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HÖLDER AND L^p ESTIMATES FOR THE SOLUTIONS OF THE $\bar{\partial}$-EQUATION IN NON-SMOOTH STRICTLY PSEUDOCONVEX DOMAINS

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Abstract

Let D a bounded strictly pseudoconvex non-smooth domain in $\mathbf{C}^n$. In this paper we prove that the estimates in L^p and Lipschitz classes for the solutions of the $\bar{\partial}$-equation with L^p-data in regular strictly pseudoconvex domains (see[2]) are also valid for D. We also give estimates of the same type for the $\bar{\partial}_b$ in the regular part of the boundary of these domains.

0. Introduction and statement of results

This paper is a continuation of [1] and deals with the estimates for the $\bar{\partial}$-equation on strictly pseudoconvex non smooth domains. By this we mean a domain D defined by the condition $D = \{r < 0\}$ where r is a strictly p.s.h. function of class C^2 defined in a neighborhood of bD. We recall that it is not assumed that the gradient of r be different of 0 in bD, and the boundary fails to be a regular submanifold of $\mathbf{C}^n$ just in a totally real set.

Henkin and Leiterer proved in [3] that the equation $\bar{\partial}u = f$ has a bounded solution u for any $(0,q)$-form f, with bounded coefficients, and such that $\bar{\partial}f = 0$. In [1] it was proved that there exists an integral operator

$$Tf = \int_D K(\zeta, z) \wedge f(\zeta)$$

mapping $L^{loc}_{(\alpha,\beta)}$ to $L^{loc}_{(\alpha,\beta-1)}$ such that $\bar{\partial}Tf = f$ if $\bar{\partial}f = 0$ and satisfiying the estimate $\|Tf\|_{L^p(D)} \leq c\|f\|_{L^p(D)}$ and also the Lip 1/2 estimate $\|Tf\|_{\mathrm{Lip}(1/2,D)} \leq c\|f\|_\infty$ in the case r is of class C^3.

Here and in the following the L^p spaces are with respect to the Lebesgue measure dm, and Lip (s, D) stands for the class of continuous functions on D having modulus of continuity $O(\delta^s)$. The L^p- norms will be abreviated by $\| \ \|_p$.

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The aim of this paper is to improve these estimates extending to the non-smooth case the optimal estimates obtained by Krantz in [2] for the regular case, for forms of arbitrary bedegree.

The paper is organized as follows. In section 1 we recall the construction of the operator T and the estimates for its kernel K that were obtained in [1]. In section 2 we prove the following three theorems:

Theorem 1.

The operator T satisfies the following L^p-estimates:

(a) $\|Tf\|_q \le c\|f\|_p$, $1 < p < 2n+2$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{2n+2}$

(b) $\|Tf\|_q \le c_q\|f\|_1$, $\forall q < 2n+2$

(c) *For $p = 2n+2$, $|Tf|$ satisfies the estimate*

$$\int_D \exp\{c|Tf(z)|^{\frac{2n+2}{2n+1}}\}dm(\zeta) < \infty$$

for some constant c depending on n and $\|f\|_{2n+2}$.

Theorem 2.

(a) *If $p > 2n+2$, T maps $L^p_{(0,1)}$ continuously into Lip $(\frac{1}{2} - \frac{n+1}{p}, D)$*

(b) *If $f \in L^p_{(\alpha,\beta)}$, $p > 2n+2$ and $\beta > 1$, Tf is continuous on $\bar{D}$.*

(c) *In case r is of class $C^{2+\gamma}$, $0 < \gamma \le 1/2$, then for $p \ge \frac{2n+2}{1-2\gamma}$, T maps $L^p_{(\alpha,\beta)}$ into Lip $(\min\{\gamma, \frac{1}{2} - \frac{n+1}{p}\}, D)$, for all α, β.*

Note in theorem 2 that if r is just C^2, Hölder estimates can only be obtained for $(0,1)$-forms. For forms of bedegree (α, β), $\beta > 1$, one needs extra assumptions on r to obtain Hölder estimates for a certain range of p's. In case r is of class $C^{2+1/2}$, then part (a) holds for forms of arbitrary bedegree.

In order to state our third result, which gives an improvement of the estimates at the boundary, we need to recall a definition from [1] and [6]: Assuming only that r is defined in an neighborhood of $\bar{D}$, we put for $\zeta, z \in \bar{D}$

$$\rho(\zeta, z) = |\langle \partial r(z), \zeta - z\rangle| + |\langle \partial r(\zeta), \zeta - z\rangle| + \|\zeta - z\|^2$$

This is a pseudodistance in the sense that triangle inequality holds with some constant C, and will be called the Koranyi pseudodistance.

We write $\mathrm{Lip}_\rho(s, D)$ for the subspace of Lip (s, D) such that $|f(w) - f(z)| = O(\rho(w,z)^s)$ for $\zeta, z \in bD$.

Theorem 3.

