \) is proportional to \( \frac{\partial}{\partial x_2} \). If \( x \in A \cap \Sigma_1 \) this means that \( X_w(x) = 0 \) because it is also proportional to \( \frac{\partial}{\partial x_1} \). Then \( \tilde{h} \) is a local extension of \( h \).

3) Let \( Fr(\Sigma_1) \) be the boundary on \( M \) of \( \Sigma_1 \). Then \( C_1 = \{v \in V / X_u(p) = 0 \) for some \( p \in Fr(\Sigma_1)\} = \bigcup_{p \in Fr(\Sigma_1)} I(p) \) is of the first category (i.e. it is contained in the union of a countable family of closed nowhere dense subsets of \( M \)).

Since \( Fr(\Sigma_1) \) can be covered by a finite family of coordinate systems \( (A, x) \) as in 2), it will be sufficient to prove that \( \bigcup_{p \in A \cap Fr(\Sigma_1)} I(p) \) is of the first category. Let \( T \) be a slice of \( A \) obtained by doing \( x_1 \) constant. As the isotropy is constant on the orbits:

\[
\bigcup_{p \in A \cap Fr(\Sigma_1)} I(p) = \bigcup_{p \in T \cap Fr(\Sigma_1)} I(p)
\]

Consider the vector bundle \( \pi : E \rightarrow T \), subbundle of \( T \times V \), given by the condition \( \pi^{-1}(x) = \{x\} \times \tilde{h}(x) \). Set \( \varphi : (x, v) \in E \rightarrow v \in V \).

The set \( \pi^{-1}(T \cap Fr(\Sigma_1)) \) is of the first category in \( E \) because \( T \cap Fr(\Sigma_1) \) is of the first category in \( T \). As \( \varphi \) is differentiable and \( E \) and \( V \) are manifolds of the same dimension, it follows that

\[
\varphi(\pi^{-1}(T \cap Fr(\Sigma_1))) = \bigcup_{p \in T \cap Fr(\Sigma_1)} I(p)
\]

is of the first category in \( V \).

4) Take now \( v \in (V - C_1 \cup C_2) \). The set \( Z(X_v) \) of the zeros of \( X_v \) is contained in \( \Sigma_1 \). On the other hand the 1–foliations given by:

(a) \( X_v \) on \( M - Z(X_v) \)

(b) the action of \( \mathbb{R}^n \) on \( \Sigma_1 \)

agree on \( (M - Z(X_v)) \cap \Sigma_1 \). Then \( M \) admits an 1–foliation and \( \mathcal{X}(M) = 0 \), contradiction.

2. Case of a connected nilpotent Lie group \( G \)

It will be sufficient to adapt the proof of the abelian case. Let \( V \) be the Lie algebra of \( G \). Since \( V \) is nilpotent every subalgebra of codimension one is an
ideal. Therefore the isotropy is constant over each 1-orbit and $C_1$ will still be of the first category.

Let $B$ be a 2-orbit. Given $p \in B$ there always exists an ideal $I$ of codimension one which contains $I(p)$. As $B$ is an orbit and $I$ an ideal then $I(q) \subset I$ for all $q \in B$. Consequently $C_2$ is contained in a finite or countable union of $(n-1)$-planes of $V$. In particular $C_1 \cup C_2 \neq V$. The rest is similar.

Example 1. See $P(2, \mathbb{R})$ as the plane $\mathbb{R}^2$ plus the infinite points. The vector fields on $\mathbb{R}^2$: $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$ and $x_1 \frac{\partial}{\partial x_2}$ can be extended, in a natural way, to $P(2, \mathbb{R})$ because they are affine. These vector fields generate an action of a 3-dimensional nilpotent group on $P(2, \mathbb{R})$, whose orbits are $\mathbb{R}^2$; the set of all points of infinity except the vertical one (i.e. the point associated to the vertical direction); and the infinite vertical point, which is the only fixed point.

Example 2. Take now $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$ and $-x_2 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_1}$. One obtains an action of a 3-dimensional solvable group with no fixed point. Their orbits are $\mathbb{R}^2$ and the set of the infinite points.

See [1] for a 2-dimensional example with no fixed point.

References

