Publicacions Matemàtiques, Vol 34 (1990), 225-239.

(h_0, h)-BOUNDEDNESS OF THE SOLUTIONS OF DIFFERENTIAL SYSTEMS WITH IMPULSES

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Abstract

In the present paper the question of boundedness of the solutions of systems of differential equations with impulses in terms of two measures is considered. In the investigations piecewise continuous auxiliary functions are used which are an analogue of the classical Lyapunov's functions. The ideas of Lyapunov's second method are combined with the newest ideas of the theory of stability and boundedness of the solutions of systems of differential equations.

1. Introduction

Systems of differential equations with impulses represent a natural apparatus for mathematical simulation of real processes and phenomena studied in biology, physics, control theory, etc. For instance, if the population of a given species is regulated by some impulsive factors acting at certain moments, then we have no reasons to expect that the process will be simulated by regular control. On the contrary, the solutions must have jumps at these moments and the jumps are given beforehand. Moreover, the mathematical theory of the systems of differential equations with impulses is much richer than the respective theory of systems without impulses. That is why in the recent years this theory is an important field of numerous investigations ([1]-[7]).

The usage of classical Lyapunov's functions in the study of the stability and boundedness of the solutions of systems of differential equations with impulses via Lyapunov's second method constricts the pliability of the method. The fact that the solutions of such systems are piecewise continuous functions shows that it is necessary to introduce analogues of Lyapunov's functions which have discontinuities of the first kind. The introduction of such functions makes the application of Lyapunov's second method for systems with impulses much more efficient ([1]-[6]).

In the present paper the boundedness of the solutions of systems of differential equations with impulses in the terms of two measures is studied. In the

The present investigation is partially supported by the Ministry of Culture, Science and Education of People's Republic of Bulgaria under Grant 61.

investigations piecewise continuous Lyapunov's functions are used which are combined by the newest ideas of the theory of stability and boundedness of the solutions of systems of differential equations.

The main results generalize theorems of Yoshizawa [8] and Hara, Yoneyama, Saitoh, Hirano [9].

2. Preliminary notes and definitions

Consider the following system of differential equations with impulses

$$(1)\qquad \begin{cases} \dot{x} = f(t,x),\ t \neq \tau_R(x); \\ \Delta x/_{t=\tau_R(x)} = I_R(x), \end{cases}$$

where $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, $\tau_R \in C[\mathbb{R}^n, \mathbb{R}]$, $I_R \in C[\mathbb{R}^n, \mathbb{R}^n]$ and $\Delta x/_{t=\tau_R(x)} = x(t^+) - x(t^-)$.

Let $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$. Denote by $x(t;t_0,x_0)$ the solution of system (1) which satisfies the initial condition $x(t_0^+;t_0,x_0) = x_0$ and by $J^+(t_0,x_0)$ denote the maximal interval of the form (t_0,ω) in which the solution $x(t;t_0,x_0)$ is defined.

The solutions $x(t) = x(t;t_0,x_0)$ of system (1) are piecewise continuous functions with points of discontinuity of the first kind, i.e. at the moment t_R when the integral curve of the solution meets the hypersurface

$$\sigma_R = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n : t = \tau_R(x)\}$$

the following relations hold

$$x(t_R^-) = x(t_R),\ \Delta x/_{t=t_R} = x(t_R^+) - x(t_R^-) = I_R(x(t_R)).$$

Henceforth we shall always assume that for all $x \in \mathbb{R}^n$ the following relations are valid

$$0 < \tau_1(x) < \tau_2(x) < \cdots < \tau_R(x) < \ldots \text{and} \lim_{R\to\infty} \tau_R(x) = \infty$$

and the integral curve of any solution $x(t) = x(t;t_0,x_0)$ of system (1) meets each hypersurface σ_R at most once [7].

In the further considerations we shall use the following classes of functions:

$$K = \{\sigma \in C[\mathbb{R}_+, \mathbb{R}_+] : \sigma \text{ is strictly increasing and } \sigma(0) = 0\}$$
$$CK = \{\sigma \in C[\mathbb{R}_+^2, \mathbb{R}_+] : \sigma(t,\cdot) \in K \text{ for any } t \in \mathbb{R}_+\}$$
$$\Gamma = \{h \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+] : \inf_{x\in\mathbb{R}^n} h(t,x) = 0 \text{ for any } t \in \mathbb{R}_+\}$$

Definition 1. Let $h_0, h \in \Gamma$. We say that the solutions of system (1) are:

a) (h_0, h)-equibounded if

$$(\forall \alpha > 0)(\forall t_0 \in \mathbb{R}_+)(\exists \beta = \beta(t_0, \alpha) > 0)(\forall x_0 \in \mathbb{R}^n, h_0(t_0, x_0) \leq \alpha) \\ (\forall t > t_0) : h(t, x(t; t_0, x_0)) < \beta.$$

b) (h_0, h)-uniformly bounded if the number β of a) does not depend on $t_0 \in \mathbb{R}_+$.