(a) *For $2n+2 < p < \infty$, T maps $L^p_{(0,1)}$ into $Lip_\rho(\frac{1}{2} - \frac{n+1}{p})$*

(b) *For $f \in L^\infty_{(0,1)}$ then*

$$|Tf(z) - Tf(w)| \le c\|f\|_\infty \rho(z,w)^{1/2}|log\rho(z,w)|$$

(c) *If r is of class $C^{2+\gamma}$, $0 < \gamma \leq 1/2$, then for $p \geq \frac{2n+2}{1-2\gamma}$, T maps $L^p_{(\alpha,\beta)}$ into $Lip_\rho(min(\gamma, \frac{1}{2} - \frac{n+1}{p}), D)$*

Since $c_1\|\zeta - z\|^2 \leq \rho(\zeta, z) \leq c_2\|\zeta - z\|$, the meaning of theorem 3 is that the solutions of the $\bar\partial$-equations will be, for the range of p indicated, twice as regular in certain directions in bD. This, in the regular case, is a reformulation of the estimates in the non-isotropic Lipschitz spaces $\Gamma_{\alpha,2\alpha}$ introduced by Stein.

Finally, section 3 contains the estimates of the integrals in the proof of the theorems.

Our technique differs from that in [2] in two aspects: first of all, of course, the non-smoothness makes more involved the estimate of the singularity of the kernels and, secondly, we use direct methods instead of interpolation results in obtaining the Hölder estimates. A main thechnical difficulty for that is that the domain being non smooth we do not have at our disposal the criteria $\nabla u(z) = O(u(z)^{s-1})$ for u to be in Lip (s, D).

1. The kernels solving $\bar\partial$

In [1] kernels are obtained to solve $\bar\partial$ in nonregular strictly pseudoconvex domains, of Henkin-Ramirez type with weight factors. Let us briefly recall their construction and main properties.

1.1. General construction.

For U a C^1 bounded domain in $\mathbf{C}^n$, let $s, Q: \bar U \times \bar U \to \mathbf{C}^n$ where s is a section of Bochner-Martinelli type, say:

$$\|s(\zeta, z)\| = O(\|\zeta - z\|)$$

for ζ,$z \in \bar U$, and

$$|\langle s(\zeta, z), \zeta - z\rangle| \geq c_L\|\zeta - z\|^2$$

whenever $\zeta \in \bar U$ and $z \in L$ compact in U, and Q is of class C^1 and holomorphic in z.

Let also G be a holomorphic function of one complex variable defined in a neigborhood of $\bar U \times \bar U$ under the map $(\zeta, z) \to 1 + \langle Q(\zeta, z), \zeta - z\rangle$ and with $G(1) = 1$.

Finally define

$$(1) \quad K(\zeta, z) = c_n \sum_{k=0}^{n-1} \frac{(n-1)!}{k!} G^{(k)}(1 + \langle Q, \zeta - z\rangle) \frac{\tilde s \wedge (d\tilde Q)^k \wedge (d\tilde s)^{n-k-1}}{\langle s, \zeta - z\rangle^{n-k}}$$

where $c_n = ((2\pi i)^n (n-1)!)^{-1}$, $\tilde s = \sum_{j=0}^n s_j d(\zeta_j - z_j)$ and $\tilde Q = \sum_{j=0}^n Q_j d(\zeta_j - z_j)$ K is a $2n - 1$ form in $d\zeta$,$d\bar\zeta$,dz and $d\bar z$ together, and for $0 \leq \alpha, \beta \leq n$, let $K_{\alpha,\beta}$ the component of bedegree (α, β) in z and $(n - \alpha, n - \beta - 1)$ in ζ.

Then:

Theorem 1.1. *Whenever $\beta \geq 1$, if $K_{\alpha,\beta}(\zeta,z)|_{\zeta\in bU}$ is 0 for $z \in U$, the operator*

$$Tf = (-1)^{\alpha+\beta} \int_U f \wedge K_{\alpha,\beta-1} \tag{2}$$

satisfies $\bar\partial Tf = f$ if $f \in \mathcal{C}^1{}_{(\alpha,\beta)}(\bar U)$.

1.2. The section and weights in the strictly pseudoconvex case.

If D is a strictly pseudoconvex (non regular domain) and r is a $\mathcal{C}^2$ defining function for D, let $U_\delta = \{r(z) < \delta\}$ and $V_\delta = \{-\delta < r(z) < \delta\}$. Then Henkin and Heffer's lemmas provides us with a family of functions Φ_j, $j = 1, ..., n$, and constants c_0,ϵ_0, δ_0 depending only on the function r (but not on its gradient, nor on the variable z), such that $\Phi_j \in \mathcal{C}^1(\bar V_{\delta_0} \times \bar U_{\delta_0})$ and are holomorphic in z, and the function $\Phi(\zeta,z) = \sum_{j=1}^n \Phi_j(\zeta,z)(\zeta_j - z_j)$ satisfies:

$$\begin{aligned}
&|\Phi(\zeta,z)| \geq c_0 \text{ if } \|\zeta - z\| \geq \epsilon_0 \\
&2\Re\Phi(\zeta,z) \geq r(\zeta) - r(z) + c_0\|\zeta - z\|^2 \text{ if } \|\zeta - z\| < \epsilon_0 \\
&d_\zeta \Phi_{|\zeta=z} = d_z\Phi_{|z=\zeta} = \partial r(z) \\
&\Phi_j(\zeta,z) = \frac{\partial r}{\partial \zeta_j}(\zeta) + O(\|\zeta - z\|) \text{ when } \|\zeta - z\| < \epsilon_0,\ |r(z)| < \delta_0
\end{aligned}$$

We define:

$$A(\zeta,z) = -r(z) + \Phi(\zeta,z)$$

and if $\chi \in \mathcal{C}_c^\infty(\mathbf{C}^n)$, $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $V_{\frac{\delta_0}{2}}$, define

$$Q_j(\zeta,z) = \chi(\zeta)\frac{\Phi_j(\zeta,z)}{A(\zeta,z)}$$

and $Q = (Q_1, ..., Q_n)$.

Write now, $V_{\epsilon,\delta} = \{(\zeta,z) : |r(\zeta)| < \delta, |r(z)| < \delta, \|\zeta - z\| < \epsilon\}$ and define $\Phi_j^*(\zeta,z) = \Phi_j(z,\zeta)$, $\Phi^*(\zeta,z) = \Phi(z,\zeta)$, $v_j(\zeta,z) = r(\zeta)\Phi_j^*(\zeta,z) + A(z,\zeta)\Phi_j(\zeta,z)$, and $v(\zeta,z) = \sum_{j=1}^n v_j(\zeta,z)(\zeta_j - z_j)$.

Finally, if $\phi \in \mathcal{C}_c^\infty(\mathbf{C}^n)$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $V_{\frac{\delta_0}{2},\frac{\epsilon_0}{2}}$, define

$$s_j(\zeta,z) = \phi(\zeta,z)v_j(\zeta,z) + (1 - \phi(\zeta,z))(\bar\zeta_j - \bar z_j)$$

$s = (s_1, ...s_n)$ and $H = \langle s, \zeta - z\rangle$.

The following estimates are crucial for the estimates of the kernel's singularity:

$$2ReA \approx -r(\zeta) - r(z) + c_0\|\zeta - z\|^2$$

$$|H| \geq c\{(r(\zeta)-r(z))^2 + (-r(\zeta)-r(z))\|\zeta - z\|^2 + \|\zeta - z\|^4 + Im\phi Im\phi^*\},$$

for $(\zeta, z) \in V_{\epsilon_0,\delta_0} \cap (\bar U \times \bar U)$. This implies that, in terms of the pseudodistance, ρ:

$$|A| \approx -r(\zeta) - r(z) + \rho(\zeta, z)$$
$$|H| \geq c\{(-r(\zeta)-r(z))\|\zeta - z\|^2 + \rho^2(\zeta, z)\}$$

1.3. The resulting kernels and their estimates.

Take in the formula (1) $G(w) = w^n$, and the section and weight introduced before. Define also:

$$\omega(\zeta, z) = \sum_{j=0}^{n} \Phi_j(\zeta, z) d(\zeta_j - z_j)$$

$$\omega^*(\zeta, z) = \sum_{j=1}^{n} \Phi_j(z, \zeta) d(\zeta_j - z_j)$$

$$A^*(\zeta, z) = A(z, \zeta)$$

$$\eta(\zeta, z) = r(\zeta)\bar\partial_z \omega^* + A^* \bar\partial_\zeta \omega$$

After a combinatoric computation one obtains in that case:

$$(3)\quad K(\zeta, z) = v \wedge \sum_{k=0}^{n-1} c_{n,k} \Big(\frac{-r(\zeta)}{A}\Big)^{n-k} \frac{1}{H^{n-k}A^k} (\bar\partial_\zeta \omega)^k \wedge \eta^{n-k-1}$$
$$+ [r(\zeta)\bar\partial_z A^* - A^* \bar\partial_\zeta r] \wedge \omega \wedge \omega^* \wedge \sum_{k=0}^{n-1} c_{n,k}(n-k-1) \Big(\frac{-r(\zeta)}{A}\Big)^{n-k} \frac{1}{H^{n-k}A^k}$$
$$(\bar\partial_\zeta \omega)^k \wedge \eta^{n-k-2} -$$
$$- r(\zeta)\bar\partial_\zeta A \wedge \omega \wedge \omega^* \wedge \sum_{k=1}^{n-1} c_{n,k} k \Big(\frac{-r(\zeta)}{A}\Big)^{n-k} \frac{1}{H^{n-k}A^{k+1}} (\bar\partial_\zeta \omega)^{k-1} \wedge \eta^{n-k-1}$$

Define now $q(\zeta, z) = \|dr(z)\| + \|\zeta - z\|$, and denote $\|dr(z)\| = \lambda(z)$

It is clear from (3) that $K(\zeta, z) = 0$ for $\zeta \in bU$ so Theorem 1.1 applies and the kernel K has the estimate (see [1], lemmas 2.4 and 2.5):

$$(4)\qquad |K(\zeta, z)| = O\Big(\frac{|A|^{n-1}}{|D|^n}\{-r(\zeta) + q^2(\zeta, z)\}\|\zeta - z\|\Big)\}$$

In terms of the pseudodistance, we have (see[1], lemma 6.2)

$$(5)\quad |K(\zeta,z)| = O\Big(\frac{1}{\|\zeta-z\|^{2n-1}} + \frac{(-r(z))^{n-1}\|\zeta-z\|q^2(\zeta,z)}{[-r(z)\|\zeta-z\|^2+\rho(\zeta,z)^2]^n} + \frac{\|\zeta-z\|q^2(\zeta,z)}{\rho(\zeta,z)^{n+1}}\Big) =^{def} O(N_1)$$

As showed in [1], lemma 6.3 ,the kernel $K(\zeta,z)$ is integrable in each variable, uniformely in the other. Using a standard regularization process, one can then show that if $f \in L^1_{loc(\alpha,\beta)}$ and $\bar\partial Tf = 0$ in the weak sense then Tf is a form in $L^1_{loc(\alpha,\beta-1)}$ and $\bar\partial Tf = f$ in the weak sense.