c) h-ultimately bounded for bound B if

$$(\forall (t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n)(\exists T = T(t_0, x_0) > 0)(\forall t \geq t_0 + T) : \\ h(t, x(t; t_0, x_0)) < B.$$

d) (h_0, h)-equi-ultimately bounded for bound B if

$$(\forall \alpha > 0)(\forall t_0 \in \mathbb{R}_+)(\exists T = T(t_0, \alpha) > 0)(\forall x_0 \in \mathbb{R}^n, h_0(t_0, x_0) \leq \alpha) \\ (\forall t \geq t_0 + T) : h(t, x(t; t_0, x_0)) < B.$$

e) (h_0, h)-uniformly ultimately bounded for bound B if the number T of d) does not depend on $t_0 \in \mathbb{R}_+$.

Definition 2. Let the function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be measurable. We say that $\lambda(t)$ is integrally positive if $\int_I \lambda(t)\, dt = \infty$ whenever $I = \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i]$, $\alpha_i < \beta_i < \alpha_{i+1}$ and $\beta_i - \alpha_i \geq \delta > 0$.

We shall introduce the class $\mathcal{V}_0$ of pieceiwse continuous auxiliary functions which are an analogue of Lyapunov's functions [**3**].

Let $\tau_0(x) = 0$ for $x \in \mathbb{R}^n$. Consider the sets

$$G_R = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : \tau_{R-1}(x) < t < \tau_R(x)\} \text{ and } G = \bigcup_{R=1}^{\infty} G_R$$

Definition 3. We say that the function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ belongs to the class $\mathcal{V}_0$ if $V(t, x)$ is continuous in G, locally Lipschitz continuous with respect to x in any of the sets G_R and for $(t_0, x_0) \in \sigma_R$, $R = 1, 2, \ldots$ there exist the limits

$$V(t_0^-, x_0) = \lim_{\substack{(t,x) \to (t_0, x_0) \\ (t,x) \in G_R}} V(t, x) \quad , \quad V(t_0^+, x_0) = \lim_{\substack{(t,x) \to (t_0, x_0) \\ (t,x) \in G_{R+1}}} V(t, x)$$

and, moreover, the equality $V(t_0^-, x_0) = V(t_0, x_0)$ holds.

Let $V \in \mathcal{V}_0$. For $(t, x) \in G$ define the function

$$\dot{V}_{(1)}(t, x) = \limsup_{h \to 0+} \frac{1}{h} [V(t + h, x + hf(t, x)) - V(t, x)].$$

We shall note that if $x = x(t)$ is a solution of system (1), then

$$\dot{V}_{(1)}(t, x(t)) = D^+ V(t, x(t)) = \limsup_{h \to 0+} \frac{1}{h}[V(t+h, x(t+h)) - V(t, x(t))]$$

for $t \neq t_R$ where $t_R = \tau_R(x(t_R))$.

Definition 4. Let $h_0, h \in \Gamma$. The function $V \in \mathcal{V}_0$ is called:

a) h-radially unbounded if there exists a function $a \in K$, $a(\gamma) \to \infty$ as $\gamma \to \infty$ and such that $V(t^+, x) \geq a(h(t, x))$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

b) h_0-decrescent if there exist $\delta > 0$ and a function $b \in K$ such that $V(t^+, x) \leq b(h_0(t, x))$ for $h_0(t, x) < \delta$.

c) weakly h_0-decrescent if there exist $\delta > 0$ and a function $b \in CK$ such that from $h_0(t, x) < \delta$ it follows that $V(t^+, x) \leq b(t, h_0(t, x))$.

Let $h, h_0 \in \Gamma$ and $V, W \in \mathcal{V}_0$. For the sake of brevity of the formulation of the main results we shall make a list of some conditions to be used in the formulation of the subsequent theorems.

A. If for the solution $x(t; t_0, x_0)$ of system (1) there exists $\delta_0 > 0$ such that $h(t, x(t; t_0, x_0)) \leq \delta_0 < \infty$ for each $t \in J^+(t_0, x_0)$, then $x(t; t_0, x_0)$ is defined in the interval (t_0, ∞).

B1. The function V is h-radially unbounded.

B2. $\dot{V}_{(1)}(t, x) \leq 0$ for $(t, x) \in G$.

B3. $\dot{V}_{(1)}(t, x) \leq -CV(t, x)$ for $(t, x) \in G$ where $C > 0$ is a constant.

B4. $\dot{V}_{(1)}(t, x) \leq -\lambda(t) C(h(t, x))$ for $(t, x) \in G$ where $\lambda(t)$ is integrally positive and $C \in K$.