Notice that when $z \in bU$, the estimate (5) implies

$$(6)\qquad |K(\zeta,z)| = O(\rho(\zeta,z)^{\frac{1}{2}-n} + \lambda(z)^2\rho(\zeta,z)^{-n-\frac{1}{2}}) =^{def} O(N_2)$$

and also the worse but symmetric estimate:

$$(7)\qquad |K(\zeta,z)| = O\Big(\frac{1}{\|\zeta-z\|^{2n-1}} + \frac{\lambda(\zeta)^2+\lambda(z)^2}{\|\zeta-z\|^{2n-3}\rho^2(\zeta,z)}\Big) =^{def} O(N_3)$$

1.4. Estimates of the differences.

Our method in proving Hölder estimates involves estimates of the differences of the kernels $|K(\zeta,z)-K(\zeta,w)|$. A suitable control can be obtained in terms of the gradient of K with respect to the second variable whenever it makes sense, that is when the coefficients of the form $K(\zeta,z)$ are C^1 in z. Formula (3) shows that all terms but those involving $\bar\partial_z\omega^*$ are C^1 in z , because $\bar\partial\omega^* = \sum\bar\partial_z\Phi_j^* \wedge d(\zeta_j - z_j)$ and $\bar\partial_z\Phi_j^*(\zeta,z) = \bar\partial_z\Phi_j(z,\zeta)$ is only continuous, since it involves second derivatives on r.

Observe also that bad terms may appear only once (because $\bar\partial_z\Phi^*\wedge\bar\partial_z\Phi^* = 0$), and in the components $K_{\alpha,0}$ they do not appear at all.

So we can write the kernel K as a sum

$$K(\zeta,z) = K_1(\zeta,z)\wedge\bar\partial_z\omega^* + K_2(\zeta,z)$$

where K_1, K_2 are C^1 in z and $K_{\alpha,0} = (K_2)_{\alpha,0}$. The kernels K_1 and K_2 satisfy the same estimates as K, that is (5) and (6).

The gradients in z of K_1 and K_2 satisfy the estimates contained in lemma 6.6 of [1] and hence it follows that there exists c such that if $\|z-w\|$ is small enough and $\|\zeta-z\| \geq c\|z-w\|^{1/2}$, $\zeta \in D$, then, for $j=1,2$

$$(8)\qquad |K_j(\zeta,z)-K_j(\zeta,w)| = O(\|z-w\|M_1(\zeta,z))$$

where

(8')

$$M_1 =^{def} \Big(\frac{\lambda}{\|\zeta-z\|^{2n+1}} + \frac{|r(z)|^n\|\zeta-z\|\lambda(z)^2}{[|r(z)|\|\zeta-z\|^2+\rho(\zeta,z)^2]^{n+1}} + \frac{\|\zeta-z\|\lambda(z)^2}{\rho(\zeta,z)^{n+2}}\Big)q(\zeta,z)$$

thus, $K_{\alpha,0}(\zeta,z)-K_{\alpha,0}(\zeta,w)$ will satisfy (8) if ζ,z,w are as above, without any further requirement on r. For the components $K_{\alpha,\beta}$ with $\beta>0$ write

$$K(\zeta,z)-K(\zeta,w)=\{K_1(\zeta,z)-K_1(\zeta,w)\}\wedge\bar{\partial}_z\omega^*+K_2(\zeta,z)-K_2(\zeta,w)\\ +K_1(\zeta,z)\wedge\{\bar{\partial}_z\omega^*-\bar{\partial}_w\omega^*\}$$

It follows that if d^2r satisfies a Lipschitz estimate of order γ, and z, w, ζ are as above, then

$$(9)\qquad |K_{\alpha,\beta}(\zeta,z)-K_{\alpha,\beta}(\zeta,w)|=O(\|z-w\|M_1(\zeta,z)+\|z-w\|^{\gamma}N_1(\zeta,z))$$

The estimates (8), (8′) and (9) will be used in the Euclidean Hölder estimates.

For the non-isotropic Hölder estimates it will be convenient to estimate the difference $K_{\alpha,\beta}(\zeta,z)-K_{\alpha,\beta}(\zeta,w)$ just in terms of ρ. In a similar way as before, but using now lemma 6.9 of [1] instead of lemma 6.6, we see that if r is just C^2 and $\rho(\zeta,z)\geq c\rho(z,w)$ $\zeta,z,w\in bU$ then:

$$(10)\qquad |K_{\alpha,0}(\zeta,z)-K_{\alpha,0}(\zeta,w)|=O(\rho(z,w)^{1/2}M_2)$$

where

$$(10')\qquad M_2=^{def}\rho(\zeta,z)^{-n}+\lambda(z)^2\rho(\zeta,z)^{-1-n}$$

and if $r\in C^{2+\gamma}$ then for $\beta>0$

$$(11)\qquad |K_{\alpha,\beta}(\zeta,z)-K_{\alpha,\beta}(\zeta,w)|=O(\rho(z,w)^{1/2}M_2+\rho(z,w)^{\gamma/2}N_2)$$

where

$$(11')\qquad N_2=\rho(\zeta,z)^{1/2-n}+\lambda(z)^2\rho(\zeta,z)^{-n-1/2}$$

2. Proof of theorems

2.1. Proof of theorem 1:

As in [2], the proof of Theorem 1 is based on the following lemmas:

Lemma. *Let $K:\mathbf{R}^m\times\mathbf{R}^n\to\mathbf{C}$ have the property that $K(x,.)$ is of weak type s as a function of y, uniformly in x, and $K(.,y)$ is of weak type s as a function of x, uniformly in y. Then the linear transformation*

$$f(x)\to Tf(x)=\int_{\mathbf{R}^n}K(x,y)f(y)dy$$

defined for $f : \mathbf{R}^n \to \mathbf{C}$, *satisfies* $\|Tf\|_{L^q} \le A_p\|f\|_{L^p}$ *whenever* $s > 1$, $1 < p < q < \infty$ *and* $\frac{1}{q} = \frac{1}{p} + \frac{1}{s} - 1$. *Moreover,* $\|Tf\|_{L^{s-\epsilon}} \le A_\epsilon\|f\|_{L^1}$ *for all* $\epsilon > 0$.

Lemma. *Suppose* K,s,f *as above and* $f \in L^{s'}$ *where* $\frac{1}{s} + \frac{1}{s'} = 1$, *suppose also that* $m = n$, $\Omega \subset\subset \mathbf{R}^n$, *and* $suppK \subseteq \bar{\Omega} \times \bar{\Omega}$. *Then*

$$\int_\Omega exp((c\frac{|Tf(z)|}{\|f\|_{L^{s'}(\Omega)}})^s)dm(z) \le M < \infty$$

and c, M *do not depend on* f, *but on* s *and* $m(\Omega)$

Acording to the lemma, we have to prove that $K(\zeta, z)$ is of $\frac{2n+2}{2n+1}$-weak type on D in each variable uniformely in the other.

For this we use the estimate (7) and, since it is symmetric, it will be enough to prove the following result:

Lemma 2.1. $m\{z : |K(\zeta,z)| > t\} = O(t^{-\frac{2n+2}{2n+1}})$, *uniformly in* $\zeta \in \bar{D}$

This will be done in section 3. ■

2.2. Proof of Theorem 2:

Let $p > 2n+2$ and $f \in L^p_{(0,1)}$. First we prove that $Tf \in \mathrm{Lip}(\frac{1}{2} - \frac{n+1}{p}, D)$

Let $z, w \in \bar{D}$, $\delta = \|z - w\|$. We define $\eta = \delta^{1/2}$ and estimate $Tf(z) - Tf(w)$, using (8), by

$$\text{(12)} \quad \int_{B_{c\eta}(z)\cap D} |f(\zeta)|N_1(\zeta,z)dm(\zeta) + \int_{B_{c\eta}(z)\cap D} |f(\zeta)|N_1(\zeta,w)dm(\zeta) + \delta\int_{D\setminus B_{c\eta}(z)} |f(\zeta)|M_1(\zeta,z)dm(\zeta)$$

In case $r \in C^{2+\gamma}$ and $f \in L^p_{(\alpha,\beta)}$, $\beta > 1$, we will have, acording to (9), one more term:

$$\delta^\gamma \int_{D\setminus B_{c\eta}(z)} |f(\zeta)|N_1(\zeta,z)dm(\zeta)$$

Using Hölder estimates we are lead to the following lemmas which will be proved in section 3:

Lemma 2.2.1. *For* $1 \le s < \frac{2n+2}{2n+1}$ *we have*

$$\{\int_{B_\eta(z)\cap D} N_1(\zeta,z)^s dm(\zeta)\}^{1/s} = O(\eta^{\frac{2n+2}{s} - 2n - 1})$$

Corollary. *For* $1 \le s < \frac{2n+2}{2n+1}$

$$\{\int_{D\setminus B_\eta(z)} N_1(\zeta,z)^s dm(\zeta)\}^{1/s} = O(1)$$

Lemma 2.2.2. *For* $1 \leq s < \frac{2n+2}{2n+1}$

$$\{\int_{D\setminus B_\eta(z)} M_1(\zeta,z)^s dm(\zeta)\}^{1/s} = O(\eta^{\frac{2n+2}{s}-2n-3})$$

Using lemma 2.2.1 the first integral in (12) is $O(\delta^{\frac{1}{2}-\frac{n+1}{p}})$. The second one is also $O(\delta^{\frac{1}{2}-\frac{n+1}{p}})$, because $B_{c\eta}(z) \subset B_{c\eta}(w)$. Finally by lemma 2.2.2, the last term in (12) is $O(\delta\eta^{\frac{2n+2}{s}-2n-3}) = O(\delta^{\frac{1}{2}-\frac{n+1}{p}})$

This proves the first part of the theorem. Under the conditions in (b) we have one more term which is O(1) according to the corollary of lemma 2.2.1. Then if $p \geq \frac{2n+2}{1-2\gamma}$ we obtain a Lipschitz condition whith exponent $\min(\gamma, \frac{1}{2}-\frac{n+1}{p})$ which gives part (b) of the theorem. Finally, for part (c) we simply write,

$$|Tf(z)-Tf(w)| \leq \|f\|_p\{\int_{B_\epsilon}(|K(\zeta,z)|+|K(\zeta,w)|)^s dm(\zeta)\}^{1/s}$$
$$+\|f\|_p\{\int_{D\setminus B_\epsilon(z)}|K(\zeta,z)-K(\zeta,w)|^s dm(\zeta)\}^{1/s}$$

where $\frac{1}{p}+\frac{1}{s}=1$

The first term can be made arbitrarily small by choosing ϵ small and the second too by choosing w close enough to z beacause the kernels are continuous off the diagonal. ■

2.3. Proof of the Theorem 3:

First, let $A_\delta(z) = \{\zeta : \rho(\zeta,z) < \delta\}$, be the Hörmander ball related to ρ. For $p > 2n+2$ and $f \in L^p_{(0,1)}$ we split now the integrals giving $Tf(z)-Tf(w)$ for z,$w \in bD$ in the form:

$$(13)\quad \int_{A_{c\delta}(z)} |f(\zeta)|N_2(\zeta,z)dm(\zeta) + \int_{A_{c\delta}(z)} |f(\zeta)|N_2(\zeta,w)dm(\zeta)$$
$$+\delta^{1/2}\int_{D\setminus A_{c\delta}(z)} |f(\zeta)|M_2(\zeta,z)dm(\zeta)$$

Here $\delta = \rho(z,w)$ and we have used (10)

In case $r \in C^{2+\gamma}$ and $f \in L^p_{(\alpha,\beta)}$, $\beta > 1$, we will have by (11) another term:

$$(14)\qquad \delta^{\gamma/2}\int_{D\setminus A_{c\delta}(z)} |f(\zeta)|N_2(\zeta,z)dm(\zeta)$$

Using Hölder's inequality as before we are lead now to the following lemmas:

Lemma 2.3.1. *For* $1 \leq s < \frac{2n+2}{2n+1}$,

$$\{\int_{A_\delta(z)} N_2(\zeta,z)^s dm(\zeta)\}^{1/s} = O(\delta^{\frac{n+1}{s}-n-\frac{1}{2}})$$

Corollary. *For* $1 \leq s < \frac{2n+2}{2n+1}$

$$\{\int_{D\setminus A_\delta(z)} N_2(\zeta,z)^s dm(\zeta)\}^{1/s} = O(1)$$

Lemma 2.3.2. *The integral*

$$\{\int_{D\setminus A_\delta(z)} M_2(\zeta,z)^s dm(\zeta)\}^{1/s}$$

is $O(\delta^{\frac{n+1}{s}-n-1})$ *for* $1 < s < \frac{2n+2}{2n+1}$ *and* $O(|log\delta|)$ *for* $s = 1$

Now, as in the proof of theorem 2, using lemma 2.3.1, we see that the first term in (13) is $O(\delta^{\frac{1}{2}-\frac{n+1}{p}})$, and the second one has the same order estimate, because $A_{c\delta}(z) \subset A_{c'\delta}(w)$ (since $\rho(z,w) = \delta$). Finally, the lemma (2.3.2) gives us the same estimate for the third one, whenever $p < \infty$. This proves part (a) of the theorem.

When $p = +\infty$, $s = 1$, and using the same lemmas we have for (13) the estimate $\delta^{\frac{1}{2}} + \delta^{\frac{1}{2}}|ln\delta|$. This gives (b).

Finally for general forms (part (c)), the corollary of lemma 2.2.1 and the same considerations as in the corresponding part in theorem 1 complete the proof of (c). ■

3. Proof of the estimates

3.1. The case of the euclidean balls.

First, let us notice that for $z \in D$ far away from the boundary, or in the set of singular points ($\lambda(z) = 0$), the only terms appearing in the integrals are those of type $\|\zeta - z\|^{-2n+1}$ and $\|\zeta - z\|^{-2n}$.

Otherwise, when $\lambda > 0$, in order to integrate in or outside euclidean balls we choose the coordinates:

$$t_1 = \Re\langle\frac{\partial r(z)}{2\lambda(z)}, \zeta - z\rangle$$

$$t_2 = \Im\langle\frac{\partial r(z)}{2\lambda(z)}, \zeta - z\rangle$$

$$t_3, \dots, t_{2n}$$

euclidean orthonormal coordinates in T_z^c, the complex tangent space to $\{r = r(z)\}$ at the point z.

The jacobian matrix associated to the change $x(\zeta - z) \to t(\zeta - z)$ is 1, and let us denote $t' = (t_3, ..., t_{2n})$, and $t = (t_1, t_2, t')$

In terms of these coordinates, $\rho(\zeta, z) \simeq \lambda(z)[|t_1| + |t_2|] + \|t\|^2$.