B5. $\dot{V}_{(1)}(t, x) \leq -C(W(t, x)) + \lambda(t)\phi(V(t, x))$ for $(t, x) \in G$ where $C(\gamma)$ is nonnegative and continuous in $\mathbb{R}$ and

$$\liminf_{\gamma \to \infty} C(\gamma) > 0 \tag{2}$$

$\lambda(t)$ is nonnegative and continuous in $\mathbb{R}_+$ and

$$\int_0^\infty \lambda(t)\, dt < \infty \tag{3}$$

$\phi(u)$ is positive and continuous in $\mathbb{R}$ and

$$\int_0^\infty \frac{du}{\phi(u)} = \infty. \tag{4}$$

B6. There exists a constant K such that

$$V(t, x) \geq K \text{ for any } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{5}$$

B7. $V(t^+, x + I_R(x)) \leq V(t, x)$ for $(t, x) \in \sigma_R$, $R = 1, 2, \ldots$

C1. $|\dot{W}_{(1)}(t,x)| \le \mu(t)\omega(W(t,x))$ for $(t,x) \in G$ where $\mu(t)$ is nonnegative and continuous in $\mathbb{R}_+$ and

$$\int_s^t \mu(\tau)\,d\tau \le m(t-s) \text{ for } t \ge s \ge 0 \tag{6}$$

where $m(\gamma) \in K$ and $\omega(u)$ is positive and continuous in $\mathbb{R}$ and

$$\int_0^\infty \frac{du}{\omega(u)} = \infty. \tag{7}$$

C2. $\dot{W}_{(1)}(t,x) \le \mu(t)\omega(W(t,x))$ for $(t,x) \in G$ where $\mu(t)$ and $\omega(u)$ are the functions of condition C1.

C3. There exists a function $m \in K$ such that for $t \ge s \ge 0$ and for any piecewise continuous in $[s,t]$ function $u(\tau)$ with points of discontinuity of the first kind t_R such that $t_R = \tau_R(u(t_R))$ at which $u(\tau)$ is continuous from the left, the following inequality holds

$$\left|\int_s^t \dot{W}_{(1)}(\tau, u(\tau))\,d\tau\right| \le m(t-s). \tag{8}$$

C4. $W(t^+, x + I_R(x)) = W(t,x)$ for $(t,x) \in \sigma_R$.

C5. $W(t,x)$ is h-radially unbounded.

3. Main results

Theorem 1. *Let condition (A) hold and function $V \in \mathcal{V}_0$ exist for which conditions B1, B2 and B7 hold. Then the solutions of system (1) are:*

1. *(h_0,h)-equibounded if V is weakly h_0-decrescent.*
2. *(h_0,h)-uniformly bounded if V is h_0-decrescent.*

Proof: Since V is h-radially unbounded, then there exists a function $a \in K$, $a(\gamma) \to \infty$ as $\gamma \to \infty$ and such that

$$V(t^+,x) \ge a(h(t,x)) \text{ for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{9}$$

1. If V is weakly h_0-decrescent, then there exist $\delta_0 > 0$ and a function $b \in CK$ such that

$$V(t^+,x) \le b(t, h_0(t,x)) \text{ for } h_0(t,x) < \delta_0 \tag{10}$$

Let $\alpha > 0$ and $t_0 \in \mathbb{R}_+ (\alpha < \delta_0)$ be given. Choose $\beta = \beta(t_0,\alpha) > 0$ so that

$$a(\beta) > b(t_0,\alpha) \tag{11}$$

Let $x_0 \in \mathbb{R}^n$, $h_0(t_0, x_0) \leq \alpha$ and let $x(t) = x(t; t_0, x_0)$. Set $v(t) = V(t, x(t))$. Since $V(t,x)$ is locally Lipschitz continuous in any of the sets G_R, then from B2 it follows that $D^+v(t) \leq 0$ for $t \in J^+(t_0, x_0)$, $t \neq t_R$ where $t_R = \tau_R(x(t_R))$. From B7 it follows that $v(t_R^+) \leq v(t_R)$. That is why the function $v(t)$ is decreasing in the interval $J^+(t_0, x_0)$. Then from (9), (10) and (11) we get

$$a(h(t, x(t)) \leq v(t^+) \leq v(t) \leq v(t_0^+) \leq b(t_0, h_0(t_0, x_0)) \leq b(t_0, \alpha) < a(\beta)$$

for $t \in J^+(t_0, x_0)$ which implies that $h(t, x(t)) < \beta$. From condition (A) it follows that $J^+(t_0, x_0) = (t_0, \infty)$.

Thus 1, is proved.

2. If V is h_0-decrescent, then (10) and (11) hold for some function $b \in K$ independent of t. Hence the number β can be chosen independent of t_0 and so that for $h_0(t_0, x_0) \leq \alpha$ we have $h(t, x(t)) < \beta$. This shows that the solutions of system (1) are (h_0, h)-uniformly bounded.