Proof of lemma 2.1:

We use (7); the estimate is clear for the term $\|\zeta - z\|^{1-2n}$. Note that since $\lambda(\zeta) = \lambda(z) + O(\|\zeta - z\|)$ we can delete $\lambda(\zeta)$ in the estimate. Now, we have

$$A = \{\frac{\lambda^2}{\|t'\|^{2n-3}(\lambda|t_1 + it_2| + \|t'\|^2)^2} > s\} \subset \{\|t'\| < (\frac{\lambda^2}{s})^{\frac{1}{2n+1}},$$
$$(\lambda|t_1 + it_2|)^2 < \frac{\lambda^2}{s\|t'\|^{2n-3}}\}$$

and then

$$m(A) \leq m\{\|t'\| < (\frac{\lambda^2}{s})^{\frac{1}{2n+1}}, (\lambda|t_1 + it_2|)^2 < \frac{\lambda^2}{s\|t'\|^{2n-3}}\}$$

$$= \int_{\{0<\|t'\|<(\frac{\lambda^2}{s})^{\frac{1}{2n+1}}\}} dm(t') \int_{\{0<|t_1+it_2|^2<\frac{1}{s\|t'\|^{2n-3}}\}} dt_1 dt_2 = O(s^{-\frac{2n+2}{2n+1}}) \quad \blacksquare$$

Proof of lemma 2.2.1:

In view of formula (5) we have to estimate, for $1 \leq s < \frac{2n+2}{2n+1}$, the following integrals:

$$\int_{D\cap B_\eta(z)} \frac{dm(\zeta)}{\|\zeta - z\|^{(2n-1)s}} \tag{15}$$

$$\int_{D\cap B_\eta(z)} (\frac{|r(z)|^{n-1}\|\zeta - z\|\lambda^2}{[|r(z)|\|\zeta - z\|^2 + \rho^2]^n})^s dm(\zeta) \tag{16}$$

$$\int_{D\cap B_\eta(z)} (\frac{\|\zeta - z\|\lambda^2}{\rho(\zeta, z)^{n+1}})^s dm(\zeta) \tag{17}$$

The first one is immediately seen to be $O(\eta^{1-(2n-1)(s-1)})$. The second, using the coordinates above, is estimated by

$$|r(z)|^{(n-1)s}\lambda^{2s} \int_{B_\eta(0)} \frac{\|t\|^s dt_1 ... dt_{2n}}{[|r(z)|^{1/2}\|t\| + \lambda(|t_1| + |t_2|) + \|t\|^2]^{2ns}}$$

and taking polar coordinates:

$$t_1 = R\cos\theta$$

$$t_2 = R\sin\theta\cos\phi$$

....

the integral above is bounded by:

$$Ct.|r(z)|^{(n-1)s}\lambda^{2(s-1)}\int_0^\eta R^{2n-3+s}dR\int_0^\pi \lambda R\sin\theta d\theta$$
$$\int_0^\pi \frac{\lambda R\sin\theta sin\varphi d\varphi}{[|r(z)|^{1/2}R+\lambda R\cos\theta+\lambda R\sin\theta\cos\varphi+R^2]^{2ns}}$$
$$\leq C|r(z)|^{(n-1)s}\lambda^{2(s-1)}\int_0^\eta \frac{R^{2n-3+s}dR}{[(|r(z)|^{1/2}+R)R]^{2ns-2}}$$
$$\leq C|r(z)|^{(n-1)s}\lambda^{2(s-1)}\int_0^\eta \frac{dR}{(|r(z)|^{1/2}+R)^{2ns-2}R^{(2n-1)(s-1)}}$$
$$\leq C\lambda^{2(s-1)}\int_0^\eta \frac{dR}{R^{(2n+1)(s-1)}}$$

this is in turn bounded by $0(\eta^{1-(2n+1)(s-1)})$.

The third one is treated in a similar way. ■

Proof of lemma 2.2.2:

Acording to the definitions (8') and easy calculations, we have,

$$M_1(\zeta,z) = O(\frac{q(\zeta,z)}{\|\zeta-z\|^{2n+1}}+\frac{|r(z)|^n\|\zeta-z\|\lambda^3}{[|r(z)|^{1/2}\|\zeta-z\|+\rho]^{2n+2}}+\frac{\lambda^3}{\rho(\zeta,z)^{n+3/2}})$$

and so we have to estimate the three integrals corresponding to the three terms on the right. The first one is estimated by

$$\int_{D\setminus B_\eta(z)}\{\frac{\lambda^s}{\|\zeta-z\|^{(2n+1)s}}+\frac{1}{\|\zeta-z\|^{2ns}}\}dm(\zeta) \tag{18}$$

and this is

$$O(\frac{1}{\eta^{(2n+1)s-2n}}+\frac{1}{\eta^{2n(s-1)}})$$

when $1 < s$ and

$$O(\frac{1}{\eta}+|ln\eta|)$$

when $s = 1$

Both are of order

$$O(\frac{1}{\eta^{1+(2n+1)(s-1)}})$$

for $1 \leq s < \frac{2n+2}{2n+1}$. For the second one we use the same coordinates as before and bound it by

$$C|r(z)|^{ns}\lambda^{3s-2}\int_\eta^{d_1} R^{2n-3+s}dR\int_0^\pi \lambda R\sin\theta d\theta\int_0^\pi \frac{\lambda R\sin\theta\sin\varphi d\varphi}{[|r(z)|^{1/2}R+R^2]^{(2n+2)s}}$$

$$\leq C|r(z)|^{ns}\lambda^{3s-2}\int_\eta^{d_1}\frac{dR}{R^{2+(2n+1)(s-1)}(|r(z)|^{1/2}+R)^{(2n+2)s-2}}$$

$$\leq C\lambda^{3s-2}\int_\eta^{d_1}\frac{dR}{R^{2+(2n+3)(s-1)}} = O(\eta^{-1-(2n+3)(s-1)})$$

for $1 \leq s < \frac{2n+2}{2n+1}$, where d_1 is the euclidean diameter of D. The third one is estimated along the same line and it is left to the reader. ■

3.2. The case of the Hörmander balls.

First let us notice that, for $z \in bD$, defining $\epsilon(z,\delta) = inf\{\delta^{1/2}, \frac{\delta}{\lambda(z)}\}$, we have that $A_\delta(z) \approx B^2_{\epsilon(z,\delta)}(z) \times B^{2n-2}_{\delta^{1/2}}(z)$, where the ball $B^{2n-2}_{\delta^{1/2}}(z)$ is $2n-2$ dimensional and is taken on the complex hyperplane T^c_z, and the ball $B^2_{\epsilon(z,\delta)}(z)$ is 2-dimensional and is taken on the orthogonal complement of T^c_z. So $A_\delta(z) \subset B^{2n}_{\delta^{1/2}}(z)$, and also $m(A_\delta(z)) \leq C\epsilon(z,\delta)^2\delta^{n-1}$

Proof of lemma 2.3.1:

By the formula (11) we have to estimate:

$$\int_{A_\delta(z)}(\frac{1}{\rho(\zeta,z)^{n-1/2}}+\frac{\lambda^2}{\rho(\zeta,z)^{n+1/2}})dm(\zeta) \leq C\int_{B_{\delta^{1/2}}(z)}\frac{dm(\zeta)}{\|\zeta-z\|^{(2n-1)s}} + \int_{A_\delta(z)}\frac{\lambda^{2s}dm(\zeta)}{\rho(\zeta,z)^{(n+\frac12)s}} \tag{19}$$

because $\rho(\zeta,z) \geq \|\zeta - z\|$ and the considerations above.

The first integral is as (15) in the proof of the lemma 2.2.1, and the second one can be evaluated by

$$\lambda^{2s}\sum_{k=0}^\infty \frac{m(A_{2^{-k}\delta}(z))}{(2^{-k}\delta)^{(n+\frac12)s}} \leq C\lambda^{2s}\sum_{k=0}^\infty\frac{\epsilon(z,2^{-k}\delta)^2(2^{-k}\delta)^{n-1}}{(2^{-k}\delta)^{(n+\frac12)s}}$$

$$\leq C\lambda^{2(s-1)}\delta^{\frac12+(n+\frac12)(s-1)}\sum_{k=0}^\infty\frac{1}{2^{k[\frac12-(n+\frac12)(s-1)]}}$$

because $\epsilon(z,\delta) \leq \frac{\delta}{\lambda}$, and the last series converges for $1 \leq s < \frac{2n+2}{2n+1}$.

So (19) is $O(\delta^{\frac{1}{2}-(n+\frac{1}{2})(s-1)})$ and this proves the estimate. ■

Proof of the lemma 2.3.2:

In view of (10), we have to estimate

$$\int_{D\setminus A_\delta(z)} \left(\frac{1}{\rho(\zeta,z)^n} + \frac{\lambda^2}{\rho(\zeta,z)^{n+1}}\right)^s dm(\zeta) \tag{20}$$

and proceeding as in lemma 2.3.1, since $\rho(\zeta,z) \geq \|\zeta - z\|^2$ and $B_{c\delta^{1/2}(z)} \subset A_\delta(z)$, we bound the first one by

$$\int_{D\setminus B_{c\delta^{1/2}}(z)} \frac{dm(\zeta)}{\|\zeta - z\|^{2ns}} = O(\delta^{-n(s-1)})$$

when $1 < s < \frac{2n}{2n-1}$ and $0(|\ln\delta|)$ when $s = 1$.
and also, for d' suficiently large, the second one by

$$\int_{D\setminus A_\delta(z)} \frac{\lambda^{2s} dm(\zeta)}{\rho(\zeta,z)^{(n+1)s}}$$

$$\leq \lambda^{2s} \sum_{k=1}^{d'} \frac{\epsilon(z,2^k\delta)^2(2^k\delta)^{n-1}}{(2^k\delta)^{(n+1)s}} = \lambda^{2(s-1)} \sum_{k=1}^{d'} \frac{2^k\delta}{(2^k\delta)^{1+(n-1)(s-1)}}$$

Wen $1 < s < \frac{2n+2}{2n+1}$, this is bounded by

$$\lambda^{2(s-1)}\delta^{-(n+1)(s-1)} \sum_{k=1}^{\infty} \frac{1}{2^{k(n+1)(s-1)}}$$

and when $s = 1$, since $2^k\delta \leq \rho(D)$, the Hörmander diameter of D,we have the bound

$$\lambda^{2(s-1)}\rho(D) \sum_{k=1}^{d'} \frac{1}{2^k\delta} \leq C \int_{\frac{1}{\delta}}^{\frac{d'}{\delta}} \frac{dx}{x} = O(|ln\delta|)$$

And the lema follows. ■

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