Theorem 1 is proved. ■

Corollary 1. *Let condition (A) hold and function $U \in \mathcal{V}_0$ exist which is h-radially unbounded and such that $\dot{U}_{(1)}(t,x) \leq \lambda(t)\phi(U(t,x))$ for $(t,x) \in G$ where the function $\lambda(t)$ is nonnegative and continuous in $\mathbb{R}_+$ and $\int_0^\infty \lambda(t)\,dt < \infty$ and $\phi(u)$ is positive and continuous in $\mathbb{R}$ and $\int_0^\infty du/\phi(u) = \infty$,*

$$U(t^+, x + I_R(x)) \leq U(t,x) \text{ for } (t,x) \in \sigma_R,\ R = 1, 2, \ldots$$

Then the solutions of system (1) are:

1. *(h_0, h)-equibounded if U is weakly h_0-decrescent.*
2. *(h_0, h)-uniformly bounded if U is h_0-decrescent.*

Proof: It is immediately verified that the function

$$V(t,x) = \exp\left\{-\int_0^t \lambda(s)\,ds + \Phi(U(t,x))\right\},\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

where $\Phi(u) = \int_0^u du/\phi(u)$ satisfies the conditions of Theorem 1. ■

Theorem 2. *Let condition (A) hold and a function $V \in \mathcal{V}_0$ exist which is weakly h_0-decrescent and for which conditions B1, B3 and B7 hold. Then the solutions of system (1) are (h_0, h)-equi-ultimately bounded.*

Proof: From Theorem 1 it follows that the solutions of system (1) are (h_0, h)-equibounded. Hence each solution $x(t) = x(t; t_0, x_0)$ of (1) is defined in the interval (t_0, ∞).

Since V is h-radially unbounded, then there exist $B > 0$ and $a \in K$, $a(\gamma) \to \infty$ as $\gamma \to \infty$ such that

$$V(t^+, x) \geq a(h(t, x)) \text{ for } h(t, x) \geq B \tag{12}$$

Since V is weakly h_0-decrescent, then there exist $\delta_0 > 0$ and $b \in CK$ such that (10) holds.

Let $\alpha > 0$ and $t_0 \in \mathbb{R}_+$ be given, $x_0 \in \mathbb{R}^n$ be such that $h_0(t_0, x_0) \leq \alpha$ and let $x(t) = x(t; t_0, x_0)$. From B3 and B7 we obtain

$$V(t, x(t)) \leq V(t_0^+, x_0)\exp[-C(t - t_0)] \text{ for } t > t_0. \tag{13}$$

Set $T = T(t_0, \alpha) > \frac{1}{C}\ln[b(t_0, \alpha)/a(B)]$. Then from (12) and (13) it follows that for $t \geq t_0 + T$ the following inequalities hold

$$\begin{aligned} a(h(t, x(t)) &\leq V(t^+, x(t^+)) \leq V(T, x(t)) \leq \\ &\leq V(t_0^+, x_0)\exp[-C(t - t_0)] \leq b(t_0, h_0(t_0, x_0))\exp(-CT) < a(B) \end{aligned}$$

Hence $h(t, x(t)) < B$ for $t \geq t_0 + T$.

Theorem 2 is proved. ■

Theorem 3. *Let condition (A) hold and a function $V \in \mathcal{V}_0$ exist which is h_0-decrescent and for which conditions B1, B4 and B7 hold. Then the solutions of system (1) are (h_0, h)-uniformly ultimately bounded.*

Proof: From Theorem 1 it follows that the solutions of system (1) are (h_0, h)-uniformly bounded. Hence each solution $x(t) = x(t; t_0, x_0)$ of (1) is defined in the interval (t_0, ∞).

Since V is h-radially unbounded, then there exist $R > 0$ and $a \in K$, $a(\gamma) \to \infty$ as $\gamma \to \infty$ such that

$$V(t^+, x) \geq a(h(t, x)) \text{ for } h(t, x) \geq R \tag{14}$$

Since V is h_0-decrescent, then there exist $\delta_0 > 0$ and $b \in K$ such that

$$V(t^+, x) \leq b(h_0(t, x)) \text{ for } h_0(t, x) < \delta_0. \tag{15}$$

Choose $B \geq R$ so that $a(B) > b(R)$. Let $\alpha \geq R$ be given. We shall prove that there exists $T = T(\alpha) > 0$ such that for any solution $x(t) = x(t; t_0, x_0)$ of system (1) for which $h_0(t_0, x_0) \leq \alpha$ and for some $\zeta \in [t_0, t_0 + T]$ the following inequality holds

$$h_0(\zeta, x(\zeta)) < R \tag{16}$$

Suppose that this is not true. Then for any $T > 0$ there exists a solution $x(t) = x(t; t_0, x_0)$ of (1) for which $h_0(t_0, x_0) \leq \alpha$ and such that for all $t \in [t_0, t_0 + T]$ we have

$$h_0(t, x(t)) \geq R \tag{17}$$

From B4 and B7 it follows that

$$\begin{aligned} V(t, x(t)) - V(t_0^+, x_0) \leq \int_{t_0}^{t} \dot{V}_{(1)}(s, x(s))\, ds \leq \\ - \int_{t_0}^{t} \lambda(s) C(h_0(s, x(s)))\, ds,\ t > t_0 \end{aligned} \tag{18}$$

But the function $V(t, x(t))$ is monotonely decreasing in the interval (t_0, ∞). Hence there exists the limit

$$\lim_{t \to \infty} V(t, x(t)) = V_0 \geq 0 \tag{19}$$

Then from (15), (17), (18) and (19) we obtain

$$\int_{t_0}^{\infty} \lambda(t) C(h_0(t, x(t))\, dt \leq b(R) - V_0$$

From the integral positivity of $\lambda(t)$ it follows that there exists $T > 0$ such that

$$\int_{t_0}^{t_0+T} \lambda(t)\, dt > \frac{b(R) - V_0 + 1}{C(R)}$$

Then

$$\begin{aligned} b(R) - V_0 \geq \int_{t_0}^{\infty} \lambda(t) C(h_0(t, x(t)))\, dt \geq \int_{t_0}^{t_0+T} \lambda(t) C(h_0(t, x(t)))\, dt \geq \\ \geq C(R) \int_{t_0}^{t_0+T} \lambda(t)\, dt > b(R) - V_0 + 1. \end{aligned}$$

The contradiction obtained shows that there exists $T = T(\alpha) > 0$ such that for any solution $x(t) = x(t; t_0, x_0)$ of (1) for which $h_0(t_0, x_0) \leq \alpha$, there exists $\zeta \in [t_0, t_0 + T]$ such that (16) holds. Then for $t \geq \zeta$ (hence for any $t \geq t_0 + T$ too) the following inequalities hold

$$\begin{aligned} a(h(t, x(t)) \leq V(t^+, x(t^+)) \leq V(t, x(t)) \leq V(\zeta^+, x(\zeta^+)) \leq \\ \leq b(h_0(\zeta, x(\zeta)) \leq b(R) < a(B). \end{aligned}$$

Hence the solutions of system (1) are (h_0, h)-uniformly ultimately bounded for bound B. ■

Theorem 4. *Let condition (A) hold and functions $V, W \in \mathcal{V}_0$ exist for which conditions B5, B6, B7, C1, C4 and C5 hold. Then:*

1. *V is h-radially unbounded.*
2. *The solutions of system (1) are h-ultimately bounded.*
3. *If V is weakly h_0-decrescent, then the solutions of system (1) are (h_0, h)-equibounded.*
4. *If V is h_0-decrescent, then the solutions of system (1) are (h_0, h)-uniformly bounded.*

Proof:

1. Assume that the assertion is not true. Then there exists $N_0 > 0$ such that for any $\gamma > 0$ there exist $\tau \in \mathbb{R}_+$ and $\overline{x} \in \mathbb{R}^n$ for which $h(\tau, \overline{x}) > \gamma$ and such that $V(\tau^+, \overline{x}) \leq N_0$.

From (2) it follows that there exist $R_1 > 0$ and $\delta > 0$ such that for any $\gamma \geq R_1$ we have $C(\gamma) \geq \delta$.

Let $L = \int_0^\infty \lambda(t)\, dt$ and $M = \sup\{\phi(u) : K \leq u \leq \Phi^{-1}(\Phi(N_0) + L)$ where $\Phi(u) = \int_0^u du/\phi(u)$.

From C5 it follows that there exists a function $a \in K$, $a(\gamma) \to \infty$ as $\gamma \to \infty$ and such that

$$W(t^+, x) \geq a(h(t,x)) \text{ for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{20}$$

From (4) and the condition $a(\gamma) \to \infty$ as $\gamma \to \infty$ it follows that there exists $R_2 > R_1$ such that $a(R_2) > R_1$ and

$$\int_{R_1}^{a(R_2)} \frac{d\gamma}{\omega(\gamma)} > m\left(\frac{N_0 - K + ML}{\delta}\right) \tag{21}$$

In the above assumption we replace γ by R_2. As a result we obtain that there exist $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$ such that $h(t_0, x_0) > R_2$ and $V(t_0^+, x_0) \leq N_0$.

From condition (A) and from C5, C1 and C4 it follows that the solution $x(t) = x(t; t_0, x_0)$ of system (1) is defined in the interval (t_0, ∞).

From B5 and B7 it follows that $\dot{V}_{(1)}(t, x(t)) \leq \lambda(t)\phi(V(t, x(t))$ for $t \neq t_R$ where $t_R = \tau_R(x(t_R))$ and $V(t_R^+, x(t_R^+)) \leq V(t_R, x(t_R))$, whence by integration we obtain

$$\Phi(V(t, x(t)) - \Phi(V(t_0^+, x_0)) \leq \int_{t_0}^t \frac{\dot{V}_{(1)}(s, x(s))}{\phi(V(s, x(s)))}\, ds \leq L$$

Hence $K \leq V(t, x(t)) \leq \Phi^{-1}(\Phi(N_0) + L)$, whence we conclude that $\phi(V(t, x(t))) \leq M$ for $t > t_0$.

Assume that $W(t, x(t)) > R_1$ for any $t > t_0$. Then from B5 and B7 it follows that

$$\dot{V}_{(1)}(t, x(t)) \leq -\delta + M\lambda(t) \text{ for } t > t_0,\ t \neq t_R \tag{22}$$

$$V(t_R^+, x(t_R^+)) \leq V(t_R, x(t_R)), \tag{23}$$

whence by integration we obtain

$$V(t,x(t)) - V(t_0^+,x_0) \le \int_{t_0}^{t} \dot V_{(1)}(s,x(s))\,ds \le -\delta(t-t_0) + M\int_{t_0}^{t}\lambda(s)\,ds$$

Then

$$V(t,x(t)) \le N_0 - \delta(t-t_0) + ML,\ t > t_0. \tag{24}$$

But the right-hand side of (24) tends to $-\infty$ as $t \to \infty$ and this contradicts B6. Hence there exist values of $t > t_0$ for which $W(t,x(t)) \le R_1$. From condition C4 it follows that the function $W(t,x(t))$ is continuous, hence there exists $\zeta > t_0$ such that $W(\zeta,x(\zeta)) = R_1$ and $W(t,x(t)) > R_1$ for $t \in (t_0,\zeta)$.

Since inequalities (22) and (23) are satisfied for $t \in (t_0,\zeta)$, then

$$V(\zeta,x(\zeta)) \le N_0 - \delta(\zeta - t_0) + ML \tag{25}$$

From (21), (20), conditions C1 and C4 and (6) we obtain

$$\begin{aligned} m\left(\frac{N_0 - K + ML}{\delta}\right) &< \int_{R_1}^{a(R_2)} d\gamma/\omega(\gamma) \le \left|\int_{W(t_0^+,x_0)}^{W(\zeta,x(\zeta))} d\gamma/\omega(\gamma)\right| \le \\ &\le \left|\int_{t_0}^{\zeta} \frac{\dot W_{(1)}(t,x(t))}{\omega(W(t,x(t)))}\,dt\right| \le \int_{t_0}^{\zeta}\mu(t)\,dt \le m(\zeta - t_0). \end{aligned}$$

Hence

$$(N_0 - K + ML)/\delta < \zeta - t_0. \tag{26}$$

From inequalities (25) and (26) we obtain that $V(\zeta,x(\zeta)) < K$ which contradicts B6. Thus assertion 1 is proved.

2. Suppose that the solutions of system (1) are not h-ultimately bounded. Then there exist $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, a solution $x(t) = x(t;t_0,x_0)$ of (1) and a sequence $\{\zeta_R\}$ such that $\zeta_R \to \infty$ as $R \to \infty$ and $h(\zeta_R,x(\zeta_R)) \ge a^{-1}(R_1)$ where R_1 is the constant defined in the proof of assertion 1. From the h-radial unboundedness of W we obtain $W(\zeta_R,x(\zeta_R)) \ge R_1$.

From (2) it follows that there exists R_0, $0 < R_0 < R_1$ such that for $\gamma \ge R_0$ we have $C(\gamma) \ge \frac{\delta}{2}$ where δ is the constant defined in the proof of assertion 1.

As in the proof of assertion 1 we can find a sequence $\{\eta_R\}$ such that $\eta_R \to \infty$ as $R \to \infty$ and $W(\eta_R,x(\eta_R)) \le R_0$. Choose subsequences of the sequences $\{\zeta_R\}$ and $\{\eta_R\}$ which we denote again by $\{\zeta_R\}$ and $\{\eta_R\}$, such that $\eta_R < \zeta_R < \eta_{R+1}, \eta_R \to \infty$ as $R \to \infty$ and

$$\begin{aligned} &W(\eta_R,x(\eta_R)) = R_0,\ W(\zeta_R,x(\zeta_R)) = R_1 \\ &R_0 \le W(t,x(t)) \le R_1 \text{ for } t \in [\eta_R,\zeta_R] \end{aligned} \tag{27}$$

We shall prove that

$$\sum_{R=1}^{\infty}(\zeta_R - \eta_R) = \infty \tag{28}$$

Indeed, if we supppose that $(\zeta_R - \eta_R) \to 0$ as $R \to \infty$, then from C1, C4 and (27) we obtain

$$0 < \int_{R_0}^{R_1} \frac{d\gamma}{\gamma} \leq \int_{\eta_R}^{\zeta_R} \frac{\dot{W}_{(1)}(t, x(t))}{\omega(W(t, x(t)))}\, dt \leq m(\zeta_R - \eta_R) \to 0$$

as $R \to \infty$. The contradiction obtained shows that (28) holds.

If we set $\tilde{M} = \sup\{\phi(u) : K \leq u \leq \Phi^{-1}(\Phi(V(t_0^+, x_0)) + L)$, as in the proof of assertion 1 we can prove that $\phi(V(t, x(t))) \leq \tilde{M}$ for $t > t_0$. Then from B5 and B7 it follows that

$$V(\zeta_n, x(\zeta_n)) - V(t_0^+, x_0) \leq \int_{t_0}^{\zeta_n} \dot{V}_{(1)}(t, x(t))\, dt \leq -\int_{t_0}^{\zeta_n} C(W(t, x(t)))\, dt + \tilde{M}L.$$

Hence

$$V(\zeta_n, x(\zeta_n)) \leq V(t_0^+, x_0) - \sum_{R=1}^{n} \int_{\eta_R}^{\zeta_R} C(W(t, x(t)))\, dt + \tilde{M}L \leq$$

$$\leq V(t_0^+, x_0) + \tilde{M}L - \frac{\delta}{2}\sum_{R=1}^{n}(\zeta_R - \eta_R).$$

From (28) it follows that the right-hand side of last inequality tends to $-\infty$ as $n \to \infty$ which contradicts B6. Hence the solution of (1) are h-ultimately bounded.

3. Let V be weakly h_0-decrescent. Then condition B5 and assertion 1 proved above show that the conditions of Corollary 1 are satisfied. Hence the solutions of system (1) are (h_0, h)-equibounded.

4. is proved in the same way.

Thus Theorem 4 is proved. ■

Theorem 5. *Let condition (A) hold and functions $V, W \in \mathcal{V}_0$ exist for which conditions B5, B6, B7, C2, C4 and C5 hold. Then the solutions of system (1) are h-ultimately bounded.*

If, moreover, V and W are weakly h_0-decrescent, then the solutions of system (1) are (h_0, h)-equibounded.

Proof: Conditions C2, C4 and C5 and (A) imply the global existence of the solutions of system (1).

The h-ultimate boundedness of the solutions of (1) is proved as in the proof of assertion 2 of Theorem 4. That is why we shall prove only the second part of Theorem 5.

Suppose that the solutions of system (1) are not (h_0, h)-equibounded. Then there exist $\alpha_0 > 0$ and $t_0 \in \mathbb{R}_+$ such that for any $\beta > 0$ there exists $\overline{x} \in \mathbb{R}^n$ for which $h_0(t_0, \overline{x}) \leq \alpha_0$, a solution $x(t; t_0, \overline{x})$ of (1) and $T > 0$ such that $h(T, x(T; t_0, \overline{x})) \geq \beta$.

Let R_1, δ and L be the constants defined in the proof of Theorem 4.

Since V and W are weakly h_0-decrescent, then there exist $b, b_1 \in CK$ such that $V(t^+, x) \leq b(t, h_0(t,x))$ and $W(t^+, x) \leq b_1(t, h_0(t,x))$. Then for $h_0(t_0, \overline{x}) \leq \alpha_0$ we have $V(t_0^+, \overline{x}) \leq b(t_0, \alpha_0)$ and $W(t_0^+, \overline{x}) \leq b_1(t_0, \alpha_0)$. We set $N = \max\{b(t_0, \alpha_0), b_1(t_0, \alpha_0)\}$ and $M = \sup\{\phi(u) : K \leq u \leq \Phi^{-1}(\Phi(N) + L)$ where $\Phi(u) = \int_0^t d\gamma/\phi(\gamma)$.

Condition C5 implies the existence of a function $a \in K$, $a(\gamma) \to \infty$ as $\gamma \to \infty$ such that $W(t^+, x) \geq a(h(t,x))$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

From (4) and the condition $a(\gamma) \to \infty$ as $\gamma \to \infty$ it follows that we can choose $\beta_0 > \alpha_0$ such that $a(\beta_0) > N$ and

$$m\left(\frac{N - K + ML}{\delta}\right) < \int_N^{a(\beta_0)} \frac{d\gamma}{\omega(\gamma)}. \tag{29}$$

We replace in the above assumption β by β_0. As a result we obtain that there exists $x_0 \in \mathbb{R}^n$ for which $h_0(t_0, x_0) \leq \alpha_0$, a solution $x(t) = x(t; t_0, x_0)$ of system (1) and $t_3 > t_0$ such that $h(t_3, x(t_3)) \geq \beta_0$. Then $W(t_3, x(t_3)) \geq a(h(t_3, x(t_3))) \geq a(\beta_0)$. Moreover, it is clear that $V(t_0^+, x_0) \leq N$ and $W(t_0^+, x_0) \leq N$.

From condition C4 it follows that the function $W(t, x(t))$ is continuous, hence there exist t_1, t_2, $t_0 < t_1 < t_2 \leq t_3$ such that $W(t_1, x(t_1)) = N$, $W(t_2, x(t_2)) = a(\beta_0)$ and $N < W(t, x(t)) < a(\beta_0)$ for $t \in (t_1, t_2)$.

As in the proof of assertion 1 of Theorem 4 it can be proved that $\phi(V(t, x(t))) \leq N$ for $t > t_0$.

Then from conditions B5 and B7 we obtain

$$V(t_2, x(t_2)) - V(t_0^+, x_0) \leq \int_{t_0}^{t_2} \dot{V}_{(1)}(t, x(t))\, dt \leq -\int_{t_0}^{t_2} C(W(t, x(t)))\, dt,$$

whence it follows that

$$V(t_2, x(t_2)) \leq N - \delta(t_2 - t_1) + ML \tag{30}$$

From conditions C2 and C4 and from (29) it follows that

$$\begin{aligned} m\left(\frac{N - K + ML}{\delta}\right) &< \int_N^{a(\beta_0)} \frac{d\gamma}{\omega(\gamma)} \leq \\ &\leq \int_{t_1}^{t_2} \frac{\dot{V}_{(1)}(t, x(t))}{\omega(W(t, x(t)))}\, dt \leq \int_{t_1}^{t_2} \mu(t)\, dt \leq m(t_2 - t_1) \end{aligned}$$

whence we obtain

$$(N - K + ML)/\delta < t_2 - t_1 \tag{31}$$

From (30) and (31) we get that $V(t_2, x(t_2)) < K$ which contradicts B6. Hence the solutions of system (1) are (h_0, h)-equibounded. ■

Theorem 6. *Let condition (A) hold and functions $V, W \in \mathcal{V}_0$ exist for which conditions B5, B6, B7, C3, C4 and C5 hold. Then assertions 1, 2, 3 and 4 of Theorem 4 are valid.*

Proof:

1. Suppose that V is not h-radially unbounded. Then there exists $N_0 > 0$ such that for any $\gamma > 0$ there exist $\tau \in \mathbb{R}_+$ and $\overline{x} \in \mathbb{R}^n$ such that $h(\tau, \overline{x}) > \gamma$ and $V(\tau^+, \overline{x}) \leq N_0$.

Let R_1, δ, L and M be the constants defined in the proof of Theorem 4 and $a \in K$, $a(\gamma) \to \infty$ as $\gamma \to \infty$ be such that (20) holds.

Choose $R_2 > R_1$ so that

$$a(R_2) > R_1 + m\left(\frac{N_0 - K + ML}{\delta}\right) \tag{32}$$

Let $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$ be such that $h(t_0, x_0) > R_2$ and $V(t_0^+, x_0) \leq N_0$ and let $x(t) = x(t; t_0, x_0)$.

As in the proof of Theorem 4 it is proved that there exists $\zeta > t_0$ such that $W(\zeta, x(\zeta)) = R_1$ and $W(t, x(t)) > R_1$ for $t \in (t_0, \zeta)$ and

$$V(\zeta, x(\zeta)) \leq N_0 - \delta(\zeta - t_0) + ML. \tag{33}$$

Moreover, $W(t_0^+, x_0) \geq a(h(t_0, x_0)) > a(R_2)$. Then from (32) and C3 it follows that

$$m\left(\frac{N_0 - K + ML}{\delta}\right) < a(R_2) - R_1 \leq W(t_0^+, x_0) - W(\zeta, x(\zeta)) \leq$$
$$\leq \left| \int_{t_0}^{\zeta} \dot{W}_{(1)}(t, x(t))\, dt \right| \leq m(\zeta - t_0).$$

Hence

$$(N_0 - K + ML)/\delta < \zeta - t_0. \tag{34}$$

From (33) and (34) it follows that $V(\zeta, x(\zeta)) < K$ which contradicts B6. Hence V is h-radially unbounded.

The proof of assertions 2, 3 and 4 is carried out as in the proof of Theorem 4. ■

Theorem 7. *Let the following conditions be fulfilled:*

a) *Condition (A) holds.*

b) *There exist functions $V, W \in V_0$ which satisfy conditions B5, B6, B7, C4 and C5.*

c) *The function h is locally Lipschitz continuous with respect to x.*

d) $|\limsup_{s\to 0^+} \frac{1}{s}[h(t+s, x+sf(t,x)) - h(t,x)]| \le \mu(t)\omega(h(t,x))$ *for* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, *where* $\mu(t)$ *and* $\omega(\gamma)$ *are the functions of condition C1.*

Then the assertions 1-4 of Theorem 4 are valid.

The proof of Theorem 7 is analogous to the proof of Theorem 4.

Theorem 8. *Let the conditions of Theorem 7 hold, condition d) being replaced by condition e):*

e) *There exists $m \in K$ such that for $t \ge s \ge 0$ and for any piecewise continuous in $[s,t]$ function $u(\tau)$ with points of discontinuity of the first kind t_R where $t_R = \tau_R(u(t_R))$ at which it is continuous from the left, the following inequality holds*

$$\left| \int_s^t \{\limsup_{s\to 0^+} \frac{1}{s}[h(\tau+s, u(\tau+s)) - h(\tau, u(\tau))]\}\, d\tau \right| \le m(t-s)$$

Then assertions 1-4 of Theorem 4 are valid.

The proof of Theorem 8 is analogous to the proof of Theorem 4.

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Rebut el 2 de Juny de 1989